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STEADY STATE MODEL OF CUSP LOSSES

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STEADY STATE MODEL OF CUSP LOSSES

by

J.B. TAYLOR

A B S T R A C T

When space charge fields can be neglected the loss of charged particles from a cusped magnetic field can be calculated exactly from a steady state statistical model without any restrictive assumptions about the form of the equilibrium configuration. Losses calculated in this way represent an upper limit for the loss from a Maxwellian plasma. In a spindle shaped cusp it is found that the loss through the line (ring) cusp corresponds, as expected, to a leak whose width is approximately a Larmor radius. However a comparable loss occurs through the point cusp, where it corresponds to a leak which is much larger than a Larmor radius. This enlargement is not very important for the spindle cusp itself, which in any case has a large loss from the ring cusp, but a similar enlargement could greatly increase the end-losses from a high-β theta-pinch.

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INTRODUCTION

The loss of plasma in a real Cusp experiment is partly determined by the electric field in space charge sheaths or boundary layers at the plasma-field interface. It is unlikely that a rigorous theoretical description can be given of this complicated situation for in addition to a self-consistent calculation of the electric field this would need a knowledge of the complex and variable electrical properties of the vacuum chamber walls under plasma bombardment. The solution may also depend on the exact way in which the plasma-field boundary was first created since it appears that even the simple plane plasma-field boundary is not uniquely determined by the state of the internal plasma but depends on the density of trapped electrons[1] which in turn depends on how the plasma formed.

On the other hand, it may be that in some circumstances the space charge electric fields are short-circuited or automatically compensated by slow electrons, in which case the losses are determined purely by the magnetic field and can be treated on an individual particle basis. The problem then becomes more straightforward and susceptible to theoretical analysis.

Several estimates have been given for the cusp losses in the absence of electric fields. In the case of a sharp plasma-field interface the effective size of the "leak" at a line Cusp was calculated by Berkowitz et. al.,[2] using a generalised adiabatic invariant, to be a Larmor radius - a result which seems intuitively obvious. This same calculation, and intuition, would lead one to expect that the leak at a point Cusp would also be determined by the Larmor radius; if so then in the axisymmetric spindle Cusp (Fig.1) the leak at the points $\sim \pi a^2$ would be negligible[3] compared to the leak at the line Cusp $\sim 2\pi Ra$.

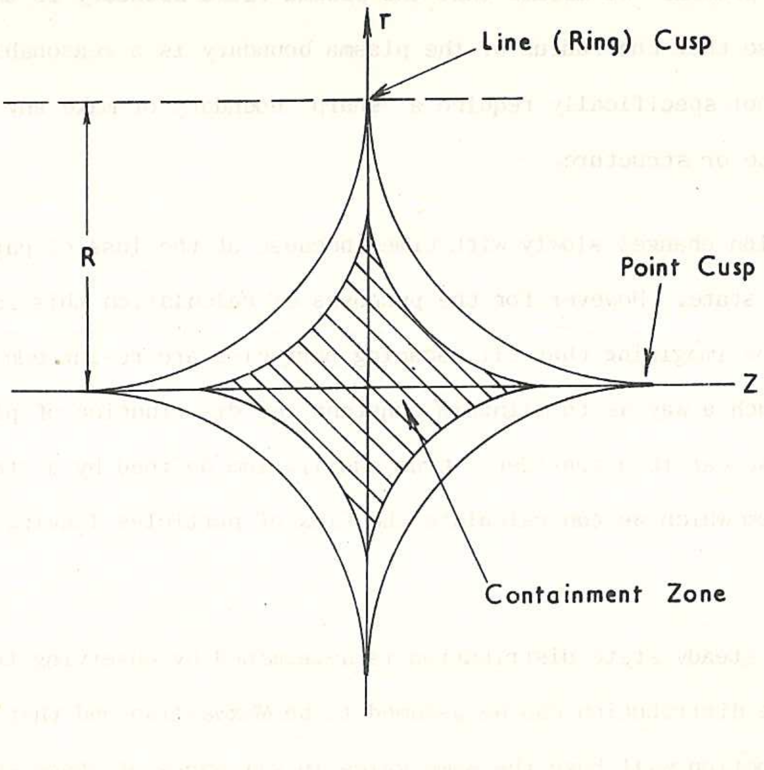

FIG. 1

However Grad[4] has also roughly estimated the cusp losses by assuming that there is a critical flux surface outside which particles behave adiabatically but within which they are strongly non-adiabatic. In this model it is this critical flux surface which determines the size of the leak which is therefore roughly the same at the point and line cusps. This concept of a critical flux tube would appear to be applicable mainly to the case where field and plasma are well intermixed but it has also been suggested[5] that approximate equality of the losses at point and line cusps is a general phenomena.

It seems in fact that there is still some uncertainty about the individual particle losses from a Cusp even when space charge effects are neglected. Accordingly it seems worth while to calculate the losses exactly using a well defined model. This can be done using a statistical model similar to that used by Robson and Taylor[6] to discuss non-adiabatic effects in mirror systems. A similar model was used by Tamor[7] for a more restricted calculation of cusp losses and there are similarities with the work of Firsov[8] who was, however, concerned with the influence of electric fields. As will become apparent, losses calculated from this model represent an upper limit to the real loss rate.

CALCULATION OF CUSP LOSSES

We consider an axisymmetric spindle Cusp (Fig.1); we make no special assumptions about the nature of the plasma equilibrium except for the existence of the ring and point Cusps and the existence of an almost field free (high β) containment zone near the centre containing most of the plasma. We assume that the plasma-field boundary is thin compared to the plasma radius (so that the radius of the plasma boundary is a reasonably well defined concept) but we do not specifically require a "sharp" boundary or make any special assumptions about its shape or structure.

The real situation changes slowly with time (because of the loss of particles) and is only a quasi steady state. However for the purposes of calculation this is replaced by a true steady state, by imagining that all escaping particles are re-introduced into the containment zone, in such a way as to maintain constant the distribution of plasma within the containment zone. We can then consider a true equilibrium defined by a stationary distribution function, from which we can calculate the flux of particles leaving the containment zone.

The appropriate steady state distribution is determined by observing that within the containment zone the distribution can be assumed to be Maxwellian and that in a steady state the distribution function will have the same value in all parts of phase space which are

INTRODUCTION

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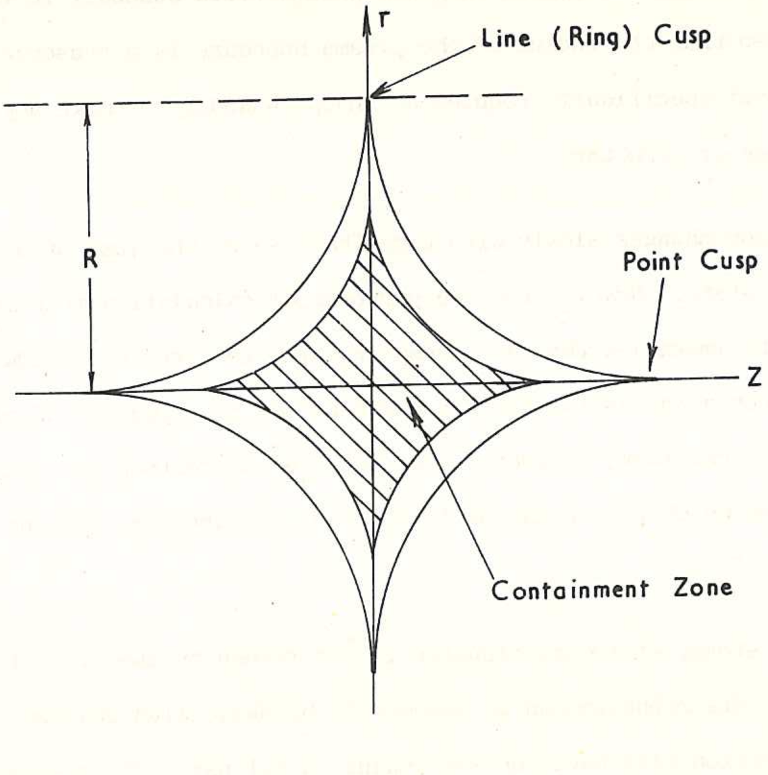

FIG. I

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The appropriate steady state distribution is determined by observing that within the containment zone the distribution can be assumed to be Maxwellian and that in a steady state the distribution function will have the same value in all parts of phase space which are accessible to each other (Lioville's Theorem). This does not, however, mean that the distribution is everywhere Maxwellian since not all of phase space is accessible to particles originating within the containment zone; outside the containment zone the distribution will be a truncated Maxwellian.

The accessible region of phase space is determined by the constants of motion; in an axisymmetric system these are the Hamiltonian

$$H = \tfrac{1}{2}\left\{p_r^2 + p_z^2 + \left(\frac{p_\theta}{r} - \frac{eA_\theta}{c}\right)^2\right\} \qquad \dots (1)$$

and the angular momentum

$$p_\theta = r\left(v_\theta + \frac{eA_\theta}{c}\right) . \qquad \dots (2)$$

[The mass of the particle is taken as unity and $A_\theta(r,z)$ is the only non-zero component of the vector potential. In the calculation it will be more convenient to work with the flux $\psi = \frac{e}{c}\, rA_\theta$ when

$$\frac{e}{c}B_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad \frac{e}{c}B_z = \frac{1}{r}\frac{\partial\psi}{\partial r} .] \qquad \dots (3)$$

Within the containment zone $\psi = 0$ and all particles originating within this region therefore satisfy the inequality

$$H > p_\theta^2/2r^2 > p_\theta^2/2R^2 \qquad \dots (4)$$

where R is the maximum radius of the containment zone. Since both H and p_θ are constants of motion only those regions of phase space satisfying

$$H > p_\theta^2/2R^2 \qquad \dots (5)$$

are accessible to particles from the containment zone.

With these preliminaries it is now a simple matter to calculate the flux through the cusps.

THE RING CUSP

The flux through one half (i.e. $z > 0$) of the ring cusp is

$$F_1 = 2\pi R \int_0^\infty dz \int_0^\infty v_r\, dv_r \int_{-\infty}^\infty dv_z \int_{-\infty}^\infty dv_\theta \exp - \frac{\beta}{2}\left(v_r^2 + v_z^2 + v_\theta^2\right) \qquad \dots (6)$$

where the integration is to be carried out at $r = R$ (we assume for simplicity that the radius of the ring cusp is the same as the maximum radius of the containment zone), and is restricted by eq.(5) to the region,

$$v_r^2 + v_z^2 > \frac{\psi^2}{R^2} + 2v_\theta \frac{\psi}{R} \; . \qquad \ldots (7)$$

Furthermore at the ring cusp $\frac{\psi}{R} = \frac{eB}{c} z$. The integrations are elementary; by the transformation $v_r = \rho \operatorname{Cos} \theta$, $v_z = \rho \operatorname{Sin} \theta$ one gets

$$F_1 = \frac{4\pi Rc}{eB} \int \frac{d\psi}{R} \int_0^\infty \rho^2 \, d\rho \int_{-\infty}^{\infty} dv_\theta \exp - \frac{\beta}{2} (\rho^2 + v_\theta^2) \qquad \ldots (8)$$

the range of integration being restricted to

$$0 < \frac{\psi}{R} < (\rho^2 + v_\theta^2)^{\frac{1}{2}} - v_\theta \; . \qquad \ldots (9)$$

The remaining integrations can now be performed directly to give

$$F_1 = \frac{2\pi Rc}{eB} \cdot \frac{3\pi}{\beta^2} \sqrt{\frac{\pi}{2\beta}} \; . \qquad \ldots (10)$$

To express this as an effective "hole size" one needs the density of plasma in the containment region

$$n = \int d^3v \, \exp \frac{-\beta}{2} \, v^2 = \left(\frac{2\pi}{\beta}\right)^{3/2} \; . \qquad \ldots (11)$$

Then equating the flux F_1 to the effusion rate ($\frac{1}{4} \, n \cdot \bar{c} \cdot 2\pi R \delta_L$) one finds an "effective leak width" at the line cusp of

$$\delta_L = \frac{3c}{2eB} \sqrt{\frac{\pi}{2\beta}} = \left(\frac{3\pi}{8}\right)^{\frac{1}{2}} a_L \qquad \ldots (12)$$

where a_L is the r.m.s. larmor radius $\frac{c}{eB}\sqrt{\frac{3}{\beta}}$. Thus, as expected, the width of the leak at the line cusp is closely equal to the larmor radius.

THE POINT CUSP

The flux through the point cusp is given by

$$F_2 = 2\pi \int_0^\infty rdr \int_0^\infty v_z \, dv_z \int_{-\infty}^{\infty} dv_r \int_{-\infty}^{\infty} dv_\theta \exp - \frac{\beta}{2} (v_r^2 + v_z^2 + v_\theta^2) \qquad \ldots (13)$$

where the integration is again restricted by eq.(5), this time to

$$v_r^2 + v_\theta^2 + v_z^2 > \frac{1}{R^2} (rv_\theta + \psi)^2 \qquad \ldots (14)$$

and at the point cusp $\psi = \frac{eBr^2}{2c}$. With the same substitutions as before eq.(13) becomes

$$F_2 = 4\pi \int rdr \int_0^\infty \rho^2 \, d\rho \int_{-\infty}^{\infty} dv_\theta \exp - \frac{\beta}{2} (\rho^2 + v_\theta^2) \; . \qquad \ldots (15)$$

If $v_\theta > 0$ the range of the r integration allowed by eq.(14) is

$$0 < \frac{eB}{c} r < - v_\theta + \sqrt{v_\theta^2 + \frac{2eBR}{c} (\rho^2 + v_\theta^2)^{\frac{1}{2}}} \qquad \ldots (16)$$

but if $v_\theta < 0$ the region

$$- v_\theta - \sqrt{v_\theta^2 - \frac{2eBR}{c} (\rho^2 + v_\theta^2)^{\frac{1}{2}}} < \frac{eBr}{c} < - v_\theta + \sqrt{v_\theta^2 - \frac{2eBR}{c} (\rho^2 + v_\theta^2)^{\frac{1}{2}}} \qquad \ldots (17)$$

is also forbidden. However eq.(17) is real only for v_θ so large as to satisfy

$$\frac{c|v_\theta|}{eB} > 2R , \qquad \ldots (18)$$

so that eq.(17) affects only particles whose larmor radius is greater than the dimensions of the system! We can therefore ignore eq.(17) and integrate over the range defined by eq.(16) for both positive and negative v_θ. The integration of eq.(15) is now straightforward and leads to

$$F_2 = \frac{2\pi Rc}{eB} \cdot \frac{3\pi}{\beta^2} \sqrt{\frac{\pi}{2\beta}} \left\{ 1 + \frac{2c}{3ReB} \sqrt{\frac{2}{\pi\beta}} \right\} \qquad \ldots (19)$$

corresponding to free effusion through a circular hole of radius

$$\delta_p = \left(\frac{3\pi}{2}\right)^{\frac{1}{4}} (a_p R)^{\frac{1}{2}} \left[1 + \left(\frac{8}{27\pi}\right)^{\frac{1}{2}} \frac{a_p}{R} \right]^{\frac{1}{2}} \qquad \ldots (20)$$

where a_p is the r.m.s. larmor radius corresponding to the field at the point cusp.

If the field strengths at line cusp and point cusp are equal then the loss at the line cusp eq.(10) and that at the point cusp eq.(19) are also almost exactly equal.

SUMMARY

Using a statistical, steady state model of the loss from an axisymmetric spindle cusp we have calculated the loss rate through line and point cusps. The loss through the line cusp is equivalent to free effusion through a slit of width $2\delta_L$, where δ_L is $(3\pi/8)^{\frac{1}{2}}$ times the r.m.s. larmor radius at the line cusp. The loss through the point cusp is almost exactly the same as that through the line cusp and corresponds approximately to free effusion through a circular hole of radius

$$\delta_p = \left(\frac{3\pi}{2}\right)^{\frac{1}{4}} (a_p R)^{\frac{1}{2}}$$

where a_p is the r.m.s. larmor radius and R the maximum radius of the confined plasma.

Our model confirms therefore, as Berkowitz et. al. assumed, that the loss at a line cusp <u>is</u> determined by the Larmor radius but at a point cusp particles penetrate much

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