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NONUNIFORM GRID GENERATION FOR BOUNDARY-LAYER PROBLEMS

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FOR

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ABSTRACT

A co-ordinate transformation is developed which gives rise to a concentration of grids in a boundary-layer without forcing too many nodes into the remainder of the region. This proposed transformation is compared with a number of existing transformations. The flexibility of the proposed scheme is demonstrated through its applications to flow over a horizontal surface and for flow in a bounded region. The transformed conservation equations can then be solved using finite difference methods in uniform rectangular grids, thereby avoiding any interpolation of the variables or their derivatives.

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INTRODUCTION

It is well known that the finite-difference schemes that employ uniform grids are the simplest and most accurate. A uniform grid, on the other hand, is not suitable for most boundary-layer problems and particularly for thin boundary layers. This is due to the fact that for adequate resolution several mesh points per boundary-layer scale are required and the physical dimensions of the system may be orders of magnitude larger than the boundary layer thickness. In this case, the computational effort for a uniform grid over the whole region will be prohibitively expensive and wasteful. A further complication can arise for time-dependent problems since the integration timestep, in an explicit method, is governed by the smallest mesh size. It is, therefore, imperative that some technique be employed which allows nonuniform grids with greatest density in the region of maximum change[1,2].

Perhaps one of the most straight forward methods is to select, a priori, a nonuniform grid by concentrating grid points within the boundary-layer and using a wider spacing outside it (see, for example, Ref.1). It has been generally believed that this nonuniform grid will result in improved accuracy of the solution without wasting computer memory space and running time. A major disadvantage of this technique, however, is that it involves interpolation of the variables or their derivatives at intermediate points and weak numerical instabilities usually arise at an interface where the grid spacing changes. Runchal, Spalding, and Wolfshtein [3] have used a nonuniform mesh and report improved accuracy when grids are concentrated in the steep-gradient region. The resulting accuracy, however, was not sufficiently adequate for practical purposes. Crowder and Dalton [4] have reported that a nonuniform grid gives a solution inferior to that for a uniform grid structure in both accuracy and computing time, at least, for the Poiseuille flow problem in a pipe. By experimenting with three different sets of nonuniform grids they concluded that, for the same number of grid points, a uniform grid gave more accurate results when compared to the exact solution.

An alternate approach that has been used with some success is based on the usage of a continuous co-ordinate transformation for the co-ordinate normal to the boundary. Thus, with an appropriate transformation, the boundary-layer structure is not present in the transformed co-ordinate system. The governing differential equation(s) is now subjected to this transformation, and the transformed equation(s) can then be solved on a uniform grid in the transformed space. The use of a stretched co-ordinate to obtain a boundary layer solution, which can be matched to an exterior solution, is a well known technique in the theoretical analysis of fluid flows. The technique there is to have a severely stretched co-ordinate in the boundary layer region and no stretching at all in the interior. The approach we shall be considering in the rest of this paper is the generation of a spatially nonuniform grid wh: produces severe stretching in the boundary layer, but which passes over smoothly and continuously to a quasi-identity transformation far from the boundary.

Let us consider, for example, a one-dimensional problem. The independent variable x is transformed to another independent variable ξ through the following equation

$$\xi = f(x) \qquad (1)$$

Various partial derivatives appearing in the Navier-Stokes equation are replaced by expressions such as

$$\frac{\partial u}{\partial x} = \frac{d\xi}{dx}\frac{\partial u}{\partial \xi} \qquad (2)$$

and

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{d\xi}{dx}\right)^2 \frac{\partial^2 u}{\partial \xi^2} + \frac{d^2\xi}{dx^2}\frac{\partial u}{\partial \xi} \qquad (3)$$

The resulting set of equations can now be solved by using a uniform grid in ξ-space. From these equations, the following constraints on the co-ordinate transformation are obtained:

1) the function must be continuous,

2) the function must be, differentiable at least twice, and

3) the function and its first two derivatives should not have any singularity.

The existing transformations as well as a more flexible one will be discussed in the following section. The resulting grid patterns are examined and results compared. Several applications are also included.

EXISTING AND PROPOSED TRANSFORMATIONS

Roberts [5] was the first to propose a highly suitable transformation in terms of a logarithmic function which resulted in the desired grid structure in x-space. His transformation for a one-dimensional flow problem with a boundary layer near $x = o$ can also be expressed as

$$\xi = 1 - \frac{\tanh^{-1}[\sqrt{1-\delta}\,(1-x)]}{\tanh^{-1}\sqrt{1-\delta}} \qquad (4)$$

where $o \le x \le 1$, $o \le \xi \le 1$, and δ is the normalized boundary layer thickness. Note that the boundary layer thickness is in the range $o < \delta \le 1$. When δ is equal to one, Eq.(4) can be seen to reduce to an identity transformation (i.e., $\xi = x$) and hence a uniform grid spacing in x-space can be used. In the case of boundary layers near both $x = -1$ and $x = 1$, i.e., for the case of flow in a bounded region (also termed a 'channel problem'), Roberts' transformation can be written as

$$\xi = \frac{\tanh^{-1}(\sqrt{1-\delta}\; x)}{\tanh^{-1}\sqrt{1-\delta}} \,. \qquad (5)$$

The grid structure for the problem of a single boundary layer near $x = o$ can be obtained from Eq.(4). Alternately, this equation can be inverted to give

$$x_i = 1 - \frac{1}{\sqrt{1-\delta}} \quad \tanh\left[\frac{N+1-i}{N} \quad \tanh^{-1}\sqrt{1-\delta}\right] \qquad (6)$$

where $i = 1, 2, \text{---}\ N + 1$ and N is the total number of meshes. Figure 1 shows grid patterns for various values of the boundary-layer thickness. It is noted that this transformation indeed results in a concentration of grids within the boundary-layer. The total number of grid points used in ξ-space will naturally depend upon the problem. For example, if it is required that there be a minimum of four meshes inside the boundary-layer (which is desirable for adequate resolution) then the total number of grids will have to be at least 22 and 30 for $\delta = 0.01$ and $\delta = 0.001$, respectively.

Recently Schumann, Grötzbach and Kleiser [6] have used the transformation suggested by Roberts. They report that, at least for their trial boundary-layer problem, the variable grid spacing technique gives better agreement with the exact solution than the fixed grid pattern in x-space. In other words, the continuously varying grid method with 60 meshes results in an accuracy of 10^{-3}, which is achieved by a fixed grid pattern with 4000 meshes. They have also reported that the finite-difference technique with variable mesh is superior to the spectral method as long as the number of degrees of freedom is no greater than 60.

An alternative transformation has been suggested by Kalnay de Rivas [7] for the boundary-layer problem near x = o. She suggests

$$\xi = \sqrt{x} \tag{7}$$

It is seen that for a uniform grid in ξ-space, this transformation gives a continuously increasing grid in x-space. Let n be the number of meshes desired in the boundary-layer of thickness δ then it is related to the total number of meshes N, in this case, by

$$N = \frac{n}{\sqrt{\delta}} \; . \tag{8}$$

Hence, for a minimum of four meshes in δ, the total number of meshes will have to be 40 and 126 for δ = 0.01 and 0.001, respectively. This transformation is thus seen to require a large number of nodes for thin boundary-layer problems. Another drawback of this transformation is that it introduces singularities, due to derivatives such as $\frac{d\xi}{dx}$ and $\frac{d^2\xi}{dx^2}$, in the governing equations near x = o. Kalnay de Rivas has also suggested the following co-ordinate transformation for the channel problem with boundary-layers at x = o and x = 1

$$\xi = \frac{2}{\pi} \sin^{-1} \sqrt{x} \tag{9}$$

For the case of a channel covering $-1 \le x \le 1$, this transformation becomes

$$\xi = \frac{4}{\pi} \sin^{-1} \sqrt{\frac{1+x}{2}} - 1 \; . \tag{10}$$

Israeli, see in Reference 2, has suggested

$$\xi = \sum_i c_i \tan^{-1} \left(\frac{x - x_i}{\delta_i} \right) \tag{11}$$

where x_i is the location of the ith boundary or interior layer, δ_i the boundary-layer thickness, and c_i is a weight factor. This transformation does not allow the number of mesh points in the boundary layer to be chosen independently of the total number of mesh points used. As an illustration, consider the simplest form of Eq.(11), viz, $\xi = c \tan^{-1}(x/\delta)$. Then, for the case of a single boundary-layer, at least half of the total number of meshes must be in the boundary layer.

We find the following transformation, for use when a single boundary layer is present adjacent to x = o, to be a flexible one:

$$\xi = \frac{\tanh mx}{\tanh m} \qquad (12)$$

where m is a number (not necessarily an integer) to be determined. This functional form meets all of the requirements noted earlier. The parameter m is calculated by requiring that there be at least n meshes, out of a total of N, in the boundary layer thickness δ. Thus, m is the solution of

$$\frac{n}{N} = \frac{\tanh m\,\delta}{\tanh m} \qquad (13)$$

For sufficiently large m (say, $m \gtrsim 4$) and thin boundary layers, m is obtained from

$$m = \frac{1}{\delta}\tanh^{-1}\left(\frac{n}{N}\right) \qquad (14)$$

Figure 2 shows the grid pattern obtained by Eq. (12) for two values of m. It is seen that for $m \ll 1$ the proposed transformation reduces to an identity transformation.

The only disadvantage of the proposed transformation is that for the case of very thin boundary-layer problems (i.e., large m) almost all of the mesh points are concentrated within the boundary-layer region ($x \sim 4\delta$) and one or two meshes remain with which to represent the main flow. This deficiency can be eliminated, following a suggestion by Eiseman [8], by adding another term to Eq.(12) We thus propose

$$\xi = c\,\frac{\tanh mx}{\tanh m} + (1-c)x^2, \qquad (15)$$

where $0 \le c \le 1$. There is a great degree of flexibility in the second term on the right hand side of Eq.(15). Eiseman [8] has used a linear term. We prefer to use a quadratic term since it gives more weight to the interior region. The free parameter c can be chosen to optimize different elements of the solution. Figure 3 shows the grid pattern for c = 0.5 and different values of m. The quadratic term contributes very little in the boundary layer. Therefore the parameter m can be related to the number of meshes n within the boundary-layer by

$$m \simeq \frac{1}{\delta}\tanh^{-1}\left(\frac{n}{cN}\right) \qquad (16)$$

This equation is valid for $m \geq 4$. For the case where the boundary-layer is expected to cover the entire range, we recommend that c should be set equal to 1. Eiseman has observed that the parameter (1-c) should be the ratio of fractional area occupied by the boundary layer to the percentage of nodes in the boundary layer.

For the fluid flow problem in a bounded region with different boundary-layer thicknesses at x = +1 and x = -1, Eq.(15) can readily be generalized to give

$$\xi = c \left[\frac{\tanh m_1(1+x)}{\tanh 2m_1} - \frac{\tanh m_2(1-x)}{\tanh 2m_2} \right] + (1-c)x \qquad (17)$$

where m_1 and m_2 are expressed, in terms of the boundary-layer thicknesses and the number of mesh desired within the layers, as

$$m_i \simeq \frac{1}{\delta_i} \tanh^{-1}\left(\frac{2n_i}{cN}\right) \qquad (i = 1,2), \qquad (18)$$

for which $m_i \geq 4$ is assumed. The parameter c is determined by the importance of the mainstream region as compared to the resolution of boundary-layers. A word of caution must be added that the parameter c must be such that the desired number of meshes within the boundary-layers out of a given N meshes is feasible. Mathematically this condition can be expressed by requiring the argument of $\tanh^{-1}$ in Eq.(18) to be less than unity, i.e., $c > 2n_i/N$.

Table I shows a compilation of various transformations that may be used to provide resolution in a thin boundary-layer at x = o in the region $o \leq x \leq 1$. Various characteristics are also noted in the table. Perhaps the two most suitable transformations are denoted by numbers five and six. In either case the number of grid points is minimized. Likewise for the problem of boundary-layers at either surface in the region $-1 \leq x \leq 1$, various transformations are summarized in Table II. Clearly, the most general and efficient one is the last one. It should be emphasized that this transformation permits an optimum choice of grids even for the case of two drastically different boundary-layer thicknesses.

APPLICATIONS

In this section, we will apply various transformations noted earlier to both the one-sided boundary-layer and two-sided boundary-layers problems. First, we consider the fluid flow problem on a surface such that boundary-layer of thickness δ is created near $x = o$. It is well known that an adequate resolution of the boundary-layers requires at least four mesh points inside the boundary-layer. Table III shows the total number of meshes required by these transformations such that a minimum of four meshes result inside the boundary-layer. The flexibility of the proposed transformations is evident. Roberts' method is considerably superior to Kalnay de Rivas particularly for thin boundary-layers. The proposed transformation is even better than the one suggested by Roberts.

We now consider the fluid flow problem in a bounded region. The applicable transformations are already summarized in Table II. Some of these are compared graphically in Fig.4. It is seen that the transformations involving hyperbolic tangent or arc hyperbolic tangents are quite favourable. The effect of an additive term in the proposed transformation is clearly shown in Fig.5. In this figure, the transformation given by Eq.(17) is sketched for three values of c (0.5, 0.8, and 1.0) and $m_1 = m_2 = 10$. Note that the region shown here is $-1 \leq x \leq o$ since the other region can be obtained by symmetry.

The proposed transformation can also be compared with the others by computing the minimum number of total meshes required to yield at least four meshes in each boundary layer. The results are given in Table IV. The superiority of hyperbolic tangent function when combined with a linear term is thus evident particularly for thin boundary-layers. It should be pointed out that the total number of mesh points is determined by the resolution required in interior, as the parameter m insures the desired resolution inside the boundary-layer.

We remark that Israeli's transformation [Eq.(11)] involving $\tan^{-1}$ could be generalized in a fashion similar to that described here for tanh. Thus the transformation $\xi = \tan^{-1}(mx)/\tan^{-1}(m)$, where m satisfies an equation similar to Eq.(14), has the same general characteristics as Eq.(12). Moreover in Eq.(17) tanh can be replaced by $(\tan^{-1})$ everywhere to give a similarly satisfactory transformation. For equations (15) and (17) and for the analogous forms of the generalized Israeli transformation, the analytic forms for $d\xi/dx$ and $d^2\xi/dx^2$ are of comparable complexity; and none can be expressed simply in terms of ξ, which would be ideal since the calculation is actually to be carried out in ξ-space. Thus the choice between the 'tanh' form which we have chosen and the '$\tan^{-1}$' form appears to be a matter of personal preference.

SUMMARY

A flexible technique has been proposed which is capable of generating a concentration of grids in and near the boundary-layer. This was accomplished by employing hyperbolic tangent functions and a linear or second order polynomial. Flexibility was provided through a relative weight parameter between the hyperbolic tangent and polynomial terms. The proposed co-ordinate transformation has been compared with other existing change of variables and the superiority as well as flexibility of the proposed method was demonstrated. For the fluid flow problem in a bounded region with drastically different boundary-layer thicknesses, the proposed method is unequalled by any other transformation considered here, save a closely related generalization of Israeli's transformation. Finally, the proposed technique is readily extendable to two and three-dimensional problems.

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TABLE I. Co-ordinate transformations for boundary-layer at $x = o$ $(o \le x \le 1)$.

Number	Transformation Definition $\xi =$	Singularity in		Number of meshes required (N) to yield n meshes in δ	Remarks	Reference
		$\frac{d\xi}{dx}$	$\frac{d^2\xi}{dx^2}$			
1	$\sqrt{x}$	at $x=o$	at $x=o$	$N = \frac{n}{\sqrt{\delta}}$	No parameter in the transformations; for thin boundary-layers required number of meshes is high	Kalnay de Rivas(1972)
2	$1 - \frac{\tanh^{-1}\sqrt{1-\delta}(1-x)}{\tanh^{-1}\sqrt{1-\delta}}$	No	No	$N \simeq \frac{n\ \ln(4/\delta)}{\ln 3}$ for $\delta \le 0.1$	Required number of meshes is medium; no parameter to adjust mesh sizes	Roberts(1971)
3	$\sum_i c_i \tan^{-1}\left(\frac{x-x_i}{\delta_i}\right)$	No	No		c_i is the weight factor x_i is the location of the ith layer	Israeli(1972)
4	$\frac{\tanh mx}{\tanh m}$	No	No	Independent of problem: m is determined from $\frac{n}{N} = \frac{\tanh m\delta}{\tanh m}$	For $m \gtrsim 4$ $m = \frac{1}{\delta}\tanh^{-1}\frac{n}{N}$ Good resolution in δ but poor resolution in inner region	Present
5	$c\frac{\tanh mx}{\tanh m} + (1-c)x$	No	No	m is determined as in (4); parameter c is determined by importance of inner region	Deficiency of (4) is removed; total number of meshes required is low	Eiseman(1978)
6	$c\frac{\tanh mx}{\tanh m} + (1-c)x^2$	No	No	Same as (5)	Importance of the interior is emphasized; total number of meshes required is low	Present

TABLE II. Co-ordinate transformations for boundary layers at $x = \pm 1$ $(-1 \le x \le 1)$

Number	Transformation Definition $\xi =$	Singularity in $\frac{d\xi}{dx}$	Singularity in $\frac{d^2\xi}{dx^2}$	Number of Meshes required (N) to yield n meshes in δ	Remarks	Reference
1	$\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{2}}$	at $x = \pm 1$	at $x = \pm 1$	$N = \frac{2n}{\sqrt{\delta}}$	No parameters in the transformation; number of required meshes is high for thin boundary layer	Present
2	$\frac{4}{\pi}\sin^{-1}\sqrt{\frac{1+x}{2}} - 1$, alternately $\frac{4}{\pi}\sin^{-1}\left[\sqrt{\frac{1+x}{4}} - \sqrt{\frac{1-x}{4}}\right]$	at $x = \pm 1$	at $x = \pm 1$	$N = n \frac{\pi}{2\sin^{-1}\sqrt{\delta/2}}$	No parameters in the transformation; number of required meshes is high for thin boundary layer	Kalnay de Rivas(1972)
3	$\frac{\tanh^{-1}(\sqrt{1-\delta}\, x)}{\tanh^{-1}\sqrt{1-\delta}}$	No	No	$N \approx 2n\frac{\ln(4/\delta)}{\ln 3}$ for $\delta \le 0.1$	Required number of meshes is medium; perhaps too many meshes in inner region	Roberts(1971)
4	$\frac{\tanh m(1+x) - \tanh m(1-x)}{\tanh 2m}$	No	No	Independent of problem; m is determined from $N \cong 2n \frac{\tanh 2m}{\tanh m\delta}$	For $m \ge 4$, $m = \frac{1}{\delta}\tanh^{-1}\left(\frac{2n}{N}\right)$ Poor resolution in interior	Present
5	$c\frac{\tanh m(1+x) - \tanh m(1-x)}{\tanh 2m} + (1-c)x$	No	No	Independent of problem; m is determined as in (4). Parameter c is determined by importance of inner region	Generalization of Eiseman's transformation; good resolution in the boundary layer as well as interior	Present
6	$c\left[\frac{\tanh m_1(1+x)}{\tanh 2m_1} - \frac{\tanh m_2(1-x)}{\tanh 2m_2}\right] + (1-c)x$	No	No	For $m_i \ge 4$ $m_i = \frac{1}{\delta_i}\tanh^{-1}\left(\frac{2n_i}{cN}\right)$	Generalization of (5): applicable when the two boundary-layers have appreciably different thicknesses	Present

TABLE III. Total number of required meshes to yield four meshes in a one-sided boundary-layer problem

Re	δ	Eq.(7)	Eq.(4)	Eq.(15) (c=0.5)
400	0.05	18	16	10
10^4	0.01	40	22	10
4×10^4	0.005	57	24	10
10^6	0.001	126	30	10

TABLE IV. Total number of required meshes to yield four meshes in each of th boundary-layers at $x = \pm 1$.

Re	δ	Eq.(9)	Eq.(5)	Eq.(17) (c=0.5)
400	0.05	28	22	20
10^4	0.01	63	30	20
4×10^4	0.005	89	34	20
10^6	0.001	199	42	20

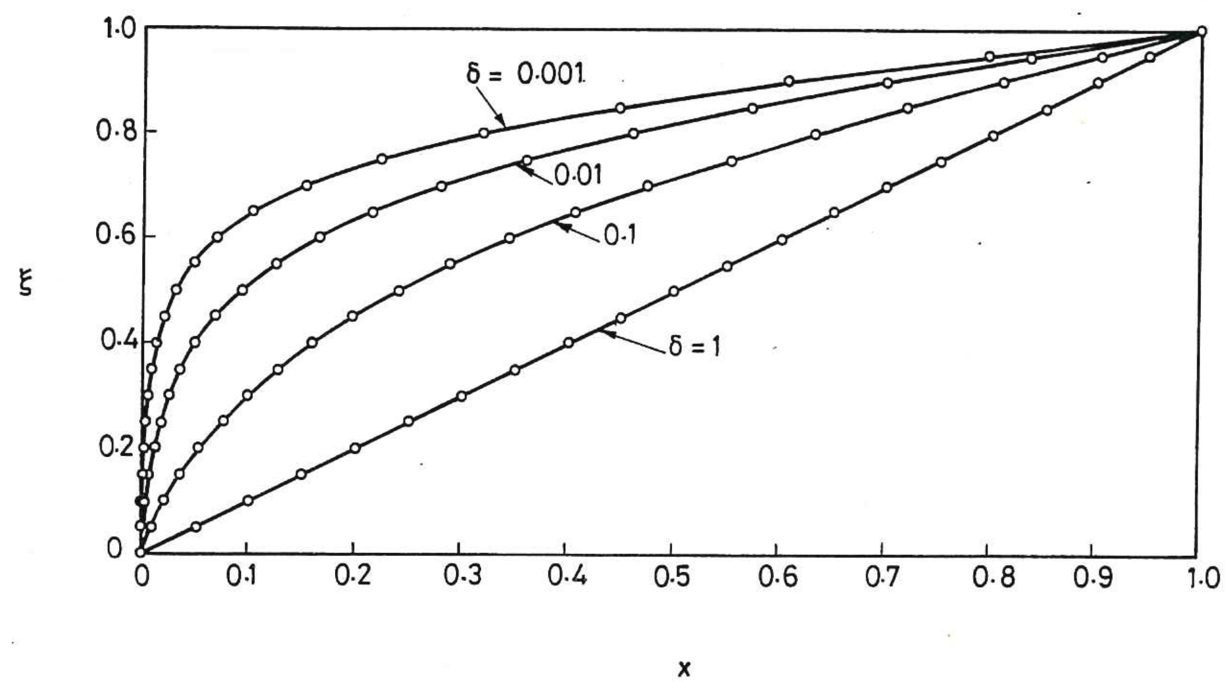

Fig.1 Roberts' co-ordinate transformation for flow over a horizontal surface (boundary-layer near x = 0).

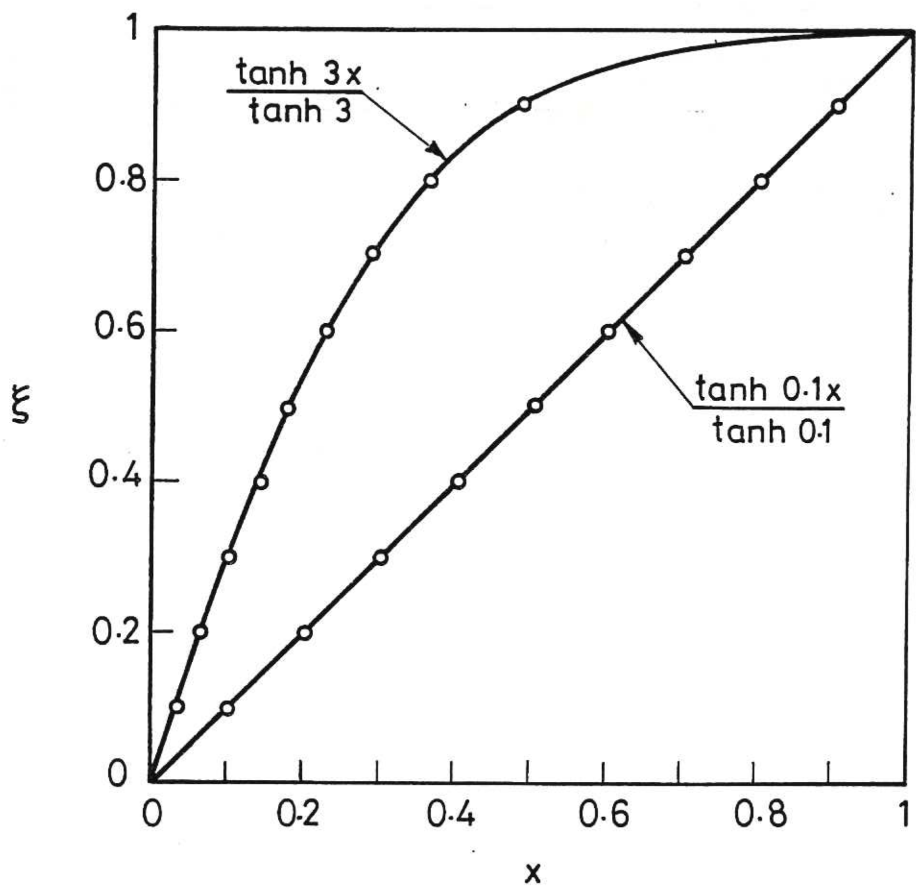

Fig.2 Hyperbolic tangent transformation for flow over a horizontal surface.

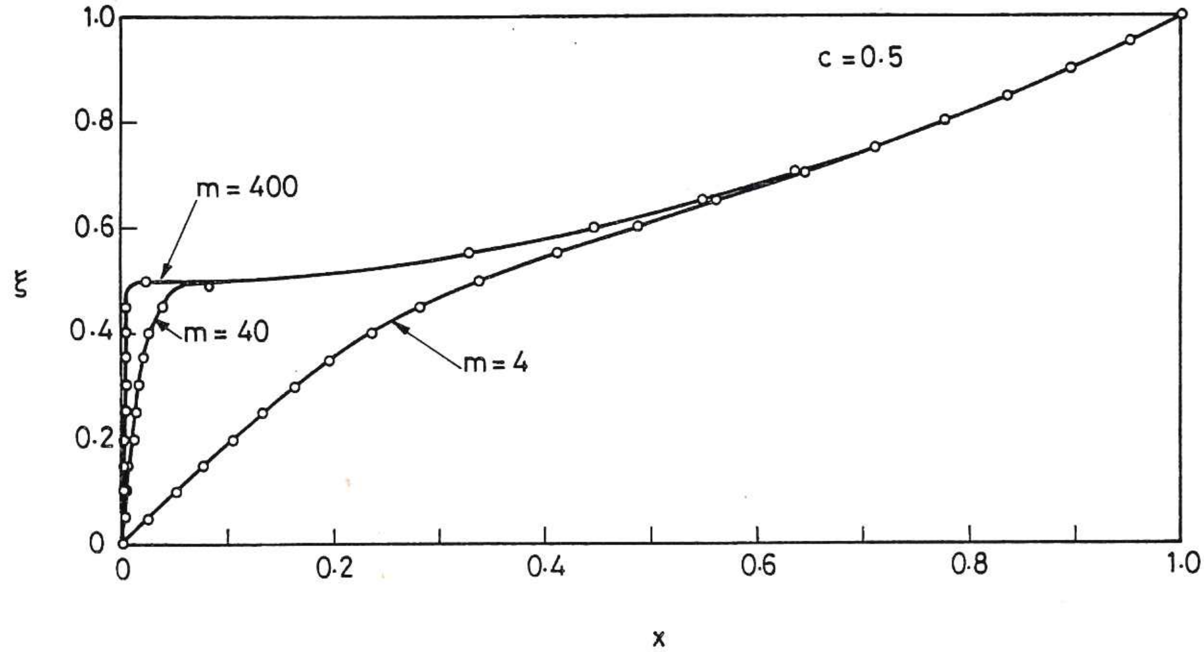

Fig.3 Hyperbolic tangent + a polynomial transformation for flow over a horizontal surface.

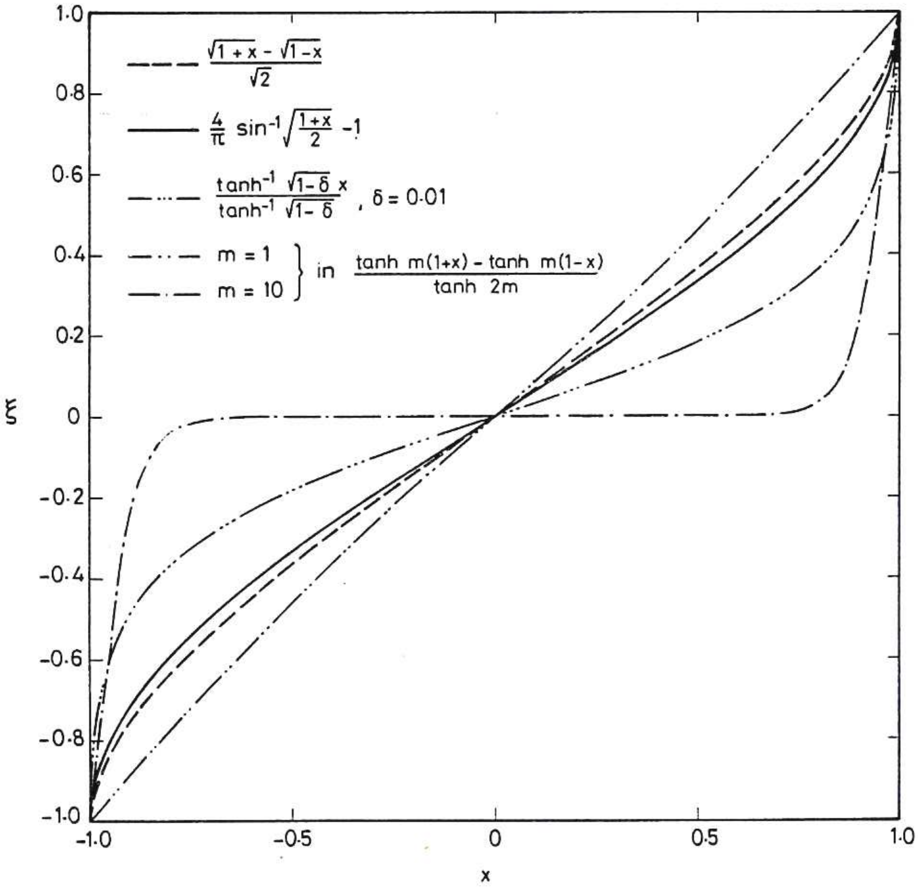

Fig.4 Comparison of various transformations for flow in a bounded region ($\delta_1 = \delta_2$).

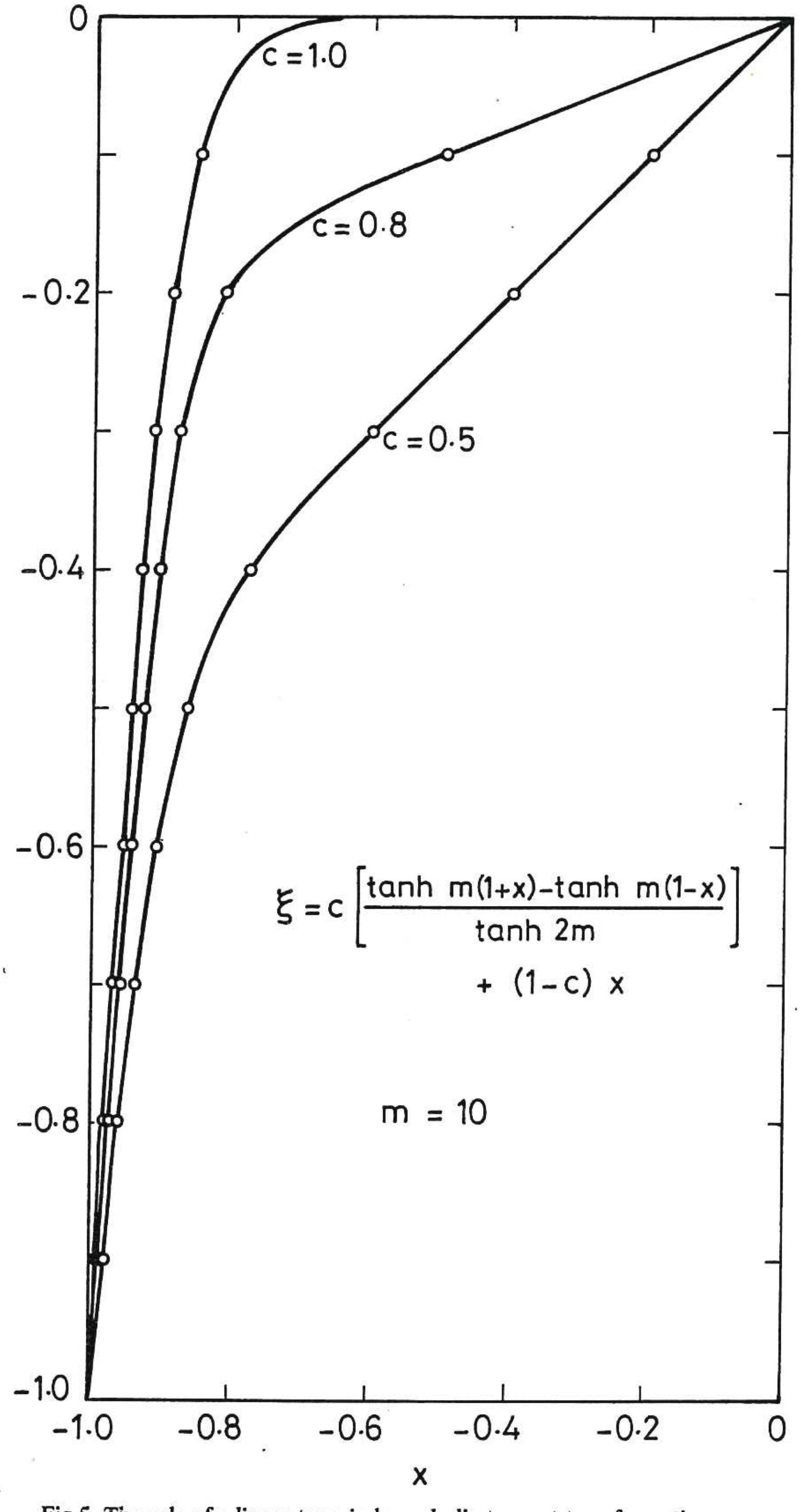

Fig.5 The role of a linear term in hyperbolic tangent transformation.