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HEAT TRANSMISSION THROUGH A LANGMUIR SHEATH IN THE PRESENCE OF ELECTRON EMISSION

by

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ABSTRACT

The steady state electrostatic sheath which forms between a plasma and a wall in order to prevent a net current flow also acts as a thermal insulator between the hot electrons and the wall. The transmitted heat flux can be written in the form $Q = \frac{1}{4} n \bar{v}_e \cdot 2kT \cdot F$, where $F = (\pi\mu/8)^{\frac{1}{2}} [5 - \ell n 2\pi\mu]$ and $\mu = m_e/m_i$. If the wall emits cold electrons due to any cause (e.g. electron, ion or photon impact) the voltage across the sheath is reduced and the thermal insulation impaired, i.e. F is increased. The maximum extent to which this occurs is limited by a saturation of the emission current due to space charge effects. In these circumstances the ratio of emission to primary electron current never exceeds $1 - 8.3\sqrt{\mu}$, the sheath voltage falls to its minimum value of 1 kT and F attains its maximum value given by $F = 0.33 + 2.2\sqrt{\mu}$.

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C O N T E N T S

1. INTRODUCTION

An inherent feature of an open-ended magnetic trap (e.g. cusp, theta-pinch, etc.) is that plasma must at some point come into contact with walls. Interest has been increasing recently in the problems of thermal conduction along the field lines to the ends of such a device[1,2,3]. If the hot plasma expands into a vacuum (if cold gas is present initially the situation may be completely different) and then strikes the walls the energy conducted along the plasma can be given up to the walls. It is tempting to apply a boundary condition of zero temperature at the wall in the belief that there will be no limitation on the heat flux at this point. However in the simplest situation a hot plasma 'in contact' with a wall is in fact insulated[3] from it by an electrostatic sheath, the Langmuir sheath[4], which prevents the bulk of the electrons striking the wall and the rate at which energy can be deposited is limited by the ion flux to the wall. Under certain circumstances, then, the net rate of energy loss by thermal conduction might be significantly reduced.

The simple theory of the Langmuir sheath[4] assumes the wall to be perfectly absorbing and non-emitting. Walls will not in general behave in such a convenient manner, particularly if the plasma is at kilovolt energies. Any deviation from this simple assumption, such as secondary electron emission or the boiling off of absorbed gas and wall material, may lead to the complete disruption of the sheath and to a consequent enhancement of the energy loss rate.

This report investigates the heat transmission properties of the sheath with particular emphasis on the part played by any cold electrons emitted by the wall. It will be shown that the actual physical processes giving rise to this emission, e.g. electron, ion, metastable atom or photon impact, are irrelevant as far as the sheath analysis is concerned and that it is only necessary to specify Γ the ratio of the emission to primary electron flux, i.e. the number of secondary electrons emitted due to all causes for every primary electron absorbed. The heat flux to the wall can then be written in the form

$$Q = \frac{1}{4} n\bar{v}_e \cdot 2kT \cdot F,$$

where n and T are the electron density and temperature in the plasma, $\bar{v}_e = (8kT/\pi m_e)^{\frac{1}{2}}$ and F, the thermal transmission factor, is a function of m_e/m_i and Γ only. In an appendix a simple thermal conduction calculation is presented which demonstrates the effect on energy containment in a plasma of using this boundary condition rather than $T = 0$.

2. THE STEADY STATE SHEATH EQUATIONS

Consider an infinite plane wall, situated at $x = 0$, in contact with a plasma filling the half space $x > 0$. Let the density of primary, i.e. plasma electrons be $n_{e1}(x)$, the density of secondary electrons be $n_{e2}(x)$ and the density of ions be $n_i(x)$. At $x = \infty$ i.e. far into the plasma, quasi-neutrality demands that

$$n_{e1}(\infty) + n_{e2}(\infty) = n_i(\infty) \equiv n_o . \qquad \ldots (1)$$

In the vicinity of the wall, i.e. in the sheath region, there will be a charge imbalance resulting in an electrostatic potential $\varphi(x)$ satisfying Poisson's equation

$$\frac{d^2\varphi}{dx^2} = 4\pi e \left[n_{e1}(x) + n_{e2}(x) - n_i(x) \right] \qquad \ldots (2)$$

and the condition

$$\varphi(\infty) = 0 .$$

If the plasma electrons have a Maxwellian velocity distribution with temperature T at $x = \infty$ then the primary electron density in the sheath region is given approximately[5] by

$$n_{e1}(x) = n_{e1}(\infty) \exp\left[\frac{e\varphi(x)}{kT}\right] . \qquad \ldots (3)$$

The mean free path between collisions is assumed to be large compared to the sheath thickness.

Electrons are assumed to be emitted from the wall with negligible energies (compared to kT) and then to fall freely through the sheath into the plasma. Their density is then given by conservation of flux

$$J_{e2} = n_{e2}(x)\, v_{e2}(x) = \text{const.} \qquad \ldots (4)$$

and their velocity by

$$\tfrac{1}{2} m_e v_{e2}^2(x) - e\varphi(x) = - e\varphi(0) . \qquad \ldots (5)$$

Equations (3) and (4) implicitly assume that $\varphi(x)$ is a monotonic function, a point that will be justified in a later paragraph.

The ions are assumed to be cold $(T_i = 0)$ and to arrive at the sheath 'edge' with a kinetic energy (in the x-direction) of $E_o = \tfrac{1}{2} m_i v_o^2$, having been accelerated towards the sheath by small potential gradients set up in the bulk of the plasma[5] (c.f. also section 3(a)). On entering the sheath they will fall freely into the wall, n_i and v_i being given by

$$J_i = n_i(x)\ v_i(x) = n_o\ v_o\ , \qquad \ldots (6)$$

$$\tfrac{1}{2}\, m_i\ v_i^2(x) + e\varphi(x) = \tfrac{1}{2}\, m_i\ v_o^2\ . \qquad \ldots (7)$$

If Γ secondary electrons are emitted per primary electron impinging on the wall, the flux of secondaries leaving the wall is given by

$$J_{e2} = \Gamma\ J_{e1}\ . \qquad \ldots (8)$$

The condition that no net current shall flow is

$$J_i = J_{e1} - J_{e2} = n_o\ v_o\ , \qquad \ldots (9)$$

hence

$$\left.\begin{aligned} J_{e1} &= \frac{1}{1-\Gamma}\ n_o\ v_o\ , \\ J_{e2} &= \frac{\Gamma}{1-\Gamma}\ n_o\ v_o\ . \end{aligned}\right\} \qquad \ldots (10)$$

Equation (4) then gives

$$n_{e2}(x) = n_o\ \frac{\Gamma}{1-\Gamma}\ \mu^{\frac{1}{2}} \left[\frac{E_o}{e(\varphi - \varphi_o)}\right]^{\frac{1}{2}}\ , \qquad \ldots (11)$$

where $\mu = m_e/m_i$ and $\varphi_o = \varphi(0)$.

From equations (1) and (3)

$$n_{e1}(x) = n_o \left[1 - \frac{\Gamma}{1-\Gamma}\ \mu^{\frac{1}{2}}\ \left(-\frac{E_o}{e\varphi_o}\right)^{\frac{1}{2}}\right] e^{\frac{e\varphi}{kT}}\ . \qquad \ldots (12)$$

Substitution into equation (2) now gives

$$\frac{d^2\varphi}{dx^2} = 4\pi n_o e \left\{ \left[1 - \frac{\Gamma}{1-\Gamma}\ \mu^{\frac{1}{2}}\ \left(-\frac{E_o}{e\varphi_o}\right)^{\frac{1}{2}}\right] e^{\frac{e\varphi}{kT}} + \frac{\Gamma}{1-\Gamma}\ \mu^{\frac{1}{2}}\ \left(\frac{E_o}{e(\varphi-\varphi_o)}\right)^{\frac{1}{2}} - \frac{1}{\sqrt{1 - e\varphi/E_o}} \right\}. \qquad \ldots (13)$$

Defining dimensionless variables by

$$\eta = -\frac{2e\varphi}{kT}\ , \qquad \eta_o = -\frac{2e\varphi_o}{kT}\ , \qquad \eta_1 = +\frac{2E_o}{kT}\ , \qquad z = \frac{x}{\lambda_D}\ ;$$

where $\lambda_D = (4\pi n_o e^2/kT)^{-\frac{1}{2}}$, the Debye length, equation (13) becomes

$$\frac{1}{2}\ \frac{d^2\eta}{dz^2} = \left(1 + \frac{\eta}{\eta_1}\right)^{-\frac{1}{2}} - \frac{\Gamma}{1-\Gamma}\ \left(\frac{\mu\eta_1}{\eta_o}\right)^{\frac{1}{2}}\ \left(1 - \frac{\eta}{\eta_o}\right)^{-\frac{1}{2}} - \left[1 - \frac{\Gamma}{1-\Gamma}\ \left(\frac{\mu\eta_1}{\eta_o}\right)^{\frac{1}{2}}\right] e^{-\eta/2} \qquad \ldots (14)$$

Multiplying through by $d\eta/dz$ and integrating from ∞ (where $d\eta/dz = 0$) to z gives

$$\frac{1}{8}\ \left(\frac{d\eta}{dz}\right)^2 = \eta_1 \left[\left(1 + \frac{\eta}{\eta_1}\right)^{\frac{1}{2}} - 1\right] + \frac{\Gamma}{1-\Gamma}\ \left(\frac{\mu\eta_1}{\eta_o}\right)^{\frac{1}{2}}\ \eta_o \left[\left(1 - \frac{\eta}{\eta_o}\right)^{\frac{1}{2}} - 1\right]$$

$$+ \left[1 - \frac{\Gamma}{1-\Gamma}\ \left(\frac{\mu\eta_1}{\eta_o}\right)^{\frac{1}{2}}\right] \left(e^{-\eta/2} - 1\right)\ . \qquad \ldots (15)$$

Taking the negative square root ($d\eta/dz < 0$), inverting and integrating from 0 to z then gives

$$z = \frac{1}{2\sqrt{2}} \int_{\eta}^{\eta_o} d\eta \left\{ \eta_1 \left[(1 + \frac{\eta}{\eta_1})^{\frac{1}{2}} - 1 \right] + \frac{\Gamma}{1 - \Gamma} (\frac{\mu\eta_1}{\eta_o})^{\frac{1}{2}} \eta_o \right.$$
$$\left. \times \left[(1 - \frac{\eta}{\eta_o})^{\frac{1}{2}} - 1 \right] + \left[1 - \frac{\Gamma}{1 - \Gamma} (\frac{\mu\eta_1}{\eta_o})^{\frac{1}{2}} \right] (e^{-\eta/2} - 1) \right\}^{-\frac{1}{2}} \quad \ldots (16)$$

3. CONSTRAINTS ON THE SOLUTION

(a) Stability

When $\Gamma = 0$, equations (14) - (16) reduce to those for the conventional Langmuir sheath. It is well known(6) that such a sheath will not form, i.e. equation (14) has no monotonic solutions, unless $\eta_1 \geqslant 1$, i.e. $E_o \geqslant kT/2$. When $\Gamma \neq 0$ the analagous stability condition is

$$\frac{\eta_o}{\eta_1} \frac{(\eta_1 - 1)}{(\eta_o + 1)} \geqslant \frac{\Gamma}{1 - \Gamma} (\frac{\mu\eta_1}{\eta_o})^{\frac{1}{2}} . \quad \ldots (17)$$

It can be argued(6) that conditions within the plasma will always adjust themselves such that ions arrive at the sheath edge with the minimum kinetic energy necessary to satisfy the stability condition, i.e. such that the equality is satisfied. This constraint of marginal stability will be used here. The equality (17) implies that, for $\Gamma < 1$, $\eta_1 \geqslant 1$, i.e. higher ion energies are required to ensure stability when electron emission is included.

(b) Zero Current

The condition that no current must flow to the wall has been used in the derivation of equation (14). In addition it can be used to determine η_o, the normalized voltage drop across the sheath. The flux of primary electrons reaching the wall is given by

$$J_{e1} = \tfrac{1}{4} n_{e1} (o) (\frac{8kT}{\pi m_e})^{\frac{1}{2}} = \frac{1}{1 - \Gamma} n_o v_o ,$$

whence, using equation (12),

$$e^{\eta_o/2} = \frac{1 - \Gamma}{\sqrt{2\pi\mu}} \eta_1^{-\frac{1}{2}} \left[1 - \frac{\Gamma}{1 - \Gamma} (\frac{\mu\eta_1}{\eta_o})^{\frac{1}{2}} \right] . \quad \ldots (18)$$

In order to determine η_o and η_1 for a given value of Γ it is in principle necessary to solve equations (17) and (18) simultaneously. However if they are rewritten in the form

$$e^{\eta_o/2} = \frac{1 - \Gamma}{\sqrt{2\pi\mu}} \eta_1^{-3/2} (1 + \frac{\eta_1 - 1}{\eta_o + 1}) , \quad \ldots (19)$$

$$\eta_1 = 1 + \frac{\eta_o + 1}{\frac{\sqrt{2\pi}}{\Gamma}\,\eta_o^{3/2}\,e^{\eta_o/2} - 1} \qquad \ldots (20)$$

and use is made of the smallness of $\sqrt{\mu}$ $(< \frac{1}{40})$ it can be seen that, provided $\Gamma \neq 1$, an approximate solution is given by

$$\left.\begin{aligned} \eta_o &\approx \ln\left\{\frac{(1-\Gamma)^2}{2\pi\mu}\right\}, \\ \eta_1 &\approx 1 . \end{aligned}\right\} \qquad \ldots (21)$$

Thus the introduction of electron emission leads to a reduction of the overall sheath potential, the associated increase in the ion energy necessary to maintain stability being negligible.

As Γ approaches unity the approximate solution (19) must break down. Before this point is reached however a new physical phenomenon becomes important. It is discussed in the next paragraph.

(c) Space Charge Limitation

The electric field at the wall is given by

$$E(0) = \frac{kT}{2e\lambda_D}\left(\frac{d\eta}{dz}\right)_o ,$$

where

$$\tfrac{1}{8}\left(\frac{d\eta}{dz}\right)_o^2 = \eta_1\left[\left(1+\frac{\eta_o}{\eta_1}\right)^{1/2} - 1\right] - \frac{\eta_o^2}{\eta_1}\,\frac{(\eta_1 - 1)}{(\eta_o + 1)} + \frac{1}{\eta_1}\,\frac{(\eta_o + \eta_1)}{(\eta_o + 1)}\left(e^{-\eta_o/2} - 1\right) \qquad \ldots (22)$$

The behaviour of this function, at least for small Γ, can be determined by substituting into (22) the approximate solution (21), giving the result

$$\tfrac{1}{8}\left(\frac{d\eta}{dz}\right)_o^2 \approx (1 + \eta_o)^{1/2} - 2 . \qquad \ldots (23)$$

Thus as Γ increases and η_o decreases, the electric field at the wall decreases. Equation (23) suggests that $E(0)$ becomes zero for $\eta_o \approx 3$, when Γ is close to, but less than unity. It therefore appears that there exists a critical value Γc of Γ at which the emission current becomes saturated and a double sheath begins to form. For any Γ in excess of Γc no monotonic solution for $\varphi(x)$ exists. A potential well, with a depth of order the emission energy, i.e. small compared to kT, forms at the origin such that all but a fraction ε of the emitted electrons are returned directly to the wall, the sheath controlling ε in such a way that $\varepsilon\,\Gamma = \Gamma c$. Thus the ratio of emission to primary electron current can never exceed Γc however many electrons are actually liberated from the wall.

In principle a well of the type described could exist for $\Gamma < \Gamma c$, reflecting back to the wall a fraction $1 - \varepsilon$ of the electrons emitted. In this case ε could lie anywhere in the range $0 < \varepsilon < 1$. Since, in the negligible emission energy limit, the well would have a very small depth and extent, the solutions would be indistinguishable from those already obtained with $\Gamma' = \varepsilon\Gamma$.

A more precise estimate of Γc can be obtained by setting $\Gamma = 1$ in equation (20) and finding its point of intersection with the curve (c.f. equation (22))

$$\eta_1 \left[(1 + \frac{\eta_0}{\eta_1})^{\frac{1}{2}} - 1 \right] - \frac{\eta_0^2}{\eta_1} \frac{(\eta_1 - 1)}{(\eta_0 + 1)} + \frac{1}{\eta_1} \frac{(\eta_0 + \eta_1)}{(\eta_0 + 1)} (e^{-\eta_0/2} - 1) = 0 \ .$$

The values of η_0 and η_1 so obtained are the limiting values for infinitely massive ions;

$$\eta_{oc} = 2.05$$

$$\eta_{1c} = 1.16 \ .$$

Substituting these values into equation (19) then gives Γc correct to first order in the small parameter $\sqrt{\mu}$, namely

$$\Gamma_c = 1 - 8.3 \sqrt{\mu} \ . \qquad \ldots (24)$$

It can be shown that an increase in η_1 above its marginal stability value results in a lower η_0, a lower $E(0)$ and consequently a lower Γc. Thus the maximum emission current is drawn from the wall when the stability condition is marginally satisfied. This result is physically reasonable since an increase in v_0 must lead to a decrease in n_i within the sheath.

It has been assumed in the present model that there are no processes producing ions in the sheath region. If ions are produced they would tend to neutralise the electron space charge and this would modify the above results.

4. THE TRANSMITTED ENERGY FLUX

Consider the flux of energy Q flowing into the wall, i.e. across the plane $x = 0$. Each primary electron striking the wall conveys on average an energy of $2kT$ (thermal effusion). Each ion carries its initial energy E_0 plus the energy $|e\varphi_0|$ gained by falling through the sheath. Due to their low initial energy the emission electrons make a negligible contribution to Q at $x = 0$.

Thus

$$Q = J_{e1} \cdot 2kT + J_i \, (E_o + |e\varphi_o|)$$

$$= \tfrac{1}{2} n_o v_o \cdot kT \left[\eta_o + \eta_1 + \frac{4}{1 - \Gamma} \right] .$$

Since $v_o = (\pi\mu/8)^{\frac{1}{2}} \eta_1^{\frac{1}{2}} \bar{v}_e$, Q may be written in the form

$$Q = \tfrac{1}{4} n_o \bar{v}_e \cdot 2kT \cdot F(\Gamma) ,$$

where

$$F(\Gamma) = (\tfrac{\pi\mu}{8})^{\frac{1}{2}} \eta_1^{\frac{1}{2}} (\eta_o + \eta_1 + \frac{4}{1 - \Gamma}) .$$

For $\Gamma < \Gamma_c$, use can be made of equations (21) giving

$$F(\Gamma) \approx (\tfrac{\pi\mu}{8})^{\frac{1}{2}} \left[\ln \left\{ \frac{(1 - \Gamma)^2}{2\pi\mu} \right\} + \frac{5 - \Gamma}{1 - \Gamma} \right] . \qquad \ldots (25)$$

The limiting value of F, correct to first order in $\sqrt{\mu}$, is given by

$$F(\Gamma_c) = 0.33 + 2.2 \sqrt{\mu} . \qquad \ldots (26)$$

If space charge effects were completely absent, i.e. if the electrons could flow freely to the wall F would be unity. Thus $F < 1$ represents an improvement, in the context of insulation, over free thermal effusion of electrons. The largest, i.e. worst, value of F is obtained for hydrogen, and is 0.38.

5. EVALUATION OF Γ

The cold electrons emitted from the wall can be grouped into three categories: those produced by primary electron impact, those produced by ion impact and those produced by any other process. The emission flux can then be written, neglecting double sheath effects.

$$J_{e2} \equiv \Gamma J_{e1} = \gamma_e J_{e1} + \gamma_i J_i + J ,$$

where γ_e is the secondary emission coefficient for electron impact and γ_i is the corresponding quantity for ion impact. The emission current due to all other processes (e.g. photo-emission) is J.

The zero current condition gives

$$J_{e1} = \frac{1}{1 - \Gamma} J_i .$$

Hence

$$\Gamma = \frac{\gamma_e + \gamma_i + J/J_i}{1 + \gamma_i + J/J_i} .$$

Since $J_i = n_o v_o = n_o (\frac{kT}{m_i})^{\frac{1}{2}} \eta_1^{\frac{1}{2}}$, this can be written

$$\Gamma = \frac{\gamma_e + \gamma_i + j\,\eta_1^{-\frac{1}{2}}}{1 + \gamma_i + j\,\eta_1^{-\frac{1}{2}}} ,$$

where

$$j = \frac{J}{n_o (\frac{kT}{m_i})^{\frac{1}{2}}} .$$

For most purposes it will be sufficiently accurate to set $\eta_1 = 1$ giving, finally,

$$\Gamma = \frac{\gamma_e + \gamma_i + j}{1 + \gamma_i + j} . \qquad \ldots (27)$$

If, when computed, this quantity is found to exceed Γ_c for the particular gas of interest then it must be replaced by Γ_c in the knowledge that such a large emission current would be partially suppressed by space charge effects.

6. SUMMARY

When a steady state electrostatic sheath forms between a plasma and a wall in order to prevent a net flow of current it also acts as a thermal insulator between the hot electrons and the wall. It is convenient to write the energy flux to the wall in the form

$$Q = \frac{1}{4} n_o \bar{v}_e \cdot 2kT \cdot F .$$

In the absence of any electron emission from the wall the potential drop across the sheath is $\frac{kT}{e} \ln \frac{1}{\sqrt{2\pi\mu}}$ and the thermal transmission factor is given by

$$F(0) = (\frac{\pi\mu}{8})^{\frac{1}{2}} \left[5 + \ln \frac{1}{2\pi\mu} \right] ,$$

where μ is the ratio of electron to ion mass.

If, due to any physical process, the wall becomes a cold electron emitter the sheath potential is lowered and the thermal insulation impaired. It is not necessary to know the details of the emission mechanism in order to determine the sheath characteristics. A knowledge of Γ, the ratio of the total emission to primary electron current is all that is required. For values of $\Gamma < \Gamma_c$, where

$$\Gamma_c = 1 - 8.3 \sqrt{\mu} ,$$

the sheath potential and thermal transmission factor are given by

$$\varphi_o(\Gamma) = \frac{kT}{e} \ln \left\{ \frac{1 - \Gamma}{\sqrt{2\pi\mu}} \right\} ,$$

$$F(\Gamma) = \left(\frac{\pi\mu}{8}\right)^{\frac{1}{2}} \left[\frac{5 - \Gamma}{1 - \Gamma} + \ln \left\{ \frac{(1 - \Gamma)^2}{2\pi\mu} \right\} \right] .$$

Values of Γ in excess of Γ_c are not possible. If the wall emits electrons in such abundance that this limit would be exceeded a shallow potential well (i.e. a double sheath) forms near the wall. This reflects back a fraction of the electrons emitted in such a way that the ratio of emission to primary electron currents is limited to Γ_c. Under these circumstances

$$\varphi_o(\Gamma_c) \approx 1.02 \frac{kT}{e}$$

$$F(\Gamma_c) \approx 0.33 + 2.2 \sqrt{\mu}.$$

Since the maximum value of $\sqrt{\mu}$ is ~ 1/43 there exists (within the assumptions of the model) an absolute maximum, i.e. worst, value for F of 0.38.

The following table contains values of interest for hydrogen and deuterium.

	Hydrogen	Deuterium
Γ_c	0.81	0.86
$e\varphi_o(0)$ $e\varphi_o(\Gamma_c)$	2.8 kT 1.0 kT	3.2 kT 1.0 kT
$E_o(0)$ $E_o(\Gamma_c)$	0.50 kT 0.58 kT	0.50 kT 0.58 kT
F(0) $F(\Gamma_c)$	0.16 0.38	0.12 0.37

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APPENDIX

AN EXAMPLE OF ENHANCED ENERGY CONTAINMENT*

In an open-ended magnetic trap energy loss due to thermal conduction along the field lines to the end walls can severely limit the temperature attainable for a given power input. In this appendix a greatly simplified example of such a situation is considered and used to demonstrate how the insulation properties of a Langmuir sheath could result in the achievement of temperatures higher than would otherwise be obtained.

Consider an infinite slab of plasma bounded by walls at $x = \pm L$. It will be assumed that there is a uniformly distributed energy input to the electrons at a steady rate W and that the ion and electron temperatures are equal. If attention is confined to the regime defined by $\lambda/L > (m_e/m_i)^{\frac{1}{2}}$, where λ is the mean free path, the effects of convection and particle loss on the temperature distribution can be neglected and a steady state situation meaningfully studied. The temperature $T(x)$ is then given by

$$\frac{d}{dx} K \frac{dT}{dx} = - W , \qquad \text{... (A1)}$$

where it is convenient to write the thermal conduction coefficient in the form

$$K = K_o \left(\frac{T}{T_o}\right)^{5/2} ,$$

with

$$K_o \simeq \frac{3}{2} n_o k \lambda_o \bar{v}_{eo} ;$$

n is the electron density and $\bar{v}_e$ the mean electron thermal speed. The subscript zero indicates that a quantity is to be evaluated at $x = 0$. If we use the value of K for a hydrogen plasma corrected for the thermo-electric effect as given by Spitzer[7], then

$$\lambda_o \approx 2 \times 10^{12} T_o^2/n_o ,$$

where, if T_o is in eV and n_o in cm^{-3}, λ_o is in cms.

Equation (A1) can be integrated directly and if an energy input time τ is defined by

$$W\tau \approx \frac{3}{2} n_o kT_o$$

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*The authors wish to acknowledge that this appendix is based on a similar calculation for theta-pinch geometry carried out jointly with Drs. F.A. Haas and I.J. Spalding.

the result can be written $$\left(\frac{T}{T_o}\right)^{7/2} = 1 - \frac{7}{4}\left(\frac{L}{\lambda_o}\right)\left(\frac{\tau_o}{\tau}\right)\left(\frac{x}{L}\right)^2 \qquad \text{... (A2)}$$

where $\tau_o = L/\bar{v}_{eo}$ and is the mean transit time for electrons at the central temperature.

Equation (A2) determines the shape of the temperature distribution but not its absolute magnitude. The latter can be determined by applying the boundary condition

$$- K_1 \left(\frac{dT}{dx}\right)_1 = \frac{1}{4} n_1 \bar{v}_{e1} \cdot 2kT_1 \cdot F , \qquad \text{... (A3)}$$

where the subscript 1 denotes evaluation at $x = L$. If pressure balance is assumed, i.e. $n_oT_o = n_1T_1$, then equation (A3) combined with the first integral of equation (A1) gives

$$\left(\frac{T_1}{T_o}\right)^{1/2} = \frac{3}{F} \frac{\tau_o}{\tau} . \qquad \text{... (A4)}$$

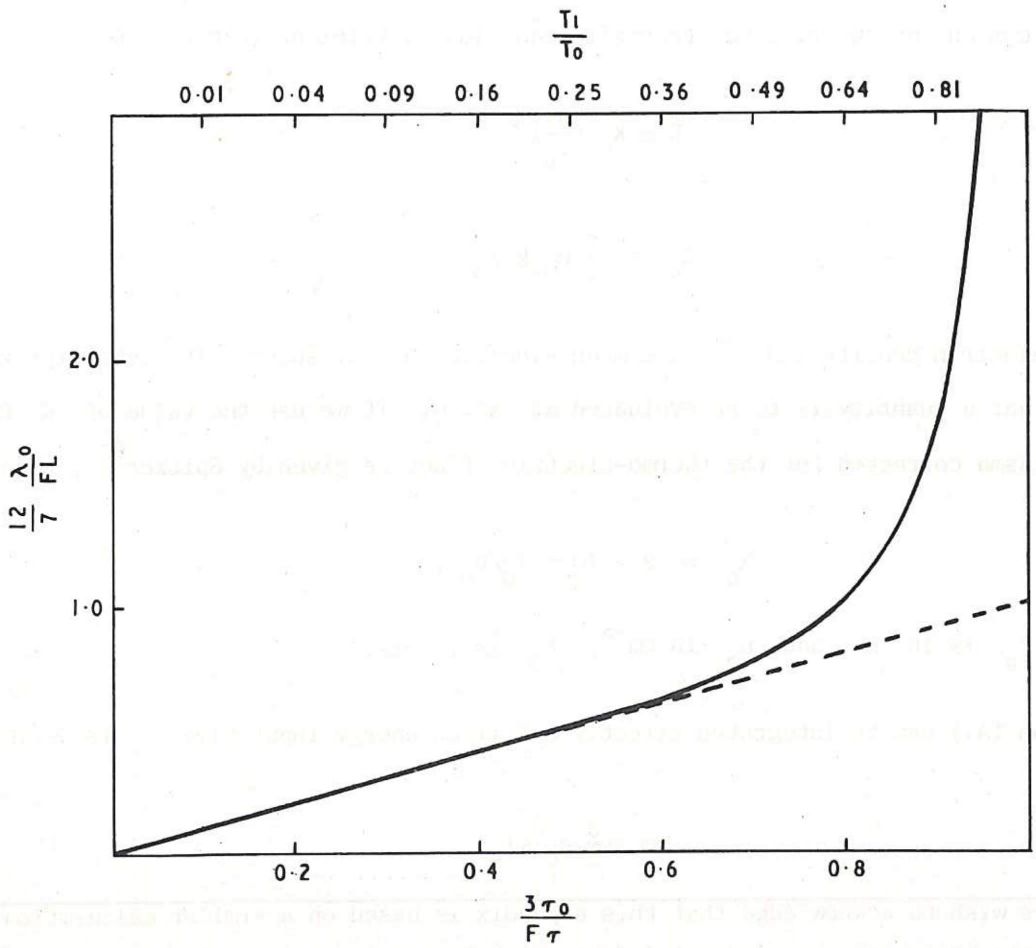
If equation (A4) is now substituted into equation (A2) evaluated at $x = L$, the following implicit equation is obtained for T_o:

$$\frac{4}{7} \frac{\lambda_o}{L} = \frac{(\tau_o/\tau)}{1 - (3\tau_o/F\tau)^7} . \qquad \text{... (A5)}$$

For hydrogen $\tau_o = 1.5 \times 10^{-8} L\, T_o^{-1/2}$, where τ_o is in secs if L is in cms and T_o in eV, hence T_o can be obtained explicitly:

$$T_o = \left[5.5 \times 10^{-2} WL^2 + \left(\frac{9 \times 10^{-8} L}{2F\tau}\right)^7\right]^{2/7} \qquad \text{... (A6)}$$

Equation (A5) is plotted in Fig.1.



The boundary condition conventionally applied at a wall is $T_1 = 0$ and the results analogous to equations (A5) and (A6) are then

$$\frac{4}{7}\,\frac{\lambda_o}{L} = \frac{\tau_o}{\tau} \qquad \ldots \text{(A7)}$$

and

$$T_o = \left[5.5 \times 10^{-2} WL^2\right]^{2/7} \qquad \ldots \text{(A8)}$$

Equation (A7) is plotted as the dashed line in Fig.1.

It is now clear that if $\tau_o/\tau \ll F/3$, i.e. if the energy input time is very long compared to the electron transit time, the energy loss rate and hence the central temperature T_o are determined by the thermal condition process alone; the wall sheath although present has no significant effect. If $\tau_o/\tau \sim F/3$, however, the sheath plays an important role in limiting the energy loss rate and much higher temperatures can be achieved than in its absence. In this regime T_o is given by

$$T_o = \left[1.87 \times 10^{-11} \frac{WL}{Fn_o}\right]^{2/3} .$$

Thus where previously (equation A8) T_o increased only as $W^{2/7}$ is now increased as $W^{2/3}$.

It should be noted that thermal conduction ceases to play an important part in determining T_o as soon as the temperature gradients in the plasma become small, i.e. as $T_1/T_o \to 1$. This occurs when $\lambda_o/L \sim F$. Thus provided F is small compared to unity, thermal conduction becomes unimportant before the physical concept becomes invalid, namely when $\lambda_o/L > 1$.

Throughout this calculation it has been implicitly assumed that any electrons emitted from one wall and then accelerated into the plasma will be thermalized. In the short mean free path limit the thermalization process will be Coulomb collisions and in the long mean free path limit it will be two stream instabilities.

The boundary condition commonly applied at a wall is $T_1 = 0$ and the results analogous to equations (35) and (36) are then

$$\frac{\sigma}{c} = \frac{\sigma_0^2}{L}\,\frac{\xi}{\zeta} \qquad \ldots (A7)$$

and

$$T_0 = [illegible]\; 2.5 \times 10^{-4} \; [illegible] \qquad \ldots (A8)$$

Equation (A7) is plotted as the dashed line in Fig. 1.

It is now clear that [illegible] in the [illegible] [illegible] [illegible] [illegible] [illegible] T_0 [illegible] [illegible] [illegible] [illegible] $T_0 \simeq 10^{-5}$, however, the [illegible] [illegible] [illegible] [illegible] [illegible]

[illegible]

[illegible]

[illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

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