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Report

ADIABATIC INVARIANTS AND THE EQUILIBRIUM OF MAGNETICALLY TRAPPED PARTICLES

2 MATHEMATICAL DETAILS

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PREFACE

In a recent paper* the relation of adiabatic invariants to the equilibrium of plasma in a magnetic field was discussed. This discussion led to a determination of the adiabatic invariants to higher order, which were found to exhibit interesting differences between the case of particles circulating round closed field lines and that of particles oscillating between mirrors. It also led to new form for the adiabatic invariant in fields with toroidal magnetic surfaces.

The details of these calculations were too lengthy for inclusion in the earlier paper, but as it is hoped to extend these calculations, notably in order to discuss stability - it seemed appropriate to record them fully. They are presented here in the form of appendices to the original paper, which is itself included for convenience and completeness.

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*Hastie, R.J., Taylor, J.B., and Haas, F.A. Adiabatic invariants and the equilibrium of magnetically trapped particles. Annals of Physics,

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ABSTRACT

This paper deals with two topics, firstly with the conditions for plasma equilibrium in an arbitrary magnetic field and their relation to the lowest order particle adiabatic invariants, secondly with the form of the higher order contributions to these adiabatic invariants. In part I the equilibrium conditions are investigated in a systematic way: as the time scale of equilibrium is increased the constraints on the distribution function become more severe until they culminate in the requirement that it be a function of the lowest order adiabatic invariants. In part II it is shown that this discussion of equilibrium leads to a convenient and novel way of generating the adiabatic invariants, not just to lowest order but including higher order contributions, for which a recurrence formula is derived. When the first correction to the longitudinal invariant $J = \oint v_{\parallel}\, ds$ is computed some interesting differences are found between the case of particles oscillating between mirrors and that of particles circulating round closed field lines. Part III discusses the effect of electric fields and the extension of the calculations to time dependent magnetic fields, leading to the third adiabatic invariant (the flux invariant). Part IV deals with the case of toroidal magnetic fields possessing magnetic surfaces and the form of longitudinal invariant appropriate in such a field. In the case of small rotational transform a modified line integral for J leads to a convenient description of particle motions in toroidal systems, including the effects of both rotational transform and guiding center drifts.

CONTENTS

PART I

1. INTRODUCTION

In discussions of low pressure plasma confined by magnetic mirrors two models have been widely used; each model leads to criteria which must be satisfied in equilibrium. In the fluid model the necessary and sufficient conditions [1] for equilibrium are

$$\frac{\partial p_{\parallel}}{\partial s} + \frac{(p_{\perp} - p_{\parallel})}{B}\frac{\partial B}{\partial s} = 0$$

and ... (1.1)

$$\int \nabla (p_{\perp} + p_{\parallel}) \cdot \frac{\underset{\sim}{B} \times \nabla B}{B^4} \, ds = 0$$

where the integral is along the line of force. On the other hand, in the guiding center model the necessary and sufficient condition for equilibrium [2,3] is that the guiding center distribution function F should depend only on μ, J, and the energy ε i.e.

$$F_{eq} \equiv F(\mu, \varepsilon, J) \quad . \qquad ...(1.2)$$

where $\mu \equiv v_{\perp}^2/2B$ and

$$J(\mu, \varepsilon, \alpha, \beta) \equiv \int_{\alpha,\beta} [2(\varepsilon - \mu B)]^{\frac{1}{2}} \, ds \qquad ... (1.3)$$

are the lowest order adiabatic invariants. [The magnetic field is $\underset{\sim}{B} = \nabla\alpha \times \nabla\beta$ and α, β label a line of force.]

The first part of the present work is an investigation of the relationship between these two equilibrium criteria and of the general rôle of adiabatic invariants in equilibria. The problem is approached through the Vlasov equation when the essential distinction between different approximate equilibria is the time for which the distribution function f can be regarded as stationary. The time scales of interest are investigated systematically by expanding f in powers of m/e, which is equivalent to an expansion in powers of the Larmor radius [4]. As expected, we find that as the time scale is lengthened, increasingly restrictive conditions are imposed on the distribution function, culminating in restrictions equivalent to those of the guiding center model. The fluid conditions do <u>not</u> emerge directly but nevertheless are shown to be appropriate to an intermediate time scale in most circumstances.

In the second part of this paper we show that this discussion of equilibrium leads naturally to a novel and powerful way of obtaining expressions for the adiabatic invariants

$\hat{\mu}$ and $\hat{J}$, not merely to lowest order in m/e but including higher order contributions[†], for which a formal recurrence relation is derived. This is possible because the invariants, while not true constants, are constant to all orders[5]. Hence, within the scope of any expansion $F(\hat{\mu}, \hat{J}, \varepsilon)$ can be regarded as an exact equilibrium and by comparing this form with the results of the direct m/e expansion one can recognise the exact particle invariants $\hat{\mu}$ and $\hat{J}$, even though the concept of such invariants was not originally introduced into the calculations. This method of obtaining invariants by first finding an equilibrium distribution circumvents the necessity for any calculation of orbits, either of particles or of guiding centres, and may have application in other problems.

Using this method we have explicitly calculated the first order corrections μ_1 and J_1 and the second order correction μ_2. An unexpected feature of our results is that the correction J_1 to the longitudinal adiabatic invariant has one form for particles which are trapped between magnetic mirrors and another form for particles which circulate unidirectionally around a closed field line.

In Part III we extend the calculation to include the effect of electric fields and time varying magnetic fields. This allows us to derive the third (flux) invariant Φ by a natural extension of the methods of Parts I and II. If the time scale for the field variation is sufficiently slow this third invariant replaces the particle energy as a "constant of the motion" and a solution of Vlasov's equation is of the form $f = f(\mu, J, \Phi)$.

Part IV deals with toroidal magnetic fields possessing magnetic surfaces and the longitudinal invariant (replacing J) appropriate in such a field. One possibility is that J should be replaced by an integral over a magnetic surface, but in the case of small rotational transform a modified line integral is more appropriate. This line integral J^{**} provides a convenient description of particle behaviour in toroidal systems.

2. PLASMA EQUILIBRIUM

In the fluid description of low pressure plasma the necessary and sufficient conditions for equilibrium are those[1] given by equations (1.1), whereas in the guiding center (g.c.) description of the plasma, an equilibrium is described[2,3] by any distribution function of the form (1.2). It may easily be shown, by direct substitution, that a g.c. equilibrium

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(†) It is important to distinguish between the quantities μ and J which are defined e.g. by equation (1.3) and the true adiabatic invariants denoted here by $\hat{\mu}$ and $\hat{J}$. The true invariants are equal to μ, J only to lowest order in the m/e expansion. It might be more appropriate to use μ_0, J_0 instead of μ, J but we wish to avoid excessive subscripts. Thus $\hat{\mu} \equiv \mu + \frac{m}{e}\mu_1 + (\frac{m}{e})^2\mu_2 \ldots$, $\hat{J} \equiv J + \frac{m}{e}J_1 + (\frac{m}{e})^2 J_2 \ldots$

distribution automatically satisfies the fluid constraints. On the other hand, there are many g.c. distributions which are not of the form (1.2) but which nevertheless lead to pressures satisfying the conditions for fluid equilibrium. Indeed BenDaniel[6] has pointed out that if the g.c. distribution leads to isotropic pressure it cannot be expressed in the form (1.2) even though the corresponding pressure distribution may satisfy (1.1). One can also construct g.c. distribution functions which lead to anisotropic pressure distributions satisfying (1.1) but which cannot be written in the form (1.2). An example, discussed in Appendix A, is the distribution function

$$F_i = H_i\ (\mu,\ \varepsilon)\ Q\ (\ \langle J \rangle\ ) \qquad \ldots\ (2.1)$$

where

$$\langle J \rangle = \sum m_i \int H_i\ (\mu,\ \varepsilon)\ J\ (\mu,\ \varepsilon,\ \alpha,\ \beta)\ d\mu\ d\varepsilon \qquad \ldots\ (2.2)$$

which cannot, in general, be expressed in the form of a g.c. equilibrium; but nevertheless leads to a pressure tensor satisfying the fluid constraints (1.1). It is clear then, that the fluid and guiding center descriptions are not equivalent and in the following section we will show that this is because they refer to different time scales.

3. LARMOR RADIUS EXPANSION

The physically interesting time scales are those set by the Larmor frequency ω_c, the frequency of motion along the lines of force, $v_{\parallel}/L$ and the frequency with which the guiding center drifts around the system, V_d/L. These three frequencies may also be expressed as

$$\omega_c \quad , \quad \frac{a}{L}\ \omega_c \quad , \quad \frac{a^2}{L^2}\ \omega_c$$

where a is the typical Larmor radius so that the criteria for equilibrium on these various time scales can be systematically examined by means of an expansion in powers of a/L. An equivalent but more convenient procedure is to expand in powers of m/e, so following and extending the procedure of Chandrasekhar, Kaufman and Watson[4]. By imposing order by order the condition that the distribution function be stationary we obtain a sequence of constraints appropriate to equilibrium on the various time scales.

The first step is to introduce an appropriate coordinate system (Appendix B). Each line of force is labelled by α, β where $\underset{\sim}{B} = \nabla\alpha \times \nabla\beta$, and the distance s is measured along each field line from some fixed plane, then $(\alpha,\ \beta,\ s)$ are used as position coordinates. Velocity is expressed in terms of $(\varepsilon,\ \mu,\ \varphi)$ where $\varepsilon \equiv \frac{1}{2}\ (v_\perp^2 + v_\parallel^2)$, $\mu \equiv v_\perp^2/2B$ and φ is the azimuthal angle about the field direction $\underset{\sim}{e}_1$. A feature of these velocity coordinates is that the transformation from $(\varepsilon, \mu, \varphi)$ to $\underset{\sim}{v}$ is two-valued, since $v_{\parallel} = \pm [2(\varepsilon - \mu B)]^{1/2}$. To deal with this we explicitly introduce an extra "coordinate" σ which takes only the values ± 1 and indicates which branch of the square root is to be taken. All quantities are thus functions of $(\alpha, \beta, s, \varepsilon, \mu, \varphi, \sigma)$. Although it has been introduced here purely as a formal device to remove an ambiguity in sign, σ will later play a much more fundamental rôle; in some respects it behaves as a constant of the motion along with ε and μ.

When expressed in these coordinates the Vlasov equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \frac{e}{m} (\underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = 0$$

can be written

$$\frac{\partial f}{\partial \varphi} = \lambda \left(D f + \frac{1}{B} \frac{\partial f}{\partial t}\right) \qquad \ldots (3.1)$$

where $\lambda \equiv m/e$ can be regarded as a formal expansion parameter equivalent to a/L. The operator D is defined by:

$$\begin{aligned} Df = \frac{1}{B} \Big\{ \sigma q \frac{\partial f}{\partial s} + c_\perp \Big(\nabla f - \underset{\sim}{V} \frac{\partial f}{\partial \mu}\Big) \cdot (\underset{\sim}{e}_2 \cos\varphi + \underset{\sim}{e}_3 \sin\varphi) + \mu\sigma q \frac{\partial f}{\partial \mu} \left[(\rho_3 - \rho_2)\cos 2\varphi + (\tau_2 - \tau_3)\sin 2\varphi\right] \Big\} \\ + \frac{1}{B} \Big\{ \sigma q (\tau_1 + \tfrac{1}{2}(\tau_2 + \tau_3)) + \Big(\frac{q^2}{c_\perp} \sigma_1 - c_\perp \sigma_2\Big)\cos\varphi + \Big(\frac{q^2}{c_\perp} \rho_1 - c_\perp \sigma_3\Big)\sin\varphi + \tfrac{1}{2} \sigma q [(\tau_2 - \tau_3)\cos 2\varphi \\ + (\rho_2 - \rho_3) \sin 2\varphi] \Big\} \frac{\partial f}{\partial \varphi} \end{aligned} \qquad \ldots (3.2)$$

where

$$q \equiv [2(\varepsilon - \mu B)]^{1/2} \quad , \quad c_\perp \equiv (2\mu B)^{1/2} \quad , \quad \underset{\sim}{V} \equiv \frac{\mu}{B} \nabla B + \frac{q^2}{B} \underset{\sim}{e}_1 \cdot \nabla \underset{\sim}{e}_1 \quad . \qquad \ldots (3.3)$$

The unit vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ are orthogonal, with $\underset{\sim}{e}_1 \equiv \underset{\sim}{B}/B$, and the coefficients ρ_i, σ_i, τ_i are related to the curvature and torsion of the lines of force as described in Appendix B. The velocity $\underset{\sim}{V}$ is related to the usual guiding center drift $\underset{\sim}{V}_d$ by

$$\frac{e}{m} \underset{\sim}{V}_d = (\underset{\sim}{e}_1 \times \underset{\sim}{V}) \quad . \qquad \ldots (3.4)$$

We shall look for solutions of (3.1) in the form

$$f = f_0 + \lambda f_1 + \lambda^2 f_2 + \ldots \qquad \ldots (3.5)$$

The time dependence of f_0 may be regarded as being of any order in λ, depending on the time scale one wishes to investigate. We shall regard $\partial f_0 / \partial t$ as negligible to successively higher orders in λ and so obtain criteria for equilibrium on successively longer time scales.

(a) Zero Order

In the lowest order we have

$$\frac{\partial f_o}{\partial \varphi} = \frac{m}{eB}\frac{\partial f_o}{\partial t} \qquad \ldots (3.6)$$

indicating merely that if f_o is to be stationary on the time scale of the Larmor period then the appropriate constraint is that it must be independent of the azimuthal angle φ.

(b) First Order

In the next order

$$\frac{\partial f_1}{\partial \varphi} = D\,f_o + \frac{1}{B}\left(\frac{\partial f_o}{\partial t}\right)_1 \qquad \ldots (3.7)$$

so that if f is again stationary we have

$$f_1 = g_1\,(\alpha,\beta,s,\mu,\varepsilon,\sigma) + \int d\varphi\, D\,f_o \qquad \ldots (3.8)$$

and, since f_1 must be single valued,

$$\langle D\,f_o \rangle = 0 \qquad \ldots (3.9)$$

where g_1 is an arbitrary function of the indicated variables only, and the angular brackets denote the average over the angle φ. Since f_o is already independent of φ (3.9) becomes

$$\langle D\,f_o \rangle \equiv \sigma\frac{q}{B}\frac{\partial f_o}{\partial s} = 0 \quad . \qquad \ldots (3.10)$$

Hence the requirement that $\partial f_o/\partial t = 0$ in this order, (corresponding to equilibrium on the time scale $L/V_{\parallel}$) imposes the constraint that f_o must be independent of both φ and s, and that f_1 be given by

$$f_1 = g_1 + \int D\,f_o \quad . \qquad \ldots (3.11)$$

(here and henceforth, if the variable of integration is not explicitly indicated it is understood to mean integration with respect to φ; other variables of integration will be explicitly indicated). Equation (3.10) indicates, through its Lagrangian subsidiary equations, that in this order, α, β, μ, ε, σ are all "constants of the motion" in accordance with the physical picture of a g.c. tied to a line of force. [But note that neither here nor elsewhere do we introduce the g.c. concept directly.]

(c) Second Order

In this order we have

$$\frac{\partial f_2}{\partial \varphi} - \frac{1}{\lambda B}\frac{\partial f_o}{\partial t} = D\,f_1 = D\,g_1 + D\!\int D\,f_o \qquad \ldots (3.12)$$

so that if f is stationary in this order we have

$$f_2 = g_2 (\alpha\ \beta\ s, \mu\ \varepsilon\ \sigma) + \int D\ g_1 + \int D \int D\ f_o\ . \qquad \ldots (3.13)$$

To ensure that f_2 be single valued we must have

$$\langle Dg_1 \rangle + \langle D \int D\ f_o \rangle = 0 \qquad \ldots (3.14)$$

or

$$\frac{\sigma q}{B} \frac{\partial g_1}{\partial s} + \langle D \int D\ f_o \rangle = 0 \qquad \ldots (3.15)$$

so that

$$g_1 = h_1 (\alpha, \beta, \mu, \varepsilon, \sigma) - \sigma \int_{s_o}^{s} \frac{Bds}{q} \langle D \int D\ f_o \rangle \qquad \ldots (3.16)$$

where h_1 is arbitrary.

Equation (3.16) leads to a new constraint on f_o; in its simplest form this would arise, e.g., from the requirement that g_1 be single valued in s and would be

$$\oint \frac{Bds}{q} \langle D \int D\ f_o \rangle = 0 \qquad \ldots (3.17)$$

where the integral is around a closed field line. If $\varepsilon > \mu B$ everywhere along the field line then the constraint does take the simple form (3.17). This is applicable when particles circulate unidirectionally around a closed field line without undergoing mirror reflection. In this case the particle streams moving in either direction, represented by $\sigma = \pm 1$, are independent and the constraint (3.17) applies to each direction individually.

When particles are reflected by magnetic mirrors the situation is less simple because $\varepsilon < \mu B$ over part of the range of s and q becomes imaginary. Physically the two streams represented by $\sigma = \pm 1$ are no longer independent and it is the coupling between them which now leads to a constraint. Mathematically this constraint arises because the two branches ($\sigma = \pm 1$) of the distribution function coincide whenever $\varepsilon = \mu B$ and f_o is independent of σ at these "turning points". However f_o does not vary with s so it must be independent of σ whenever it refers to particles trapped between mirrors.

Similarly g_1 is independent of σ at a turning point (but not elsewhere) and the change in g_1 between turning points must be the same for both $\sigma = \pm 1$. This leads, from (3.16) to the condition that

$$+\int_A^B \frac{Bds}{q} \Big[\langle D \int D\ f_o \rangle \Big]_{\sigma=+1} = -\int_A^B \frac{Bds}{q} \Big[\langle D \int D\ f_o \rangle \Big]_{\sigma=-1} \qquad \ldots (3.18)$$

or

$$\sum_{\sigma=\pm 1} \int \frac{B ds}{q} \left\langle D \int D f_o \right\rangle = 0 \qquad \text{... (3.19)}$$

where the integral is between turning points. For brevity we introduce the operator

$$L f \equiv \left\langle D \int D f \right\rangle \qquad \text{... (3.20)}$$

then we can summarize the second order constraints on f_o as:

(i) For particles which circulate round a closed field line:

$$\left[\oint \frac{B ds}{q} L f_o \right]_{\sigma=+1} = \left[\oint \frac{B ds}{q} L f_o \right]_{\sigma=-1} = 0 \quad . \qquad \text{... (3.21)}$$

(ii) For particles trapped between mirrors:

$$\sum_{\sigma=\pm 1} \left[\int \frac{B ds}{q} L f_o \right] = 0 \quad \text{and} \quad \left[f_o \right]_{\sigma=+1} = \left[f_o \right]_{\sigma=-1} . \qquad \text{... (3.22)}$$

In fact the operator L is independent of σ but we leave these constraints in the general form so that we may refer to them in connection with other operators which will arise later.

The operator L is discussed in the Appendix; when operating on a function such as f_o (μ, ε, α, β), i.e. one independent of s, φ it can be expressed as

$$L f_o = \frac{e V_d \cdot \nabla f_o}{m B} + \mu \left(\frac{\partial f_o}{\partial \mu} \right)_{\alpha\beta} \frac{q}{B} \frac{\partial}{\partial s} \left(\frac{q \, e_1 \cdot \text{curl} \, e_1}{B} \right) \qquad \text{... (3.23)}$$

where $\dfrac{eV_d}{m}$ is the drift velocity defined earlier. Inserting this expression for $L f_o$ into the constraints (3.21) and (3.22) one finds that either constraint can be concisely expressed in the α, β, coordinate system as

$$\left(\frac{\partial f_o}{\partial \alpha} \frac{\partial J}{\partial \beta} - \frac{\partial f_o}{\partial \beta} \frac{\partial J}{\partial \alpha} \right) = 0 \qquad \text{... (3.24)}$$

where J is defined by

$$J\,(\alpha, \beta, \mu, \varepsilon) = \oint q \; ds \quad , \qquad \text{... (3.25)}$$

the integration being around a closed field line or between turning points as appropriate.

According to equation (3.24) f_o does not depend on α, β individually but only on J (α, β, μ, ε); the lower order constraints already make f_o independent of φ and s and we therefore conclude that it must be of the g.c. equilibrium form.

So far, then, we have shown that if equilibrium is to persist on the L/V_d time scale:-

$$f_o \equiv f_o (J, \mu, \varepsilon, \sigma) \qquad \ldots (3.26)$$

$$f_1 = h_1 (\alpha, \beta, \mu, \varepsilon, \sigma) - \sigma \int_{s_o}^{s} \frac{Bds}{q} \; L f_o + \int D f_o \qquad \ldots (3.27)$$

$$f_2 = g_2 (\alpha, \beta, s, \mu, \varepsilon, \sigma) + \int D f_1 \qquad \ldots (3.28)$$

where f_o, h_1 and g_2 are arbitrary. For particles trapped between mirrors f_o is independent of σ and it is convenient to take the lower limit of integration, s_o, at a turning point; then h_1 is also independent of σ. For particles which are not reflected by mirrors (circulating particles) the choice of s_o must be left arbitrary.

If we had retained time dependence of f_o in this order we would have obtained, instead of (3.24), the equation

$$\frac{e}{m} \frac{\partial f_o}{\partial t} \frac{\partial J}{\partial \varepsilon} + \left(\frac{\partial f_o}{\partial \alpha} \frac{\partial J}{\partial \beta} - \frac{\partial f_o}{\partial \beta} \frac{\partial J}{\partial \alpha} \right) = 0 \quad . \qquad \ldots (3.29)$$

and the Lagrangian subsidiary equations then give the time derivatives

$$\bar{\dot{\alpha}} = + \frac{m}{e} \left(\frac{\partial J}{\partial \beta}\right) \left(\frac{\partial J}{\partial \varepsilon}\right)^{-1} \quad , \quad \bar{\dot{\beta}} = - \frac{m}{e} \left(\frac{\partial J}{\partial \alpha}\right) \left(\frac{\partial J}{\partial \varepsilon}\right)^{-1} \qquad \ldots (3.30)$$

and so $\dot{J} = 0$. (The $\bar{\dot{\alpha}}$ and $\bar{\dot{\beta}}$ are of course, just the first order guiding center drift velocities averaged over the oscillation between mirrors or around the line of force.) Consequently μ, J and ε are the appropriate constants of the motion on the drift time scale and f_o is a function of these constants. For circulating particles σ is also a "constant of the motion" and so also appears in f_o.

(d) Third Order

So far in our analysis we have shown that as we increase the order, i.e. lengthen the time scale of equilibrium, we must impose increasingly severe constraints on f_o: however this process does not continue indefinitely as we shall now demonstrate.

In the third order, with $\partial f_o/\partial t = 0$, the initial form of the constraint condition is, as in previous orders,

$$\langle D f_2 \rangle = 0 \qquad \ldots (3.31)$$

or

$$\frac{\sigma q}{B} \frac{\partial g_2}{\partial s} = - L h_1 + \sigma L \int_{s_o}^{s} \frac{Bds}{q} L f_o - \left\langle D \int D \int D f_o \right\rangle \quad . \qquad \ldots (3.32)$$

This equation determines g_2 and leads, by precisely the same arguments as applied to equation (3.15) for g_1, to constraints similar to (3.21) or (3.22). All that is necessary to obtain these new constraints is to replace the operator L in (3.21) or (3.22) by the right hand side of (3.32). For particles which are not reflected at mirrors this yields;

$$\oint \frac{Bds}{q} \left\{ L\, h_1 - \sigma L \int \frac{Bds}{q}\, L\, f_o + \left\langle D \int D \int D\, f_o \right\rangle \right\} = 0 \qquad \ldots (3.33)$$

while for mirror trapped particles this must be summed over $\sigma = \pm 1$.

Now at this stage h_1 is a function of the same form as was f_o in equations (3.21) and (3.22) so that when $L\, h_1$ is evaluated and expressed in α, β coordinates, equation (3.33) becomes

$$\left(\frac{\partial h_1}{\partial \alpha} \frac{\partial J}{\partial \beta} - \frac{\partial h_1}{\partial \beta} \frac{\partial J}{\partial \alpha} \right) = \oint \frac{Bds}{q} \left\{ \sigma L \int \frac{Bds}{q}\, L\, f_o - \left\langle D \int D \int D\, f_o \right\rangle \right\} \equiv H\, f_o \qquad \ldots (3.34)$$

This is an equation of a type we have not met before. The terms in h_1 constitute the derivative along the direction $J = \text{constant}$ so that

$$h_1 = k_1 (\mu, \varepsilon, J) + \int_{J = \text{constant}} \frac{d\alpha}{\partial J / \partial \beta}\, H\, f_o \qquad \ldots (3.35)$$

which determines h_1, and hence f_1, up to an arbitrary function of μ, ε, J. At first sight it may appear that (3.35) does impose an additional constraint on f_o if the surfaces $J = \text{constant}$ (which correspond to precessional drift surfaces) are closed. In this event h_1 can be single valued only if f_o satisfies

$$\oint_{J = \text{constant}} \frac{d\alpha}{\partial J / \partial \beta}\, H\, f_o = 0 \quad . \qquad \ldots (3.36)$$

However when the operator H (Appendix C) is evaluated in full, a lengthy calculation shows that this constraint is automatically satisfied by any function of the form $f_o \equiv f_o(\mu, \varepsilon, J)$; in the case of mirror trapped particles one finds that $H\, f_o$ is identically zero while for circulating particles it can be expressed in the form

$$H\, f_o = \frac{\partial J}{\partial \beta} \cdot \frac{\partial}{\partial \alpha} \left(P \frac{\partial f_o}{\partial J} \right) - \frac{\partial J}{\partial \alpha} \cdot \frac{\partial}{\partial \beta} \left(P \frac{\partial f_o}{\partial J} \right) \qquad \ldots (3.37)$$

(where P is defined in Appendix C). In either case, therefore, the loop integral (3.36) vanishes identically.

Another form for (3.37) is

$$H\, f_o = \left(\frac{\partial P}{\partial \alpha} \frac{\partial f_o}{\partial \beta} - \frac{\partial P}{\partial \beta} \frac{\partial f_o}{\partial \alpha} \right) \qquad \ldots (3.38)$$

which indicates that the velocities defined by

$$\bar{\dot{\alpha}}_2 = (\frac{m}{e})^2 \frac{\partial P}{\partial \beta} (\frac{\partial J}{\partial \varepsilon})^{-1} \qquad \bar{\dot{\beta}}_2 = -(\frac{m}{e})^2 \frac{\partial P}{\partial \alpha} (\frac{\partial J}{\partial \varepsilon})^{-1} \qquad \text{... (3.39)}$$

must represent the second order drift velocity, averaged over the motion along the line of force, just as (3.30) represented the average of the first order drift velocity. The vanishing of (3.36) shows that the second order drifts produce no <u>cumulative</u> displacement from the first order drift surfaces.

However the most important feature of this section is that on carrying the expansion to a higher order <u>we have not on this occasion needed to impose any new restriction on f_o</u>. Instead we find that f_1 is now determined apart from a function of the same form as f_o and which could be absorbed into f_o if desired. We will later indicate why no further restrictions on f_o are to be expected even if we were to calculate to still higher orders and will show that restricting f_o to be of the form $f_o(\mu, \varepsilon, J)$ is sufficient for equilibrium to all orders. For the moment, however, we anticipate this result and turn our attention to the question of the fluid constraints.

4. THE FLUID EQUATIONS

It has been shown that if one requires equilibrium to persist for increasingly longer times then successively more stringent constraints must be imposed on f_o, culminating in the requirement that it be of the g.c. form (1.2); however, the fluid constraints (1.1) have not appeared in the intermediate time scales, as might have been expected. This is because we have so far been concerned only with the particle distribution function and have not considered the electromagnetic fields.

Electromagnetic fields

When the electric field is included a new physical time scale is introduced - by the plasma frequency ω_p - and the m/e expansion must be extended to incorporate this. If ω_p is comparable with ω_c this can be done most simply by formally regarding the m/e expansion as one in which $e \to \infty$ (m finite) for then both $\omega_p \to \infty$ and $\omega_c \to \infty$ but ω_p/ω_c remains finite. In a similar way the case $\omega_p \ll \omega_c$ can be dealt with by regarding the expansion as one in which $m \to 0$ (e finite), for then $\omega_p \to \infty$, $\omega_c \to \infty$ but $\omega_p/\omega_c \to 0$.

The condition for the electric fields to be stationary is

$$\frac{\partial \rho}{\partial t} = \sum_i e_i \int \frac{\partial f_i}{\partial t} \, d^3v = 0 \qquad \text{... (4.1)}$$

and because e, but not m, appears in this equation there are differences between the theory with $\omega_p \sim \omega_c$ and that with $\omega_p \ll \omega_c$. In a situation where $\omega_p \ll \omega_c$ the charge e is treated as finite; then (4.1) shows that $\partial\rho/\partial t$ will vanish to any order so long as $\partial f/\partial t$ does so. Consequently there is nothing to be added to the discussion of equilibrium criteria and all the conclusions reached in Section 3 are unchanged. On the other hand, in a situation where $\omega_p \sim \omega_c$, the charge e must be treated as a large quantity; then equation (4.1) indicates that $\partial\rho/\partial t$ is one order lower than $\partial f/\partial t$. Consequently, the vanishing of $\partial f/\partial t$ to a given order only ensures that $\partial\rho/\partial t$ vanishes to one order lower and equilibrium can only be ensured by making both $\partial f/\partial t$ and $\partial\rho/\partial t$ vanish to the appropriate order.

In the case of equilibrium on the drift time scale, when f_o is already constrained to be of the g.c. form $f_o\ (\mu, \varepsilon, J)$, no new constraint is needed to ensure that $\partial\rho/\partial t = 0$, for it has been shown that if $f_o \equiv f_o\ (\mu, \varepsilon, J)$ the distribution is stationary not merely on the drift time scale but also to one order higher (indeed to all orders, as we shall see later). Consequently the criteria for equilibrium on the drift time scale are unaltered by the inclusion of electric fields.

However, in discussing equilibria on the intermediate time scale $v_{\|}/L$ an alteration is necessary; not only is it necessary that $f_o \equiv f_o\ (\mu, \varepsilon, \alpha, \beta)$, so making $\partial f/\partial t$ zero to order λ, but $\partial\rho/\partial t$ must also be zero to order λ. This leads to an extra constraint which is easily found by retaining the time dependence of f_o in equation (3.15) which then becomes

$$\frac{e}{mB}\ \frac{\partial f_o}{\partial t} + \frac{\sigma q}{B}\ \frac{\partial g_1}{\partial s} + \left\langle D \int D\, f_o \right\rangle = 0 \qquad \ldots (4.2)$$

The constraint $\partial\rho/\partial t = 0$ is therefore

$$\frac{\partial\psi}{\partial s} + \sum_{i,\sigma} m_i \int \frac{B\, d\mu\, d\varepsilon}{q} \left\langle D \int D\, f_o^i \right\rangle = 0 \qquad \ldots (4.3)$$

(where

$$\psi = \sum_{i,\sigma} \sigma \int m_i\, g_1^i\, d\mu \quad d\varepsilon \qquad \ldots (4.4)$$

is essentially the current parallel to B). The existence of a single valued ψ which vanishes in the vacuum surrounding the plasma requires the integral over s of the second term in (4.3) to vanish. When the appropriate form (3.23) is inserted for $\left\langle D \int D\, f_o \right\rangle$ the resulting constraint can be written entirely in terms of <u>macroscopic</u> quantities as

$$\int \frac{ds}{B^3}\ \nabla(p_{\perp} + p_{\|}) \cdot \underset{\sim}{B} \times \underset{\sim}{\rho} = 0 \qquad \ldots (4.5)$$

which will be recognised as an alternative expression for the fluid constraint (1.1). (This form (4.5) is applicable at finite pressure whereas (1.1) applies only in the low pressure limit.)

From this discussion we conclude that when the time dependence of electromagnetic fields is considered the guiding center constraint remains sufficient and necessary for equilibrium to all orders. However the weaker constraint $f_o = f_o(\mu, \varepsilon, \alpha, \beta)$, which was previously adequate for equilibrium on the time scale $L/V_{\parallel}$, now needs to be supplemented by the fluid constraint (1.1) unless the plasma density is so low that $\omega_p \ll \omega_c$.

PART II

5. ADIABATIC INVARIANTS AND EQUILIBRIA IN HIGHER ORDERS

In Part I it was shown that as the time scale of equilibrium was lengthened by going to higher orders in λ, the restrictions on f_o became increasingly severe until it was restricted to the guiding center form. However, once this point had been reached, an extension by another order imposed no extra restrictions on f_o; instead the restrictions affected f_1 which was thereby determined in terms of f_o (apart from an arbitrary function of μ, ε, J, which could be regarded as part of f_o). It was suggested that no matter how far the calculation was pursued no further restrictions on f_o would be found.

That this conclusion is correct is indicated by the following; it is well known[5] that invariant quantities $\hat{\mu}$ and $\hat{J}$ exist which are constant to all orders in m/e and which are identical in lowest order with the μ, J defined in Part I. Therefore, within the framework of any m/e expansion scheme, a distribution such as

$$\psi = \psi\ (\hat{\mu}, \hat{J}, \varepsilon) \qquad \ldots (5.1)$$

can be regarded as an exact equilibrium, and if we put

$$\hat{\mu} = \mu_o + \lambda\,\mu_1 + \lambda^2\,\mu_2 \ldots. \qquad \ldots (5.2)$$

$$\hat{J} = J_o + \lambda\,J_1 + \lambda^2\,J_2 \ldots. \qquad \ldots (5.3)$$

(where for emphasis we now write μ_o, J_o for the zero order invariants μ, J) we see that an equilibrium correct to all orders can be expressed in the form

$$\psi\left(\mu_o, J_o, \varepsilon\right) + \lambda\left(\mu_1 \frac{\partial\psi}{\partial\mu_o} + J_1 \frac{\partial\psi}{\partial J_o}\right) + \lambda^2 \ldots. \qquad \ldots (5.4A)$$

This general equilibrium thus contains one arbitrary function of three variables and is completely defined once its lowest order form is given. However if ψ is regarded as explicitly dependent on λ (5.4A) can be written in the form

$$\psi_o\ (\mu_o, J_o, \varepsilon) + \lambda\ \left(\mu_1 \frac{\partial\psi_o}{\partial\mu_o} + J_1 \frac{\partial\psi_o}{\partial J_o} + \psi_1\ (\mu_o, \varepsilon, J_o)\right) + \lambda^2 \ldots \qquad \ldots (5.4B)$$

though this corresponds only to a relabelling of the equilibria and by summing $\psi_o + \lambda\,\psi_1 + \lambda^2\,\psi_2 \ldots$ (5.4B) can always be re-cast into the form (5.4A).

Clearly, the lowest order term in (5.4) must be identified with the f_o of Part I; it is then apparent that even if one demands equilibrium to all orders f_o remains an arbitrary function of μ_o, J_o, ε .

It is also clear the higher order corrections $\mu_1, J_1; \mu_2, J_2; \ldots$ can be obtained by identifying higher order terms of the series $f_o + \lambda f_1 + \lambda^2 f_2 \ldots$ which have already been calculated in Part I, with the corresponding terms in (5.4). There are marked differences between the case of particles trapped between mirrors and those which circulate round a toroidal system and we first consider only particles trapped between mirrors.

Mirror-trapped particles

Collecting together results from Part I we have

$$f_1 = k_1 (\mu_o, J_o, \varepsilon) + \int Df_o - \sigma \int_{s_o}^{s} \frac{Bds}{q} Lf_o + \int_{J = \text{const}} \frac{d\alpha}{\partial J/\partial \beta} Hf_o \qquad \ldots (5.5)$$

where the various operators have been introduced in Part I and Appendix C. For convenience* we are taking s_o to be a turning point $(q = 0)$ as this simplifies the evaluation of H, – in fact it then vanishes identically for trapped particles. When the operators D and L are explicitly introduced into (5.5) one finds, significantly, that f_1 depends only on derivatives $\partial f_o/\partial \mu_o$ and $\partial f_o/\partial J_o$ although higher derivatives appear in the individual operators. Consequently f_1 as given by (5.5) is, indeed, of the functional form indicated by (5.4) and can be completely identified with (5.4) by setting

$$\mu_1 = -\frac{1}{B}\left[\underset{\sim}{v}_\perp \cdot \underset{\sim}{W}_d + \frac{\underset{\sim}{v} \cdot \underset{\sim}{e}_1}{4}\left\{\underset{\sim}{v}_\perp \cdot (\underset{\sim}{a} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{e}_1 + \underset{\sim}{a} \cdot (\underset{\sim}{v}_\perp \cdot \underset{\sim}{\nabla})\, \underset{\sim}{e}_1 + 4\mu_o\, (\underset{\sim}{e}_1 \cdot \underset{\sim}{\nabla} \times \underset{\sim}{e}_1)\right\}\right] \qquad \ldots (5.6)$$

and

$$J_1 = \underset{\sim}{a} \cdot \nabla J_o + \mu_1 \left(\frac{\partial J_o}{\partial \mu_o}\right)_{\alpha\beta\varepsilon} - \sigma \int_{s_o}^{s} \frac{ds}{q} \underset{\sim}{W}_d \cdot \nabla J_o \qquad \ldots (5.7)$$

where

$$J_o = \oint [2(\varepsilon - \mu_o B)]^{1/2} ds \quad , \qquad \ldots (5.8)$$

$$\underset{\sim}{W}_d = \frac{\underset{\sim}{B}}{B^2} \times (q^2 \underset{\sim}{\rho} + \mu \nabla B) = \frac{e}{m} \underset{\sim}{V}_d \qquad \ldots (5.9)$$

$$\underset{\sim}{a} = \frac{\underset{\sim}{v}_\perp \times \underset{\sim}{B}}{B^2} \quad , \qquad \ldots (5.10)$$

$\underset{\sim}{e}_1$ is the unit vector along $\underset{\sim}{B}$ and $\underset{\sim}{\rho}$ is the curvature of the field. All quantities in (5.6) and (5.7) are referred to the position of the particle not to the guiding center which we have nowhere introduced, similarly the integrals in (5.7) and (5.8) are along the line of force through the instantaneous position of the particle.

* Any other choice leads, of course, to the same final result, any change in the explicit s integration being compensated by a corresponding change in the operator H which implicitly depends on s_o .

The expressions for μ_1 and J_1 agree with those given by Kruskal[7] and by Northrop, Liu and Kruskal[8] respectively. The second invariant J_1 can be cast into more convenient form by observing that the term $\underset{\sim}{a} \cdot \nabla J_o$ in (5.7) is simply the change in J_o which would be introduced if the path of integration in (5.8) were transferred to the field line through the instantaneous guiding center. Similarly $\mu_1\, \partial J_o/\partial\mu_o$ is the change introduced if we replace μ_o by $(\mu_o + \mu_1)$ in the integrand of (5.8). Hence if we collect together zero and first order contributions to J and take all integrals along the field line through the guiding center, instead of through the particle, we can write

$$J_o + \frac{m}{e} J_1 = \oint \left[2(\varepsilon - (\mu_o + \frac{m}{e}\mu_1) B) \right]^{\frac{1}{2}} ds - \sigma \int_{s_o}^{s} \frac{ds}{q} \underset{\sim}{V}_d \cdot \nabla J_o \quad . \qquad \ldots (5.11)$$

This can be written in yet another form which is important for the later discussion of circulating particles. We introduce the <u>instantaneous</u> drift velocity in α, β space by defining

$$\dot{\alpha} \equiv \underset{\sim}{V}_d \cdot \nabla\alpha \quad , \quad \dot{\beta} \equiv \underset{\sim}{V}_d \cdot \nabla\beta \qquad \ldots (5.12)$$

and recall that the average drifts $\bar{\dot{\alpha}}$ and $\bar{\dot{\beta}}$ are related to $\partial J/\partial\beta$ and $\partial J/\partial\alpha$ by (3.30); then (5.11) can be written

$$(J_o + \frac{m}{e} J_1) = \oint \left[2(\varepsilon - (\mu_o + \frac{m}{e}\mu_1) B) \right]^{\frac{1}{2}} ds + \sigma \frac{m}{e} \oint \frac{ds''}{q(s'')} \int_{s_o}^{s} G(s', s'') \frac{ds'}{q(s')} \qquad \ldots (5.13)$$

where $G(s', s'')$ is a zero order quantity defined by

$$G(s', s'') \equiv \frac{e^2}{m^2} \left[\dot{\alpha}(s')\, \dot{\beta}(s'') - \dot{\alpha}(s'')\, \dot{\beta}(s') \right] . \qquad \ldots (5.14)$$

The factor σ which appears in the last term of (5.11) or (5.13) is due to our convention that ds is measured in the direction of $\underset{\sim}{e}_1$ irrespective of whether the particle is moving to left or right. If instead the integration is always taken in the direction of motion then the σ is unnecessary. Equation (5.11) can then be interpreted by observing that the last term represents the change in J_o since the particle left the turning point due to the drift of the guiding center and so is exactly the amount which must be added to J_o to ensure that $(J_o + \frac{m}{e} J_1)$ retains its original value as one follows the particle. (However note that $(J_o + \frac{m}{e} J_1)$ is entirely a local quantity which can be computed from the magnetic field and the instantaneous position of the particle; it is <u>not</u> necessary to calculate the orbit of particle or guiding center.)

In a simple axi-symmetric magnetic field the symmetry ensures that $\underset{\sim}{V}_d \cdot \nabla J_o$ is identically zero so that in such a field J is given correctly through first order by

$$J = \oint \left[2(\varepsilon - (\mu_o + \frac{m}{e}\mu_1) B) \right]^{\frac{1}{2}} ds \ , \qquad \ldots (5.15)$$

the path of integration being along the field line through the guiding center.

<u>Circulating particles</u>

For circulating particles i.e. those not subject to mirror reflection, f_1 is again given by

$$f_1 = k_1 (\sigma, \mu_o, \varepsilon, J_o) + \int Df_o - \sigma \int_{s_o}^{s} \frac{Bds}{q} Lf_o + \int_{J = \text{const}} \frac{d\alpha}{\partial J/\partial \beta} Hf_o \qquad \ldots (5.16)$$

and

$$Hf_o = \oint \frac{Bds'}{q} \left\{ \sigma L \int_{s_o}^{s'} \frac{Bds}{q} Lf_o - \left\langle D \int D \int Df_o \right\rangle \right\} . \qquad \ldots (5.17)$$

but there is now no natural choice for the lower limit s_o and the operator H must be evaluated for an arbitrary s_o. As a result, <u>H no longer vanishes</u>; at first sight this appears to mean that f_1 involves an integral over the precessional drift surfaces J_o = constant. Fortunately, however, it is possible (see Appendix) to express the operator H as a total derivative along J = constant, i.e.

$$Hf_o = \frac{\partial J}{\partial \beta} \cdot \frac{\partial}{\partial \alpha} \left(P \frac{\partial f_o}{\partial J} \right) - \frac{\partial J}{\partial \alpha} \frac{\partial}{\partial \beta} \left(P \frac{\partial f_o}{\partial J} \right) \qquad \ldots (5.18)$$

where P is given by

$$P = \sigma \mu_o \oint ds' \left[\tau_1 + \tfrac{1}{2} (\tau_2 + \tau_3)\right] - \frac{\sigma}{2} \oint \frac{ds''}{q(s'')} \int_{s_o}^{s''} G(s',s'') \frac{ds'}{q(s')} \qquad \ldots (5.19)$$

the function G being again defined by (5.14). The coefficients τ_1, τ_2, τ_3 are related to the torsion of the field line and $\dot{\alpha}$, $\dot{\beta}$ are again the instantaneous drift velocities. Consequently

$$f_1 = k_1 + \int Df_o - \sigma \int_{s_o}^{s} \frac{Bds}{q} Lf_o + P \frac{\partial f_o}{\partial J} \qquad \ldots (5.20)$$

which once again involves only the first derivatives of f_o and so is of the form (5.4). Direct comparison with (5.4) now yields the invariants for circulating (non-reflected) particles as

$$\mu_1 = -\frac{1}{B} \left[\underset{\sim}{v}_{\perp} \cdot \underset{\sim}{W}_d + \frac{\underset{\sim}{e}_1 \cdot \underset{\sim}{v}}{4} \left\{ \underset{\sim}{v}_{\perp} \cdot (\underset{\sim}{a} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{e}_1 + \underset{\sim}{a} \cdot (\underset{\sim}{v}_{\perp} \cdot \underset{\sim}{\nabla})\, e_1 + 4\mu_o\, \underset{\sim}{e}_1 \cdot \nabla \times \underset{\sim}{e}_1 \right\} \right] \qquad \ldots (5.21)$$

and

$$J_1 = \underset{\sim}{a} \cdot \nabla J_o + \mu_1 \frac{\partial J_o}{\partial \mu_o} + \sigma \mu_o \oint ds' \left[\tau_1 + \tfrac{1}{2}(\tau_2 + \tau_3)\right]$$

$$+ \sigma \oint \frac{ds''}{q(s'')} \int_{s_o}^{s} \frac{ds'}{q(s')} G(s', s'') - \frac{\sigma}{2} \oint \frac{ds''}{q(s'')} \int_{s_o}^{s''} \frac{ds'}{q(s')} G(s', s'') . \qquad \ldots (5.22)$$

The second invariant can again be simplified by changing the path of integration to the line of force through the guiding center, then

$$J = J_o + \frac{m}{e} J_1 = \oint \left[2(\varepsilon - (\mu_o + \frac{m}{e}\mu_1) B)\right]^{\frac{1}{2}} ds + \frac{\sigma m}{e} \mu_o \oint [\tau_1 + \tfrac{1}{2}(\tau_2 + \tau_3)]\, ds'$$

$$+ \frac{\sigma m}{e} \oint \frac{ds''}{q(s'')} \int_{s_o}^{s} \frac{ds'}{q(s')} G(s', s'') - \frac{\sigma}{2}\frac{m}{e} \oint \frac{ds''}{q(s'')} \int_{s_o}^{s''} \frac{ds'}{q(s')} G(s', s'') \quad \ldots (5.23)$$

The invariant μ_1 is identical with that for particles trapped between mirrors but the second invariant is of a different form*. Some difference was to be expected since the expression for $(J_o + \frac{m}{e} J_1)$ in the mirror case involved an integral whose lower limit s_o was a definite physical point - the mirror reflection point - which does not exist for circulating particles. If J had been given by the same expression therefore, it would now have involved an arbitrary value of s_o. The extra double integral in (5.23) rectifies this; both integrals of $G(s', s'')$ depend on the arbitrary point s_o but their sum is independent of s_o and can, indeed, be written as

$$\tfrac{1}{2}\sigma \oint \frac{ds''}{q(s'')} \int_{s''}^{s} \frac{ds'}{q(s')} G(s', s'') \quad \ldots (5.24)$$

in which there is no arbitrary quantity.

Another difference between J for circulating and oscillating particles - which was not foreseen - is the term involving the integral of the torsion around the closed line of force. As this term is one of the terms arising from the operator H it can be interpreted as one of the consequences of the second order drift velocities (3.39). It may seem surprising that one can relate part of a first order quantity J_1 to a second order drift, but just as

$$\int_{s_o}^{s} \underset{\sim}{V}_d \cdot \nabla J_o \frac{ds}{q} \quad \ldots (5.25)$$

can be regarded as the accumulated change in J_o due to the first order drifts over times of order L/q or $L/V_\parallel$, so can

$$\int_{J_o = \text{const.}} \underset{\sim}{V}_d^{(2)} \cdot \nabla J_o \frac{d\alpha}{\partial J/\partial\beta} \quad \ldots (5.26)$$

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* Northrop, Liu and Kruskal[8] have calculated J_1, for mirror trapped particles only, and give a 'direct derivation' of the invariant which would seem to apply whether the particle is trapped between mirrors or not. This derivation amounts to showing that if $J = (J_o + \lambda J_1)$ is given by (5.11) then dJ/dt is of order λ^2. However this is also the case when J is given by (5.23) and would be true if J_1 were replaced by $[J_1 + Q(\alpha,\beta,\mu,\varepsilon)]$ where Q is any function, so that this argument alone cannot determine the invariant. To be a first order invariant not only must dJ/dt be of order $(m/e)^2$ but its average must vanish to higher order so that the error in J remains of order $(m/e)^2$ as $t \to \infty$.

be regarded as the accumulated change in J_o due to second order drifts, over times of order $L/V_d^{(1)}$ or $\lambda^{-1} L/V_\parallel$. In (5.26) the drift is an order smaller than in (5.25) but the time for which it acts is an order longer so that (5.26) still yields a first order quantity. The second order drift changes sign with $V_\parallel$ so its accumulated value for oscillating particles is always zero. It must be emphasised again that this is merely an interpretation of the result of a rigorous calculation, it is certainly not necessary to invoke second order (or even first order) drifts in order to determine J_1.

By the same token there is no necessity for the drift surface to be closed (i.e. for the drift motion to be periodic). The invariant $\hat{J}$ exists as a consequence of the periodicity of the motion along the lines of force and does not depend on any periodicity in the drift motion. If the drift motion is periodic it gives rise to a further invariant[3] - the flux invariant Φ (i.e. the total flux through a drift surface). In a static situation such as has been discussed thus far, this invariant is redundant since constancy of μ, J, ε inevitably ensures constancy of Φ. However Φ can be obtained by our procedure provided one includes appropriately slow variations of the magnetic field and this will be discussed in part 3. For the moment we return to $\hat{\mu}$ and $\hat{J}$ and consider the general n^{th} order term in their expansions.

6. HIGHER ORDER CORRECTIONS

A recursion formula for μ_n and J_n involving only the operators already introduced can be obtained as follows. When the equilibrium constraint $\langle Df_n \rangle$ was applied to f_o, f_1, f_2, we found that f_1 could be expressed in terms of f_o, for example

$$f_1 = k_1 + \int Df_o - \sigma \int_{s_o}^{s} \frac{Bds}{q} L f_o + \int \frac{d\alpha}{\partial J/\partial\beta} Hf_o \quad . \qquad \ldots (6.1)$$

If we carry out exactly the same calculation, but consider instead of f_o, f_1, f_2 the general consecutive terms f_n, f_{n+1}, f_{n+2} we obtain

$$f_{n+1} = k_{n+1} + \int Df_n - \sigma \int_{s_o}^{s} \frac{Bds}{q} L f_n + \int \frac{d\alpha}{\partial J/\partial\beta} H f_n \quad . \qquad \ldots (6.2)$$

Multiplying this by λ^{n+1} and summing we have

$$f = k + \lambda K f \qquad \ldots (6.3)$$

where K represents the sum of the three integral operators in (6.2) and $k(\mu,\varepsilon,J) \equiv \Sigma \lambda^n k_n$. Consequently f can be written as

$$f = (1 - \lambda K)^{-1} k(\mu, \varepsilon, J) \qquad \ldots (6.4)$$

which involves a single arbitrary function and generates f in the standard form (5.4A): it can therefore be directly identified with $\psi(\hat{\mu}, \hat{J}, \varepsilon)$. If now, we choose $\psi(\hat{\mu}, \hat{J}, \varepsilon) \equiv \hat{\mu}$ then the n^{th} term in the expansion (5.4A) of ψ is just μ_n; similarly we may choose $\psi \equiv \hat{J}$ and generate a series whose n^{th} term is J_n. Comparing these with the solution generated by (6.4) allows one to write down a recursion formula for μ_n and J_n :-

$$\left.\begin{matrix} J_{n+1} \\ \\ \mu_{n+1} \end{matrix}\right\} = \int D \begin{Bmatrix} J_n \\ \mu_n \end{Bmatrix} - \sigma \int_{s_0}^{s} \frac{Bds}{q} L \begin{Bmatrix} J_n \\ \mu_n \end{Bmatrix} + \int \frac{d\alpha}{\partial J/\partial\beta} H \begin{Bmatrix} J_n \\ \mu_n \end{Bmatrix}. \quad \ldots (6.5)$$

Using this recursion formula we have determined μ_2 in an arbitrary magnetic field. This can be written

$$\mu_2 = c_o + \sum_{1}^{4} (c_n \cos n\varphi + s_n \sin n\varphi) \quad \ldots (6.6)$$

where

$$\tan \varphi = \underset{\sim}{e}_3 \cdot \underset{\sim}{v} / \underset{\sim}{e}_2 \cdot \underset{\sim}{v} \quad . \quad \ldots (6.7)$$

In a general magnetic field the coefficients c_n, s_n are very lengthy but for a vacuum magnetic field they simplify somewhat and putting $\eta \equiv (\rho_3 - \rho_2)$ and $\nu \equiv (\tau_3 - \tau_2)$ the coefficients for a vacuum magnetic field can be written* :

$$c_o = -\frac{1}{2}\frac{q^4}{B^3}\rho^2 + \frac{q^2\mu B}{B^3}\left\{ \operatorname{div} \underset{\sim}{\rho} - \frac{1}{8}(\eta^2 + \nu^2)\right\} + \frac{(\mu B)^2}{B^3}\left\{ \frac{3}{4}(\operatorname{div} \underset{\sim}{\rho} - \underset{\sim}{\rho}^2) + \frac{1}{8}(\underset{\sim}{\nabla} \cdot \underset{\sim}{e}_1)^2 \right\} \quad \ldots (6.8)$$

$$c_4 = \frac{1}{16}(\eta^2 - \nu^2)\frac{(\mu B)^2}{B^3} \quad \ldots (6.9)$$

$$s_4 = -\frac{(\mu B)^2}{B^3}\frac{1}{8}\eta\nu \quad \ldots (6.10)$$

$$c_3 = \frac{qc_\perp \mu B}{12 B^3}\left\{ -\frac{\partial\nu}{\partial y} - \frac{\partial\eta}{\partial x} + \eta(2\sigma_3 - \rho_1) - \nu(2\sigma_2 - \sigma_1) \right\} \quad \ldots (6.11)$$

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* A special case of μ_2 - its value on the median plane of an axisymmetric vacuum field - was calculated some time ago by C. Gardner and is quoted by T. Northrop[7]. Our result does not agree with this formula but we understand from Dr. Gardner that there is an error in the formula of ref. 7 and his latest calculation agrees with ours.

$$s_3 = \frac{qc_\perp \mu B}{12\ B^3} \left\{ -\frac{\partial \eta}{\partial y} + \frac{\partial \nu}{\partial x} - \nu(2\sigma_3 - \rho_1) - \eta(2\sigma_2 - \sigma_1) \right\} \quad \ldots (6.12)$$

$$c_2 = +\frac{3q^2\mu B}{4B^3} \left\{ 2\nu\tau_1 - B\frac{\partial}{\partial s}\left(\frac{\eta}{B}\right) \right\} + \frac{(\mu B)^2}{2B^3} \left\{ -B\frac{\partial}{\partial s}\left(\frac{\eta}{B}\right) + 2\nu\tau_1 - \rho_1^2 + \sigma_1^2 - \frac{1}{2}\eta(\underset{\sim}{\nabla}\cdot \underset{\sim}{e}_1) \right\} \quad \ldots (6.13)$$

$$s_2 = \frac{3q^2\mu B}{4B^3} \left\{ 2\eta\tau_1 + B\frac{\partial}{\partial s}\left(\frac{\nu}{B}\right) \right\} + \frac{(\mu B)^2}{2B^3} \left\{ B\frac{\partial}{\partial s}\left(\frac{\nu}{B}\right) + 2\eta\tau_1 + 2\rho_1\sigma_1 + \frac{1}{2}\nu\ (\underset{\sim}{\nabla}\cdot \underset{\sim}{e}\ ) \right\} \quad \ldots (6.14)$$

$$c_1 = \frac{c_\perp q\ \mu B}{4B^3} \left\{ \frac{\partial \nu}{\partial y} - \frac{\partial \eta}{\partial x} + \nu(\sigma_1 - 2\sigma_2) + \eta(\rho_1 - 2\sigma_3) \right\} - \frac{qc_\perp}{B^3}\ \sigma_1\tau_1\ (\mu B + q^2) + \frac{q}{B}\ \frac{\partial}{\partial s}\left[\frac{c_\perp}{B^2}\ \rho_1(q^2 + \mu B)\right] \quad \ldots (6.15)$$

$$s_1 = \frac{c_\perp q\mu B}{4B^3} \left\{ \frac{\partial \eta}{\partial y} + \frac{\partial \nu}{\partial x} + \eta(\sigma_1 - 2\sigma_2) - \nu(\rho_1 - 2\sigma_3) \right\} - \frac{qc_\perp}{B^3}\ \rho_1\tau_1\ (q^2 + \mu B) - \frac{q}{B}\ \frac{\partial}{\partial s}\left[\frac{c_\perp}{B^2}\sigma_1\ (q^2 + \mu B)\right] \quad \ldots (6.16)$$

PART III

7. TIME DEPENDENT FIELDS, ELECTRIC FIELDS AND THE THIRD INVARIANT

In parts I and II only static magnetic fields were considered; this led to the two invariants μ, J which together with the energy ε form three "constants of the motion". In time dependent magnetic fields ε itself is no longer a constant but if the field variations are sufficiently slow it is known(3) that there is still a third invariant quantity, namely the flux Φ through a drift surface J = constant. We now consider how this third invariant arises from our present viewpoint and at the same time discuss the related effect of electric fields. These calculations are very similar to those of parts I and II so that we need give only an outline of the arguments involved.

To investigate these effects we first transform the Vlasov equation to a velocity frame moving with the field lines. For this we choose a velocity

$$\underset{\sim}{U} = \left\{ \frac{\partial \alpha}{\partial t} \nabla\beta - \frac{\partial \beta}{\partial t} \nabla\alpha \right\} \times \frac{\underset{\sim}{B}}{B^2} \qquad \ldots (7.1)$$

so that

$$\frac{\partial \alpha}{\partial t} + \underset{\sim}{U} \cdot \nabla\alpha = 0 \quad : \quad \frac{\partial \beta}{\partial t} + \underset{\sim}{U} \cdot \nabla\beta = 0 \ . \qquad \ldots (7.2)$$

This velocity $\underset{\sim}{U}$ is <u>not</u> the same as the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift, in fact with

$$E = - \frac{\partial \underset{\sim}{A}}{\partial t} - \nabla\varphi \qquad \ldots (7.3)$$

and $\underset{\sim}{A} = \alpha \underset{\sim}{\nabla}\beta$ the velocity U is

$$\underset{\sim}{U} = \frac{(\underset{\sim}{E} + \underset{\sim}{\nabla}\psi) \times \underset{\sim}{B}}{B^2} \qquad \ldots (7.4)$$

where $\psi = (\alpha\ \partial\beta/\partial t + \varphi)$ and $\nabla\psi$ does not, in general, vanish.

After transforming to this frame of reference the variables μ, ε, φ are introduced as in parts I and II when the Vlasov equation can be written

$$\frac{\partial f}{\partial \varphi} = \lambda \left(\frac{1}{B} \frac{\partial f}{\partial t} + Df + Gf \right) \qquad \ldots (7.5)$$

where D is the operator introduced earlier and G is a new operator, given in Appendix C, which depends explicitly on $\underset{\sim}{U}$ and on $\nabla\psi$. In fact G may be split into two parts each depending only on either $\underset{\sim}{U}$ or $\nabla\psi$.

$$G \equiv \lambda^{-1} G^{\psi} + G^{U} \ . \qquad \ldots (7.6)$$

The calculation of f proceeds order by order just as in section 3, so that we can omit all details. The effects of electrostatic fields and time dependent magnetic fields of various magnitudes are introduced by treating G as being of the appropriate order in λ. For example, to reveal the flux invariant the time dependence of the fields must be taken to be one order higher than that of the drifts V_d/L, that is

$$\frac{1}{\alpha}\frac{\partial\alpha}{\partial t} \sim \frac{U}{L} \sim \frac{a}{L}\cdot\frac{V_d}{L} , \qquad \dots (7.7)$$

which makes the operators G^{ψ} and G^{U} of order λ^2. Then $\frac{1}{B}\frac{\partial f}{\partial t}$ and $G^{U}f$ are of the same order and $\frac{1}{B}\frac{\partial}{\partial t}$ can be absorbed into G^{U}.

The zero and first order calculations are unaffected by the additional operator G and, as before, constrain f_o to be independent of φ and s, respectively.

Second order

It is in second order that the operator G first affects the calculation and in place of (3.12) one now finds

$$\frac{\partial f_2}{\partial\varphi} = Df_1 + G^{\psi}f_o \qquad \dots (7.8)$$

leading to

$$f_2 = g_2 + \int Dg_1 + \int D\int Df_o + \int G^{\psi}f_o \qquad \dots (7.9)$$

and the associated constraint

$$\langle Dg_1\rangle + \langle D\int Df_o\rangle + \langle G^{\psi}f_o\rangle = 0 . \qquad \dots (7.10)$$

When the last term in this equation is evaluated it has a similar form to the first, with which it can be combined so that (7.10) can be written

$$\sigma\frac{q}{B}\frac{\partial}{\partial s}\left(g_1 - \psi\frac{\partial f_o}{\partial\varepsilon}\right) + \langle D\int Df_o\rangle = 0 . \qquad \dots (7.11)$$

Consequently the constraint in this order is still just

$$\oint\frac{Bds}{q}\langle D\int Df_o\rangle = 0 \qquad \dots (7.12)$$

indicating that f_o is of the form $f_o(\mu, \varepsilon, J, t)$. There is however a change in g_1 which is now given by

$$g_1 = -\sigma\int_{s_o}^{s}\frac{Bds}{q} Lf_o + \frac{\partial f_o}{\partial\varepsilon}\psi + h_1 (\mu, \varepsilon, \alpha, \beta, \sigma) \qquad \dots (7.13)$$

instead of by equation (3.16).

Third Order

In this order, both G^{ψ} and G^{U} enter the calculation, and we have

$$\frac{\partial f_3}{\partial\varphi} = Df_2 + G^{\psi}f_1 + G^{U}f_o \qquad \dots (7.14)$$

leading to the constraint

$$\langle Df_2 \rangle + \langle G^{\psi} f_1 \rangle + \langle G^U f_o \rangle = 0 \quad . \qquad \ldots (7.15)$$

When f_2 and f_1 are introduced in terms of f_o one finds, as before, an equation for g_2, similar to equation (3.32). For brevity this will be written as

$$\sigma \frac{q}{B} \frac{\partial g_2}{\partial s} = - Lh_1 + \sigma L \int \frac{Bds}{q} Lf_o - \langle D \int D \int Df_o \rangle + \sigma \frac{q}{B} \frac{\partial}{\partial s} \left(\psi \frac{\partial h_1}{\partial \varepsilon} \right) + Mf_o \quad \ldots (7.16)$$

where Mf_o represents all the several terms arising from G^{ψ} and G^U which appear in (7.15) and the operator M is given in Appendix C. Integration of equation (7.16) over s leads to a generalisation of (3.33) and (3.34), namely;

$$\left(\frac{\partial h_1}{\partial \alpha} \frac{\partial J}{\partial \beta} - \frac{\partial h_1}{\partial \beta} \frac{\partial J}{\partial \alpha} \right) = Hf_o + \oint \frac{Bds}{q} Mf_o \quad . \qquad \ldots (7.17)$$

Now, as we observed in section 3 the expression on the left of (7.17) is the derivative of h_1 along the precessional drift surfaces J = constant, and if these precessional surfaces are closed (7.17) may itself lead to a constraint on f_o namely;

$$\oint_{J = \text{constant}} \frac{d\alpha}{\partial J/\partial \beta} Hf_o + \oint_{J = \text{constant}} \frac{d\alpha}{\partial J/\partial \beta} \oint \frac{Bds}{q} Mf_o = 0 \quad . \qquad \ldots (7.18)$$

where the integrals are taken around the precessional drift surfaces.

It was noted in part II that the first term is automatically zero, so that (7.18) leads to no new constraints in the time independent case. However in the present situation there __is__ a constraint. To find this we must evaluate M; this is a tedious calculation given elsewhere[9] and we content ourselves here with the result which is

$$\oint_{J = \text{const}} \frac{d\alpha}{\partial J/\partial \beta} \left[\left(\frac{\partial f_o}{\partial t} \right) \left(\frac{\partial J}{\partial \varepsilon} \right)_{\alpha\beta} - \left(\frac{\partial f_o}{\partial \varepsilon} \right) \left(\frac{\partial J}{\partial t} \right)_{\alpha\beta} \right] = 0 \quad . \qquad \ldots (7.19)$$

[Here, $\partial J/\partial t$ at constant α, β means the rate of change following the field line labelled by the __numbers__ (α, β) (i.e. the Lagranian derivative moving with the velocity U,) the __functions__ $\alpha(x)$ and $\beta(x)$ must of course vary if the magnetic field is to change.]

Equation (7.19) can be written

$$\left(\frac{\partial \Phi}{\partial t} \frac{\partial f_o}{\partial \varepsilon} - \frac{\partial \Phi}{\partial \varepsilon} \frac{\partial f_o}{\partial t} \right) = 0 \qquad \ldots (7.20)$$

where

$$\Phi \equiv \oint_{J = \text{const.}} \alpha \, d \beta \qquad \ldots (7.21)$$

is the flux contained within the precessional drift surface J = constant. Equation (7.20) shows that f_o is of the form $f_o(\mu, J, \Phi)$. Hence in time varying fields, although the energy ε is no longer a constant it is replaced as an invariant by the flux Φ and there are still three "constants of motion" μ, J, Φ.

PART IV

TOROIDAL SYSTEMS WITH SMALL ROTATIONAL TRANSFORM

8. MAGNETIC SURFACES

In our discussion of adiabatic invariants in part II, two different cases were distinguished, that of mirror-trapped (i.e. oscillating) particles and that of particles circulating round a closed line of force in a toroidal field. Closure of the lines is a very special circumstance and more usually in a toroidal system the lines of force are not closed. Instead, as e.g. in a stellarator with small rotational transform, the field lines generate toroidal magnetic surfaces. The structure of such fields has been considered in detail by Kruskal and Kulsrud[10] whose notation will be closely followed.

A magnetic field possessing magnetic surfaces can be represented by

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\psi \times \underset{\sim}{\nabla}\nu \qquad \ldots(8.1)$$

where ψ is a single valued function which is constant on each toroidal magnetic surface and ν is a multiple-valued function. By a suitable choice of scale, ψ can be made equal to the longitudinal magnetic flux inside the magnetic surface ψ. Then ν is an angle like variable which increases by unity during one loop encircling the magnetic axis and increases by $\iota/2\pi$ during one circuit around the torus.

For the moment we ignore any complexity introduced by the multivalued nature of ν e.g. by introducing appropriate "cuts" across the torus. Then we can use ψ, ν in exactly the same way that we used α, β in parts I and II, and the equilibrium constraints can be determined by the same procedure. We consider only the $\sigma = +1$ stream, the changes necessary for $\sigma = -1$ are obvious.

In zero order the constraint is again that f_o be independent of φ and in first order that it satisfy

$$\langle Df_o \rangle = \frac{q}{B}\frac{\partial f_o}{\partial s} = 0 \ . \qquad \ldots(8.2)$$

This implies that f_o is constant along a line of force and as each line generates its magnetic surface this is usually interpreted to mean that f_o must be constant over a magnetic surface. This is also indicated directly by the alternative form of (8.2), namely

$$\langle Df_o \rangle = \frac{q}{B^2}(B \cdot \nabla f_o) = 0 \ . \qquad \ldots(8.3)$$

For the present we adopt this interpretation and examine whether any further constraints appear in next order.

Second Order

The second order equation is

$$\frac{\partial f_2}{\partial \varphi} = Df_1 = Dg_1 + D\int Df_o \qquad \ldots (8.4)$$

leading to the usual equation for g_1,

$$\frac{q}{B}\frac{\partial g_1}{\partial s} + \left\langle D\int Df_o \right\rangle = 0 \ , \qquad \ldots (8.5)$$

but because the field lines are no longer assumed to close on themselves this no longer leads directly to the constraint (3.17) which was obtained by integrating around a closed field line. Nevertheless there is a constraint on f_o implied in (8.5) as can be seen by writing it as

$$\underset{\sim}{B}\cdot\nabla g_1 + \frac{B^2}{q}\left\langle D\int Df_o \right\rangle = 0 \qquad \ldots (8.6)$$

and then annihilating the first term by multiplying by $|\nabla\psi|^{-1}$ and integrating over a magnetic surface. Then the first term vanishes identically and f_o must satisfy

$$\iint \frac{dS}{|\nabla\psi|}\,\frac{B^2}{q}\left\langle D\int Df_o \right\rangle = 0 \ . \qquad \ldots (8.7)$$

From this it may be deduced[10] that

$$f_o \equiv f_o(J^*, \mu, \varepsilon) \qquad \ldots (8.8)$$

where J^* is a surface adiabatic invariant

$$J^* = \iint \frac{Bq\,dS}{|\nabla\psi|} \qquad \ldots (8.9)$$

which is an obvious generalisation of the simple invariant J obtained by replacing

$$\lim_{s\to\infty}\int ds \rightarrow \int \frac{BdS}{|\nabla\psi|} \qquad \ldots (8.10)$$

as is appropriate if the line of force covers a ψ surface ergodically. Since J^* is by definition constant over a magnetic surface (8.9) merely confirms what had been concluded from the first order calculation, namely that f_o is constant over a magnetic surface. Under certain circumstances however (8.8) remains true in the time dependent situation[11].

Although these results appear to be the natural extension of those in part II they may not be really appropriate, particularly when we recall our observations in part I about the relation of the constraints on f_o to the time scale of the corresponding equilibrium. In particular it is clear that J can only be a relevant quantity on time scales much longer than $L/V_\parallel$, (where L is of the order of the circumference of the torus). Similarly J^*, or even the magnetic surface ψ itself can only be relevant on time scales long compared to the time taken for a particle to "sample" the whole of a magnetic surface which, especially in the case of small rotational transform, may be very long compared to $L/V_{\parallel}$. Consequently the following argument would seem more suitable.

9. SMALL ROTATIONAL TRANSFORM

In the case of small rotational transform one should introduce the transform itself as another small parameter and include it in the expansion (or ordering) procedure. This can be simply done if one regards the field as composed of two parts; a large field which possesses closed field lines and a smaller additional field which produces the rotational transform and the ergodic behaviour of the field lines. (This is, in fact, the conventional way of treating stellarator fields.) Thus we can write

$$\underset{\sim}{B} = \nabla\psi \times \nabla\nu_o + \nabla\psi \times \nabla\nu_1 \qquad \text{... (9.1)}$$

where (ψ, ν_o) label the closed field lines of the dominant field and ν_1 corresponds to the small rotational transform. The final results do not depend on the exact way in which ν is split into its component parts ν_o and ν_1. Corresponding to the splitting of the magnetic field we can formally regard D as split into a dominant part D_o and a small D_1, however, as will be seen there is no need to explicitly determine D_1. The calculation proceeds like all its predecessors. In zero order there is no change; in first order there is the constraint $\langle D_o f_o \rangle = 0$ indicating that f_o is a function only of $(\psi, \nu_o, \varepsilon, \mu)$, and f_1 is given by

$$f_1 = g_1 (\psi, \nu_o, \varepsilon, \mu, s) + \int Df_o \;. \qquad \text{... (9.2)}$$

In second order there is an essential change from section 8. The second order equation is now

$$\frac{\partial f_2}{\partial \varphi} = D_o f_1 + \lambda^{-1} D_1 f_o \qquad \text{... (9.3)}$$

so providing the constraint

$$\langle D_o g_1 \rangle + \langle D_o \int D_o f_o \rangle + \lambda^{-1} \langle D_1 f_o \rangle = 0 \;. \qquad \text{... (9.4)}$$

The first two terms are very familiar; it is only necessary to note that they refer entirely to the field corresponding to ν_o which has closed field lines. The last term is easily found without the need to explicitly determine D_1 by observing that, since both $\langle D_o f_o \rangle$ and $B_o \cdot \nabla f_o$ are zero,

$$\langle D_1 f_o \rangle = \langle D f_o \rangle = \frac{q}{B^2} (\underset{\sim}{B} \cdot \nabla f_o) = \frac{q}{B^2} (\underset{\sim}{B}_1 \cdot \nabla f_o) \;. \qquad \text{... (9.5)}$$

Hence equation (9.4) can be written

$$\frac{q}{B}\left(\frac{\partial g_1}{\partial s}\right)_o + \left\langle D_o \int D_o f_o \right\rangle + \lambda^{-1}\,\frac{q}{B^2}\,(\underset{\sim}{B}\cdot\nabla f_o) = 0 \qquad \ldots (9.6)$$

where $(\partial g_1/\partial s)_o$ must be taken along the direction of $\underset{\sim}{B}_o$. The constraint which arises from (9.6) and the requirement that g_1 be single valued is

$$\oint \frac{Bds}{q}\left\langle D\int D f_o\right\rangle + \lambda^{-1}\oint \frac{\underset{\sim}{B}\cdot\nabla f_o}{B_o}\,ds = 0 \;. \qquad \ldots (9.7)$$

The familiar first term in (9.7) becomes

$$\left(\frac{\partial f_o}{\partial\psi}\frac{\partial J}{\partial\nu} - \frac{\partial f_o}{\partial\nu}\frac{\partial J}{\partial\psi}\right) \qquad \ldots (9.8)$$

where J is defined by the line integral

$$J \equiv \oint [2(\varepsilon - \mu B)]^{\frac{1}{2}}\,ds \qquad \ldots (9.9)$$

around the closed line corresponding to the dominant part of the field. The second term in (9.7) is unfamiliar but can be reduced to a more transparent form by introducing $\underset{\sim}{B} = (\nabla\psi \times \nabla\nu)$ and

$$\nabla f_o = \nabla\psi\,\frac{\partial f_o}{\partial\psi} + \nabla\nu_o\,\frac{\partial f_o}{\partial\nu_o} \qquad \ldots (9.10)$$

then

$$\oint \frac{\underset{\sim}{B}\cdot\nabla f_o}{B_o}\,ds = -\frac{\partial f_o}{\partial\nu_o}\oint \frac{(\underset{\sim}{B}_o\cdot\underset{\sim}{\nabla\nu})}{B_o}\,ds \equiv -\frac{\partial f_o}{\partial\nu_o}\oint \nabla\nu\cdot\underset{\sim}{ds} \qquad \ldots (9.11)$$

the last identity following because the path of integration is along the closed field line $\underset{\sim}{B}_o$. Now recalling that the vector potential $\underset{\sim}{A}$ can be written $\underset{\sim}{A} = \psi\nabla\nu$ some further manipulation allows us to write

$$\oint \underset{\sim}{\nabla\nu}\cdot\underset{\sim}{ds} = \frac{\partial}{\partial\psi}\oint \underset{\sim}{A}\cdot\underset{\sim}{ds} \qquad \ldots (9.12)$$

and to show that

$$\frac{\partial}{\partial\nu}\oint \underset{\sim}{A}\cdot\underset{\sim}{ds} = 0 \;. \qquad \ldots (9.13)$$

Collecting these results together the final form of the constraint (9.7) can be written;

$$\left(\frac{\partial f_o}{\partial\psi}\frac{\partial J^{**}}{\partial\nu} - \frac{\partial f_o}{\partial\nu}\frac{\partial J^{**}}{\partial\psi}\right) = 0 \qquad \ldots (9.14)$$

so that

$$f_o \equiv f_o\,(\mu, \varepsilon, J^{**}) \qquad \ldots (9.15)$$

where J^{**} is a new form of invariant defined by

$$J^{**} = \oint (\underset{\sim}{q} + \frac{e}{m}\underset{\sim}{A})\cdot\underset{\sim}{ds} \;. \qquad \ldots (9.16)$$

This is a result which could have been anticipated from the form of the canonical angular momentum in a magnetic field. However the value and simplicity of introducing J^{**} does not seem to have been appreciated. By its use one automatically incorporates the effect of both rotational transform and that of g.c. drifts, whereas previous calculations[12] have achieved this only by a direct calculation. For example the average motion of a particle under the combined effect of transform and drift is given by

$$\bar{\dot{\psi}} = \left(\frac{\partial J^{**}}{\partial \nu}\right)\left(\frac{\partial J}{\partial \varepsilon}\right)^{-1} \qquad \bar{\dot{\nu}} = -\left(\frac{\partial J^{**}}{\partial \psi}\right)\left(\frac{\partial J}{\partial \varepsilon}\right)^{-1} \qquad \ldots (9.17)$$

and the particle remains on a surface of constant J^{**}, not on a magnetic surface of constant ψ. Similarly the equilibrium distribution f_o is not constant over a magnetic surface but over a J^{**} surface. This is because the actual path of a particle is due to a combination of the drifts, which divert the particle from the magnetic surface and the rotational transform which generates the surface. Both these effects are included in J^{**}; $\oint q\, ds$ represents the effect of drifts and $\oint A\, ds$ that of the rotational transform. When the latter is the dominant effect (small drift during one circuit compared to $\iota/2\pi$) then

$$J^{**} \rightarrow \frac{e}{m} \oint A\, ds$$

which <u>is</u> constant over a magnetic surface. Hence the results of section 8 can be recovered in the appropriate limit.

The invariant J^{**} is different according to whether the particle is moving parallel or antiparallel to the field. This asymmetry arises because the rotational transform and drifts are additive in the one case and subtractive in the other, this asymmetry in J^{**} has, therefore, no connection with the asymmetry in J_1 (part II) which is a rather subtle consequence of the dynamics of a charged particle in a magnetic field.

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APPENDIX A

As an example of a guiding center distribution which is not of the equilibrium form $F(\mu, \varepsilon, J)$ but which nevertheless leads to anisotropic pressures satisfying the fluid equilibrium constraints, one may take the separable distribution (for each species).

$$F_i \,(\mu, \varepsilon, \alpha, \beta) = H_i \,(\mu, \varepsilon)\, Q(\alpha, \beta) \quad . \qquad \text{... A.1}$$

Clearly (A.1) is not in general a guiding center equilibrium. The pressure resulting from (A.1) can be expressed in the form

$$\begin{aligned} p_\perp &= R_\perp(B)\ Q(\alpha, \beta) \\ p_\parallel &= R_\parallel(B)\ Q(\alpha, \beta) \end{aligned} \qquad \text{... A.2}$$

where

$$R_\perp\,(B) = \sum_i m_i \int \frac{\mu B^2}{q} H_i\,(\mu, \varepsilon)\, d\mu\, d\varepsilon \qquad \text{... A.3}$$

$$R_\parallel\,(B) = \sum_i m_i \int q\, B\, H_i\,(\mu, \varepsilon)\, d\mu\, d\varepsilon \ . \qquad \text{... A.4}$$

It can be seen by direct substitution that A.2 satisfies the first fluid constraint

$$\frac{\partial p_\parallel}{\partial s} + \frac{(p_\perp - p_\parallel)}{B} \frac{\partial B}{\partial s} = 0 \ . \qquad \text{... A.5}$$

When (A.2) is substituted into the second fluid constraint

$$\int \nabla\,(p_\perp + p_\parallel)\, \frac{(\underset{\sim}{B} \times \nabla B)}{B^4}\, ds = 0 \qquad \text{... A.6}$$

the result can be written as

$$\int d\chi\, \frac{(R_\parallel(B) + R_\perp(B))}{B^3} \left\{ \frac{\partial Q}{\partial \alpha} \frac{\partial B}{\partial \beta} - \frac{\partial Q}{\partial \beta} \frac{\partial B}{\partial \alpha} \right\} = 0 \qquad \text{... A.7}$$

However, on differentiating (A.4),

$$\frac{\partial}{\partial \beta} \left(\frac{R_\parallel}{B^2} \right) = - \frac{(R_\parallel(B) + R_\perp(B))}{B^3} \frac{\partial B}{\partial \beta} \qquad \text{... A.8}$$

so that (A.7) becomes

$$\frac{\partial Q}{\partial \alpha} \frac{\partial}{\partial \beta} \left(\int R_\parallel(B) \frac{d\chi}{B^2} \right) - \frac{\partial Q}{\partial \beta} \frac{\partial}{\partial \alpha} \left(\int R_\parallel(B) \frac{d\chi}{B^2} \right) = 0 \qquad \text{... A.9}$$

The second equilibrium condition (A.6) can therefore be satisfied by making $Q(\alpha, \beta)$ a function of $\int R_\parallel(B) \frac{ds}{B}$. Another form for this, obtained from (A.4) is

$$\int R_\parallel(B) \frac{ds}{B} = \sum_i m_i \int H_i\,(\mu, \varepsilon)\, J(\mu, \varepsilon, \alpha, \beta)\, d\mu\, d\varepsilon \equiv \langle J \rangle \qquad \text{... A.10}$$

so the distribution (A.1) can be expressed as

$$F_i\,(\mu, \varepsilon, \alpha, \beta) = H_i\,(\mu, \varepsilon)\ Q(\langle J \rangle) \ . \qquad \text{... A.11}$$

Although this is not a g.c. equilibrium, it nevertheless satisfies both of the fluid equilibrium conditions (A.5), (A.6).

APPENDIX B

COORDINATE SYSTEMS

(a) Orthogonal Coordinate Systems

The operators introduced in Part I and discussed in Appendix C are most conveniently evaluated in one of the following coordinate systems.

The first is a generalisation of that discussed by Chandrasekhar, Kaufman and Watson[4]. Three orthogonal unit vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ are chosen with $\underset{\sim}{e}_1 = \underset{\sim}{B}/B$. We do not necessarily choose $\underset{\sim}{e}_2$ in the direction of the principal normal to a line of force as C.K.W. do, for, as will be seen later, other choices of $\underset{\sim}{e}_2$ prove to be more useful. Then if ds, dx, dy denote elements of arc length in the directions $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ respectively.

$$\underset{\sim}{\nabla} = \underset{\sim}{e}_1 \frac{\partial}{\partial s} + \underset{\sim}{e}_2 \frac{\partial}{\partial x} + \underset{\sim}{e}_3 \frac{\partial}{\partial y} \qquad \text{... (B.1)}$$

and

$$\frac{\partial \underset{\sim}{e}_1}{\partial s} = \rho_1 \underset{\sim}{e}_2 - \sigma_1 \underset{\sim}{e}_3$$

$$\frac{\partial \underset{\sim}{e}_2}{\partial s} = - (\tau_1 \underset{\sim}{e}_3 + \rho_1 \underset{\sim}{e}_1) \qquad \text{... (B.2)}$$

$$\frac{\partial \underset{\sim}{e}_3}{\partial s} = \tau_1 \underset{\sim}{e}_2 + \sigma_1 \underset{\sim}{e}_1$$

where ρ_1, and $-\sigma_1$ are the components of the curvature of a line of force in the $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ directions, and τ_1 is related to the torsion. [If we choose $\underset{\sim}{e}_2$ along the principal normal τ_1 <u>is</u> the torsion, otherwise the torsion is equal to $\tau_1 + \frac{\partial}{\partial s} (\tan^{-1} \sigma_1 / \rho_1)$] . The other derivatives are

$$\frac{\partial \underset{\sim}{e}_1}{\partial x} = \rho_2 \underset{\sim}{e}_2 - \tau_2 \underset{\sim}{e}_3 \qquad \frac{\partial \underset{\sim}{e}_1}{\partial y} = \tau_3 \underset{\sim}{e}_2 + \rho_3 \underset{\sim}{e}_3$$

$$\frac{\partial \underset{\sim}{e}_2}{\partial x} = \sigma_2 \underset{\sim}{e}_3 - \rho_2 \underset{\sim}{e}_1 \qquad \frac{\partial \underset{\sim}{e}_2}{\partial y} = \sigma_3 \underset{\sim}{e}_3 - \tau_3 \underset{\sim}{e}_1 \qquad \text{... (B.3)}$$

$$\frac{\partial \underset{\sim}{e}_3}{\partial x} = \tau_2 \underset{\sim}{e}_1 - \sigma_2 \underset{\sim}{e}_2 \qquad \frac{\partial \underset{\sim}{e}_3}{\partial y} = - \rho_3 \underset{\sim}{e}_1 - \sigma_3 \underset{\sim}{e}_2$$

where the ρ_i, σ_i are the curvatures of the $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$ axes, and the τ_i are related to their torsions.

Because s, x, y are not true curvalinear coordinates the derivatives are non-commuting; (in fact the operation $\frac{\partial}{\partial x}$ must be regarded as shorthand for $\underset{\sim}{e}_2 \cdot \underset{\sim}{\nabla}$ etc.).

The commutators are required extensively in the evaluation of the operators in Appendices C and D, and are derived below:

$$\frac{\partial^2 Q}{\partial x \partial y} - \frac{\partial^2 Q}{\partial y \partial x} = (\mathbf{e}_2 \cdot \nabla)\,(\mathbf{e}_3 \cdot \nabla Q) - (\mathbf{e}_3 \cdot \nabla)\,(\mathbf{e}_2 \cdot \nabla Q)$$

$$= \nabla Q \cdot \left\{ \frac{\partial \mathbf{e}_3}{\partial x} - \frac{\partial \mathbf{e}_2}{\partial y} \right\} \qquad \ldots \text{(B.4)}$$

$$= (\tau_2 + \tau_3)\frac{\partial Q}{\partial s} - \sigma_2 \frac{\partial Q}{\partial x} - \sigma_3 \frac{\partial Q}{\partial y}$$

$$\frac{\partial^2 Q}{\partial s \partial x} - \frac{\partial^2 Q}{\partial x \partial s} = \nabla Q \cdot \left\{ \frac{\partial \mathbf{e}_2}{\partial s} - \frac{\partial \mathbf{e}_1}{\partial x} \right\}$$

$$\ldots \text{(B.5)}$$

$$= (\tau_2 - \tau_1)\frac{\partial Q}{\partial y} - \rho_2 \frac{\partial Q}{\partial x} - \rho_1 \frac{\partial Q}{\partial s}$$

$$\frac{\partial^2 Q}{\partial s \partial y} - \frac{\partial^2 Q}{\partial y \partial s} = \nabla Q \cdot \left\{ \frac{\partial \mathbf{e}_3}{\partial s} - \frac{\partial \mathbf{e}_1}{\partial y} \right\}$$

$$\ldots \text{(B.6)}$$

$$= (\tau_1 - \tau_3)\frac{\partial Q}{\partial x} - \rho_3 \frac{\partial Q}{\partial y} + \sigma_1 \frac{\partial Q}{\partial s}$$

In evaluating the operators of Appendices C and D, we will also require the results of differentiating the curvature and torsion coefficients, ρ_i, σ_i, τ_i, in the s, x and y directions. We derive one of these results here as an example and list the others which we will require, and which can be established in similar fashion.

$$\frac{\partial \rho_1}{\partial y} - \frac{\partial \tau_3}{\partial s} = \mathbf{e}_3 \cdot \nabla \left[\mathbf{e}_2 \cdot (\mathbf{e}_1 \cdot \nabla)\mathbf{e}_1 \right] - \mathbf{e}_1 \cdot \nabla \left[\mathbf{e}_2 \cdot (\mathbf{e}_3 \cdot \nabla)\mathbf{e}_1 \right]$$

$$= \frac{\partial \mathbf{e}_2}{\partial y} \cdot \frac{\partial \mathbf{e}_1}{\partial s} + \mathbf{e}_2 \cdot \left(\frac{\partial \mathbf{e}_1}{\partial y} \cdot \nabla \right) \mathbf{e}_1 - \frac{\partial \mathbf{e}_2}{\partial s} \cdot \frac{\partial \mathbf{e}_1}{\partial y} - \mathbf{e}_2 \cdot \left(\frac{\partial \mathbf{e}_3}{\partial s} \cdot \nabla \right) \mathbf{e}_1 \qquad \ldots \text{(B.7)}$$

$$= \tau_3(\rho_2 + \rho_3) + \tau_1(\rho_3 - \rho_2) - \sigma_1(\rho_1 + \sigma_3)$$

Similarly

$$\frac{\partial \sigma_1}{\partial x} - \frac{\partial \tau_2}{\partial s} = \tau_2(\rho_2 + \rho_3) - \tau_1(\rho_3 - \rho_2) + \rho_1(\sigma_1 + \sigma_2) \qquad \ldots \text{(B.8)}$$

$$\frac{\partial \rho_1}{\partial x} - \frac{\partial \rho_2}{\partial s} = \tau_1(\tau_3 - \tau_2) - \tau_2\tau_3 + \rho_1^2 + \rho_2^2 - \sigma_1\sigma_2 \qquad \ldots \text{(B.9)}$$

$$\frac{\partial \sigma_1}{\partial y} + \frac{\partial \rho_3}{\partial s} = \tau_1(\tau_3 - \tau_2) + \tau_2\tau_3 - \sigma_1^2 - \rho_3^2 + \rho_1\sigma_3 \qquad \ldots \text{(B.10)}$$

$$\frac{\partial \tau_3}{\partial x} - \frac{\partial \rho_2}{\partial y} = \sigma_2(\rho_3 - \rho_2) + \sigma_3(\tau_2 - \tau_3) + \rho_1(\tau_2 + \tau_3) \qquad \ldots \text{(B.11)}$$

$$\frac{\partial\tau_2}{\partial y} + \frac{\partial\rho_3}{\partial x} = \sigma_2(\tau_2 - \tau_3) - \sigma_3(\rho_3 - \rho_2) - \sigma_1(\tau_2 + \tau_3) \qquad \ldots (B.12)$$

$$\frac{\partial\tau_1}{\partial y} + \frac{\partial\sigma_3}{\partial s} = \sigma_2(\tau_1 - \tau_3) - \sigma_1(\tau_1 + \tau_3) - \rho_3(\rho_1 + \sigma_3) \qquad \ldots (B.13)$$

$$\frac{\partial\tau_1}{\partial x} + \frac{\partial\sigma_2}{\partial s} = \sigma_3(\tau_2 - \tau_1) + \rho_1(\tau_1 + \tau_2) - \rho_2(\sigma_1 + \sigma_2) \qquad \ldots (B.14)$$

$$\frac{\partial}{\partial x}(\rho_2 + \rho_3) + \frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial x}\right) = \rho_1(\rho_2 + \rho_3) - \frac{\rho_2}{B}\frac{\partial B}{\partial x} + (\tau_2 - \tau_1)\frac{1}{B}\frac{\partial B}{\partial y} \qquad \ldots (B.15)$$

$$\frac{\partial}{\partial y}(\rho_2 + \rho_3) + \frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial y}\right) = -\sigma_1(\rho_2 + \rho_3) - \frac{\rho_3}{B}\frac{\partial B}{\partial y} + (\tau_1 - \tau_3)\frac{1}{B}\frac{\partial B}{\partial x} \qquad \ldots (B.16)$$

Some further useful identities are

$$\underset{\sim}{\nabla}\cdot \underset{\sim}{e}_1 = -\frac{1}{B}\frac{\partial B}{\partial s} = \rho_2 + \rho_3$$

$$\text{curl}\ \underset{\sim}{e}_1 = \sigma_1\,\underset{\sim}{e}_2 + \rho_1\,\underset{\sim}{e}_3 - \underset{\sim}{e}_1\,(\tau_2 + \tau_3) \qquad \ldots (B.17)$$

$$\frac{J_{\parallel}}{B} = \underset{\sim}{e}_1\cdot \text{curl}\ \underset{\sim}{e}_1 = -(\tau_2 + \tau_3)$$

Rotations of the $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ axes

In the final statement of results it is usually convenient to take $\underset{\sim}{e}_2$ in the direction of the principal normal to a line of force. However, in the evaluation of the operators L and H of Appendix C it is more convenient to take another choice of axes, obtained by rotating the given $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ axes about the direction $\underset{\sim}{e}_1$. Consider the new axes

$$\hat{\underset{\sim}{e}}_2 = \eta\underset{\sim}{e}_2 + \Upsilon\underset{\sim}{e}_3 \quad ; \quad \hat{\underset{\sim}{e}}_3 = \underset{\sim}{e}_1 \times \hat{\underset{\sim}{e}}_2 = \eta\underset{\sim}{e}_3 - \Upsilon\underset{\sim}{e}_2 \qquad \ldots (B.18)$$

where $\eta^2 + \Upsilon^2 = 1$.

Then in the new coordinate frame we have

$$(\hat{\underset{\sim}{e}}_2\cdot\underset{\sim}{\nabla})\underset{\sim}{e}_1 = \hat{\rho}_2\,\hat{\underset{\sim}{e}}_2 - \hat{\tau}_2\,\hat{\underset{\sim}{e}}_3 = \eta(\underset{\sim}{e}_2\cdot\underset{\sim}{\nabla})\underset{\sim}{e}_1 + \Upsilon(\underset{\sim}{e}_3\cdot\underset{\sim}{\nabla})\underset{\sim}{e}_1 \qquad \ldots (B.19)$$

Then using the equations (B.4) and (B.18) we may solve (B.19) for $\hat{\rho}_2$, $\hat{\tau}_2$ to obtain

$$\begin{aligned}\hat{\rho}_2 &= \eta^2\rho_2 + \Upsilon^2\rho_3 + \Upsilon\eta(\tau_3 - \tau_2)\\ \hat{\tau}_2 &= \eta^2\tau_2 + \Upsilon^2\tau_3 + \Upsilon\eta(\rho_2 - \rho_3)\end{aligned} \qquad \ldots (B.20)$$

In similar fashion we may obtain expressions for all the $\hat{\rho}_i$, $\hat{\sigma}_i$, $\hat{\tau}_i$ in terms of ρ_i, σ_i τ_i and η, Υ. For example

$$\begin{aligned}\hat{\rho}_3 &= \eta^2\rho_3 + \Upsilon^2\rho_2 + \eta\Upsilon(\tau_2 - \tau_3)\\ \hat{\tau}_3 &= \eta^2\tau_3 + \Upsilon^2\tau_2 + \eta\Upsilon(\rho_3 - \rho_2)\end{aligned} \qquad \ldots (B.21)$$

It is clear from (B.20), (B.21) that certain combinations of the ρ's and τ's are invariant under rotations. For example

$$\hat{\tau}_2 + \hat{\tau}_3 = \tau_2 + \tau_3 \;; \qquad \hat{\rho}_2 + \hat{\rho}_3 = \rho_2 + \rho_3$$

This is consistent with their interpretations as $-\underset{\sim}{e}_1 \cdot \text{curl}\, \underset{\sim}{e}_1$ and $\underset{\sim}{\nabla} \cdot \underset{\sim}{e}_1$ respectively, which are clearly independent of the orientation of the $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ axes. However the factors $(\tau_2 - \tau_3)$ and $(\rho_3 - \rho_2)$ which appear in D_{2S}, A_{2C} and D_{2C}, A_{2S}(see Appendix C) respectively are not invariant under rotations of these axes, and one or other (but not both simultaneously) can be made to vanish. Thus

$$\hat{\tau}_2 - \hat{\tau}_3 = (\tau_2 - \tau_3)\,(\eta^2 - \Upsilon^2) + 2\Upsilon\eta(\rho_2 - \rho_3) \qquad \text{... (B.22)}$$

and this is zero if

$$\eta^2 = \frac{1}{2}\left[\,1 \pm \frac{(\rho_2 - \rho_3)}{\{(\tau_2 - \tau_3)^2 + (\rho_2 - \rho_3)^2\}^{\frac{1}{2}}}\,\right] .$$

With this choice of new axes, D_{2S} and A_{2C} vanish, and considerable simplification occurs in the complicated H operator

(b) <u>The α, β, s Coordinate System</u>

In this coordinate system the gradient operator is given by

$$\underset{\sim}{\nabla} = \underset{\sim}{\nabla}\alpha \frac{\partial}{\partial\alpha} + \underset{\sim}{\nabla}\beta \frac{\partial}{\partial\beta} + \underset{\sim}{\nabla}s \frac{\partial}{\partial s} \qquad \text{... (B.23)}$$

where $\underset{\sim}{B} = \underset{\sim}{\nabla}\alpha \times \underset{\sim}{\nabla}\beta$, and the three base vectors are in general non-orthogonal. Thus

$$\frac{\partial}{\partial\alpha} = -\frac{\underset{\sim}{\nabla}s \times \underset{\sim}{\nabla}\beta}{B} \cdot \underset{\sim}{\nabla} \;; \quad \frac{\partial}{\partial\beta} = \frac{\underset{\sim}{\nabla}s \times \underset{\sim}{\nabla}\alpha}{B} \cdot \underset{\sim}{\nabla} \;; \quad \frac{\partial}{\partial s} = \frac{\underset{\sim}{\nabla}\alpha \times \underset{\sim}{\nabla}\beta}{B} \cdot \underset{\sim}{\nabla} \equiv \underset{\sim}{e}_1 \cdot \underset{\sim}{\nabla} \qquad \text{... (B.24)}$$

and the three derivatives commute.

However in the present problem, derivatives in other directions in α, β, s space occur naturally. These are

$$\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\alpha}{B} \cdot \underset{\sim}{\nabla} = \frac{\partial}{\partial\beta_\perp} \;; \quad -\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta}{B} \cdot \underset{\sim}{\nabla} = \frac{\partial}{\partial\alpha_\perp} \qquad \text{... (B.25)}$$

As anticipated in the above notation these are derivatives taken at right angles to field lines, and therefore not at constant s. Consequently they do not commute with $\frac{\partial}{\partial s}$, nor with each other. In fact, writing

$$\frac{\partial}{\partial\alpha_\perp} = X_\alpha \frac{\partial}{\partial\alpha} + X_\beta \frac{\partial}{\partial\beta} + X_s \frac{\partial}{\partial s}$$

and operating on α, β, and s respectively with $\frac{\partial}{\partial\alpha_\perp}$, we find

$$X_\alpha = 1 \;; \quad X_\beta = 0 \;; \quad X_s = \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta \cdot \underset{\sim}{\nabla}s}{B}$$

so that

$$\frac{\partial}{\partial\alpha_\perp} = \frac{\partial}{\partial\alpha} - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta \cdot \underset{\sim}{\nabla}s}{B} \frac{\partial}{\partial s} \quad . \qquad \text{... (B.26)}$$

Similarly

$$\frac{\partial}{\partial\beta_\perp} = \frac{\partial}{\partial\beta} + \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\alpha \cdot \underset{\sim}{\nabla}s}{B} \frac{\partial}{\partial s} \quad . \qquad \text{... (B.27)}$$

A consequence of these results is that for any function $A = A(\alpha, \beta, s)$,

$$\frac{\partial}{\partial\beta_\perp} \int^s A ds = \int^s \frac{\partial A}{\partial\beta} ds + \left(A \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\alpha \cdot \underset{\sim}{\nabla}s}{B} \right)$$

$$\frac{\partial}{\partial\alpha_\perp} \int^s A ds = \int^s \frac{\partial A}{\partial\alpha} ds - \left(A \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta \cdot \underset{\sim}{\nabla}s}{B} \right) \quad . \qquad \text{... (B.28)}$$

We note that for any function $Q = Q(\alpha, \beta)$ which is independent of s,

$$\frac{\partial Q}{\partial\alpha_\perp} = \frac{\partial Q}{\partial\alpha} \quad ; \quad \frac{\partial Q}{\partial\beta_\perp} = \frac{\partial Q}{\partial\beta}$$

Finally we will require the resolution of the curvature of a line of force $\underset{\sim}{\rho}$, in α, β space. This is

$$\underset{\sim}{\rho} = \rho_\alpha \underset{\sim}{\nabla}\alpha + \rho_\beta \underset{\sim}{\nabla}\beta$$

and we now obtain expressions for ρ_α and ρ_β which will be useful in the main text.

Consider

$$\frac{\partial}{\partial s}\left(\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta \cdot \underset{\sim}{\nabla}s}{B} \right) = \underset{\sim}{\nabla}s \cdot \frac{\partial}{\partial s}\left(\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta}{B} \right) - \underset{\sim}{\nabla}s \cdot \left(\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta}{B} \cdot \underset{\sim}{\nabla} \right) \underset{\sim}{e}_1 \quad ;$$

then writing $\frac{\partial}{\partial s} = \underset{\sim}{e}_1 \cdot \underset{\sim}{\nabla}$ and using the vector identity

$$(\underset{\sim}{a} \cdot \underset{\sim}{\nabla})\underset{\sim}{b} - (\underset{\sim}{b} \cdot \underset{\sim}{\nabla})\underset{\sim}{a} = \underset{\sim}{a}(\underset{\sim}{\nabla} \cdot \underset{\sim}{b}) - \underset{\sim}{b}(\underset{\sim}{\nabla} \cdot \underset{\sim}{a}) - \underset{\sim}{\nabla} \times (\underset{\sim}{a} \times \underset{\sim}{b})$$

this reduces to

$$\frac{\partial}{\partial s}\left(\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta \cdot \underset{\sim}{\nabla}s}{B} \right) = \underset{\sim}{\nabla}s \cdot \left[- \text{curl}\left(\frac{\underset{\sim}{\nabla}\beta}{B} \right) + \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta}{B} \underset{\sim}{\nabla} \cdot \underset{\sim}{e}_1 - \underset{\sim}{e}_1 \underset{\sim}{\nabla} \cdot \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta}{B} \right]$$

$$= - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta \cdot \underset{\sim}{\nabla}s}{B} \left(\frac{1}{B}\frac{\partial B}{\partial s} \right) + \frac{1}{B}\frac{\partial B}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha_\perp} - \frac{1}{B} \underset{\sim}{\nabla}\beta \cdot \text{curl } \underset{\sim}{e}_1 \quad .$$

The first three terms vanish by (B.26), and using $\underset{\sim}{\rho} = - \underset{\sim}{e}_1 \times \text{curl } \underset{\sim}{e}_1$ in the last term we obtain

$$\frac{\partial}{\partial s}\left(\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta \cdot \underset{\sim}{\nabla}s}{B} \right) = + \rho_\alpha \quad . \qquad \text{... (B.29)}$$

Similarly we have the result

$$\frac{\partial}{\partial s}\left(\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\alpha \cdot \underset{\sim}{\nabla}s}{B} \right) = - \rho_\beta \qquad \text{... (B.30)}$$

Finally we construct the commutators of the $\frac{\partial}{\partial\alpha_\perp}$, $\frac{\partial}{\partial\beta_\perp}$ derivatives with each other and with $\frac{\partial}{\partial s}$.

Thus

$$\frac{\partial^2}{\partial s \partial \alpha_\perp} - \frac{\partial^2}{\partial \alpha_\perp \partial s} = \frac{\partial^2}{\partial s \partial \alpha} - \frac{\partial}{\partial s} \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta \cdot \underset{\sim}{\nabla} s}{B} \frac{\partial}{\partial s} - \frac{\partial^2}{\partial \alpha \partial s} + \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta \cdot \underset{\sim}{\nabla} s}{B} \frac{\partial^2}{\partial s^2} = - \rho_\alpha \frac{\partial}{\partial s} \quad , \qquad \text{... (B.31)}$$

and similarly

$$\frac{\partial^2}{\partial s \partial \beta_\perp} - \frac{\partial^2}{\partial \beta_\perp \partial s} = \rho_\beta \frac{\partial}{\partial s} \quad , \qquad \text{... (B.32)}$$

$$\frac{\partial^2}{\partial \alpha_\perp \partial \beta_\perp} - \frac{\partial^2}{\partial \beta_\perp \partial \alpha_\perp} = - \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta}{B} \cdot \underset{\sim}{\nabla} \left[\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\alpha}{B} \cdot \underset{\sim}{\nabla} \right] + \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\alpha}{B} \cdot \underset{\sim}{\nabla} \left[\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta}{B} \cdot \underset{\sim}{\nabla} \right]$$

$$= \left(\frac{1}{B} \frac{\partial B}{\partial \beta_\perp} \frac{\partial}{\partial \alpha_\perp} - \frac{1}{B} \frac{\partial B}{\partial \alpha_\perp} \frac{\partial}{\partial \beta_\perp} \right)$$

$$+ \frac{1}{B^2} \Big\{ \text{curl} \left[(\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta) \times (\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\alpha) \right] - (\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta) \, \text{div} \, (\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\alpha)$$

$$+ (\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\alpha) \, \text{div} \, (\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta) \Big\} \cdot \underset{\sim}{\nabla} \quad ,$$

where the identity for curl $(\underset{\sim}{a} \times \underset{\sim}{b})$ has been used.

Now using

$$(\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta) \times (\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\alpha) = - \underset{\sim}{B} \; , \quad \text{div} \, (\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\alpha) = \underset{\sim}{\nabla}\alpha \cdot \text{curl} \, \underset{\sim}{e}_1$$

and

$$\text{div} \, (\underset{\sim}{e}_1 \times \underset{\sim}{\nabla}\beta) = \underset{\sim}{\nabla}\beta \cdot \text{curl} \, \underset{\sim}{e}_1 \quad ,$$

the above result reduces to

$$\frac{\partial^2}{\partial \alpha_\perp \partial \beta_\perp} - \frac{\partial^2}{\partial \beta_\perp \partial \alpha_\perp} = - \frac{(\underset{\sim}{e}_1 \cdot \text{curl} \, \underset{\sim}{e}_1)}{B} \frac{\partial}{\partial s} \qquad \text{... (B.33)}$$

APPENDIX C

TRANSFORMATION OF THE VLASOV EQUATION TO μ, ε, σ, φ VARIABLES, AND THE DERIVATION OF THE D OPERATOR

In the body of this paper we have shown that the equilibrium conditions and the adiabatic invariants, are determined by the operators D, L, H and G, and their integrals. In this, and the following appendices, the various operators are explicitly evaluated and the results collected.

The Vlasov equation for a distribution of particles $f(\underset{\sim}{r}, \underset{\sim}{v}, t)$ in a general stationary magnetic field, and in the absence of electric fields, is

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f + \frac{e}{m} \underset{\sim}{v} \times \underset{\sim}{B} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = 0 \qquad \text{... (C.1)}$$

This equation is put in a more convenient form by the following transformations of the velocity variables $\underset{\sim}{v}$.

First we resolve $\underset{\sim}{v}$ in the directions $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ so that

$$\underset{\sim}{v} = v_{\parallel}\, \underset{\sim}{e}_1 + v_x\, \underset{\sim}{e}_2 + v_y\, \underset{\sim}{e}_3 \qquad \text{... (C.2)}$$

Under this change of variable, the various terms of (C.1) transform as follows

$$\left(\frac{\partial}{\partial t}\right)_{\underset{\sim}{v}} = \left(\frac{\partial}{\partial t}\right)_{v_{\parallel} v_x v_y} \qquad \text{... (C.3)}$$

$$\frac{\partial}{\partial \underset{\sim}{v}} = \underset{\sim}{e}_1 \frac{\partial}{\partial v_{\parallel}} + \underset{\sim}{e}_2 \frac{\partial}{\partial v_x} + \underset{\sim}{e}_3 \frac{\partial}{\partial v_y} \qquad \text{... (C.4)}$$

$$(\underset{\sim}{v} \cdot \underset{\sim}{\nabla})_{v_i} = (\underset{\sim}{v} \cdot \underset{\sim}{\nabla})_{v_{\parallel} v_x v_y} + \underset{\sim}{v}\underset{\sim}{v} : \nabla \underset{\sim}{e}_1 \frac{\partial}{\partial v_{\parallel}} + \underset{\sim}{v}\underset{\sim}{v} : \nabla \underset{\sim}{e}_2 \frac{\partial}{\partial v_x} + \underset{\sim}{v}\underset{\sim}{v} : \nabla \underset{\sim}{e}_3 \frac{\partial}{\partial v_y} \quad ,\text{... (C.5)}$$

where the dyadic notation means, $\underset{\sim}{A}\underset{\sim}{B} : \nabla \underset{\sim}{C} \equiv \underset{\sim}{A} \cdot [(\underset{\sim}{B} \cdot \underset{\sim}{\nabla})\underset{\sim}{C}]$, and where $\underset{\sim}{v}$ on the right hand side of (C.5) is given by (C.2). Using the definitions (B.2,3) for the ρ_i, σ_i, τ_i, the right hand side of (C.5) can be written in terms of $v_x\, v_y\, v_{\parallel}$. For example, we find

$$\underset{\sim}{v}\underset{\sim}{v} : \nabla \underset{\sim}{e}_1 = v_{\parallel}(v_x \rho_1 - v_y\, \sigma_1) + v_x^2 \rho_2 + v_x v_y (\tau_3 - \tau_2) + v_y^2 \rho_3 \quad , \qquad \text{... (C.6)}$$

and similar results hold for $\underset{\sim}{v}\underset{\sim}{v} : \nabla \underset{\sim}{e}_2$ and $\underset{\sim}{v}\underset{\sim}{v} : \nabla \underset{\sim}{e}_3$.

In the plane perpendicular to $\underset{\sim}{B}$, we introduce polar variables defined by

$$\begin{aligned} v_x &= v_{\perp} \cos\varphi \\ v_y &= v_{\perp} \sin\varphi \end{aligned} \qquad \text{... (C.7)}$$

so that

$$\begin{aligned} \frac{\partial}{\partial v_x} &= \cos\varphi \frac{\partial}{\partial v_{\perp}} - \frac{\sin\varphi}{v_{\perp}} \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial v_y} &= \sin\varphi \frac{\partial}{\partial v_{\perp}} + \frac{\cos\varphi}{v_{\perp}} \frac{\partial}{\partial \varphi} \end{aligned} \qquad \text{... (C.8)}$$

Thus, in terms of the new variables $v_\parallel$, $v_\perp$, φ, we have

$$\left(\frac{\partial}{\partial t}\right)_{v_i} = \left(\frac{\partial}{\partial t}\right)_{v_\parallel, v_\perp, \varphi} \qquad \ldots \text{(C.9)}$$

$$\frac{\partial}{\partial \underset{\sim}{v}} = 2\underset{\sim}{v} \frac{\partial}{\partial v_\perp^2} + \frac{\underset{\sim}{e}_1 \times \underset{\sim}{v}_\perp}{v_\perp^2} \frac{\partial}{\partial \varphi} + \underset{\sim}{e}_1 \frac{\partial}{\partial v_\parallel} \quad , \qquad \ldots \text{(C.10)}$$

where the vector $\underset{\sim}{v}_\perp \equiv v_\perp (\underset{\sim}{e}_2 \cos\varphi + \underset{\sim}{e}_3 \sin\varphi)$, so that the last term of (C.1) has been reduced to the familiar form

$$\frac{e}{m} \underset{\sim}{v} \times \underset{\sim}{B} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = - \frac{eB}{m} \frac{\partial f}{\partial \varphi} \quad , \qquad \ldots \text{(C.11)}$$

and

$$\begin{aligned}(\underset{\sim}{v} \cdot \underset{\sim}{\nabla})_{v_i} = {} & (\underset{\sim}{v} \cdot \underset{\sim}{\nabla})_{v_\parallel v_\perp \varphi} + v_\parallel [v_\perp^2(\rho_2 + \rho_3) + 2v_\parallel v_\perp \rho_1 \cos\varphi - 2v_\parallel v_\perp \sigma_1 \sin\varphi \\ & + v_\perp^2(\tau_3 - \tau_2) \sin 2\varphi + v_\perp^2(\rho_2 - \rho_3)\cos 2\varphi] \left(\frac{\partial}{\partial v_\parallel^2} - \frac{\partial}{\partial v_\perp^2}\right) \\ & + [v_\parallel(\tau_1 + \tfrac{1}{2}(\tau_2 + \tau_3)) + \cos\varphi \left(\frac{v_\parallel^2}{v_\perp} \sigma_1 - v_\perp \sigma_2\right) + \sin\varphi \left(\frac{v_\parallel^2}{v_\perp} \rho_1 - \sigma_3 v_\perp\right) \\ & + \sin 2\varphi \, \tfrac{1}{2} v_\parallel (\rho_3 - \rho_2) + \cos 2\varphi \, \tfrac{1}{2} v_\parallel (\tau_2 - \tau_3)] \frac{\partial}{\partial \varphi}\end{aligned} \qquad \ldots \text{(C.12)}$$

Finally the $v_\parallel$, $v_\perp$ variables are replaced by ε, μ, σ, i.e. the particle energy, magnetic moment, and direction of flight relative to $\underset{\sim}{B}$. These are defined by,

$$\begin{aligned}\varepsilon &= \frac{1}{2} (v_\parallel^2 + v_\perp^2) \quad ; \\ \mu &= \frac{1}{2} \frac{v_\perp^2}{B} \\ \sigma[2(\varepsilon - \mu B)]^{1/2} &= v_\parallel \quad .\end{aligned} \qquad \ldots \text{(C.13)}$$

It follows that $\sigma^2 = +1$ and therefore that σ takes the values ± 1 depending on the sign of $\underset{\sim}{v} \cdot \underset{\sim}{B}$.

Under this final transformation the various derivatives become

$$\begin{aligned}&\left(\frac{\partial}{\partial t}\right)_{v_\parallel v_\perp} = \left(\frac{\partial}{\partial t}\right)_{\mu\varepsilon} \quad ; \quad \left(\frac{\partial}{\partial \varphi}\right)_{v_\parallel v_\perp} = \left(\frac{\partial}{\partial \varphi}\right)_{\mu\varepsilon} \\ &\frac{\partial}{\partial v_\perp^2} = \frac{1}{2} \left(\frac{1}{B} \frac{\partial}{\partial \mu} + \frac{\partial}{\partial \varepsilon}\right) \\ &\frac{\partial}{\partial v_\parallel^2} = \frac{1}{2} \frac{\partial}{\partial \varepsilon} \\ &(\underset{\sim}{v} \cdot \underset{\sim}{\nabla})_{v_\parallel v_\perp} = (\underset{\sim}{v} \cdot \underset{\sim}{\nabla})_{\mu\varepsilon} - \frac{\underset{\sim}{v} \cdot \underset{\sim}{\nabla} B}{B} \frac{\partial}{\partial \mu} \quad .\end{aligned} \qquad \ldots \text{(C.14)}$$

Thus the Vlasov equation may be written in terms of the new velocity-space variables μ, ε, φ, σ, in the form

$$\frac{1}{B}\frac{\partial f}{\partial t} + Df = \frac{e}{m}\frac{\partial f}{\partial \varphi} \qquad \ldots \text{(C.15)}$$

with $f = f(\mu, \varepsilon, \varphi, \sigma, \underset{\sim}{r}, t)$, and

$$\begin{aligned} D = {} & D_o + \cos\varphi\, D_c + \sin\varphi\, D_s + \cos 2\varphi\, D_{2c} + \sin 2\varphi\, D_{2s} \\ & + \left[A_o + \cos\varphi\, A_c + \sin\varphi\, A_s + \cos 2\varphi\, A_{2c} + \sin 2\varphi\, A_{2s}\right] \frac{\partial}{\partial \varphi} \end{aligned} \qquad \ldots \text{(C.16)}$$

where the D's are differential operators, the A's merely coefficients and both are independent of φ. They are given by

$$D_o = \frac{\sigma q}{B}\frac{\partial}{\partial s} \;\; ; \;\; D_c = \frac{c_\perp}{B}\left(\frac{\partial}{\partial x} - V_x \frac{\partial}{\partial \mu}\right) \;\; ; \;\; D_s = \frac{c_\perp}{B}\left(\frac{\partial}{\partial y} - V_y \frac{\partial}{\partial \mu}\right)$$

$$D_{2c} = \frac{\mu\sigma q}{B}(\rho_3 - \rho_2)\frac{\partial}{\partial \mu} \;\; ; \;\; D_{2s} = \frac{\mu\sigma q}{B}(\tau_2 - \tau_3)\frac{\partial}{\partial \mu}$$

$$A_o = \frac{\sigma q}{B}\left[\tau_1 + \frac{1}{2}(\tau_2 + \tau_3)\right] \;\; ; \;\; A_c = \frac{1}{B}\left[\frac{q^2\sigma_1}{c_\perp} - c_\perp\sigma_2\right] \;\; ; \;\; A_s = \frac{1}{B}\left[\frac{q^2\rho_1}{c_\perp} - c_\perp\sigma_3\right]$$

$$A_{2c} = \frac{\sigma q}{2B}(\tau_2 - \tau_3) \;\; ; \;\; A_{2s} = \frac{\sigma q}{2B}(\rho_2 - \rho_3) \, , \qquad \ldots \text{(C.17)}$$

where q and $c_\perp$ are to be regarded as functions of μ, ε defined by

$$q = + \left[2(\varepsilon - \mu B)\right]^{\frac{1}{2}}$$

and $\qquad \ldots$ (C.18)

$$c_\perp = + (2\mu B)^{\frac{1}{2}} \, .$$

The derivatives $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are shorthand for $\underset{\sim}{e}_1 \cdot \underset{\sim}{\nabla}$ etc., and V_x, V_y are given by

$$V_x = \frac{\mu}{B}\frac{\partial B}{\partial x} + \frac{q^2}{B}\rho_1 \;\; ; \;\; V_y = \frac{\mu}{B}\frac{\partial B}{\partial y} - \frac{q^2}{B}\sigma_1 \, . \qquad \ldots \text{(C.19)}$$

An important property of the D operator is now apparent, namely, that terms which are functions of an even multiple of φ (i.e. D_o, D_{2c}, D_{2s}, A_o, A_{2c}, A_{2s}) are odd in σ, whereas terms which involve odd multiples of φ (i.e. D_c, D_s, A_c, A_s) are even in σ. This property leads to important consequences concerning the L and H operators. In fact these are particular cases of a more general result which will now be proved. The basic operation in the construction of the various operators is

$$\langle D \textstyle\int D \int \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \int D\, Q \rangle \equiv K_m Q \, . \qquad \ldots \text{(C.20)}$$
$$\text{(m-1) } \textstyle\int D \text{ operations}$$

Theorem

For any function $Q(\mu, \varepsilon, \sigma, \underset{\sim}{r})$ which is an even function of σ (i.e. for which $Q(\sigma) = Q(-\sigma)$), $K_m Q$ is even or odd in σ according as m is an even or odd integer.

Proof

From the foregoing remarks on the form of D , it follows that we may write D as

$$D = \sum_{n=-2}^{+2} \sigma^{n+1} e^{in\varphi} \left[\tilde{D}_n + \tilde{A}_n \frac{\partial}{\partial\varphi} \right] \qquad \text{... (C.21)}$$

where the $\tilde{D}_n$ and $\tilde{A}_n$ are complex.

Then in $K_m Q$ the only terms for which the φ average is non-zero are those for which

$$e^{i \sum_{k=1}^{m} n_k \varphi} = 1$$

i.e. those for which $\sum_{k=1}^{m} n_k = 0$. However the σ dependence of the general term in (C.20) is

$$\sigma^{\left[\sum_{k=1}^{m} (1+n_k)\right]},$$

and is therefore of the form σ^m for all terms which are non-zero after averaging over φ. This proves the result. Functions which are odd in σ, may be dealt with by writing them as a product of σ with an even function. Thus if $P(\sigma)$ is odd in σ,

$$K_m P(\sigma) = \sigma K_m P(|\sigma|), \text{ and } P(|\sigma|) \text{ is even .}$$

From (C.16) and (C.17) we may immediately write down the result of operating on $f_0(\mu, \varepsilon, \sigma, \underset{\sim}{r})$ with D. Thus

$$\langle Df_0 \rangle = \langle D_0 f_0 \rangle \equiv \frac{\sigma q}{B} \frac{\partial f_0}{\partial s} \qquad \text{... (C.22)}$$

[The oddness in σ is a particular case of the theorem just proved, for this is just $K_1 f_0$]

In solving the Vlasov equation, an integration over φ has to be performed at each order, and this leads to a sequence of arbitrary constants g_1, g_2, etc. Thus in any φ integration arising in the operators L, H etc, the lower limit of integration has already been accounted for. Thus for example we interpret $\int Df_0$ as

$$\int^{\varphi} Df_0 = \sin\varphi\, D_c f_0 - \cos\varphi\, D_s f_0 + \frac{1}{2}\sin 2\varphi\, D_{2c} f_0 - \frac{1}{2}\cos 2\varphi\, D_{2s} f_0 \qquad \text{... (C.23)}$$

It will be noted that (C.23) contains no secular terms, these having vanished as a consequence of the fact that f_1 must be a single-valued function of φ, i.e. $\langle Df_0 \rangle = 0$. In fact the requirement that f is single-valued leads to the secular terms disappearing from each order of the problem. This enables us to treat each integral $\int^{\varphi} A d\varphi$, where A is any harmonically dependent function of φ, as if it were $\int^{\varphi} [A - \langle A \rangle] d\varphi$. If $\langle A \rangle = 0$ this is clearly correct, but if $\langle A \rangle \neq 0$ the procedure is still permissible, since somewhere in the problem there must be a term of the form $-\langle A \rangle$ which ensures the cancellation of the secular terms.

APPENDIX D

THE L OPERATOR

The L operator is defined by

$$LQ = \left\langle D \int^{\varphi} DQ \right\rangle . \qquad \ldots (D.1)$$

In accordance with our remarks in Appendix C, we ignore secular terms arising from $\int^{\varphi} DQ$. We shall require the result of LQ not only for $Q = f_o\,(\alpha, \beta, \mu, \varepsilon, \sigma)$ but also for an s dependent Q. This more general Q is considered here. First we note that L is just the operator K_m discussed in Appendix C (C.20), with $m = 2$, and consequently LQ is even or odd in σ according as Q is even or odd. From (C.16),

$$\begin{aligned} LQ = {} & \langle (D_o+\cos\varphi D_c+\sin\varphi D_s+\cos 2\varphi D_{2c}+\sin 2\varphi D_{2s}) \\ & \quad (\sin\varphi D_c-\cos\varphi D_s+\tfrac{1}{2}\sin 2\varphi D_{2c}-\tfrac{1}{2}\cos 2\varphi D_{2s})Q\rangle \\ & + \langle A_o+\cos\varphi A_c+\sin\varphi A_s+\cos 2\varphi A_{2c}+\sin 2\varphi A_{2s}) \\ & \quad (\cos\varphi D_c+\sin\varphi D_s+\cos 2\varphi D_{2c}+\sin 2\varphi D_{2s})Q\rangle \end{aligned} \qquad \ldots (D.2)$$

On taking the average over φ, $\langle \cos^2\varphi\rangle = \langle \sin^2\varphi\rangle = \langle \cos^2 2\varphi\rangle = \langle \sin^2 2\varphi\rangle = \frac{1}{2}$ and all other averages are zero, so that D.2 becomes

$$LQ = \tfrac{1}{2}\,(D_sD_c-D_cD_s)Q+\tfrac{1}{2}(A_cD_c+A_sD_s)Q+\tfrac{1}{4}(D_{2s}D_{2c}-D_{2c}D_{2s})Q+\tfrac{1}{2}(A_{2s}D_{2s}+A_{2c}D_{2c})Q \qquad \ldots (D.3)$$

Inspection of the explicit forms of the D's and A's reveals that

$$\begin{aligned} D_{2s}\,D_{2c} - D_{2c}\,D_{2s} &\equiv 0 \\ A_{2s}\,D_{2s} + A_{2c}\,D_{2c} &\equiv 0 \end{aligned} \qquad \ldots (D.4)$$

The evaluation of the remaining terms in D.3 involves the use of the commutator (B.4) operating on Q and on B, the results B.7 and B.8 to replace $\frac{\partial\rho_1}{\partial y}$ and $\frac{\partial\sigma_1}{\partial x}$, and B.17. The final result is

$$LQ = \frac{\underset{\sim}{W}_d \cdot \underset{\sim}{\nabla}Q}{B} + \frac{\mu}{B}\left(\frac{\partial Q}{\partial\mu}\right) q\,\frac{\partial}{\partial s}\left(\frac{q\,\underset{\sim}{e}_1 \cdot \operatorname{curl} \underset{\sim}{e}_1}{B}\right) + \frac{\mu}{B}\,(\underset{\sim}{e}_1 \cdot \operatorname{curl} \underset{\sim}{e}_1)\,\frac{\partial Q}{\partial s} \qquad \ldots (D.5)$$

In part I, $\oint \frac{Bds}{q}\, Lf_o$ (appropriately interpreted for a mirror system) is required, and when put equal to zero gives rise to the constraint (3.17) on f_o. It follows by integrating (D.5) that

$$\oint \frac{Bds}{q}\, Lf_o = \oint \frac{ds}{q}\, \underset{\sim}{W}_d \cdot \underset{\sim}{\nabla} f_o, \qquad \ldots (D.6)$$

where the last term of (D.5) has vanished through $\frac{\partial f_o}{\partial s} = 0$. In the case of a closed line system the middle term of (D.5) disappears by virtue of an integration round a closed path, whereas, for a mirror system the fact that $q = 0$ at the limits of integration again ensures that this term makes no contribution. Thus the second order constraint on f_o (3.21) is

$$\oint \frac{ds}{q} \underset{\sim}{W}_d \cdot \underset{\sim}{\nabla} f_o \equiv \oint \frac{ds}{q} \frac{\underset{\sim}{e}_1}{B} \times (q^2 \underset{\sim}{\rho} + \mu \underset{\sim}{\nabla} B) \cdot \underset{\sim}{\nabla} f_o = 0 \qquad \ldots \text{(D.7)}$$

The reduction of this equation to the simple form (3.24) in terms of α, β coordinates proceeds as follows.

In terms of α, β coordinates

$$\begin{aligned} \underset{\sim}{\nabla} f_o &= \frac{\partial f_o}{\partial \alpha} \underset{\sim}{\nabla} \alpha + \frac{\partial f_o}{\partial \beta} \underset{\sim}{\nabla} \beta \, , \\ \underset{\sim}{\nabla} B &= \frac{\partial B}{\partial \alpha} \underset{\sim}{\nabla} \alpha + \frac{\partial B}{\partial \beta} \underset{\sim}{\nabla} \beta + \frac{\partial B}{\partial s} \underset{\sim}{\nabla} s \\ \underset{\sim}{\rho} &= \rho_\alpha \cdot \underset{\sim}{\nabla} \alpha + \rho_\beta \, \underset{\sim}{\nabla} \beta \, , \end{aligned} \qquad \ldots \text{(D.8)}$$

where as shown in Appendix B (B.29), (B.30)

$$\begin{aligned} \rho_\alpha &= \frac{\partial}{\partial s} \left(\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} \beta \cdot \underset{\sim}{\nabla} s}{B} \right) \\ \rho_\beta &= - \frac{\partial}{\partial s} \left(\frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} \alpha \cdot \underset{\sim}{\nabla} s}{B} \right) \end{aligned} \qquad \ldots \text{(D.9)}$$

and D.7 becomes

$$\frac{\partial f_o}{\partial \alpha} \oint \left[\frac{\partial q}{\partial \beta} + \frac{\partial}{\partial s} \left(q \, \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} \alpha \cdot \underset{\sim}{\nabla} s}{B} \right) \right] ds - \frac{\partial f_o}{\partial \beta} \oint \left[\frac{\partial q}{\partial \alpha} - \frac{\partial}{\partial s} \left(q \, \frac{\underset{\sim}{e}_1 \times \underset{\sim}{\nabla} \beta \cdot \underset{\sim}{\nabla} s}{B} \right) \right] ds = 0 \quad \ldots \text{(D.10)}$$

The $\partial/\partial s$ terms of the integrands do not contribute for either oscillatory or circulatory particles, so that D.10 becomes

$$\frac{\partial f_o}{\partial \alpha} \frac{\partial J}{\partial \beta} - \frac{\partial f_o}{\partial \beta} \frac{\partial J}{\partial \alpha} = 0, \qquad \text{where} \qquad J = \oint q \, ds$$

That is;

$$\oint \frac{Bds}{q} \, Lf_o = \left(\frac{\partial f_o}{\partial \alpha} \frac{\partial J}{\partial \beta} - \frac{\partial f_o}{\partial \beta} \frac{\partial J}{\partial \alpha} \right) \qquad \ldots \text{(D.11)}$$

THE H OPERATOR

The H operator (3.34) is defined by

$$Hf_o = \oint \frac{Bds}{q} \left\{ \sigma L \int_{s_o}^{s} \frac{Bds}{q} \, Lf_o - \left\langle D \int D \int Df_o \right\rangle \right\} \qquad \ldots \text{(D.12)}$$

where the operator is required only for a function $f_o = f_o(\mu, \varepsilon, J, \sigma)$. Hitherto, although different interpretations have had to be adopted for some of the operators, depending on whether the particle distribution is contained by mirrors or by a toroidal topology (e.g. 3.21, 3.22), the results have been independent of these considerations (e.g. D.11). With Hf_o this ceases to be the case.

(a) Oscillatory Periodicity

For this case the zero order distribution f_o must be even in σ(3.22). In section (a) of this appendix it was observed that L is just the operator K_2. Consequently Lf_o and therefore $L\int_{s_o}^{s}\frac{Bds}{q}\,Lf_o$ are both even in σ. Similarly $\left\langle D\int D\int Df_o\right\rangle$ is just K_3f_o and must be odd in σ, so that the whole integrand of (D.12) is an odd function of σ and therefore vanishes when integrated $\oint\frac{Bds}{q}\equiv\sum_{\sigma=\pm1}\int_A^B\frac{Bds}{q}$ between reflection points in a mirror system. That is

$$Hf_o = 0 \qquad \ldots \text{(D.13)}$$

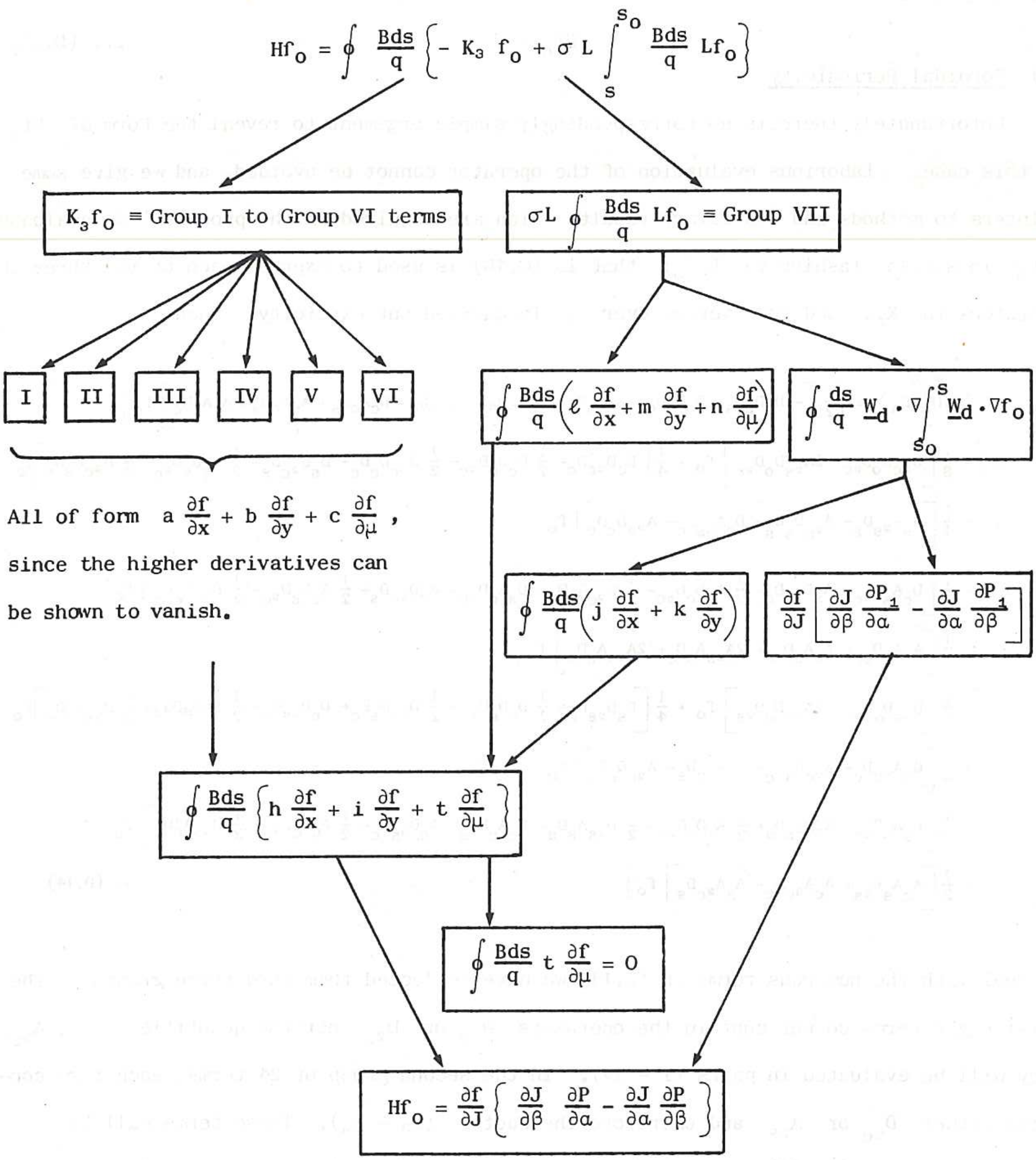
(b) Toroidal Periodicity

Unfortunately there is no correspondingly simple argument to reveal the form of Hf_o in this case. Laborious evaluation of the operator cannot be avoided, and we give some pointers to methods and subsidiary results which are required in the process. We evaluate K_3f_o in similar fashion to K_2f_o; that is (C.16) is used to express each of the three D operators in K_3, and the average over φ is carried out explicity. Then

$$\begin{aligned}
K_3f_o \equiv \left\langle D\int^{\varphi} D\int^{\varphi} Df_o\right\rangle = &\left\{\frac{1}{2}\left[-D_sD_oD_s - D_cD_oD_c + A_sD_oD_c - A_cD_oD_s + D_sA_oD_c - D_cA_oD_s + A_sA_oD_s + A_cA_oD_c\right]f_o\right\}\\
&+\left\{-\frac{1}{8}\left[D_{2c}D_oD_{2c} - 2A_{2s}D_oD_{2c}\right]f_o + \frac{1}{4}\left[D_cD_{2c}D_c - \frac{1}{2}D_cD_cD_{2c} - \frac{1}{2}D_{2c}D_cD_c - D_sD_{2c}D_s + \frac{1}{2}D_sD_sD_{2c} + \frac{1}{2}D_{2c}D_sD_s\right]f_o\right.\\
&\quad+\frac{1}{4}\left[D_sA_{2s}D_s - A_{2s}D_sD_s - D_cA_{2s}D_c + A_{2s}D_cD_c\right]f_o\\
&\quad+\frac{1}{4}\left[D_cA_sD_{2c} - A_sD_{2c}D_c + \frac{1}{2}A_sD_cD_{2c} - \frac{1}{2}D_{2c}A_sD_c + D_sA_cD_{2c} - A_cD_{2c}D_s + \frac{1}{2}A_cD_sD_{2c} - \frac{1}{2}D_{2c}A_cD_s\right]f_o\\
&\quad\left.+\frac{1}{4}\left[A_cA_cD_{2c} - A_sA_sD_{2c} + 2A_{2s}A_sD_c + 2A_{2s}A_cD_s\right]f_o\right\}\\
&+\left\{-\frac{1}{8}\left[D_{2s}D_oD_{2s} + 2A_{2c}D_oD_{2s}\right]f_o + \frac{1}{4}\left[D_sD_{2s}D_c - \frac{1}{2}D_sD_cD_{2s} - \frac{1}{2}D_{2s}D_sD_c + D_cD_{2s}D_s - \frac{1}{2}D_cD_sD_{2s} - \frac{1}{2}D_{2s}D_cD_s\right]f_o\right.\\
&\quad+\frac{1}{4}\left[D_sA_{2c}D_c - A_{2c}D_sD_c + D_cA_{2c}D_s - A_{2c}D_cD_s\right]f_o\\
&\quad+\frac{1}{4}\left[D_sA_sD_{2s} - A_sD_{2s}D_s + \frac{1}{2}A_sD_sD_{2s} - \frac{1}{2}D_{2s}A_sD_s - D_cA_cD_{2s} + A_cD_{2s}D_c - \frac{1}{2}A_cD_cD_{2s} + \frac{1}{2}D_{2s}A_cD_c\right]f_o\\
&\quad\left.+\frac{1}{2}\left[A_cA_sD_{2s} + A_cA_{2c}D_c - A_sA_{2c}D_s\right]f_o\right\} \qquad \ldots \text{(D.14)}
\end{aligned}$$

To deal with the numerous terms in (D.14) we have collected them into three groups. The first eight terms do not contain the operators D_{2c} or D_{2s} nor the quantities A_{2c}, A_{2s}. They will be evaluated in pairs (I - IV). In the second group of 24 terms, each term contains either D_{2c} or A_{2s} and therefore the factor $(\rho_3 - \rho_2)$. These terms will be evaluated as one unit (V), as will the remaining terms VI, all of which contain either D_{2s} or A_{2c} and therefore the factor $(\tau_2 - \tau_3)$. Finally, to complete the calculation of Hf_o, we must evaluate $\sigma L\int_{s_o}^{s}\frac{Bds}{q}\,Lf_o$ which we denote by term VII. In evaluating terms I - VII we shall find that no third derivatives of f_o appear, and that the few second derivatives which are found, will integrate out under the s integration. The remaining terms will then be expressed in the form $Hf_o = a\frac{\partial f_o}{\partial J} + b\frac{\partial f_o}{\partial \mu}$. Since the algebra is very involved the reader may find the appended block diagram of assistance in following the details of the calculation.

The following block-diagram illustrates the order in which the intermediate steps in the evaluation of H are carried out:-



In this diagram the symbols a,b,c,l,m,n,j,k,h,i and t are intended only to indicate the general form of the results at each stage, and hence, are not used elsewhere in the report.

We commence with $K_3 f_o$:

I. $\underline{-\frac{1}{2}\,(D_s D_o D_s + D_c D_o D_c)}$

Using the forms for D_o, D_s, D_c given in C.17, and the commutation relations B.5, B.6 the following results may be obtained:-

$$D_o D_c f_o = \frac{\sigma g}{B^2}\, c_\perp \left\{ \frac{1}{2}\,(\rho_3 - \rho_2)\,\frac{\partial f_o}{\partial x} + (\tau_2 - \tau_1)\,\frac{\partial f_o}{\partial y} \right\} - \frac{\partial f_o}{\partial \mu}\, D_o \left(\frac{c_\perp V_x}{B} \right) \qquad \text{... (D.15)}$$

$$D_o D_s f_o = \frac{\sigma g}{B^2}\, c_\perp \left\{ \frac{1}{2}\,(\rho_2 - \rho_3)\,\frac{\partial f_o}{\partial y} + (\tau_1 - \tau_3)\,\frac{\partial f_o}{\partial x} \right\} - \frac{\partial f_o}{\partial \mu}\, D_o \left(\frac{c_\perp V_y}{B} \right) \qquad \text{... (D.16)}$$

where for simplicity we retain $\frac{\partial f_o}{\partial J}\frac{\partial J}{\partial x}$ and $\frac{\partial f_o}{\partial J}\frac{\partial J}{\partial y}$ as $\frac{\partial f_o}{\partial x}$, $\frac{\partial f_o}{\partial y}$ respectively.

Now to obtain I, we operate on D.15 and D.16 with D_c and D_s respectively and use the following identities:-

$$\frac{\partial}{\partial s}\frac{1}{B}\frac{\partial^2 f}{\partial x^2} = \frac{(\rho_3 - \rho_2)}{B}\,\frac{\partial^2 f}{\partial x^2} + \frac{(\tau_2 - \tau_1)}{B}\left[\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial x \partial y}\right]$$
$$+ \frac{1}{B}\frac{\partial f}{\partial y}\left[\frac{\partial}{\partial x}\,(\tau_2 - \tau_1) - \rho_1(\tau_2 - \tau_1)\right] - \frac{1}{B}\frac{\partial f}{\partial x}\left[\frac{\partial \rho_2}{\partial x} - \rho_1 \rho_2\right] \qquad \text{... (D.17)}$$

$$\frac{\partial}{\partial s}\frac{1}{B}\frac{\partial^2 f}{\partial y^2} = -\frac{(\rho_3 - \rho_2)}{B}\,\frac{\partial^2 f}{\partial y^2} + \frac{(\tau_1 - \tau_3)}{B}\left[\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial x \partial y}\right]$$
$$+ \frac{1}{B}\frac{\partial f}{\partial x}\left[\frac{\partial}{\partial y}\,(\tau_1 - \tau_3) + \sigma_1\,(\tau_1 - \tau_3)\right] - \frac{1}{B}\frac{\partial f}{\partial y}\left[\frac{\partial \rho_3}{\partial y} + \sigma_1 \rho_3\right] \qquad \text{... (D.18)}$$

which follow from double application of B.5 and B.6. Finally, after much manipulation one obtains

$$-\frac{1}{2}\left[D_c D_o D_c + D_s D_o D_s\right] f_o = -\frac{1}{2}\frac{\sigma g}{B}\frac{\partial}{\partial s}\frac{\mu}{B}\left\{(V_x^2 + V_y^2)\,\frac{\partial^2 f_o}{\partial \mu^2} - 2\left(V_x\,\frac{\partial^2 f_o}{\partial \mu \partial x} + V_y\,\frac{\partial^2 f_o}{\partial \mu \partial y}\right) + \left(\frac{\partial^2 f_o}{\partial x^2} + \frac{\partial^2 f_o}{\partial y^2}\right)\right\}$$
$$+ \frac{1}{2}\frac{\partial f_o}{\partial \mu}\left[D_c D_o\left(\frac{c_\perp V_x}{B}\right) + D_s D_o\left(\frac{c_\perp V_y}{B}\right)\right]$$
$$+ \frac{1}{2}\frac{\mu \sigma g}{B^2}\frac{\partial f_o}{\partial x}\left\{\sigma_2(\tau_2 + \tau_3) + \rho_1 \rho_2 + \sigma_1(\tau_1 - \tau_3) - 2\tau_1 \sigma_2 - \frac{\partial \rho_3}{\partial x} + \frac{\partial}{\partial y}(\tau_3 - \tau_1) + 2(\tau_1 - \tau_3)\left(\sigma_1 + \frac{2}{B}\frac{\partial B}{\partial y}\right) + (\rho_3 - \rho_2)\left(\frac{2}{B}\frac{\partial B}{\partial x} - \rho_1\right)\right\}$$
$$+ \frac{1}{4}\frac{\sigma g^3}{B^3}\frac{\partial f_o}{\partial x}\left[\rho_1(\rho_3 - \rho_2) + 2\sigma_1(\tau_3 - \tau_1)\right]$$
$$+ \frac{1}{2}\frac{\mu \sigma g}{B^2}\frac{\partial f_o}{\partial y}\left\{\sigma_3(\tau_2 + \tau_3) - \sigma_1 \rho_3 + \rho_1(\tau_1 - \tau_2) - 2\sigma_3 \tau_1 - \frac{\partial \rho_2}{\partial y} + \frac{\partial}{\partial x}(\tau_1 - \tau_2) + 2(\tau_2 - \tau_1)\left(\frac{2}{B}\frac{\partial B}{\partial x} - \rho_1\right) - (\rho_3 - \rho_2)\left(\frac{2}{B}\frac{\partial B}{\partial y} + \sigma_1\right)\right\}$$
$$+ \frac{1}{4}\frac{\sigma g^3}{B^3}\frac{\partial f_o}{\partial y}\left[\sigma_1(\rho_3 - \rho_2) + 2\rho_1(\tau_2 - \tau_1)\right] \qquad \text{... (D.19)}$$

II. $+\frac{1}{2}(A_sD_oD_c - A_cD_oD_s)f_o$

Using D.15 and D.16 again this result follows at once

$$\frac{1}{2}\left[A_sD_oD_c - A_cD_oD_s\right] f_o = \frac{\partial f_o}{\partial x}\left\{\frac{1}{4}(\rho_3-\rho_2)\frac{\sigma q}{B^3}\left[q^2\rho_1 - 2\mu B\sigma_3\right] + \frac{1}{2}(\tau_3-\tau_1)\frac{\sigma q}{B^3}\left[q^2\sigma_1 - 2\mu B\sigma_2\right]\right\}$$
$$+\frac{\partial f_o}{\partial y}\left\{\frac{1}{4}(\rho_3-\rho_2)\frac{\sigma q}{B^3}\left[q^2\sigma_1 - 2\mu B\sigma_2\right] + \frac{1}{2}(\tau_2-\tau_1)\frac{\sigma q}{B^3}\left[q^2\rho_1 - 2\mu B\sigma_3\right]\right\}$$
$$+\frac{\partial f_o}{\partial \mu}\frac{1}{2}\left\{A_cD_o\left(\frac{c_\perp V_y}{B}\right) - A_sD_o\left(\frac{c_\perp V_x}{B}\right)\right\} \qquad \text{... (D.20)}$$

III. $\frac{1}{2}(D_sA_oD_c - D_cA_oD_s)f_o$

Again C.17 and the commutation result B.4 are used. The result follows after some algebraic manipulation

$$\frac{1}{2}(D_sA_oD_c - D_cA_oD_s)f_o = \frac{\partial f_o}{\partial x}\frac{\mu\sigma q}{B^2}\left\{\frac{\partial T}{\partial y} - T\left(\sigma_1 - \sigma_2 + \frac{2}{B}\frac{\partial B}{\partial y}\right)\right\}$$
$$+\frac{1}{2}\frac{\partial f_o}{\partial x}\frac{\sigma q^3}{B^3}\sigma_1 T - \frac{\partial f_o}{\partial y}\frac{\mu\sigma q}{B^2}\left\{\frac{\partial T}{\partial x} + T\left(\rho_1 - \sigma_3 - \frac{2}{B}\frac{\partial B}{\partial x}\right)\right\}$$
$$+\frac{1}{2}\frac{\partial f_o}{\partial y}\frac{\sigma q^3}{B^3}\rho_1 T + \frac{\partial f_o}{\partial \mu}\left\{-\frac{1}{2}\frac{\sigma q^3}{B^3}T(\rho_1 V_y + \sigma_1 V_x)\right.$$
$$\left.+\frac{\mu\sigma q}{B^2}\left[V_y\frac{\partial T}{\partial x} - V_y T\left(\rho_1 + \frac{1}{B}\frac{\partial B}{\partial x}\right) - V_x\frac{\partial T}{\partial y} + TV_x\left(\frac{1}{B}\frac{\partial B}{\partial y} - \sigma_1\right) - T\left(\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x}\right)\right]\right\}$$
$$\text{... (D.21)}$$

where $$T \equiv \tau_1 + \frac{1}{2}(\tau_2 + \tau_3)$$

IV. $\frac{1}{2}(A_oA_cD_c + A_oA_sD_s)f_o$

From C.17 this may be easily simplified to the form

$$\frac{1}{2}(A_oA_cD_c + A_oA_sD_s)f_o = \frac{1}{2}\frac{\partial f_o}{\partial x}\frac{\sigma q}{B^3}T(q^2\sigma_1 - 2\mu B\sigma_2)$$
$$+\frac{1}{2}\frac{\partial f_o}{\partial y}\frac{\sigma q}{B^3}T(q^2\rho_1 - 2\mu B\sigma_3)$$
$$-\frac{1}{2}\frac{\partial f_o}{\partial \mu}\frac{\sigma q}{B^3}T\left\{V_x(q^2\sigma_1 - 2\mu B\sigma_2) + V_y(q^2\rho_1 - 2\mu B\sigma_3)\right\} \qquad \text{... (D.23)}$$

V, VI. See (D.14) for the form of these terms.

Of the group V terms, only $D_{2c}D_oD_{2c}$ contains second derivaties of f_o. To prove this we use the <u>lemma</u>: The commutator of two linear differential operators is itself a linear differential operator.

Thus, for example we may write the term

$$\left[D_cD_{2c}D_c - \frac{1}{2}D_cD_cD_{2c} - \frac{1}{2}D_{2c}D_cD_c - D_sD_{2c}D_s + \frac{1}{2}D_sD_sD_{2c} + \frac{1}{2}D_{2c}D_sD_s\right]$$

of group V in the form

$$\frac{1}{2} D_c \left[D_{2c}D_c - D_cD_{2c} \right] - \frac{1}{2}\left[D_{2c}D_c - D_cD_{2c} \right] D_c$$

$$+ \; \frac{1}{2} D_s \left[D_sD_{2c} - D_{2c}D_s \right] - \frac{1}{2}\left[D_sD_{2c} - D_{2c}D_s \right] D_s$$

where, by the lemma, $D_{2c}D_c - D_cD_{2c}$ is a linear differential operator and consequently so is its commutator with D_c and similarly for $D_{2c}D_s - D_sD_{2c}$ and D_s. Thus this term of group V produces only first derivatives when operating on f_o. The same can be shown to hold for the remaining terms of group V, and an analogous situation arises in the terms of group VI, where the only second derivaties occur in $D_{2s}D_oD_{2s}$.

V. <u>See (D.14).</u>

$$-\frac{1}{8}\left[D_{2c}D_oD_{2c} \quad 2A_{2s}D_oD_{2c} \right] f_o = -\frac{1}{16}\frac{\sigma g}{B}\frac{\partial}{\partial s}\frac{\partial}{\partial \mu}\frac{\mu^2 q^2(\rho_3-\rho_2)^2}{B^2}\frac{\partial f_o}{\partial \mu} \qquad \text{... (D.24)}$$

As noted above the remaining terms in group V contain only first derivatives of f_o i.e. $\frac{\partial f_o}{\partial \mu}$, $\frac{\partial f_o}{\partial x}$ and $\frac{\partial f_o}{\partial y}$ terms. This knowledge enables us to write down these terms immediately as

$$\begin{aligned}
&\frac{\partial f_o}{\partial x}\left\{ \frac{1}{4} D_cD_{2c}\left(\frac{c_\perp}{B}\right) - \frac{1}{8} D_{2c}D_c\left(\frac{c_\perp}{B}\right) - \frac{1}{4} D_c\left(A_{2s}\frac{c_\perp}{B}\right) + \frac{1}{4} A_{2s}D_c\left(\frac{c_\perp}{B}\right) \right.\\
&\qquad\qquad \left. - \frac{1}{4} A_sD_{2c}\left(\frac{c_\perp}{B}\right) - \frac{1}{8} D_{2c}\left(A_s\frac{c_\perp}{B}\right) + \frac{1}{2} A_{2s}A_s\frac{c_\perp}{B} \right\}\\
+ \; &\frac{\partial f_o}{\partial y}\left\{ -\frac{1}{4} D_sD_{2c}\left(\frac{c_\perp}{B}\right) + \frac{1}{8} D_{2c}D_s\left(\frac{c_\perp}{B}\right) + \frac{1}{4} D_s\left(A_{2s}\frac{c_\perp}{B}\right) - \frac{1}{4} A_{2s}D_s\left(\frac{c_\perp}{B}\right) \right.\\
&\qquad\qquad \left. - \frac{1}{4} A_cD_{2c}\left(\frac{c_\perp}{B}\right) - \frac{1}{8} D_{2c}\left(A_c\frac{c_\perp}{B}\right) - \frac{1}{2} A_{2s}A_c\frac{c_\perp}{B} \right\}\\
+ \; &\frac{\partial f_o}{\partial \mu}\left\{ -\frac{1}{4} D_cD_{2c}\left(\frac{c_\perp V_x}{B}\right) + \frac{1}{4} D_cD_c(\mu A_{2s}) + \frac{1}{8} D_{2c}D_c\left(\frac{c_\perp V_x}{B}\right) + \frac{1}{4} D_sD_{2c}\left(\frac{c_\perp V_y}{B}\right) \right.\\
&\quad - \frac{1}{4} D_sD_s(\mu A_{2s}) - \frac{1}{8} D_{2c}D_s\left(\frac{c_\perp V_y}{B}\right) - \frac{1}{4} D_s\left(A_{2s}\frac{c_\perp V_y}{B}\right) + \frac{1}{4} A_{2s}D_s\left(\frac{c_\perp V_y}{B}\right)\\
&\quad + \frac{1}{4} D_c\left(A_{2s}\frac{c_\perp V_x}{B}\right) - \frac{1}{4} A_{2s}D_c\left(\frac{c_\perp V_x}{B}\right) - \frac{1}{2} D_c(\mu A_sA_{2s}) + \frac{1}{4} A_sD_{2c}\left(\frac{c_\perp V_x}{B}\right)\\
&\quad - \frac{1}{4} A_sD_c(\mu A_{2s}) + \frac{1}{8} D_{2c}\left(A_s\frac{c_\perp V_x}{B}\right) - \frac{1}{2} D_s(\mu A_cA_{2s}) + \frac{1}{4} A_cD_{2c}\left(\frac{c_\perp V_y}{B}\right)\\
&\quad \left. - \frac{1}{4} A_cD_s(\mu A_{2s}) + \frac{1}{8} D_{2c}\left(A_c\frac{c_\perp V_y}{B}\right) - \frac{1}{2} A_cA_c\mu A_{2s} + \frac{1}{2} A_sA_s\mu A_{2s} - \frac{1}{2} A_{2s}A_s\frac{c_\perp V_x}{B} - \frac{1}{2}A_{2s}A_c\frac{c_\perp V_y}{B} \right\}
\end{aligned}$$

... (D.25)

where we have used the fact that

$$D_{2c} = -2A_{2s}\mu\frac{\partial}{\partial \mu} \qquad \text{... (D.26)}$$

<u>VI. See (D.14)</u>

In similar fashion we may account for the terms of group VI. First, the only second derivatives occur in

$$-\frac{1}{8}\left[D_{2s}D_oD_{2s} + 2A_{2c}D_oD_{2s} \right] f_o = -\frac{1}{16}\frac{\sigma g}{B}\frac{\partial}{\partial s}\frac{\partial}{\partial \mu}\frac{\mu^2 q^2(\tau_2-\tau_3)^2}{B^2}\frac{\partial f_o}{\partial \mu} \qquad \text{... (D.27)}$$

The rest of the group VI terms contain only first derivatives of f_o, and can be written down (as in D.25 for group V) at once as

$$\frac{\partial f_o}{\partial x}\left\{+\frac{1}{4}D_sD_{2s}\left(\frac{c_\perp}{B}\right)-\frac{1}{8}D_{2s}D_s\left(\frac{c_\perp}{B}\right)+\frac{1}{4}D_s\left(A_{2c}\frac{c_\perp}{B}\right)-\frac{1}{4}A_{2c}D_s\left(\frac{c_\perp}{B}\right)\right.$$
$$\left.+\frac{1}{4}A_cD_{2s}\left(\frac{c_\perp}{B}\right)+\frac{1}{8}D_{2s}\left(A_c\frac{c_\perp}{B}\right)+\frac{1}{2}A_cA_{2c}\frac{c_\perp}{B}\right\}$$
$$+\frac{\partial f_o}{\partial y}\left\{\frac{1}{4}D_cD_{2s}\left(\frac{c_\perp}{B}\right)-\frac{1}{8}D_{2s}D_c\left(\frac{c_\perp}{B}\right)+\frac{1}{4}D_c\left(A_{2c}\frac{c_\perp}{B}\right)-\frac{1}{4}A_{2c}D_c\left(\frac{c_\perp}{B}\right)\right.$$
$$\left.-\frac{1}{4}A_sD_{2s}\left(\frac{c_\perp}{B}\right)-\frac{1}{8}D_{2s}\left(A_s\frac{c_\perp}{B}\right)-\frac{1}{2}A_sA_{2c}\frac{c_\perp}{B}\right\}$$
$$+\frac{\partial f_o}{\partial \mu}\left\{-\frac{1}{4}D_sD_{2s}\left(\frac{c_\perp V_x}{B}\right)-\frac{1}{4}D_sD_c(\mu A_{2c})+\frac{1}{8}D_{2s}D_s\left(\frac{c_\perp V_x}{B}\right)-\frac{1}{4}D_cD_{2s}\left(\frac{c_\perp V_y}{B}\right)\right.$$
$$-\frac{1}{4}D_cD_s(\mu A_{2c})+\frac{1}{8}D_{2s}D_c\left(\frac{c_\perp V_y}{B}\right)-\frac{1}{4}D_s\left(A_{2c}\frac{c_\perp V_x}{B}\right)+\frac{1}{4}A_{2c}D_s\left(\frac{c_\perp V_x}{B}\right)$$
$$-\frac{1}{4}D_c\left(A_{2c}\frac{c_\perp V_y}{B}\right)+\frac{1}{4}A_{2c}D_c\left(\frac{c_\perp V_y}{B}\right)+\frac{1}{2}D_s(A_sA_{2c}\mu)+\frac{1}{4}A_sD_{2s}\left(\frac{c_\perp V_y}{B}\right)$$
$$+\frac{1}{4}A_sD_s(\mu A_{2c})+\frac{1}{8}D_{2s}\left(A_s\frac{c_\perp V_y}{B}\right)-\frac{1}{2}D_c(A_cA_{2c}\mu)-\frac{1}{4}A_cD_{2s}\left(\frac{c_\perp V_x}{B}\right)$$
$$\left.-\frac{1}{4}A_cD_c(\mu A_{2c})-\frac{1}{8}D_{2s}\left(A_c\frac{c_\perp V_x}{B}\right)+A_cA_sA_{2c}\mu-\frac{1}{2}A_cA_{2c}\frac{c_\perp V_x}{B}+\frac{1}{2}A_sA_{2c}\frac{c_\perp V_y}{B}\right\} \quad \ldots \text{(D.28)}$$

where we have used the fact that

$$D_{2s} = 2A_{2c}\mu\frac{\partial}{\partial\mu} \quad \ldots \text{(D.29)}$$

A simple relationship exists between the $\frac{\partial f_o}{\partial x}$ terms of group V and the $\frac{\partial f_o}{\partial y}$ terms of group VI. In fact the group IV terms in D.28 may be obtained from the group V terms by the substitution $D_{2c}\rightarrow D_{2s}$; $A_{2s}\rightarrow -A_{2c}$ that is by the substitution $(\rho_3-\rho_2)\rightarrow(\tau_2-\tau_3)$. Similarly the $\frac{\partial f_o}{\partial x}$ terms of group VI in D.28 may be obtained from the $\frac{\partial f_o}{\partial y}$ terms of group V in D.25 by the substitution

$$D_{2c}\rightarrow -D_{2s} \quad ; \quad A_{2s}\rightarrow A_{2c}$$

that is by the substitution

$$(\rho_3-\rho_2)\rightarrow(\tau_3-\tau_2)$$

Thus if the form of D.25 is

$$\frac{\partial f_o}{\partial x}X(\rho_3-\rho_2)+\frac{\partial f_o}{\partial y}Y(\rho_3-\rho_2)+\frac{\partial f_o}{\partial\mu}Z(\rho_3-\rho_2), \quad \ldots \text{(D.30)}$$

then D.28 may be written in the form

$$-\frac{\partial f_o}{\partial x}Y(\tau_2-\tau_3)+\frac{\partial f_o}{\partial y}X(\tau_2-\tau_3)+\frac{\partial f_o}{\partial\mu}Z'(\tau_2-\tau_3) \quad \ldots \text{(D.31)}$$

where there is a more complex relationship between the functions Z and Z′ which we shall not use.

We proceed to collect up the various terms in $\frac{\partial f_o}{\partial x}$, $\frac{\partial f_o}{\partial y}$ from V and VI as given in

D.25 and D.28, expressing them in terms of the ρ_i, σ_i, τ_i of appendix B.

Substituting from C.17 in D.25, and after some straightforward algebra the following expressions are obtained for the $\frac{\partial f_o}{\partial x}$ and $\frac{\partial f_o}{\partial y}$ terms of V:-

$$\frac{1}{2}\frac{\partial f_o}{\partial x}\frac{\mu\sigma q}{B^2}\left\{\frac{\partial}{\partial x}(\rho_3-\rho_2)+(\rho_3-\rho_2)\left(2\sigma_3+\rho_1-\frac{1}{B}\frac{\partial B}{\partial x}\right)\right\}-\frac{1}{2}\frac{\partial f_o}{\partial x}\frac{\sigma q^3}{B^3}\rho_1(\rho_3-\rho_2)$$

$$+\frac{1}{2}\frac{\partial f_o}{\partial y}\frac{\mu\sigma q}{B^2}\left\{-\frac{\partial}{\partial y}(\rho_3-\rho_2)+(\rho_3-\rho_2)\left(2\sigma_2+\sigma_1+\frac{1}{B}\frac{\partial B}{\partial y}\right)\right\}-\frac{1}{2}\frac{\partial f_o}{\partial y}\frac{\sigma q^3}{B^3}\sigma_1(\rho_3-\rho_2) \quad \ldots \text{(D.32)}$$

As observed above, a knowledge of the expression (D.32) enables us to write down immediately (from D.31) the corresponding group VI terms in D.28. These are

$$\frac{1}{2}\frac{\partial f_o}{\partial x}\frac{\mu\sigma q}{B^2}\left\{\frac{\partial}{\partial y}(\tau_2-\tau_3)-(\tau_2-\tau_3)\left(2\sigma_2+\sigma_1+\frac{1}{B}\frac{\partial B}{\partial y}\right)\right\}+\frac{1}{2}\frac{\partial f_o}{\partial x}\frac{\sigma q^3}{B^3}\sigma_1(\tau_2-\tau_3)$$

$$+\frac{1}{2}\frac{\partial f_o}{\partial y}\frac{\mu\sigma q}{B^2}\left\{\frac{\partial}{\partial x}(\tau_2-\tau_3)+(\tau_2-\tau_3)\left(2\sigma_3+\rho_1-\frac{1}{B}\frac{\partial B}{\partial x}\right)\right\}-\frac{1}{2}\frac{\partial f_o}{\partial y}\frac{\sigma q^3}{B^3}\rho_1(\tau_2-\tau_3) \quad \ldots \text{(D.33)}$$

Before attempting to evaluate or simplify the $\frac{\partial f_o}{\partial \mu}$ terms of V or VI, which is a task of great complexity, we will consider the final term VII of Hf_o.

VII. $\sigma L\int_{s_o}^{s}\frac{B\,ds}{q}\,Lf_o$

First we note that

$$\int_{s_o}^{s}\frac{B\,ds}{q}\,Lf_o=\int_{s_o}^{s}\frac{ds}{q}\,\underset{\sim}{W}_d\cdot\nabla f_o+\mu\frac{\partial f_o}{\partial\mu}\,q\,\frac{(\underset{\sim}{e}_1\cdot\text{curl}\,\underset{\sim}{e}_1)}{B}\equiv Q\,(\alpha,\beta,s,\mu,\varepsilon)\,, \quad \ldots \text{(D.34)}$$

where the lower-limit of the second term on the right-hand side has been absorbed into the arbitrary constant h_1 (i.e. $h_1 = h_1\,(\alpha,\beta,\mu,\varepsilon,\sigma,s_o)$). From D.5 we have the result

$$LQ=\frac{\underset{\sim}{W}_d\cdot\nabla Q}{B}-\mu\frac{q}{B^2}(\underset{\sim}{e}_1\cdot\text{curl}\,\underset{\sim}{e}_1)\frac{\partial}{\partial\mu}\left(q\frac{\partial Q}{\partial s}\right)+\frac{q}{B}\frac{\partial}{\partial s}\left[\mu\frac{\partial Q}{\partial\mu}\,q\,\frac{(\underset{\sim}{e}_1\cdot\text{curl}\,\underset{\sim}{e}_1)}{B}\right] \quad \ldots \text{(D.35)}$$

where, with Q given by D.34, the first term is

$$\frac{\underset{\sim}{W}_d\cdot\nabla}{B}\int_{s_o}^{s}\frac{ds}{q}\,\underset{\sim}{W}_d\cdot\nabla f_o+\frac{\underset{\sim}{W}_d\cdot\nabla}{B}\left(\mu\frac{\partial f_o}{\partial\mu}\,q\,\frac{\underset{\sim}{e}_1\cdot\text{curl}\,\underset{\sim}{e}_1}{B}\right) \quad \ldots \text{(D.36)}$$

and the second term in D.35 reduces to

$$-\frac{1}{2}\frac{q}{B}\frac{\partial}{\partial s}\left\{\mu\frac{\partial f_o}{\partial\mu}\cdot\frac{q^2}{B^2}(\underset{\sim}{e}_1\cdot\text{curl}\,\underset{\sim}{e}_1)^2\right]\Bigg\}-\frac{\mu q}{B^2}(\underset{\sim}{e}_1\cdot\text{curl}\,\underset{\sim}{e}_1)\frac{\partial}{\partial\mu}(\underset{\sim}{W}_d\cdot f_o) \quad \ldots \text{(D.37)}$$

The first term in D.37 and the third term in D.35 give zero contributions to Hf_o when the $\oint\frac{B\,ds}{q}$ integration is performed. Note however that both these terms do form part of g_2, and consequently contribute to μ_2 and J_2. Combining the second term of D.36 with that of D.37 leads to cancellation of the double derivative $(\nabla\frac{\partial f_o}{\partial\mu})$ terms, leaving the following $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial \mu}$ terms:-

$$- \frac{\partial f_o}{\partial \mu} \left\{ \frac{\mu q^3}{B^2} \left[\sigma_1 \frac{\partial}{\partial x} \left(\frac{\tau_2+\tau_3}{B}\right) + \rho_1 \frac{\partial}{\partial y} \left(\frac{\tau_2+\tau_3}{B}\right) \right] \right.$$

$$+ \frac{\mu^2 q}{B^2} \left[\frac{\partial B}{\partial x} \frac{\partial}{\partial y} \left(\frac{\tau_2+\tau_3}{B}\right) - \frac{\partial B}{\partial y} \frac{\partial}{\partial x} \left(\frac{\tau_2+\tau_3}{B}\right) \right]$$

$$\left. - \frac{\mu^2 q}{B^3} (\tau_2+\tau_3) \left[\sigma_1 \frac{\partial B}{\partial x} + \rho_1 \frac{\partial B}{\partial y} \right] \right\} \quad \text{... (D.38)}$$

$$- \frac{\partial f_o}{\partial x} \frac{\mu q}{B^2} (\tau_2+\tau_3)(2\sigma_1 + \frac{1}{B}\frac{\partial B}{\partial y}) + \frac{\partial f_o}{\partial y} \frac{\mu q}{B^2} (\tau_2+\tau_3)(\frac{1}{B}\frac{\partial B}{\partial x} - 2\rho_1) \quad \text{... (D.39)}$$

In order to evaluate the remaining term in D.36 it is advantageous to introduce a change of coordinates. It is common practice to work in terms of the general curvilinear set α, β, s which is described in Appendix B. In the present work however, it is more convenient to introduce the coordinates $\alpha_\perp$, $\beta_\perp$, s which are related to α, β, s through the transformations B.26 and B.27. Thus the integral $\oint \frac{ds}{q} \underset{\sim}{W}_d \cdot \nabla \int_{s_o}^{s} \frac{ds}{q} \underset{\sim}{W}_d \cdot \nabla f_o$ becomes,

$$- \oint \frac{ds}{q} \underset{\sim}{W}_d \cdot \nabla \int_{s_o}^{s} \frac{ds}{q} \left[\frac{\partial f_o}{\partial \alpha_\perp} (q^2 \rho_\beta + \mu \frac{\partial B}{\partial \beta_\perp}) - \frac{\partial f_o}{\partial \beta_\perp} (q^2 \rho_\alpha + \mu \frac{\partial B}{\partial \alpha_\perp}) \right] \quad \text{... (D.40)}$$

Using the results,

$$\frac{\partial}{\partial \alpha_\perp} \int^{s} A ds = \int^{s} \left[\frac{\partial A}{\partial \alpha_\perp} - A\rho_\alpha \right] ds$$

$$\frac{\partial}{\partial \beta_\perp} \int^{s} A ds = \int^{s} \left[\frac{\partial A}{\partial \beta_\perp} - A\rho_\beta \right] ds$$

which may be obtained from B.26, B.27 and B.28, it is straightforward to write D.40 in the form

$$\oint \frac{ds}{q} \underset{\sim}{W}_d \cdot \nabla \left(\frac{\partial f_o}{\partial \alpha_\perp} \cdot \frac{\partial I}{\partial \beta_\perp} - \frac{\partial f_o}{\partial \beta_\perp} \cdot \frac{\partial I}{\partial \alpha_\perp} \right) \quad \text{... (D.41)}$$

where $I = \int_{s_o}^{s} q \, ds$. Some further algebra gives

$$\oint ds \left\{ (I_\beta)' \left[f_{\alpha\alpha} I_\beta + f_\alpha I_{\alpha\beta} - f_{\alpha\beta} I_\alpha - f_\beta I_{\alpha\alpha} \right] - (I_\alpha)' \left[f_{\beta\alpha} I_\beta + f_\alpha I_{\beta\beta} - f_{\beta\beta} I_\alpha - f_\beta I_{\beta\alpha} \right] \right\},$$

where the suffices α, β denote (unless otherwise stated) differentiation with respect to $\alpha_\perp$, $\beta_\perp$ respectively, and the prime denotes differentiation with respect to s.

For $f_o = f_o(\mu, \varepsilon, J)$ we have relations of the form

$$f_\alpha = f_J J_\alpha, \; f_{\alpha\alpha} = f_J J_{\alpha\alpha} + f_{JJ} J_\alpha^2 \; f_{\alpha\beta} = f_{\beta\alpha} = f_J J_{\alpha\beta} + f_{JJ} J_\alpha J_\beta \text{ etc.}$$

and integral properties of the form

$$\oint ds \, (I_\alpha)' = J_\alpha, \oint ds \, (I_\alpha I_\beta)' = J_\alpha J_\beta, \oint ds \, (I_\alpha^2)' = J_\alpha^2 \text{ etc.}$$

Use of these results together with the commutation relation B.33 leads to

$$\frac{1}{2} J_\beta f_J \oint ds \left[I_\beta (I_{\alpha\alpha})' + (I_\alpha)' I_{\alpha\beta} - (I_\beta)' I_{\alpha\alpha} - I_\alpha (I_{\alpha\beta})' \right]$$
$$+ \frac{1}{2} J_\alpha f_J \oint ds \left[I_\alpha (I_{\beta\beta})' + (I_\beta)' I_{\beta\alpha} - (I_\alpha)' I_{\beta\beta} - I_\beta (I_{\beta\alpha})' \right]$$
$$+ J_\alpha f_J \oint ds\, (I_\beta)' \frac{q}{B} (\tau_2 + \tau_3) - J_\beta f_J \oint ds\, (I_\alpha)' \frac{q}{B} (\tau_2 + \tau_3), \qquad \ldots \text{(D.42)}$$

where we have taken cognizance of the fact that

$$\oint ds \left[(I_\beta)' I_{\alpha\alpha} - (I_\alpha)' I_{\alpha\beta} \right] = \frac{1}{2} \oint ds \left[(I_\beta)' I_{\alpha\alpha} - I_\beta (I_{\alpha\alpha})' - (I_\alpha)' I_{\alpha\beta} + I_\alpha (I_{\alpha\beta})' \right]$$
$$+ \frac{1}{2} J_\beta J_{\alpha\alpha} - \frac{1}{2} J_\alpha J_{\alpha\beta} \quad ,$$

together with a similar result for α and β interchanged. It can be further shown that

$$\frac{\partial}{\partial \beta_\perp} \int^s ds \left[I_\alpha (I_\beta)' - I_\beta (I_\alpha)' \right]$$
$$= \int^s ds \left[I_\alpha (I_{\beta\beta})' - (I_\alpha)' I_{\beta\beta} + (I_\beta)' I_{\beta\alpha} - I_\beta (I_{\beta\alpha})' \right] \qquad \ldots \text{(D.43)}$$

where α, β may again be interchanged. We now use (D.43) to write D.42 in its final form

$$\frac{\partial f_o}{\partial J} \left[\frac{\partial J}{\partial \beta_\perp} \frac{\partial P}{\partial \alpha_\perp} - \frac{\partial J}{\partial \alpha_\perp} \frac{\partial P}{\partial \beta_\perp} \right] + \oint ds \frac{\tau_2 + \tau_3}{B^2} \left\{ \frac{\partial f_o}{\partial y} (q^2 \rho_1 + \mu \frac{\partial B}{\partial x}) - \frac{\partial f_o}{\partial x} (- q^2 \sigma_1 + \mu \frac{\partial \beta}{\partial y}) \right\} ,$$

where $\ldots$ (D.44)

$$P = \frac{1}{2} \oint ds \left[\frac{\partial}{\partial s} (\frac{\partial I}{\partial \alpha_\perp}) \frac{\partial I}{\partial \beta_\perp} - \frac{\partial}{\partial s} (\frac{\partial I}{\partial \beta_\perp}) \frac{\partial I}{\partial \alpha_\perp} \right] \qquad \ldots \text{(D.45)}$$

Since J and P are independent of s it follows that $\alpha_\perp$ and $\beta_\perp$ in D.44 may now be replaced by α and β respectively (see Appendix B). Thus term VII of the H operator has given the following result after the $\oint \frac{Bds}{q}$ has been carried out round the closed path.

$$\oint \frac{Bds}{q} L \int_{s_o}^{s} \frac{Bds}{q} Lf_o = \frac{\partial f_o}{\partial J} \left(\frac{\partial J}{\partial \beta} \frac{\partial P}{\partial \alpha} - \frac{\partial J}{\partial \alpha} \frac{\partial P}{\partial \beta} \right)$$
$$+ \oint \frac{Bds}{q} \frac{(\tau_2 + \tau_3)}{B^3} \frac{\partial f_o}{\partial y} \left[q^3 \rho_1 + 2\mu q \left(\frac{1}{B} \frac{\partial B}{\partial x} - \rho_1 \right) \right]$$
$$- \oint \frac{Bds}{q} \frac{(\tau_2 + \tau_3)}{B^3} \frac{\partial f_o}{\partial x} \left[- q^3 \sigma_1 + 2\mu q \left(\frac{1}{B} \frac{\partial B}{\partial y} + \sigma_1 \right) \right]$$
$$- \frac{\partial f_o}{\partial \mu} \oint \frac{Bds}{q} \left\{ \frac{\mu q^3}{B^2} \left[\sigma_1 \frac{\partial}{\partial x} \frac{(\tau_2 + \tau_3)}{B} + \rho_1 \frac{\partial}{\partial y} \frac{(\tau_2 + \tau_3)}{B} \right] \right.$$
$$+ \frac{\mu^2 q}{B^2} \left[\frac{\partial B}{\partial x} \frac{\partial}{\partial y} \frac{(\tau_2 + \tau_3)}{B} - \frac{\partial B}{\partial y} \frac{\partial}{\partial x} \frac{(\tau_2 + \tau_3)}{B} \right]$$
$$\left. - \frac{\mu^2 q}{B^3} (\tau_2 + \tau_3) \left[\sigma_1 \frac{\partial B}{\partial x} + \rho_1 \frac{\partial B}{\partial y} \right] \right\} \qquad \ldots \text{(D.46)}$$

where P is given by D.45, and $\frac{\partial f_o}{\partial x}$ is understood to mean $\frac{\partial f_o}{\partial J} \frac{\partial J}{\partial x}$ and similarly for $\frac{\partial f_o}{\partial y}$.

This completes the evaluation of the $\frac{\partial f_o}{\partial x}, \frac{\partial f_o}{\partial y}$ terms in Hf_o so that we are now in a position to calculate the coefficient of $\frac{\partial f_o}{\partial J}$ in Hf_o.

We note first that all second derivatives of f_o which have arisen so far have indeed integrated to zero in Hf_o. These terms occur in (D.19) (term I), (D.24) (term V), (D.27) (term VI) and (D.35), (D.37) (term VII).

We now form the complete coefficient of $\frac{\partial f_o}{\partial x}$ by adding together the contributions from all sources, namely (D.19), (D.20), (D.21), (D.23), (D.32), (D.33) and (D.46). The same sources give the complete $\frac{\partial f_o}{\partial y}$ coefficient, and the following result for these contributions to Hf_o is:-

$$Hf_o = \oint \frac{Bds}{q}\, \frac{1}{2}\, \frac{\mu\sigma g}{B}\, \frac{\partial}{\partial s} \left\{ \frac{1}{B} \frac{\partial f_o}{\partial x} \left(\frac{1}{B} \frac{\partial B}{\partial x} - \sigma_3 \right) + \frac{1}{B} \frac{\partial f_o}{\partial y} \left(\frac{1}{B} \frac{\partial B}{\partial y} + \sigma_2 \right) \right\}$$

$$+ \oint \frac{Bds}{q}\, \frac{\mu\sigma g}{B^2}\, \underset{\sim}{\nabla} f_o \cdot \text{curl}\, \underset{\sim}{T} + \frac{\partial f_o}{\partial J} \left(\frac{\partial J}{\partial \beta} \frac{\partial P}{\partial \alpha} - \frac{\partial J}{\partial \alpha} \frac{\partial P}{\partial \beta} \right) + b\, \frac{\partial f_o}{\partial \mu} \qquad \text{... (D.47)}$$

where

$$\underset{\sim}{T} = \left[\tau_1 + \frac{1}{2} (\tau_2 + \tau_3) \right] \underset{\sim}{e}_1 - \sigma_2 \underset{\sim}{e}_2 - \sigma_3 \underset{\sim}{e}_3$$

$$= T\underset{\sim}{e}_1 - \sigma_2 \underset{\sim}{e}_3 - \sigma_3 \underset{\sim}{e}_3 \,, \qquad \text{... (D.48)}$$

P is given by (D.45) and b, the coefficient of $\frac{\partial f_o}{\partial \mu}$ in Hf_o, has yet to be determined. In obtaining (D.47) we have made use of the relations (B.7) to (B.16). The first term in (D.47) vanishes because the path of integration is closed. To reduce the second term to a more familiar form we write

$$\text{curl}\, \underset{\sim}{T} = T \left[\rho_1 \underset{\sim}{e}_3 + \sigma_1 \underset{\sim}{e}_2 - \tau_2 \underset{\sim}{e}_1 - \tau_3 \underset{\sim}{e}_1 \right] - \sigma_2 \left[\tau_1 \underset{\sim}{e}_2 + \sigma_2 \underset{\sim}{e}_1 + \rho_2 \underset{\sim}{e}_3 - \tau_3 \underset{\sim}{e}_2 \right]$$

$$- \sigma_3 \left[\tau_1 \underset{\sim}{e}_3 - \tau_2 \underset{\sim}{e}_3 - \rho_3 \underset{\sim}{e}_2 + \sigma_3 \underset{\sim}{e}_1 \right] - \underset{\sim}{e}_3 \left[\frac{\partial \sigma_2}{\partial s} - \frac{\partial T}{\partial x} \right]$$

$$+ \underset{\sim}{e}_2 \left[\frac{\partial T}{\partial y} + \frac{\partial \sigma_3}{\partial s} \right] + \underset{\sim}{e}_1 \left[\frac{\partial \sigma_2}{\partial y} - \frac{\partial \sigma_3}{\partial x} \right] \qquad \text{... (D.49)}$$

Then using (B.5) and (B.6) it follows that

$$\oint \frac{ds}{B}\, \underset{\sim}{\nabla} f_o \cdot \text{curl}\, \underset{\sim}{T} = \oint ds\, \frac{\partial}{\partial s} \left(\frac{\sigma_3}{B} \frac{\partial f_o}{\partial x} - \frac{\sigma_2}{B} \frac{\partial f_o}{\partial y} \right)$$

$$+ \oint ds \left\{ T\, \frac{\underset{\sim}{B} \times \underset{\sim}{\rho} \cdot \underset{\sim}{\nabla} f_o}{B^2} - \frac{\underset{\sim}{B} \times \underset{\sim}{\nabla} T \cdot \underset{\sim}{\nabla} f_o}{B^2} \right\} \qquad \text{... (D.50)}$$

Transforming to α, β coordinates, and noting that the closed path of integration causes the first integral to vanish we obtain

$$\oint \frac{ds}{B}\, \underset{\sim}{\nabla} f_o \cdot \text{curl}\, T = \frac{\partial f_o}{\partial J} \left\{ \frac{\partial J}{\partial \alpha} \frac{\partial}{\partial \beta} \oint T ds - \frac{\partial J}{\partial \beta} \frac{\partial}{\partial \alpha} \oint T ds \right\} \qquad \text{... (D.51)}$$

Hence

$$Hf_o = b\frac{\partial f_o}{\partial \mu} + \frac{\partial f_o}{\partial J}\left\{\frac{\partial J}{\partial \beta}\frac{\partial P}{\partial \alpha} - \frac{\partial J}{\partial \alpha}\frac{\partial P}{\partial \beta}\right\} \qquad \ldots (D.52)$$

where P is now given by:-

$$P = \frac{1}{2}\oint ds\left\{\frac{\partial}{\partial s}\left(\frac{\partial I}{\partial \alpha_\perp}\right)\frac{\partial I}{\partial \beta_\perp} - \frac{\partial}{\partial s}\left(\frac{\partial I}{\partial \beta_\perp}\right)\frac{\partial I}{\partial \alpha_\perp} + 2\mu\sigma T\right\} \qquad \ldots (D.53)$$

<u>Calculation of the Coefficient of $\frac{\partial f_o}{\partial \mu}$ in Hf_o</u>

It has been observed that the result for Hf_o contains only first order derivatives, and since the D operator contains no derivatives with respect to ε, and since $f_o = f_o(\mu, \varepsilon, J)$, that

$$Hf_o = a\frac{\partial f_o}{\partial J} + b\frac{\partial f_o}{\partial \mu} \qquad \ldots (D.54)$$

must be a correct representation. In (D.52) we have given the result for the function a above. Now it remains to evaluate b also. As before there are seven sources of $\frac{\partial f_o}{\partial \mu}$ terms (the terms I-VII). We shall commence with the most formidable groups.

<u>$\frac{\partial f_o}{\partial \mu}$ terms of V and VI</u>

We must now simplify the $\frac{\partial f_o}{\partial \mu}$ coefficients which appear in (D.25) and (D.28). In accordance with our remarks in Appendix B, we now consider the problem in a special co-ordinate system, obtained by a suitable rotation of the $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ axes. In this system the torsion coefficients $\hat{\tau}_i$, have the special property that $\hat{\tau}_2 = \hat{\tau}_3$ at all points. In consequence the terms in VI vanish.

In the following, the superscript $\hat{}$ will be dropped from the ρ_i, σ_i, τ_i, but it is to be remembered that $\tau_2 \equiv \tau_3$ throughout this section.

Using the definitions of the A's and D's as given in (C.17), we have the following results for the terms arising in (D.25):-

$$-\frac{1}{2}A_{2s}A_c\frac{c_\perp V_y}{B} = \frac{q^5}{B^4}\left[-\frac{1}{4}\sigma_1^2(\rho_3 - \rho_2)\right] + \frac{q^3\mu B}{B^4}\left[\frac{1}{4}\sigma_1(\rho_3 - \rho_2)\left(\frac{1}{B}\frac{\partial B}{\partial y} + 2\sigma_2\right)\right]$$
$$- \frac{q(\mu B)^2}{B^4}\left[\frac{1}{2}(\rho_3 - \rho_2)\sigma_2\frac{1}{B}\frac{\partial B}{\partial y}\right]$$

$$-\frac{1}{2}A_{2s}A_s\frac{c_\perp V_x}{B} = \frac{q^5}{B^4}\left[\frac{1}{4}\rho_1^2(\rho_3 - \rho_2)\right] + \frac{q^3\mu B}{B^4}\left[\frac{1}{4}\rho_1(\rho_3 - \rho_2)\left(\frac{1}{B}\frac{\partial B}{\partial x} - 2\sigma_3\right)\right]$$
$$- \frac{q(\mu B)^2}{B^4}\left[\frac{1}{2}(\rho_3 - \rho_2)\sigma_3\frac{1}{B}\frac{\partial B}{\partial x}\right]$$

$$-\frac{1}{4}A_{2s}D_c\left(\frac{c_\perp V_x}{B}\right) = \frac{q^5}{B^4}\left[-\frac{\rho_1^2}{8}(\rho_3 - \rho_2)\right] + \frac{q^3\mu B}{B^4}\left[\frac{1}{8}(\rho_3 - \rho_2)\left(4\rho_1^2 + 2\frac{\partial \rho_1}{\partial x} - 7\rho_1\frac{1}{B}\frac{\partial B}{\partial x}\right)\right]$$
$$+ \frac{q(\mu B)^2}{B^4}\left[\frac{1}{4}(\rho_3 - \rho_2)\left(\frac{1}{B}\frac{\partial^2 B}{\partial x^2} - 3\left(\frac{1}{B}\frac{\partial B}{\partial x}\right)^2\right)\right]$$

$$
\begin{aligned}
\frac{1}{4} A_{2s} D_s \left(\frac{c_\perp V_y}{B^4}\right) &= \frac{q^5}{B^4}\left[\frac{\sigma_1^2}{8}(\rho_3 - \rho_2)\right] - \frac{q^3 \mu B}{B^4}\left[\frac{1}{8}(\rho_3 - \rho_2)\left(4\sigma_1^2 - 2\,\frac{\partial \sigma_1}{\partial y} + 7\sigma_1 \frac{1}{B}\frac{\partial B}{\partial y}\right)\right] \\
&\quad - \frac{q(\mu B)^2}{B^4}\left[\frac{1}{4}(\rho_3 - \rho_2)\left(\frac{1}{B}\frac{\partial^2 B}{\partial y^2} - 3\left(\frac{1}{B}\frac{\partial B}{\partial y}\right)^2\right)\right]
\end{aligned}
$$

$$
-\frac{1}{2} A_c A_c \mu A_{2s} = \frac{q^5}{B^4}\left[\frac{\sigma_1^2(\rho_3 - \rho_2)}{8}\right] - \frac{q^3 \mu B}{B^4}\left[\frac{1}{2}(\rho_3 - \rho_2)\sigma_1\sigma_2\right] + \frac{q(\mu B)^2}{B^4}\left[\frac{1}{2}(\rho_3 - \rho_2)\sigma_2^2\right]
$$

$$
+\frac{1}{2} A_s A_s \mu A_{2s} = -\frac{q^5}{B^4}\left[\frac{\rho_1^2(\rho_3 - \rho_2)}{8}\right] + \frac{q^3 \mu B}{B^4}\left[\frac{1}{2}(\rho_3 - \rho_2)\rho_1\sigma_3\right] - \frac{q(\mu B)^2}{B^4}\left[\frac{1}{2}(\rho_3 - \rho_2)\sigma_3^2\right]
$$

$$
\begin{aligned}
-\frac{1}{4} A_c D_s(\mu A_{2s}) &= \frac{q^5}{B^4}\left[\frac{\sigma_1^2}{8}(\rho_3 - \rho_2)\right] + \frac{q^3 \mu B}{B^4}\frac{1}{8}\left[\sigma_1 \frac{\partial}{\partial y}(\rho_3 - \rho_2) - (\rho_3 - \rho_2)\sigma_1\left(\sigma_1 + \frac{2}{B}\frac{\partial B}{\partial y}\right)\right. \\
&\quad \left. -2(\rho_3 - \rho_2)\sigma_1\sigma_2\right] + \frac{q(\mu B)^2}{B^4}\frac{1}{4}\left[-\sigma_2 \frac{\partial}{\partial y}(\rho_3 - \rho_2) + (\rho_3 - \rho_2)\sigma_2\left(\sigma_1 + \frac{2}{B}\frac{\partial B}{\partial y}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{4} A_s D_c(\mu A_{2s}) &= -\frac{q^5}{B^4}\left[\frac{\rho_1^2}{8}(\rho_3 - \rho_2)\right] + \frac{q^3 \mu B}{B^4}\frac{1}{8}\left[\rho_1 \frac{\partial}{\partial x}(\rho_3 - \rho_2) + (\rho_3 - \rho_2)\rho_1\left(\rho_1 - \frac{2}{B}\frac{\partial B}{\partial x}\right)\right. \\
&\quad \left. +2(\rho_3 - \rho_2)\rho_1\sigma_3\right] + \frac{q(\mu B)^2}{B^4}\frac{1}{4}\left[-\sigma_3 \frac{\partial}{\partial x}(\rho_3 - \rho_2) + (\rho_3 - \rho_2)\sigma_3\left(-\rho_1 + \frac{2}{B}\frac{\partial B}{\partial x}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{4} A_c D_{2c}\left(\frac{c_\perp V_y}{B}\right) &= -\frac{q^5}{B^4}\left[\frac{\sigma_1^2}{8}(\rho_3 - \rho_2)\right] + \frac{q^3 \mu B}{B^4}\left[\frac{1}{8}(\rho_3 - \rho_2)\left(2\sigma_1\sigma_2 + 3\sigma_1 \frac{1}{B}\frac{\partial B}{\partial y} + 4\sigma_1^2\right)\right] \\
&\quad - \frac{q(\mu B)^2}{B^4}\left[\frac{1}{4}\sigma_2(\rho_3 - \rho_2)\left(\frac{3}{B}\frac{\partial B}{\partial y} + 4\sigma_1\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{4} A_s D_{2c}\left(\frac{c_\perp V_x}{B}\right) &= \frac{q^5}{B^4}\left[\frac{\rho_1^2}{8}(\rho_3 - \rho_2)\right] + \frac{q^3 \mu B}{B^4}\left[\frac{1}{8}(\rho_3 - \rho_2)\left(-2\rho_1\sigma_3 + 3\rho_1 \frac{1}{B}\frac{\partial B}{\partial x} - 4\rho_1^2\right)\right] \\
&\quad - \frac{q(\mu B)^2}{B^4}\left[\frac{1}{4}\sigma_3(\rho_3 - \rho_2)\left(\frac{3}{B}\frac{\partial B}{\partial x} - 4\rho_1\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{8} D_{2c}\left(\frac{A_s c_\perp V_x}{B}\right) &= \frac{q^3 \mu B}{B^4}\left[\frac{1}{8}(\rho_3 - \rho_2)\left(\rho_1 \frac{1}{B}\frac{\partial B}{\partial x} - 4\rho_1^2 - 2\rho_1\sigma_3\right)\right] \\
&\quad + \frac{q(\mu B)^2}{B^4}\left[\frac{1}{8}(\rho_3 - \rho_2)\left(4\sigma_3\rho_1 - 4\sigma_3 \frac{1}{B}\frac{\partial B}{\partial x} - 2\,\frac{1}{B}\frac{\partial B}{\partial x}\rho_1\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{8} D_{2c}\left(\frac{A_c c_\perp V_y}{B}\right) &= \frac{q^3 \mu B}{B^4}\left[\frac{1}{8}(\rho_3 - \rho_2)\left(\sigma_1 \frac{1}{B}\frac{\partial B}{\partial y} + 4\sigma_1^2 + 2\sigma_1\sigma_2\right)\right] \\
&\quad + \frac{q(\mu B)^2}{B^4}\left[\frac{1}{8}(\rho_3 - \rho_2)\left(-4\sigma_1\sigma_2 - 4\sigma_2 \frac{1}{B}\frac{\partial B}{\partial y} - 2\sigma_1 \frac{1}{B}\frac{\partial B}{\partial y}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{8} D_{2c} D_c\left(\frac{c_\perp V_x}{B}\right) &= \frac{q^3 \mu B}{B^4}\left[\frac{1}{8}(\rho_3 - \rho_2)\left(-7\rho_1 \frac{1}{B}\frac{\partial B}{\partial x} + 2\,\frac{\partial \rho_1}{\partial x} + 8\rho_1^2\right)\right] \\
&\quad - \frac{q(\mu B)^2}{B^4}\left[\frac{1}{4}(\rho_3 - \rho_2)\left(6\left(\frac{1}{B}\frac{\partial B}{\partial x}\right)^2 - 2\left(\frac{1}{B}\frac{\partial^2 B}{\partial x^2}\right) - 7\rho_1 \frac{1}{B}\frac{\partial B}{\partial x} + 2\,\frac{\partial \rho_1}{\partial x} + 4\rho_1^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{8} D_{2c} D_s\left(\frac{c_\perp V_y}{B}\right) &= -\frac{q^3 \mu B}{B^4}\left[\frac{1}{8}(\rho_3 - \rho_2)\left(7\sigma_1 \frac{1}{B}\frac{\partial B}{\partial y} - 2\,\frac{\partial \sigma_1}{\partial y} + 8\sigma_1^2\right)\right] \\
&\quad + \frac{q(\mu B)^2}{B^4}\left[\frac{1}{4}(\rho_3 - \rho_2)\left(6\left(\frac{1}{B}\frac{\partial B}{\partial y}\right)^2 - 2\left(\frac{1}{B}\frac{\partial^2 B}{\partial y^2}\right) + 7\sigma_1 \frac{1}{B}\frac{\partial B}{\partial y} - 2\,\frac{\partial \sigma_1}{\partial y} + 4\sigma_1^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{2} D_c(\mu A_s A_{2s}) &= -\frac{q^5}{B^4}\left[\frac{1}{8}\rho_1^2(\rho_3 - \rho_2)\right] \\
&\quad + \frac{q^3 \mu B}{B^4}\frac{1}{4}\left[\rho_1 \frac{\partial}{\partial x}(\rho_3 - \rho_2) + (\rho_3 - \rho_2)\left(\frac{\partial \rho_1}{\partial x} + 3\rho_1^2 + 3\rho_1\sigma_3 - 3\rho_1 \frac{1}{B}\frac{\partial B}{\partial x}\right)\right] \\
&\quad - \frac{q(\mu B)^2}{B^4}\left[(\rho_3 - \rho_2)\frac{\partial \sigma_3}{\partial x} + \sigma_3 \frac{\partial}{\partial x}(\rho_3 - \rho_2) + (\rho_3 - \rho_2)\left(\rho_1 - \frac{3}{B}\frac{\partial B}{\partial x}\right)\sigma_3\right]
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{2}D_s(\mu A_c A_{2s}) &= +\frac{q^5}{B^4}\left[\frac{1}{8}\sigma_1^2(\rho_3-\rho_2)\right] \\
&+\frac{q^3\mu B}{B^4}\frac{1}{4}\left[\sigma_1\frac{\partial}{\partial y}(\rho_3-\rho_2)+(\rho_3-\rho_2)\left(\frac{\partial\sigma_1}{\partial y}-3\sigma_1^2-3\sigma_1\sigma_2-3\sigma_1\frac{1}{B}\frac{\partial B}{\partial y}\right)\right] \\
&-\frac{q(\mu B)^2}{B^4}\frac{1}{2}\left[(\rho_3-\rho_2)\frac{\partial\sigma_2}{\partial y}+\sigma_2\frac{\partial}{\partial y}(\rho_3-\rho_2)+(\rho_3-\rho_2)\left(-\sigma_1-\frac{3}{B}\frac{\partial B}{\partial y}\right)\sigma_2\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{4}D_c\left(A_{2s}\frac{c_\perp V_x}{B}\right) &= \frac{q^5}{B^4}\left[\frac{1}{8}\rho_1^2(\rho_3-\rho_2)\right] \\
&-\frac{q^3\mu B}{B^4}\frac{1}{4}\left[\rho_1\frac{\partial}{\partial x}(\rho_3-\rho_2)+(\rho_3-\rho_2)\left(\frac{\partial\rho_1}{\partial x}+3\rho_1^2-\frac{9}{2}\rho_1\frac{1}{B}\frac{\partial B}{\partial x}\right)\right] \\
&-\frac{q(\mu B)^2}{B^4}\frac{1}{4}\left[\frac{1}{B}\frac{\partial B}{\partial x}\frac{\partial}{\partial x}(\rho_3-\rho_2)+(\rho_3-\rho_2)\left(\frac{1}{B}\frac{\partial^2 B}{\partial x^2}-4\left(\frac{1}{B}\frac{\partial B}{\partial x}\right)^2+\rho_1\frac{1}{B}\frac{\partial B}{\partial x}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{4}D_s\left(A_{2s}\frac{c_\perp V_y}{B}\right) &= -\frac{q^5}{B^4}\left[\frac{\sigma_1^2}{8}(\rho_3-\rho_2)\right] \\
&-\frac{q^3\mu B}{B^4}\frac{1}{4}\left[\sigma_1\frac{\partial}{\partial y}(\rho_3-\rho_2)+(\rho_3-\rho_2)\left(\frac{\partial\sigma_1}{\partial y}-3\sigma_1^2-\frac{9}{2}\sigma_1\frac{1}{B}\frac{\partial B}{\partial y}\right)\right] \\
&+\frac{q(\mu B)^2}{B^4}\frac{1}{4}\left[\frac{1}{B}\frac{\partial B}{\partial y}\frac{\partial}{\partial y}(\rho_3-\rho_2)+(\rho_3-\rho_2)\left(\frac{1}{B}\frac{\partial^2 B}{\partial y^2}-4\left(\frac{1}{B}\frac{\partial B}{\partial y}\right)^2-\sigma_1\frac{1}{B}\frac{\partial B}{\partial y}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{4}D_cD_c(\mu A_{2s}) &= -\frac{q^5}{B^4}\frac{1}{8}\rho_1^2(\rho_3-\rho_2) \\
&+\frac{q^3\mu B}{B^4}\frac{1}{8}\left[3\rho_1\frac{\partial}{\partial x}(\rho_3-\rho_2)+2\frac{\partial}{\partial x}\rho_1(\rho_3-\rho_2)+(\rho_3-\rho_2)\left(9\rho_1^2-12\rho_1\frac{1}{B}\frac{\partial B}{\partial x}\right)\right] \\
+\frac{q}{B^4}(\mu B)^2\frac{1}{4}&\left\{\left(\frac{5}{B}\frac{\partial B}{\partial x}-2\rho_1\right)\frac{\partial}{\partial x}(\rho_3-\rho_2)-\frac{\partial^2}{\partial x^2}(\rho_3-\rho_2)-(\rho_3-\rho_2)\left[\frac{\partial\rho_1}{\partial x}-\frac{2}{B}\frac{\partial^2 B}{\partial x^2}+\rho_1^2\right.\right. \\
&\left.\left.-5\rho_1\frac{1}{B}\frac{\partial B}{\partial x}+8\left(\frac{1}{B}\frac{\partial B}{\partial x}\right)^2\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{4}D_sD_s(\mu A_{2s}) &= \frac{q^5}{B^4}\frac{1}{8}\sigma_1^2(\rho_3-\rho_2) \\
&+\frac{q^3\mu B}{B^4}\frac{1}{8}\left[3\sigma_1\frac{\partial}{\partial y}(\rho_3-\rho_2)+2\frac{\partial}{\partial y}(\sigma_1(\rho_3-\rho_2))-(\rho_3-\rho_2)\left(9\sigma_1^2+12\sigma_1\frac{1}{B}\frac{\partial B}{\partial y}\right)\right] \\
-\frac{q}{B^4}(\mu B)^2\cdot\frac{1}{4}&\left\{\left(\frac{5}{B}\frac{\partial B}{\partial y}+2\sigma_1\right)\frac{\partial}{\partial y}(\rho_3-\rho_2)-\frac{\partial^2}{\partial y^2}(\rho_3-\rho_2)-(\rho_3-\rho_2)\left[-\frac{\partial\sigma_1}{\partial y}-\frac{2}{B}\frac{\partial^2 B}{\partial y^2}+\sigma_1^2\right.\right. \\
&\left.\left.+5\sigma_1\frac{1}{B}\frac{\partial B}{\partial y}+8\left(\frac{1}{B}\frac{\partial B}{\partial y}\right)^2\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{4}D_cD_{2c}\left(\frac{c_\perp V_x}{B}\right) &= \frac{1}{4}D_c\left(\frac{A_{2s}c_\perp V_x}{B}\right)+\frac{q^3}{B^4}\mu B\frac{3}{4}(\rho_3-\rho_2)\left(\frac{\rho_1}{B}\frac{\partial B}{\partial x}-2\rho_1^2\right) \\
&+\frac{q}{B^4}(\mu B)^2\left[\frac{1}{2}(\rho_3-\rho_2)\left(2\frac{\partial\rho_1}{\partial x}-\frac{1}{B}\frac{\partial^2 B}{\partial x^2}+4\left(\frac{1}{B}\frac{\partial B}{\partial x}\right)^2+2\rho_1^2-7\frac{\rho_1}{B}\frac{\partial B}{\partial x}\right)\right] \\
&-\frac{q(\mu B)^2}{B^4}\left[\frac{1}{2}\left(\frac{1}{B}\frac{\partial B}{\partial x}-2\rho_1\right)\frac{\partial}{\partial x}(\rho_3-\rho_2)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{4}D_sD_{2c}\left(\frac{c_\perp V_y}{B}\right) &= -\frac{1}{4}D_s\left(\frac{A_{2s}c_\perp V_y}{B}\right)+\frac{q^3}{B^4}\mu B\frac{3}{4}(\rho_3-\rho_2)\left(\frac{\sigma_1}{B}\frac{\partial B}{\partial y}+2\sigma_1^2\right) \\
&-\frac{q(\mu B)^2}{B^4}\left[\frac{1}{2}(\rho_3-\rho_2)\left(-2\frac{\partial\sigma_1}{\partial y}-\frac{1}{B}\frac{\partial^2 B}{\partial y^2}+4\left(\frac{1}{B}\frac{\partial B}{\partial y}\right)^2+2\sigma_1^2+7\frac{\sigma_1}{B}\frac{\partial B}{\partial y}\right)\right] \\
&+\frac{q(\mu B)^2}{B^4}\left[\frac{1}{2}\left(\frac{1}{B}\frac{\partial B}{\partial y}+2\sigma_1\right)\frac{\partial}{\partial y}(\rho_3-\rho_2)\right] \qquad \ldots \text{(D.55)}
\end{aligned}
$$

If we now sum the terms in (D.55) we find that the coefficient of $\frac{\partial f_o}{\partial\mu}$ may be written in the form,

$$\frac{1}{8}\frac{q^5}{B^4}(\rho_3-\rho_2)\Big[\quad\Big]_1 + \frac{q^3\mu B}{B^4}\Big[\quad\Big]_2 + \frac{q(\mu B)^2}{B^4}\Big[\quad\Big]_3 . \qquad \text{... (D.56)}$$

Tedious but straightforward algebra yields the following results for the coefficients $\Big[\quad\Big]_1$, $\Big[\quad\Big]_2$ and $\Big[\quad\Big]_3$, :-

$$\Big[\quad\Big]_1 = 0 \qquad \text{... (D.57)}$$

$$\begin{aligned}\Big[\quad\Big]_2 &= \frac{1}{2}\rho_1\frac{\partial}{\partial x}(\rho_3-\rho_2) + \frac{1}{2}(\rho_3-\rho_2)\left\{\frac{\partial\rho_1}{\partial x} + \rho_1\sigma_3 - \rho_1^2 - \frac{\rho_1}{B}\frac{\partial B}{\partial x}\right\} \\ &+ \frac{1}{2}\sigma_1\frac{\partial}{\partial y}(\rho_3-\rho_2) + \frac{1}{2}(\rho_3-\rho_2)\left\{\frac{\partial\sigma_1}{\partial y} - \sigma_1\sigma_2 + \sigma_1^2 - \sigma_1\frac{1}{B}\frac{\partial B}{\partial y}\right\}\end{aligned} \qquad \text{... (D.58)}$$

$$\begin{aligned}\Big[\quad\Big]_3 &= \frac{1}{4}\left\{2\rho_1 + \frac{1}{B}\frac{\partial B}{\partial x} - 3\sigma_3\right\}\frac{\partial}{\partial x}(\rho_3-\rho_2) - \frac{1}{4}\frac{\partial^2}{\partial x^2}(\rho_3-\rho_2) \\ &+ \frac{1}{4}(\rho_3-\rho_2)\left\{\frac{\partial}{\partial x}\left[\rho_1 + \frac{1}{B}\frac{\partial B}{\partial x} - 2\sigma_3\right] - 2\sigma_3^2 + 3\rho_1\sigma_3 + \sigma_3\frac{1}{B}\frac{\partial B}{\partial x} - 5\rho_1\frac{1}{B}\frac{\partial B}{\partial x} - \rho_1^2\right\} \\ &+ \frac{1}{4}\left\{2\sigma_1 - \frac{1}{B}\frac{\partial B}{\partial y} - 3\sigma_2\right\}\frac{\partial}{\partial y}(\rho_3-\rho_2) + \frac{1}{4}\frac{\partial^2}{\partial y^2}(\rho_3-\rho_2) \\ &+ \frac{1}{4}(\rho_3-\rho_2)\left\{\frac{\partial}{\partial y}\left[\sigma_1 - \frac{1}{B}\frac{\partial B}{\partial y} - 2\sigma_2\right] + 2\sigma_2^2 - 3\sigma_1\sigma_2 + \sigma_2\frac{1}{B}\frac{\partial B}{\partial y} - 5\sigma_1\frac{1}{B}\frac{\partial B}{\partial y} + \sigma_1^2\right\}\end{aligned} \qquad \text{... (D.59)}$$

<u>$\frac{\partial f_o}{\partial\mu}$ terms of I - IV.</u>

We now simplify the $\frac{\partial f_o}{\partial\mu}$ terms which arise in I - IV, and which are given in (D.19), (D.20), (D.21) and (D.23) respectively. [Note that we shall evaluate these terms for $\tau_2 \neq \tau_3$, although of course it will eventually be necessary to take $\tau_2 = \tau_3$ in order to conform with the evaluation of the other $\frac{\partial f_o}{\partial\mu}$ terms].

IV. Using (C.19) the $\frac{\partial f_o}{\partial\mu}$ term in (D.23) can be written as

$$\begin{aligned}&-\frac{1}{2}\frac{q^3\mu B}{B^4}\left[\tau_1 + \frac{1}{2}(\tau_2+\tau_3)\right]\left(\sigma_1\frac{1}{B}\frac{\partial B}{\partial x} + \rho_1\frac{1}{B}\frac{\partial B}{\partial y} + 2\sigma_1\sigma_3 - 2\rho_1\sigma_2\right) \\ &+ \frac{q(\mu B)^2}{B^4}\left[\tau_1 + \frac{1}{2}(\tau_2+\tau_3)\right]\left(\sigma_2\frac{1}{B}\frac{\partial B}{\partial x} + \sigma_3\frac{1}{B}\frac{\partial B}{\partial y}\right)\end{aligned} \qquad \text{... (D.60)}$$

III. Using (C.19) again, together with (B.4) and (B.17) the $\frac{\partial f_o}{\partial\mu}$ term in (D.21) becomes:-

$$\begin{aligned}&-\frac{q^3\mu B}{B^4}\left\{\sigma_1\frac{\partial}{\partial x}\left[\tau_1 + \frac{1}{2}(\tau_2+\tau_3)\right] + \rho_1\frac{\partial}{\partial y}\left[\tau_1 + \frac{1}{2}(\tau_2+\tau_3)\right]\right. \\ &\qquad\left. + \left[\tau_1 + \frac{1}{2}(\tau_2+\tau_3)\right]\left(\frac{\partial\rho_1}{\partial y} + \frac{\partial\sigma_1}{\partial x} - \frac{3}{2B}\left(\sigma_1\frac{\partial B}{\partial x} + \rho_1\frac{\partial B}{\partial y}\right)\right)\right\} \\ &+ \frac{q(\mu B)^2}{B^4}\left\{\frac{1}{B}\frac{\partial B}{\partial y}\frac{\partial}{\partial x}\left[\tau_1 + \frac{1}{2}(\tau_2+\tau_3)\right] - \frac{1}{B}\frac{\partial B}{\partial x}\frac{\partial}{\partial y}\left[\tau_1 + \frac{1}{2}(\tau_2+\tau_3)\right]\right. \\ &\qquad\left. + \left[\tau_1 + \frac{1}{2}(\tau_2+\tau_3)\right]\left(\frac{\sigma_1}{B}\frac{\partial B}{\partial x} + \frac{\rho_1}{B}\frac{\partial B}{\partial y} - \frac{\sigma_2}{B}\frac{\partial B}{\partial x} - \frac{\sigma_3}{B}\frac{\partial B}{\partial y} - (\tau_2+\tau_3)(\rho_2+\rho_3)\right)\right\}\end{aligned} \qquad \text{... (D.61)}$$

II. We now consider the $\frac{\partial f_o}{\partial\mu}$ term arising in (D.20). By (C.19) this is:-

$$\frac{1}{2}\frac{q}{B^3}\left\{\left[q^2\sigma_1 - 2\mu B\sigma_2\right]\left(\frac{\partial V_y}{\partial s} + \frac{1}{2}(\rho_2+\rho_3)V_y\right) - \left[q^2\rho_1 - 2\mu B\sigma_3\right]\left(\frac{\partial V_x}{\partial s} + \frac{1}{2}(\rho_2+\rho_3)V_x\right)\right\} \quad \text{... (D.62)}$$

From (C.19) it also follows that

$$\frac{\partial V_y}{\partial s} = \mu\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial y}\right) - q^2\frac{\partial}{\partial s}\left(\frac{\sigma_1}{B}\right) + \frac{\sigma_1}{B}2\mu\frac{\partial B}{\partial s}$$

and

$$\frac{\partial V_x}{\partial s} = \mu\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial x}\right) + q^2\frac{\partial}{\partial s}\left(\frac{\rho_1}{B}\right) - \frac{\rho_1}{B}2\mu\frac{\partial B}{\partial s} \quad \text{... (D.63)}$$

Using (D.63) straightforward algebra leads to (D.62) being expressed in the form

$$\begin{aligned}
&-\frac{1}{4B}q^5\frac{\partial}{\partial s}\left(\frac{\sigma_1^2+\rho_1^2}{B^3}\right) + \frac{1}{2}\frac{q^3(\mu B)}{B^2}\left\{2\sigma_2\frac{\partial}{\partial s}\left(\frac{\sigma_1}{B}\right) + \frac{\sigma_1\sigma_2}{B}(\rho_2+\rho_3)\right.\\
&+\frac{\sigma_1}{B}\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial y}\right) - 2\sigma_1(\rho_2+\rho_3) + \frac{1}{2}\frac{1}{B}\frac{\partial B}{\partial y}(\rho_2+\rho_3)\right] + 2\sigma_3\frac{\partial}{\partial s}\left(\frac{\rho_1}{B}\right) + \frac{\rho_1\sigma_3(\rho_2+\rho_3)}{B}\\
&\left.-\frac{\rho_1}{B}\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial x}\right) + 2\rho_1(\rho_2+\rho_3) + \frac{1}{2B}\frac{\partial B}{\partial x}(\rho_2+\rho_3)\right]\right\}\\
&+\frac{q(\mu B)^2}{B^4}\left\{\sigma_3\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial x}\right) + 2\rho_1(\rho_2+\rho_3) + \frac{1}{2}\frac{1}{B}\frac{\partial B}{\partial x}(\rho_2+\rho_3)\right]\right.\\
&\left.-\sigma_2\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial y}\right) - 2\sigma_1(\rho_2+\rho_3) + \frac{1}{2}(\rho_2+\rho_3)\frac{1}{B}\frac{\partial}{\partial y}\right]\right\}
\end{aligned} \quad \text{... (D.64)}$$

Since the $\frac{\partial f_o}{\partial\mu}$ terms are inside the integral $\oint\frac{ds}{q}B$ it is advantageous to express the q^5 term in (D.64) in the form

$$-\frac{1}{4}\frac{q}{B}\frac{\partial}{\partial s}\frac{q^4(\sigma_1^2+\rho_1^2)}{B^3} + \frac{q^3(\mu B)}{B^4}(\sigma_1^2+\rho_1^2)(\rho_2+\rho_3) \quad \text{... (D.65)}$$

so that the first term will not contribute to Hf_o. The final form for group II is thus

$$\begin{aligned}
&-\frac{1}{4}\frac{q}{B}\frac{\partial}{\partial s}\frac{q^4(\sigma_1^2+\rho_1^2)}{B^3}\\
&-\frac{1}{2}\frac{q^3(\mu B)}{B^4}\left\{2\sigma_2 B\frac{\partial}{\partial s}\left(\frac{\sigma_1}{B}\right) + \sigma_1\sigma_2(\rho_2+\rho_3) + \sigma_1\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial y}\right) + \frac{1}{2}\frac{1}{B}\frac{\partial B}{\partial y}(\rho_2+\rho_3)\right]\right.\\
&\left.+2\sigma_3 B\frac{\partial}{\partial s}\left(\frac{\rho_1}{B}\right) + \rho_1\sigma_3(\rho_2+\rho_3) - \rho_1\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial x}\right) + \frac{1}{2}\frac{1}{B}\frac{\partial B}{\partial x}(\rho_2+\rho_3)\right]\right\}\\
&+\frac{q(\mu B)^2}{B^4}\left\{\sigma_3\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial x}\right) + 2\rho_1(\rho_2+\rho_3) + \frac{1}{2}\frac{1}{B}\frac{\partial B}{\partial x}(\rho_2+\rho_3)\right]\right.\\
&\left.-\sigma_2\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial y}\right) - 2\sigma_1(\rho_2+\rho_3) + \frac{1}{2}\frac{1}{B}\frac{\partial B}{\partial y}(\rho_2+\rho_3)\right]\right\}
\end{aligned} \quad \text{... (D.66)}$$

I. We now consider the $\frac{\partial f_o}{\partial\mu}$ term arising in (D.19). Simple but tedious albegra gives the result:-

$$\begin{aligned}
&\frac{q^3\mu B}{B^4}\left\{\left(\rho_1 - \frac{5}{2}\frac{1}{B}\frac{\partial B}{\partial x}\right)\left[\frac{\partial\rho_1}{\partial s} + \frac{3}{2}\rho_1(\rho_2+\rho_3)\right] + \left(\sigma_1 + \frac{5}{2}\frac{1}{B}\frac{\partial B}{\partial y}\right)\left[\frac{\partial\sigma_1}{\partial s} + \frac{3}{2}\sigma_1(\rho_2+\rho_3)\right]\right.\\
&\qquad + \frac{\partial}{\partial x}\left[\frac{\partial\rho_1}{\partial s} + \frac{3}{2}\rho_1(\rho_2+\rho_3)\right] - \frac{\partial}{\partial y}\left[\frac{\partial\sigma_1}{\partial s} + \frac{3}{2}\sigma_1(\rho_2+\rho_3)\right]
\end{aligned}$$

$$+ \rho_1 \left[\frac{\partial}{\partial s}\left(2\rho_1 - \frac{1}{B}\frac{\partial B}{\partial x}\right) + \frac{1}{2}(\rho_2+\rho_3)\left(2\rho_1 - \frac{1}{B}\frac{\partial B}{\partial x}\right)\right]$$

$$+ \sigma_1 \left[\frac{\partial}{\partial s}\left(2\sigma_1 + \frac{1}{B}\frac{\partial B}{\partial y}\right) + \frac{1}{2}(\rho_2+\rho_3)\left(2\sigma_1+\frac{1}{B}\frac{\partial B}{\partial y}\right)\right]\Bigg\}$$

$$+\frac{q(\mu B)^2}{B^4}\Bigg\{\left(\rho_1 - \frac{3}{2}\frac{1}{B}\frac{\partial B}{\partial x}\right)\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial x}\right) + \frac{1}{2}\frac{1}{B}\frac{\partial B}{\partial x}(\rho_2+\rho_3) + 2\rho_1(\rho_2+\rho_3)\right]$$

$$+\frac{\partial}{\partial x}\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial x}\right) + \frac{1}{2}\frac{1}{B}\frac{\partial B}{\partial x}(\rho_2+\rho_3) + 2\rho_1(\rho_2+\rho_3)\right]$$

$$-\frac{2}{B}\frac{\partial B}{\partial x}\left[\frac{\partial\rho_1}{\partial s} + \frac{3}{2}\rho_1(\rho_2+\rho_3)\right]$$

$$-\left(\sigma_1+\frac{3}{2}\frac{1}{B}\frac{\partial B}{\partial y}\right)\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial y}\right) + \frac{1}{2}\frac{1}{B}\frac{\partial B}{\partial y}(\rho_2+\rho_3) - 2\sigma_1(\rho_2+\rho_3)\right]$$

$$+\frac{\partial}{\partial y}\left[\frac{\partial}{\partial s}\left(\frac{1}{B}\frac{\partial B}{\partial y}\right) + \frac{1}{2}\frac{1}{B}\frac{\partial B}{\partial y}(\rho_2+\rho_3) - 2\sigma_1(\rho_2+\rho_3)\right]$$

$$+\frac{2}{B}\frac{\partial B}{\partial y}\left[\frac{\partial\sigma_1}{\partial s}+\frac{3}{2}\sigma_1(\rho_2+\rho_3)\right]$$

$$+\frac{1}{B}\frac{\partial B}{\partial x}\left[\frac{\partial}{\partial s}\left(2\rho_1 - \frac{1}{B}\frac{\partial B}{\partial x}\right) + \frac{1}{2}(\rho_2+\rho_3)\left(2\rho_1 - \frac{1}{B}\frac{\partial B}{\partial x}\right)\right]$$

$$-\frac{1}{B}\frac{\partial B}{\partial y}\left[\frac{\partial}{\partial s}\left(2\sigma_1 + \frac{1}{B}\frac{\partial B}{\partial y}\right) + \frac{1}{2}(\rho_2+\rho_3)\left(2\sigma_1 + \frac{1}{B}\frac{\partial B}{\partial y}\right)\right]\Bigg\}$$

$$-\frac{q}{4B}\frac{\partial}{\partial s}\left(\frac{V_x^2+V_y^2}{B}\right) \qquad \text{... (D.67)}$$

Before collecting up the $\frac{\partial f_o}{\partial \mu}$ terms from all sources it is convenient to combine the contributions from (D.21) and (D.23). Thus adding (D.60) and (D.61) we have:-

$$-\frac{q^3\mu B}{B^4}\Bigg\{\sigma_1\frac{\partial}{\partial x}\left[\tau_1+\frac{1}{2}(\tau_2+\tau_3)\right] + \rho_1\frac{\partial}{\partial y}\left[\tau_1+\frac{1}{2}(\tau_2+\tau_3)\right] + \left(\tau_1+\frac{1}{2}(\tau_2+\tau_3)\right)\left[\frac{\partial\rho_1}{\partial y}+\frac{\partial\sigma_1}{\partial x}+\sigma_1\sigma_3-\rho_1\sigma_2\right]$$

$$-\left(\tau_1+\frac{1}{2}(\tau_2+\tau_3)\right)\left(\frac{\sigma_1}{B}\frac{\partial B}{\partial x}+\frac{\rho_1}{B}\frac{\partial B}{\partial y}\right)\Bigg\}$$

$$+\frac{q(\mu B)^2}{B^4}\Bigg\{\frac{1}{B}\frac{\partial B}{\partial y}\frac{\partial}{\partial x}\left[\tau_1+\frac{1}{2}(\tau_2+\tau_3)\right] - \frac{1}{B}\frac{\partial B}{\partial x}\frac{\partial}{\partial y}\left[\tau_1+\frac{1}{2}(\tau_2+\tau_3)\right]$$

$$+\left[\tau_1+\frac{1}{2}(\tau_2+\tau_3)\right]\left(\frac{\sigma_1}{B}\frac{\partial B}{\partial x}+\frac{\rho_1}{B}\frac{\partial B}{\partial y} - (\tau_2+\tau_3)(\rho_2+\rho_3)\right)\Bigg\} \qquad \text{... (D.68)}$$

Using (B.7),(B.8) and (B.17), then

$$-\frac{q^3\mu B}{B^4}\left[\tau_1+\frac{1}{2}(\tau_2+\tau_3)\right]\left(\frac{\partial\rho_1}{\partial y}+\frac{\partial\sigma_1}{\partial x}+\sigma_1\sigma_3-\rho_1\sigma_2\right) = -\frac{q^3(\mu B)}{B^4}B\tau_1\frac{\partial}{\partial s}\left(\frac{\tau_2+\tau_3}{B}\right)$$

$$-\frac{1}{4}\frac{\mu q^3}{B}\frac{\partial}{\partial s}\left(\frac{\tau_2+\tau_3}{B}\right)^2, \qquad \text{... (D.69)}$$

and as before this is more conveniently expressed as

$$-\frac{1}{4}\frac{q}{B}\frac{\partial}{\partial s}\mu q^2\left(\frac{\tau_2+\tau_3}{B}\right)^2 - \frac{q^3(\mu B)}{B^4}\tau_1 B\frac{\partial}{\partial s}\left(\frac{\tau_2+\tau_3}{B}\right) + \frac{1}{2}\frac{q(\mu B)^2}{B^4}(\tau_2+\tau_3)^2(\rho_2+\rho_3)$$

The final expression for the $\frac{\partial f_o}{\partial \mu}$ terms from (D.21) and (D.23) is:

$$- \frac{q^3 \mu B}{B^4} \left\{ \sigma_1 \frac{\partial}{\partial x}\left[\tau_1 + \frac{1}{2}(\tau_2 + \tau_3)\right] + \rho_1 \frac{\partial}{\partial y}\left[\tau_1 + \frac{1}{2}(\tau_2+\tau_3)\right] + \tau_1 B \frac{\partial}{\partial s}\left(\frac{\tau_2+\tau_3}{B}\right)\right]\right\}$$

$$+ \frac{q^3 \mu B}{B^4}\left[\tau_1 + \frac{1}{2}(\tau_2+\tau_3)\right]\left(\frac{\sigma_1}{B}\frac{\partial B}{\partial x} + \frac{\rho_1}{B}\frac{\partial B}{\partial y}\right) - \frac{1}{4}\frac{q}{B}\frac{\partial}{\partial s}\mu q^2 \left(\frac{\tau_2+\tau_3}{B}\right)^2$$

$$+ \frac{q(\mu B)^2}{B^4}\left\{ \frac{1}{B}\frac{\partial B}{\partial y}\frac{\partial}{\partial x}(\tau_1 + \tfrac{1}{2}(\tau_2+\tau_3)) - \frac{1}{B}\frac{\partial B}{\partial x}\frac{\partial}{\partial y}(\tau_1 + \frac{1}{2}(\tau_2+\tau_3)) \right.$$

$$\left. + (\tau_1 + \frac{1}{2}(\tau_2+\tau_3))\left(\frac{\sigma_1}{B}\frac{\partial B}{\partial x} + \frac{\rho_1}{B}\frac{\partial B}{\partial y}\right) - \tau_1(\tau_2+\tau_3)(\rho_2+\rho_3)\right\} \qquad \text{... (D.70)}$$

<u>Sum Total of $\frac{\partial f_o}{\partial \mu}$ Terms.</u>

In this section we sum all the $\frac{\partial f_o}{\partial \mu}$ terms which have arisen in (D.38),(D.58), (D.59), (D.66), (D.67) and (D.70). Using (B.5), (B.6), (B.11) (B.12), (B.13), (B.14) and (B.17), it is possible to reduce the total coefficient of $\frac{q^3(\mu B)}{B^4}$ to:

$$B^2 \frac{\partial}{\partial s}\left[\frac{1}{B^2}\left\{ \frac{\partial \rho_1}{\partial x} - \frac{\partial \sigma_1}{\partial y} + \sigma_1\sigma_2 + \rho_1\sigma_3 + 2(\rho_1^2 + \sigma_1^2) + \frac{5}{2}\left(\frac{\sigma_1}{B}\frac{\partial B}{\partial y} - \frac{\rho_1}{B}\frac{\partial B}{\partial x}\right) - \frac{1}{4}(\tau_2+\tau_3)^2\right\}\right] \qquad \text{... (D.71)}$$

Carrying through the integration $\oint \frac{B ds}{q}$ leads to

$$-2\oint \frac{ds}{q}\frac{q(\mu B)^2}{B^3}(\rho_2+\rho_3)\left\{\frac{\partial \rho_1}{\partial x} - \frac{\partial \sigma_1}{\partial y} + \sigma_1\sigma_2 + \rho_1\sigma_3 + 2(\rho_1^2+\sigma_1^2) + \frac{5}{2}\left(\frac{\sigma_1}{B}\frac{\partial B}{\partial y} - \frac{\rho_1}{B}\frac{\partial B}{\partial x}\right) - \frac{1}{4}(\tau_2+\tau_3)^2\right\} \qquad \text{... (D.72)}$$

Using (B.5), (B.6), (B.9), (B.10), (B.11), (B.12), (B.13), (B.15) and (B.16), and taking $\tau_2 = \tau_3$ we obtain the following coefficient for $\frac{q(\mu B)^2}{B^4}$:-

$$- \frac{5}{4} B \frac{\partial}{\partial s}\left\{\frac{1}{B}\left[\left(\frac{1}{B}\frac{\partial B}{\partial x}\right)^2 + \left(\frac{1}{B}\frac{\partial B}{\partial y}\right)^2\right]\right\} + \frac{3}{4} B \frac{\partial}{\partial s}\left\{\frac{1}{B}\left[\frac{\partial}{\partial x}\left(\frac{1}{B}\frac{\partial B}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{B}\frac{\partial B}{\partial y}\right)\right]\right\}$$

$$+ \frac{1}{2} B \frac{\partial}{\partial s}\left(\frac{\rho_2\rho_3}{B}\right) + \frac{3}{4} B \frac{\partial}{\partial s}\left\{\frac{1}{B}\left[\frac{\sigma_3}{B}\frac{\partial B}{\partial x} - \frac{\sigma_2}{B}\frac{\partial B}{\partial y}\right]\right\} + \frac{3}{2} B \frac{\partial}{\partial s}\left(\frac{\tau_2^2}{B}\right), \qquad \text{... (D.73)}$$

where the contribution from (D.72) has also been included.

We now observe that on integrating (D.73) by $\oint \frac{B ds}{q}$ the coefficient of $\frac{\partial f_o}{\partial \mu}$ vanishes.

<u>CONCLUSIONS ARISING FROM APPENDIX D</u>

In this appendix we have been concerned with evaluating Hf_o. For a system which has oscillatory periodicity, it was sufficient to note that the operators K_2 and K_3, are even and odd in σ respectively, and this led to the result $Hf_o = 0$. However, for a system with toroidal perodicity we have shown that

$$Hf_o = \frac{\partial f_o}{\partial J}\left\{ \frac{\partial J}{\partial \beta}\frac{\partial P}{\partial \alpha} - \frac{\partial J}{\partial \alpha}\frac{\partial P}{\partial \beta}\right\},$$

where P is given by

$$P = \frac{1}{2}\oint ds \left\{ \frac{\partial}{\partial s}\left(\frac{\partial I}{\partial \alpha_\perp}\right)\frac{\partial I}{\partial \beta_\perp} \quad \frac{\partial}{\partial s}\left(\frac{\partial I}{\partial \beta_\perp}\right)\frac{\partial I}{\partial \alpha_\perp} + 2\mu\sigma T \right\}.$$

APPENDIX E

In this Appendix we are concerned with the extension of the method outlined in Appendix C for the derivation of the D operator, to the time dependent case with electric fields present.

We have now to deal with the equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f + \frac{e}{m}\left[\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} \right] \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = 0 \quad , \qquad \ldots \text{(E.1)}$$

where

$$\underset{\sim}{E} = - \frac{\partial \underset{\sim}{A}}{\partial t} - \underset{\sim}{\nabla}\, \varphi \qquad \ldots \text{(E.2)}$$

and

$$\underset{\sim}{A} = \alpha \underset{\sim}{\nabla} \beta \quad .$$

It follows from (E.2) that

$$(\underset{\sim}{E} + \underset{\sim}{\nabla}\Psi) = \frac{\partial \beta}{\partial t} \underset{\sim}{\nabla}\alpha - \frac{\partial \alpha}{\partial t} \underset{\sim}{\nabla}\beta \ , \qquad \ldots \text{(E.3)}$$

where

$$\Psi = \alpha \frac{\partial \beta}{\partial t} + \varphi \quad , \qquad \ldots \text{(E.4)}$$

and therefore that

$$\frac{\partial \alpha}{\partial t} + \underset{\sim}{U} \cdot \underset{\sim}{\nabla}\alpha = 0 \quad ; \quad \frac{\partial \beta}{\partial t} + \underset{\sim}{U} \cdot \underset{\sim}{\nabla}\beta = 0 \quad , \qquad \ldots \text{(E.5)}$$

where

$$\underset{\sim}{U} = \frac{(\underset{\sim}{E} + \underset{\sim}{\nabla}\Psi)}{B^2} \times \underset{\sim}{B} \quad . \qquad \ldots \text{(E.6)}$$

That is, the velocity $\underset{\sim}{U}$ is a flux-preserving velocity. Previous authors[4,12] have found it convenient to consider (E.1) in the coordinate system moving with the particle drift velocity $\dfrac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2}$. However, in the present work, we shall choose a frame of reference which moves with the magnetic field lines, i.e. moving with the velocity

$$\underset{\sim}{U} = \frac{(\underset{\sim}{E} + \underset{\sim}{\nabla}\Psi)}{B^2} \times \underset{\sim}{B} \quad .$$

Taking

$$\underset{\sim}{v} = \underset{\sim}{U} + \underset{\sim}{c}$$

in (E.1) results in the following form for the Vlasov equation:-

$$\frac{\partial f}{\partial t} - \frac{\partial \underset{\sim}{U}}{\partial t} \cdot \frac{\partial f}{\partial \underset{\sim}{c}} + (\underset{\sim}{U} + \underset{\sim}{c}) \cdot \underset{\sim}{\nabla} f - \frac{\partial f}{\partial \underset{\sim}{c}} (\underset{\sim}{U} + \underset{\sim}{c}) : \underset{\sim}{\nabla}\underset{\sim}{U} + \frac{e}{m}\left[\underset{\sim}{c} \times \underset{\sim}{B} - \nabla\Psi \right] \cdot \frac{\partial f}{\partial \underset{\sim}{c}} = 0 \qquad \ldots \text{(E.7)}$$

As in Appendix C we introduce $\mu, \varepsilon, \varphi, \sigma$ to replace $\underset{\sim}{c} = (\underset{\sim}{v} - \underset{\sim}{U})$ as velocity variables. The various terms in (E.7) transform as follows:-

$$\left(\frac{\partial f}{\partial t}\right)_{\underset{\sim}{c}} = \left(\frac{\partial f}{\partial t}\right)_{\mu\varepsilon\varphi} - \mu \frac{\partial f}{\partial \mu}\frac{1}{B}\frac{\partial B}{\partial t} - \frac{\sigma q}{B}\frac{\partial f}{\partial \mu}\underset{\sim}{c}_{\perp}\cdot\frac{\partial \underset{\sim}{e}_1}{\partial t} - \frac{\sigma q}{2\mu B}\frac{\partial f}{\partial \varphi}\frac{\partial \underset{\sim}{e}_1}{\partial t}\cdot \underset{\sim}{e}_1 \times \underset{\sim}{c}_{\perp} \quad \ldots \text{(E.8)}$$

$$- \frac{\partial f}{\partial \underset{\sim}{c}}\cdot\left[\frac{\partial \underset{\sim}{U}}{\partial t} + (\underset{\sim}{U}\cdot\underset{\sim}{\nabla})\underset{\sim}{U} + \sigma q \frac{\partial \underset{\sim}{U}}{\partial s} + (\underset{\sim}{c}_{\perp}\cdot\underset{\sim}{\nabla})\underset{\sim}{U}\right] = - \frac{\partial f}{\partial \varepsilon}\left[\sigma q\,\underset{\sim}{e}_1\,\frac{d\underset{\sim}{U}}{dt} - q^2\underset{\sim}{U}\cdot\underset{\sim}{\rho} + \sigma q\,\underset{\sim}{e}_1\underset{\sim}{c}_{\perp} : \underset{\sim}{\nabla}\,\underset{\sim}{U}\right]$$

$$- \left(\frac{\partial f}{\partial \varepsilon} + \frac{1}{B}\frac{\partial f}{\partial \mu}\right)\left[\underset{\sim}{c}_{\perp}\cdot\frac{d\underset{\sim}{U}}{dt} + \sigma q\,\underset{\sim}{c}_{\perp}\cdot\frac{\partial \underset{\sim}{U}}{\partial s} + \underset{\sim}{c}_{\perp}\,\underset{\sim}{c}_{\perp} : \underset{\sim}{\nabla}\,\underset{\sim}{U}\right]$$

$$- \frac{1}{2\mu B}\frac{\partial f}{\partial \varphi}\left[\underset{\sim}{e}_1 \times \underset{\sim}{c}_{\perp}\cdot\frac{d\underset{\sim}{U}}{dt} + \sigma q\,\underset{\sim}{e} \times \underset{\sim}{c}_{\perp}\cdot\frac{\partial \underset{\sim}{U}}{\partial s} + (\underset{\sim}{e}_1 \times \underset{\sim}{c}_{\perp})\underset{\sim}{c}_{\perp} : \underset{\sim}{\nabla}\,\underset{\sim}{U}\right]$$

$$\ldots \text{(E.9)}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{U}\cdot\underset{\sim}{\nabla} \;,$$

$$\underset{\sim}{U}\cdot\underset{\sim}{\nabla} f\big|_{\underset{\sim}{c}} = \underset{\sim}{U}\cdot\underset{\sim}{\nabla} f\big|_{\mu\varepsilon\varphi} - \mu\frac{\partial f}{\partial \mu}\frac{\underset{\sim}{U}\cdot\underset{\sim}{\nabla} B}{B} - \frac{\sigma q}{B}\frac{\partial f}{\partial \mu}\underset{\sim}{c}_{\perp}\,\underset{\sim}{U} : \underset{\sim}{\nabla}\,\underset{\sim}{e}_1 - \frac{\sigma q}{2\mu B}\frac{\partial f}{\partial \varphi}(\underset{\sim}{e}_1 \times \underset{\sim}{c}_{\perp})\,\underset{\sim}{U} : \underset{\sim}{\nabla}\,\underset{\sim}{e}_1 \quad \ldots \text{(E.10)}$$

$$\underset{\sim}{c}\cdot\underset{\sim}{\nabla} f = BDf \;, \quad \ldots \text{(E.11)}$$

where D is the operator derived in Appendix C,

$$\frac{e}{m}\underset{\sim}{c} \times \underset{\sim}{B}\cdot\frac{\partial f}{\partial \underset{\sim}{c}} = - \frac{eB}{m}\frac{\partial f}{\partial \varphi} \quad \ldots \text{(E.12)}$$

and

$$- \frac{e}{m}\underset{\sim}{\nabla}\Psi\cdot\frac{\partial f}{\partial \underset{\sim}{c}} = - \frac{e}{m}\sigma q\frac{\partial \Psi}{\partial s}\frac{\partial f}{\partial \varepsilon} - \left(\frac{1}{B}\frac{\partial f}{\partial \mu} + \frac{\partial f}{\partial \varepsilon}\right)\frac{e}{m}\underset{\sim}{c}_{\perp}\cdot\underset{\sim}{\nabla}\Psi - \frac{e}{m}\underset{\sim}{\nabla}\Psi\cdot\frac{\underset{\sim}{e}_1 \times \underset{\sim}{c}_{\perp}}{2\mu B}\frac{\partial f}{\partial \varphi} \,. \quad \ldots \text{(E.13)}$$

To replace the time derivatives of B and $\underset{\sim}{e}_1$ we use the Maxwell equation

$$\frac{\partial \underset{\sim}{B}}{\partial t} = - \text{curl}\,\underset{\sim}{E}$$

$$= - \text{curl}\left[- \underset{\sim}{U} \times \underset{\sim}{B} - \underset{\sim}{\nabla}\Psi\right] = \text{curl}\,(\underset{\sim}{U} \times \underset{\sim}{B}) \;, \quad \ldots \text{(E.14)}$$

a result which would not hold if $\underset{\sim}{U}$ were the particle drift velocity $\frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2}$. From (E.14) we have

$$\frac{1}{B}\frac{dB}{dt} = - \underset{\sim}{\nabla}\cdot\underset{\sim}{U} + \underset{\sim}{e}_1\,\underset{\sim}{e}_1 : \underset{\sim}{\nabla}\underset{\sim}{U} \equiv - (\underset{\sim}{\nabla}\cdot\underset{\sim}{U})_{\perp} \quad \ldots \text{(E.15)}$$

and

$$\frac{\partial \underset{\sim}{e}_1}{\partial t} = \frac{\partial \underset{\sim}{U}}{\partial s} - (\underset{\sim}{U}\cdot\underset{\sim}{\nabla})\,\underset{\sim}{e}_1 + \underset{\sim}{e}_1\,(\underset{\sim}{U}\cdot\underset{\sim}{\rho}) \quad \ldots \text{(E.16)}$$

If (E.8), (E.9), (E.10), (E.11), (E.12) and (E.13) are substituted into (E.7) the resultant form for the Vlasov equation is

$$BDf - \frac{e}{m}\left\{ \sigma q \frac{\partial\Psi}{\partial s}\frac{\partial f}{\partial\varepsilon} + \left(\frac{1}{B}\frac{\partial f}{\partial\mu} + \frac{\partial f}{\partial\varepsilon}\right) \underset{\sim}{c}_\perp \cdot \underset{\sim}{\nabla}\Psi + \frac{\underset{\sim}{\nabla}\Psi \cdot \underset{\sim}{e}_1 \times \underset{\sim}{c}_\perp}{2\mu B}\frac{\partial f}{\partial\varphi}\right\}$$

$$+ \frac{\partial f}{\partial\varepsilon}\left\{ \sigma q \frac{\partial}{\partial s}\left(\frac{1}{2}U^2\right) + q^2\underset{\sim}{U}\cdot\underset{\sim}{\rho} - \mu B(\underset{\sim}{\nabla}\cdot\underset{\sim}{U})_\perp - \sigma q\, \underset{\sim}{e}_1\, c_\perp : \nabla\underset{\sim}{U} - \underset{\sim}{c}_\perp \cdot \frac{d\underset{\sim}{U}}{dt}\right.$$

$$\left. - \sigma q \underset{\sim}{c}_\perp \cdot \frac{\partial\underset{\sim}{U}}{\partial s} + \cos 2\varphi\mu B\left(\underset{\sim}{e}_3\cdot\frac{\partial\underset{\sim}{U}}{\partial y} - \underset{\sim}{e}_2\cdot\frac{\partial\underset{\sim}{U}}{\partial x}\right) - \sin 2\varphi\mu B\left(\underset{\sim}{e}_2\cdot\frac{\partial\underset{\sim}{U}}{\partial y} + \underset{\sim}{e}_3\cdot\frac{\partial\underset{\sim}{U}}{\partial x}\right)\right\}$$

$$+ \frac{df}{dt} - \frac{1}{B}\frac{\partial f}{\partial\mu}\left\{ \underset{\sim}{c}_\perp\cdot\frac{d\underset{\sim}{U}}{dt} + 2\sigma q\, \underset{\sim}{c}_\perp\cdot\frac{\partial\underset{\sim}{U}}{\partial s} - \cos 2\varphi\mu B\left(\underset{\sim}{e}_3\cdot\frac{\partial\underset{\sim}{U}}{\partial y} - \underset{\sim}{e}_2\cdot\frac{\partial\underset{\sim}{U}}{\partial x}\right)\right.$$

$$\left. + \sin 2\varphi\mu B\left(\underset{\sim}{e}_2\cdot\frac{\partial\underset{\sim}{U}}{\partial y} + \underset{\sim}{e}_3\cdot\frac{\partial\underset{\sim}{U}}{\partial x}\right)\right\}$$

$$- \frac{1}{2\mu B}\frac{\partial f}{\partial\varphi}\left\{ \underset{\sim}{e}_1\times\underset{\sim}{c}_\perp\cdot\frac{d\underset{\sim}{U}}{dt} + 2\sigma q\, \underset{\sim}{e}_1\times\underset{\sim}{c}_\perp\cdot\frac{\partial\underset{\sim}{U}}{\partial s} + (\underset{\sim}{e}_1\times\underset{\sim}{c}_\perp)\, \underset{\sim}{c}_\perp : \underset{\sim}{\nabla}\,\underset{\sim}{U}\right\} = \frac{eB}{m}\frac{\partial f}{\partial\varphi}, \qquad \ldots (E.17)$$

which may be written

$$\frac{\partial f}{\partial\varphi} = G^\Psi f + \lambda\left(\frac{1}{B}\frac{\partial f}{\partial t} + G^U f + Df\right) \qquad \ldots (E.18)$$

where

$$G^\Psi = -\frac{1}{B}\left\{ \sigma q\frac{\partial\Psi}{\partial s}\frac{\partial}{\partial\varepsilon} + \underset{\sim}{c}_\perp\cdot\underset{\sim}{\nabla}\Psi\left(\frac{1}{B}\frac{\partial}{\partial\mu} + \frac{\partial}{\partial\varepsilon}\right) + \frac{\underset{\sim}{e}_1\times\underset{\sim}{c}_\perp}{2\mu B}\cdot\underset{\sim}{\nabla}\Psi\frac{\partial}{\partial\varphi}\right\}, \qquad \ldots (E.19)$$

and incorporating $\frac{\partial}{\partial t}$ into G^U

$$G^U = \frac{1}{B}\frac{d}{dt} + \frac{1}{B}\left\{ \sigma q\frac{\partial}{\partial s}\left(\frac{1}{2}U^2\right) + q^2\underset{\sim}{U}\cdot\underset{\sim}{\rho} - \mu B(\underset{\sim}{\nabla}\cdot\underset{\sim}{U})_\perp - \sigma q\underset{\sim}{e}_1\underset{\sim}{c}_\perp : \nabla\underset{\sim}{U} - \underset{\sim}{c}_\perp\cdot\frac{d\underset{\sim}{U}}{dt}\right.$$

$$\left. - \sigma q\underset{\sim}{c}_\perp\cdot\frac{\partial\underset{\sim}{U}}{\partial s} + \cos 2\varphi\mu B\left(\underset{\sim}{e}_3\cdot\frac{\partial\underset{\sim}{U}}{\partial y} - \underset{\sim}{e}_2\cdot\frac{\partial\underset{\sim}{U}}{\partial x}\right) - \sin 2\varphi\mu B\left(\underset{\sim}{e}_2\cdot\frac{\partial\underset{\sim}{U}}{\partial y} + \underset{\sim}{e}_3\cdot\frac{\partial\underset{\sim}{U}}{\partial x}\right)\right\}\frac{\partial}{\partial\varepsilon}$$

$$- \frac{1}{B^2}\left\{\underset{\sim}{c}_\perp\cdot\frac{d\underset{\sim}{U}}{dt} + 2\sigma q\,\underset{\sim}{c}_\perp\cdot\frac{\partial\underset{\sim}{U}}{\partial s} - \mu B\left(\underset{\sim}{e}_3\cdot\frac{\partial\underset{\sim}{U}}{\partial y} - \underset{\sim}{e}_2\cdot\frac{\partial\underset{\sim}{U}}{\partial x}\right)\cos 2\varphi + \mu B\left(\underset{\sim}{e}_2\cdot\frac{\partial\underset{\sim}{U}}{\partial y} + \underset{\sim}{e}_3\cdot\frac{\partial\underset{\sim}{U}}{\partial x}\right)\sin 2\varphi\right\}\frac{\partial}{\partial\mu}$$

$$- \frac{1}{2\mu B^2}\,\underset{\sim}{e}_1\times\underset{\sim}{c}_\perp\cdot\left\{\frac{d\underset{\sim}{U}}{dt} + 2\sigma q\frac{\partial\underset{\sim}{U}}{\partial s} + (\underset{\sim}{c}_\perp\cdot\nabla)\,\underset{\sim}{U}\right\}\frac{\partial}{\partial\varphi} \qquad \ldots (E.20)$$

In the main text (7.10) we are required to evaluate $\langle G^\Psi f_o\rangle$. From inspection of (E.19) we may write this down at once. Thus

$$\langle G^\Psi f_o\rangle = -\frac{\sigma q}{B}\frac{\partial}{\partial s}\left(\Psi\frac{\partial f_o}{\partial\varepsilon}\right), \qquad \ldots (E.21)$$

provided f_o is independent of φ and s. (This follows since $\langle \underset{\sim}{c}_\perp\rangle = 0$.)

<u>The M Operator</u>

The operator M is obtained by substituting (7.9) and (7.13) into (7.15) and collecting all the terms involving G^Ψ or G^U. It is defined to be

$$Mf_o = \frac{1}{2}\frac{\sigma q}{B}\frac{\partial}{\partial s}\left(\Psi^2\frac{\partial^2 f_o}{\partial\varepsilon^2}\right) - \left\langle D\int G^\Psi f_o\right\rangle - \left\langle G^\Psi\int Df_o\right\rangle$$
$$- L\left(\frac{\partial f_o}{\partial\varepsilon}\Psi\right) + \sigma\left\langle G^\Psi\int_{s_o}^{s}\frac{B\,ds}{q}\,Lf_o\right\rangle - \langle G^U f_o\rangle \qquad \ldots (E.22)$$

and is required only in the form $\oint \frac{Bds}{q} Mf_o$, when the first term of (E.22) contributes nothing.

Of the remaining five terms we may write down the last on inspection of (E.20) and again using $\langle \underset{\sim}{c}_\perp \rangle = 0$. It is

$$-\oint \frac{Bds}{q} \langle G^U f_o \rangle = -\oint \frac{ds}{q}\left[\frac{df_o}{dt} + \frac{\partial f_o}{\partial \varepsilon}(q^2 \underset{\sim}{U} \cdot \underset{\sim}{\rho} - \mu B(\underset{\sim}{\nabla} \cdot \underset{\sim}{U})_\perp)\right] \qquad \ldots \text{(E.23)}$$

It is straightforward, using (E.16), to show that

$$\frac{\partial}{\partial s}\frac{df_o}{dt} = 0 \qquad \ldots \text{(E.24)}$$

In fact in terms of α, β, s coordinates

$$\frac{df_o}{dt} \equiv \left(\frac{\partial f_o}{\partial t}\right)_{x_i} - \frac{\partial f_o}{\partial \alpha}\frac{\partial \alpha}{\partial t} - \frac{\partial f_o}{\partial \beta}\frac{\partial \beta}{\partial t} = \left(\frac{\partial f_o}{\partial t}\right)_{\alpha\beta} \qquad \ldots \text{(E.25)}$$

(where (E.5) has been used to replace $\underset{\sim}{U} \cdot \underset{\sim}{\nabla}\alpha$, $\underset{\sim}{U} \cdot \underset{\sim}{\nabla}\beta$); also

$$\begin{aligned}\left(\frac{\partial J}{\partial t}\right)_{\alpha\beta} &= \oint ds \left(\frac{\partial q}{\partial t}\right)_{\alpha\beta s} \\ &= \oint ds \left\{\left(\frac{\partial q}{\partial t}\right)_{x_i} - \frac{\partial q}{\partial \alpha}\frac{\partial \alpha}{\partial t} - \frac{\partial q}{\partial \beta}\frac{\partial \beta}{\partial t} - \frac{\partial q}{\partial s}\frac{\partial s}{\partial t}\right\}\end{aligned} \qquad \ldots \text{(E.26)}$$

In this expression $\frac{\partial \alpha}{\partial t}$, $\frac{\partial \beta}{\partial t}$ are replaced using (E.5) and $\frac{\partial s}{\partial t}$ is found with the aid of (E.16) as follows :

$$\frac{\partial}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial}{\partial t}(\underset{\sim}{e}_1 \cdot \underset{\sim}{\nabla}s) - \frac{\partial \underset{\sim}{e}_1}{\partial t} \cdot \underset{\sim}{\nabla}s = -\underset{\sim}{U} \cdot \underset{\sim}{\rho} - \frac{\partial}{\partial s}(\underset{\sim}{U} \cdot \underset{\sim}{\nabla}s) \qquad \ldots \text{(E.27)}$$

Thus

$$\begin{aligned}\left(\frac{\partial J}{\partial t}\right)_{\alpha\beta} &= \oint ds \left\{\left(\frac{\partial q}{\partial t}\right)_{x_i} + \frac{\partial q}{\partial \alpha}\underset{\sim}{U} \cdot \underset{\sim}{\nabla}\alpha + \frac{\partial q}{\partial \beta}\underset{\sim}{U} \cdot \underset{\sim}{\nabla}\beta + \frac{\partial q}{\partial s}\underset{\sim}{U} \cdot \underset{\sim}{\nabla}s - q\underset{\sim}{U} \cdot \underset{\sim}{\rho}\right\} \\ &= \oint ds \left\{\frac{dq}{dt} - q\,\underset{\sim}{U} \cdot \rho\right\} \\ &= -\oint \frac{ds}{q}\left\{q^2 \underset{\sim}{U} \cdot \underset{\sim}{\rho} - \mu B(\underset{\sim}{\nabla} \cdot \underset{\sim}{U})_\perp\right\}\end{aligned} \qquad \ldots \text{(E.28)}$$

Now on substituting (E.25) and (E.28) in (E.23) and noting that $\oint \frac{ds}{q} = \frac{\partial J}{\partial \varepsilon}$ we obtain the result

$$-\oint \frac{Bds}{q}\langle G^U f_o \rangle = \frac{\partial J}{\partial t}\frac{\partial f_o}{\partial \varepsilon} - \frac{\partial J}{\partial \varepsilon}\frac{\partial f_o}{\partial t}, \qquad \ldots \text{(E.29)}$$

where J and f_o are each to be regarded as functions of $\mu, \varepsilon, \alpha, \beta, t$. In fact $f_o = f_o(\mu, \varepsilon, J, t)$, and when the derivatives with respect to ε, t are taken at constant J, the same equation (E.29) holds.

Returning to (E.22) we evaluate

$$\oint \frac{Bds}{q}\,\sigma \left\langle G^{\Psi} \int_{s_o}^{s} \frac{Bds}{q}\, Lf_o \right\rangle = -\frac{\partial}{\partial \varepsilon} \oint ds\, \frac{\partial \Psi}{\partial s} \int_{s_o}^{s} \frac{Bds}{q}\, Lf_o$$

$$= +\frac{\partial}{\partial \varepsilon} \oint \frac{Bds}{q}\, \Psi\, Lf_o$$

$$= \frac{\partial}{\partial \varepsilon} \oint \frac{ds}{q}\, \Psi\, \underset{\sim}{W}_d \cdot \underset{\sim}{\nabla} f_o + \frac{\partial}{\partial \varepsilon}\, \mu\, \frac{\partial f_o}{\partial \mu} \oint ds\, \Psi\, \frac{\partial}{\partial s}\, \frac{q(\underset{\sim}{e}_1 \cdot \text{curl}\ \underset{\sim}{e}_1)}{B} \quad \ldots \text{(E.30)}$$

and

$$-\oint \frac{Bds}{q}\, L\left(\frac{\partial f_o}{\partial \varepsilon}\,\Psi\right) = -\oint \frac{ds}{q}\, \underset{\sim}{W}_d \cdot \underset{\sim}{\nabla}\left(\frac{\partial f_o}{\partial \varepsilon}\,\Psi\right) + \mu \oint ds\, \frac{(\underset{\sim}{e}_1 \cdot \text{curl}\ \underset{\sim}{e}_1)}{B}\, \frac{\partial}{\partial \mu}\, q\, \frac{\partial}{\partial s}\left(\frac{\partial f_o}{\partial \varepsilon}\,\Psi\right) \quad \ldots \text{(E.31)}$$

The evaluation of Mf_o will be completed when we have

$$-\oint \frac{Bds}{q}\left\{ \left\langle G^{\Psi} \int Df_o \right\rangle + \left\langle D \int G^{\Psi} f_o \right\rangle \right\}$$

Now G^{Ψ} is of the form,

$$G^{\Psi} = G_o + \cos\varphi\, G_c + \sin\varphi\, G_s + (g_c \cos\varphi + g_s \sin\varphi)\, \frac{\partial}{\partial \varphi} \quad \ldots \text{(E.32)}$$

Hence using (C.16) we find

$$\left\langle G^{\Psi} \int Df_o \right\rangle = \tfrac{1}{2}(G_s D_c - G_c D_s) f_o + \tfrac{1}{2}(g_c D_c + g_s D_s) f_o$$

$$\ldots \text{(E.33)}$$

$$\left\langle D \int G^{\Psi} f_o \right\rangle = \tfrac{1}{2}(D_s G_c - D_c G_s) f_o + \tfrac{1}{2}(A_c G_c + A_s G_s) f_o$$

where

$$G_c = -\frac{c_\perp}{B}\frac{\partial \Psi}{\partial x}\left(\frac{1}{B}\frac{\partial}{\partial \mu} + \frac{\partial}{\partial \varepsilon}\right) \quad ; \quad G_s = -\frac{c_\perp}{B}\frac{\partial \Psi}{\partial y}\left(\frac{1}{B}\frac{\partial}{\partial \mu} + \frac{\partial}{\partial \varepsilon}\right)$$

$$\ldots \text{(E.34)}$$

$$g_c = -\frac{1}{2\mu B^2}\frac{\partial \Psi}{\partial y} \quad ; \quad g_s = +\frac{1}{2\mu B}\frac{\partial \Psi}{\partial x}$$

Substituting (C.17) and (E.34) into (E.33) gives the result

$$-\oint \frac{Bds}{q}\left\{ \left\langle G^{\Psi} \int Df_o \right\rangle + \left\langle D \int G^{\Psi} f_o \right\rangle \right\} = \oint \frac{ds}{q}\left\{ \frac{1}{B}\left(\frac{\partial f_o}{\partial x}\frac{\partial \Psi}{\partial y} - \frac{\partial f_o}{\partial y}\frac{\partial \Psi}{\partial x}\right) \right.$$

$$+ \mu\, (\underset{\sim}{e}_1 \cdot \text{curl}\ \underset{\sim}{e}_1)\, \frac{\partial \Psi}{\partial s}\left(\frac{1}{B}\frac{\partial f_o}{\partial \mu} + \frac{\partial f_o}{\partial \varepsilon}\right) \quad \ldots \text{(E.35)}$$

$$\left. + \frac{\partial f_o}{\partial \varepsilon}\left(V_x \frac{\partial \Psi}{\partial y} - V_y \frac{\partial \Psi}{\partial x}\right) \right\}$$

Since the first terms of (E.22) integrates out under the $\oint \frac{Bds}{q}$ we have from (E.29), (E.30), (E.31) and (E.35)

$$\oint \frac{Bds}{q}\, Mf_o = \left(\frac{\partial J}{\partial t}\frac{\partial f_o}{\partial \varepsilon} - \frac{\partial J}{\partial \varepsilon}\frac{\partial f_o}{\partial t}\right) + \frac{\partial}{\partial \varepsilon} \oint \frac{ds}{q}\, \Psi\, \underset{\sim}{W}_d \cdot \underset{\sim}{\nabla} f_o$$

$$\ldots \text{(E.36)}$$

$$-\oint \frac{ds}{q}\, \underset{\sim}{W}_d \cdot \underset{\sim}{\nabla}\left(\frac{\partial f_o}{\partial \varepsilon}\,\Psi\right) + \frac{\partial f_o}{\partial \varepsilon} \oint \frac{ds}{q}\, \underset{\sim}{W}_d \cdot \underset{\sim}{\nabla}\Psi - \oint \frac{ds}{q}\, \frac{1}{B}\, \underset{\sim}{e}_1 \cdot \underset{\sim}{\nabla}\Psi \times \underset{\sim}{\nabla} f_o \quad .$$

We finally simplify this expression by noting that

$$\frac{\partial f_o}{\partial\alpha}\frac{\partial}{\partial\beta}\oint\Psi\frac{ds}{q}-\frac{\partial f_o}{\partial\beta}\frac{\partial}{\partial\alpha}\oint\Psi\frac{ds}{q}=\frac{\partial f_o}{\partial\alpha}\frac{\partial}{\partial\varepsilon}\frac{\partial}{\partial\beta_\perp}\oint\Psi\, q\, ds-\frac{\partial f_o}{\partial\beta}\frac{\partial}{\partial\varepsilon}\frac{\partial}{\partial\alpha_\perp}\oint\Psi\, q\, ds \qquad \ldots\text{(E.37)}$$

$$=-\oint\frac{ds}{q}\frac{\underset{\sim}{e}_1\times\underset{\sim}{\nabla}\Psi\cdot\nabla f_o}{B}+\frac{\partial}{\partial\varepsilon}\oint\Psi\frac{ds}{q}\underset{\sim}{W}_d\cdot\underset{\sim}{\nabla}f_o-\oint\Psi\frac{ds}{q}W_d\cdot\underset{\sim}{\nabla}\left(\frac{\partial f_o}{\partial\varepsilon}\right)$$

Consequently the final form is

$$\oint\frac{Bds}{q}Mf_o=\left(\frac{\partial J}{\partial t}\frac{\partial f_o}{\partial\varepsilon}-\frac{\partial J}{\partial\varepsilon}\frac{\partial f_o}{\partial t}\right)+\left(\frac{\partial f_o}{\partial\alpha}\frac{\partial\overline{\Psi}}{\partial\beta}-\frac{\partial f_o}{\partial\beta}\frac{\partial\overline{\Psi}}{\partial\alpha}\right), \qquad \ldots\text{(E.38)}$$

where

$$\overline{\Psi}=\oint\Psi\frac{ds}{q}\ .$$

When this last expression is integrated around a closed drift surface J = constant, the second part contributes nothing to the integral and leaves us with the result quoted in equation (7.20).

APPENDIX F

Calculation of μ_2

In this appendix we derive an expression for μ_2 from the recursion formula (6.5)

$$\mu_{n+1} = \int^{\varphi} D\mu_n - \sigma \int_{s_0}^{s} \frac{Bds}{q} L\mu_n + \int^{\alpha} d\alpha / \frac{\partial J}{\partial \beta} \, H\mu_n \qquad \text{... (F.1)}$$

The third term of this equation is independent of the coordinate s. Consequently this term remains constant as a particle moves along its field line. In a region of straight and constant magnetic field however $\hat{\mu} \equiv \mu_0$ so that all the s-independent terms of each μ_n must vanish.

That is

$$H\mu_n = 0 \qquad \text{... (F.2)}$$

This result has been proved explicitly in Appendix D for $H\mu_0$.

Thus the simplified formula for μ_{n+1} is

$$\mu_{n+1} = \int^{\varphi} D\mu_n - \sigma \int_{s}^{s} \frac{Bds}{q} L\mu_n \qquad \text{... (F.3)}$$

Double application of (F.3) gives μ_2 in terms of μ_0 :

$$\mu_2 = \int^{\varphi} D \int^{\varphi} D\mu_0 - \sigma\int^{\varphi} D \frac{\mu_0 q}{B} \left(\underset{\sim}{e}_1 \cdot \text{curl } \underset{\sim}{e}_1\right) + \int_{s_0}^{s} \frac{Bds}{q} \left\{ \sigma L \int_{s_0}^{s} \frac{Bds}{q} L\mu_0 - \left\langle D\!\int\! D\!\int\! D\mu_0 \right\rangle \right\} \qquad \text{... (F.4)}$$

where (D.5) has been used to evaluate $L\mu_0$. Now using (C.16) we have

$$\int^{\varphi} D \int^{\varphi} D\mu_0 - \sigma \int^{\varphi} D \frac{\mu_0 q}{B} \left(\underset{\sim}{e}_1 \cdot \text{ curl } \underset{\sim}{e}_1\right) = \sum_{n=1}^{4} (c_n \cos n\varphi + s_n \sin n\varphi) \qquad \text{... (F.5)}$$

where

$$c_1 = \left\{ -(D_0 D_c + A_0 D_s) + \frac{1}{2}\left[\frac{1}{2} D_c D_{2c} + D_{2s} D_s - A_c D_{2s} - A_{2s} D_c - \frac{1}{2} D_s D_{2s} + D_{2c} D_c + A_s D_{2c} + A_{2c} D_s\right]\right\} \mu_0$$

$$+ D_s \left(\frac{\mu_0 \sigma q}{B} \underset{\sim}{e}_1 \cdot \text{curl } \underset{\sim}{e}_1 \right) \qquad \text{... (F.6)}$$

$$s_1 = \left\{ (-D_0 D_s + A_0 D_c) + \frac{1}{2}\left[\frac{1}{2} D_s D_{2c} + D_{2s} D_c + A_s D_{2s} + A_{2s} D_s - \frac{1}{2} D_c D_{2s} - D_{2c} D_s + A_c D_{2c} + A_{2c} D_c \right]\right\} \mu_0$$

$$- D_c \left(\frac{\mu_0 \sigma q}{B} \underset{\sim}{e}_1 \cdot \text{curl } \underset{\sim}{e}_1 \right) \qquad \text{... (F.7)}$$

$$c_2 = -\frac{1}{4}\left[D_0 D_{2c} + D_c D_c - D_s D_s + 2A_0 D_{2s} + A_c D_s + A_s D_c \right] \mu_0 + \frac{1}{2} D_{2s} \left(\frac{\mu_0 \sigma q}{B} \underset{\sim}{e}_1 \cdot \text{curl } \underset{\sim}{e}_1 \right) \qquad \text{... (F.8)}$$

$$s_2 = -\frac{1}{4}\left[D_oD_{2s} + D_cD_s + D_sD_c - 2A_oD_{2c} - A_cD_c + A_sD_s\right]\mu_o - \frac{1}{2}D_{2c}\left(\frac{\mu_o\sigma q}{B}\,\underset{\sim}{e}_1 \cdot \text{curl}\,\underset{\sim}{e}_1\right) \quad \ldots \text{(F.9)}$$

$$c_3 = -\frac{1}{6}\left[-\frac{1}{2}D_sD_{2s} + D_{2c}D_c + A_sD_{2c} + A_{2c}D_s + \frac{1}{2}D_cD_{2c} - D_{2s}D_s + A_cD_{2s} + A_{2s}D_c\right]\mu_o \quad \ldots \text{(F.10)}$$

$$s_3 = -\frac{1}{6}\left[\frac{1}{2}D_sD_{2c} + D_{2s}D_c + A_sD_{2s} + A_{2s}D_s + \frac{1}{2}D_cD_{2s} + D_{2c}D_s - A_cD_{2c} - A_{2c}D_c\right]\mu_o \quad \ldots \text{(F.11)}$$

$$c_4 = -\frac{1}{16}\left[D_{2c}D_{2c} - D_{2s}D_{2s} + 2A_{2s}D_{2c} + 2A_{2c}D_{2s}\right]\mu_o \quad \ldots \text{(F.12)}$$

$$s_4 = -\frac{1}{16}\left[D_{2c}D_{2s} + D_{2s}D_{2c} - 2A_{2c}D_{2c} + 2A_{2s}D_{2s}\right]\mu_o \quad \ldots \text{(F.13)}$$

The last term of (F.4) is independent of φ, and has effectively been calculated already in obtaining Hf_o (Appendix D). The evaluation of this term is therefore carried out for $\tau_2 = \tau_3$ and consists of collecting the s derivatives in (D.24), (D.35), (D.37), (D.66), (D.67) and (D.70), summing the remaining terms to get (D.71) and (D.73) and integrating $\int^s \frac{Bds}{q}$, with f_o replaced by μ_o throughout.

Denoting this term by c_o, we find,

$$\sigma c_o = -\frac{1}{8}\mu\frac{(\rho_3 - \rho_2)^2}{B^2}(q^2 - \mu B) - \frac{1}{4}\frac{(V_x^2 + V_y^2)}{B} - \frac{1}{4}\frac{q^4}{B^3}(\sigma_1^2 + \rho_1^2) + \frac{1}{4}\frac{q^2\mu B}{B^3}(\underset{\sim}{e}_1 \cdot \text{curl}\,\underset{\sim}{e}_1)^2$$

$$+ \frac{q^2\mu B}{B^3}\left\{\underset{\sim}{\nabla}\cdot\underset{\sim}{\rho} + \frac{1}{2}\underset{\sim}{\rho}\cdot\left(6\rho - 5\frac{\underset{\sim}{\nabla}B}{B}\right) - \frac{1}{4}(\underset{\sim}{e}_1 \cdot \text{curl}\,\underset{\sim}{e}_1)^2\right\} + \frac{(\mu B)^2}{B^3}\left\{\frac{3}{4}\,\text{div}\left(\frac{1}{B}\underset{\sim}{\nabla}B\right) + \frac{3}{4}\frac{\partial}{\partial s}(\rho_2 + \rho_3)\right.$$

$$\left. + \frac{3}{4}(\rho_2 + \rho_3)^2 + \frac{3}{4}\underset{\sim}{\rho}\cdot\frac{\underset{\sim}{\nabla}B}{B} + \frac{1}{2}(\rho_2\rho_3 + \tau_2\tau_3) + \frac{1}{4}(\underset{\sim}{e}_1 \cdot \text{curl}\,\underset{\sim}{e}_1)^2 - \frac{5}{4}\left[\left(\frac{1}{B}\frac{\partial B}{\partial x}\right)^2 + \left(\frac{1}{B}\frac{\partial B}{\partial y}\right)^2\right]\right\} \quad \ldots \text{(F.14)}$$

Returning to Appendix B, it is possible to show from (B.20) and (B.21) that the quantities $\left\{(\rho_3 - \rho_2)^2 + (\tau_2 - \tau_3)\right\}$ and $(\rho_2\rho_3 + \tau_2\tau_3)$ are invariant under rotations of the $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ axes. Consequently if we express (F.14) entirely in terms of such invariant quantities it is clear that the form of the first term in an arbitrary coordinate system must be

$$-\frac{1}{8}\frac{\mu}{B^2}(q^2 - \mu B)\left\{(\rho_3 - \rho_2)^2 + (\tau_2 - \tau_3)^2\right\}$$

The final form for μ_2 is therefore

$$\mu_2 = \sum_{n=0}^{4}(c_n \cos n\varphi + s_n \sin n\varphi) \quad \ldots \text{(F.15)}$$

where the c's and s's are given by (F.6) - (F.14). For the case of a vacuum field these quantities are given in equations (6.8) - (6.16).

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