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Report

HIGH FREQUENCY ELECTROSTATIC WAVES IN A NON-UNIFORM PLASMA IN THE PRESENCE OF A MAGNETIC FIELD

C. N. LASHMORE-DAVIES

Culham Laboratory
Abingdon Berkshire

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HIGH FREQUENCY ELECTROSTATIC WAVES IN A NON-UNIFORM PLASMA IN THE PRESENCE OF A MAGNETIC FIELD

by

C.N. Lashmore-Davies

ABSTRACT

High frequency electrostatic waves which only propagate in the presence of a density gradient and a magnetic field are described. The analysis is made in cylindrical geometry and both conducting wall and vacuum boundary conditions are considered. When the wave velocity has a component along the steady magnetic field there are two branches of the dispersion relation. The two branches are compared with the two branches of the low frequency drift waves. Landau damping is shown to be negligible for these high frequency waves provided the wavelength along the magnetic field is much greater than the Debye length, but collisional damping of the waves is described. Finally, in two appendices, the use of the fluid approximation and the electrostatic approximation are justified.

U.K.A.E.A. Research Group,
Culham Laboratory,
Abingdon,
Berks.

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APPENDICES

List of Symbols used

$\underset{\sim}{E}$	–	electric field
$\underset{\sim}{H}$	–	magnetic field
$\underset{\sim}{B}$	–	magnetic induction
$\underset{\sim}{A}$	–	vector potential
φ	–	scalar potential
$\underset{\sim}{J}$	–	current density
$\underset{\sim}{v}$	–	average velocity of electrons
$\underset{\sim}{u}$	–	velocity of an electron
n	–	number density
m	–	mass of electron
$-e$	–	charge of electron
κ	–	Boltzmann's constant
T	–	electron temperature
$v_T = \sqrt{\kappa T/m}$	–	mean electron thermal velocity
ν	–	collision frequency
ω	–	wave frequency
ω_{ci}	–	ion cyclotron frequency
ω_{pi}	–	ion plasma frequency
ω_{ce}	–	electron cyclotron frequency
ω_{pe}	–	electron plasma frequency
ℓ	–	mode number
k	–	axial wave vector
a	–	plasma radius
ε_o	–	dielectric constant of free space
μ_o	–	magnetic permeability of free space
c	–	velocity of light in free space
c_S	–	ion sound velocity
c_A	–	Alfvén velocity
β	–	ratio of plasma pressure to magnetic pressure
r	–	radial coordinate
θ	–	azimuthal coordinate
z	–	axial coordinate
t	–	time

1. INTRODUCTION

The study of wave phenomena in non-uniform plasmas and in the presence of a steady magnetic field has attained great importance since the discovery of the low frequency ($\omega \ll \omega_{ci}$) drift waves[1]. Such waves can only propagate in a non-uniform plasma and become unstable under a wide range of physical conditions[2].

This report describes the modifications which drift waves undergo when $\omega \gg \omega_{ci} > \omega_{pi}$ and only the electrons can respond to the rapid variations of the wave fields. In the special case when the plasma density is uniform the wave becomes that described by Trivelpiece and Gould[9] for $\omega_{ce} > \omega_{pe}$. However, in contrast with the latter, the present wave retains its electrostatic character in the limit of propagation perpendicular to $\underset{\sim}{B}_o$.

The aim of this work is to provide a theory for comparison with an experimental study of the high frequency limit of drift waves in the frequency range where ion motion can be neglected. We consider a cylindrical column of plasma of infinite length and with a radial density gradient. The dispersion equation is first obtained which together with the boundary conditions specify the dispersion relation for the wave. The propagation and damping of the wave are then described.

2. THE DISPERSION EQUATION

We consider a frequency range ($\omega_{ci} \ll \omega \ll \omega_{ce}$) such that the motion of the ions can be neglected. The ions are simply assumed to provide a background of positive charge so that in equilibrium the plasma is neutral and there are no electric fields. The motion of the electrons is described by the electron fluid equation with constant temperature. The use of the fluid equation is justified in Appendix 1 where it is shown that the effect of Landau damping is negligible due to the high phase velocity of the waves parallel to $\underset{\sim}{B}_o$. It is also shown that the fluid description is valid so long as the wavelength along the magnetic field is much larger than the electron Debye length.

The equations needed for a complete description of the wave are the following:

$$\frac{dv}{dt} + \frac{\kappa T}{nm} \underset{\sim}{\nabla} n + \nu \underset{\sim}{v} = -\frac{e}{m} \underset{\sim}{E} - \frac{e}{m} \underset{\sim}{v} \times \underset{\sim}{B} , \qquad \ldots (1)$$

$$-e \frac{\partial n}{\partial t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{J} = 0 , \qquad \ldots (2)$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \frac{-e}{\varepsilon_o} (n - n_o) , \qquad \ldots (3)$$

where equations (1), (2) and (3) are respectively, the electron fluid equation, the continuity equation and Poisson's equation. As shown in Appendix 2, it is justified under the conditions we consider to make the electrostatic approximation and assume that the electric field is derivable from a scalar potential φ,

$$\underset{\sim}{E} = - \underset{\sim}{\nabla} \varphi \, . \qquad \ldots (4)$$

We choose a system of coordinates in which $\underset{\sim}{B}_o$ is parallel to the z-axis and consider a plasma with cylindrical symmetry. Perturbations of quantities about their equilibrium values are assumed to have the following form

$$F = f(r) \exp\{i(kz + \ell\theta - \omega t)\} \, . \qquad \ldots (5)$$

We may write the relevant variables of the problem as a sum of a constant part, suffix 0 and a perturbation, suffix 1:

$$\underset{\sim}{v} = \underset{\sim}{v}_o + \underset{\sim}{v}_1 \, ,$$

where $\underset{\sim}{v}_o = \frac{-\kappa T}{eB_o} \, \frac{1}{n_o} \, \frac{dn_o}{dr} \, \hat{\iota}_\theta$ (the diamagnetic drift velocity) and $\hat{\iota}_\theta$ is the unit vector in the θ-direction

$$n = n_o + n_1 \, ,$$

$$\underset{\sim}{E} = \underset{\sim}{E}_1 \, ,$$

$$\underset{\sim}{B} = \underset{\sim}{B}_o \, ,$$

The last equation follows because of equation (4). In order to obtain the dispersion equation we express n and $\underset{\sim}{v}$ in terms of $\underset{\sim}{E}$ from equations (1) and (3) and then substitute these expressions into equation (2). Taking the vector product of both sides of equation (1) with $\underset{\sim}{B}_o$ and neglecting terms of order ν/ω_{ce} and ω/ω_{ce}, we obtain the component of $\underset{\sim}{v}$ perpendicular to $\underset{\sim}{B}_o$

$$\underset{\sim}{v}_\perp = \frac{\underset{\sim}{E} \times \underset{\sim}{B}_o}{B_o^2} + \frac{\kappa T}{neB_o^2} \, \underset{\sim}{\nabla} n \times \underset{\sim}{B}_o \, . \qquad \ldots (6)$$

Similarly, taking the scalar product of both sides of equation (1) with $\underset{\sim}{B}_o$, we obtain

$$v_z = \left(\frac{e}{m} E_z + \frac{\kappa T}{nm} ikn_1\right) / i\omega \left(1 + \frac{i\nu}{\omega}\right) . \qquad \ldots (7)$$

From equations (6) and (7) and the definition of current density $\underset{\sim}{J} = -en\underset{\sim}{v}$ we can obtain the current densities parallel and perpendicular to $\underset{\sim}{B}_o$. At this point we also make the linearization approximation assuming that all perturbed quantities are small and that we may neglect second order terms (and higher) in these quantities. Thus we obtain

$$\underset{\sim}{J}_{1\perp} = en_o \frac{(\underset{\sim}{\nabla} \varphi \times \underset{\sim}{B}_o)}{B_o^2} - \frac{\kappa T}{B_o^2} \, \underset{\sim}{\nabla} n_1 \times \underset{\sim}{B}_o \, , \qquad \ldots (8)$$

and

$$J_{1z} = \left(\frac{n_o e^2}{m}\, ik\varphi - \frac{\kappa T}{m}\, eikn_1\right) / i\omega \left(1 + \frac{i\nu}{\omega}\right), \qquad \ldots (9)$$

where we have also used equation (4) and the fact that n_o depends only on r, the radial coordinate. Substituting equations (8) and (9) into equation (2),

$$- i\omega n_1 \quad - \underset{\sim}{\nabla} \cdot \left(\frac{n_o \underset{\sim}{\nabla}\varphi \times \underset{\sim}{B}_o}{B_o^2}\right) + \underset{\sim}{\nabla} \cdot \left(\frac{\kappa T}{eB_o^2}\, \underset{\sim}{\nabla} n_1 \times \underset{\sim}{B}_o\right),$$

$$+ ik \left(\frac{-n_o e}{m}\, k\varphi + \frac{\kappa T}{m}\, kn_1\right) / \omega \left(1 + \frac{i\nu}{\omega}\right) = 0 . \qquad \ldots (10)$$

From equation (3) we have the relation,

$$n_1 = \frac{\varepsilon_o}{e}\, \nabla^2 \varphi . \qquad \ldots (11)$$

Substituting this into equation (10), using the fact that $\underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla}\psi \times \underset{\sim}{V}) \equiv 0$, where ψ is a scalar and $\underset{\sim}{V}$ a constant vector, we obtain

$$\nabla^2\varphi + \frac{\left(1 + \frac{i\nu}{\omega}\right)}{\left(1 + \frac{i\nu}{\omega} - \frac{k^2 v_T^2}{\omega^2}\right)}\, \frac{e\ell}{\omega\varepsilon_o B_o}\, \frac{1}{r}\, \frac{dn_o}{dr}\, \varphi + \frac{1}{\left(1 + \frac{i\nu}{\omega} - \frac{k^2 v_T^2}{\omega^2}\right)}\, \frac{k^2 n_o e^2}{\varepsilon_o m\omega^2}\, \varphi = 0 . \qquad \ldots (12)$$

This is the required dispersion equation. We now assume a specific form for the variation of n_o with r, i.e.

$$n_o(r) = N_o(1 - x\, r^2/a^2) = N_o f(\tfrac{r}{a}) , \qquad \ldots (13)$$

where x is an arbitrary dimensionless parameter whose value lies between 0 and 1. With this form for n_o equation (12) can be written

$$\nabla^2\varphi - \frac{\left(1 + \frac{i\nu}{\omega}\right)}{\left(1 + \frac{i\nu}{\omega} - \frac{k^2 v_T^2}{\omega^2}\right)}\, \frac{2\ell x}{a^2}\, \frac{\omega_p^2}{\omega|\omega_{ce}|}\, \varphi + \frac{k^2\omega_p^2\, f(r/a)}{\omega^2\left(1 + \frac{i\nu}{\omega} - \frac{k^2 v_T^2}{\omega^2}\right)}\, \varphi = 0 , \qquad \ldots (14)$$

where ω_p is the electron plasma frequency referred to the centre of the plasma cylinder. Equation (14) can be solved analytically only in certain special cases e.g. for propagation perpendicularly to $\underset{\sim}{B}_o$. In other circumstances (14) may be solved numerically and for this purpose it is convenient to transform equation (14) by introducing the dimensionless variable ξ where $\xi a = r$. We then obtain

$$\nabla^2\varphi - \frac{\left(1 + \frac{i\nu}{\omega}\right)}{\left(1 + \frac{i\nu}{\omega} - \frac{k^2 v_T^2}{\omega^2}\right)}\, 2\ell x\, \frac{\omega_p^2}{\omega|\omega_{ce}|}\, \varphi + \frac{k^2 a^2 \omega_p^2\, f(\xi)}{\omega^2\left(1 + \frac{i\nu}{\omega} - \frac{k^2 v_T^2}{\omega^2}\right)}\, \varphi = 0 , \qquad \ldots (15)$$

where φ is now a function of ξ and

$$\nabla^2 \equiv \frac{1}{\xi}\frac{d}{d\xi}\left(\xi \frac{d}{d\xi}\right) - \frac{\ell^2}{\xi^2} - k^2a^2 .$$

Equation (15) is the final form of the dispersion equation in terms of the variable ξ .

3. THE BOUNDARY CONDITIONS

Plasma Bounded by a Perfectly Conducting Wall

In this case the requirement that the electric fields parallel to the wall should vanish at the boundary gives the condition

$$\varphi(\xi) = 0 \quad \text{at} \quad \xi = 1 \ . \qquad \ldots (16)$$

Plasma Bounded by a Vacuum

In this case the boundary conditions depend on whether or not there is a surface charge: they are[3]

$$E_z \ \text{and} \ E_\theta \ \text{continuous} ,$$

and

$$(E_r)_{vac.} - (E_r)_p = \frac{\sigma}{\varepsilon_o} ,$$

where σ is the surface charge density at the boundary and $(E_r)_{vac.}$ and $(E_r)_p$ are the radial components of the electric field in the vacuum and plasma respectively. The surface charge density σ is obtained in terms of φ by first expressing n_1 in terms of φ from equation (10) and then integrating this over a small volume at the surface (Stratton[3]). The vacuum boundary conditions are then expressed in terms of φ as

$$\varphi_{vac.} = \varphi_p \quad \text{at} \quad \xi = 1 , \qquad \ldots (18a)$$

$$\frac{d\varphi}{d\xi}{}_{vac.}\bigg|_{\xi=1} - \frac{d\varphi_p}{d\xi}\bigg|_{\xi=1} = \frac{\ell\omega_p^2}{\omega|\omega_{ce}|} \frac{(1-x)\left(1+\frac{i\nu}{\omega}\right)}{\left(1+\frac{i\nu}{\omega}-\frac{k^2v_T^2}{\omega^2}\right)} \varphi(1) \ . \qquad \ldots (18b)$$

There is only a surface charge density if the equilibrium number density becomes zero discontinuously at the boundary, i.e. $x \neq 1$. When $x = 1$ the surface charge vanishes and equation (18b) shows that the radial electric field is also continuous across the boundary.

In order to specify the problem completely we must also obtain the solution to equation (15) in vacuum. This equation then reduces to Bessel's equation with imaginary argument and the solution can be written

$$\varphi = B\, K_\ell\, (ka\xi) , \qquad \ldots (19)$$

where K_ℓ is the modified Bessel function of the second kind of order ℓ and we have rejected the other solution since it diverges at infinity. With the aid of (19) the two conditions (18) can be expressed as one condition:

$$\frac{d\varphi_p}{d\xi}\Bigg|_{\xi=1} = \left\{ -|\ell| - ka \frac{K_{\ell-1}(ka)}{K_\ell(ka)} - \frac{\ell\omega_p^2}{\omega|\omega_{ce}|} \frac{(1-x)\left(1+\frac{i\nu}{\omega}\right)}{\left(1+\frac{i\nu}{\omega}-\frac{k^2v_T^2}{\omega^2}\right)} \right\} \varphi(1) \ . \quad \ldots (20)$$

This is the final form for the general boundary condition when the plasma is bounded by a vacuum. The solution of equation (15) which satisfies the boundary conditions given by either equation (16) or (20) yields the dispersion relation between ω and k.

4. THE EIGEN-FREQUENCIES

$\underset{\sim}{k} \cdot \underset{\sim}{B}_o = 0$

This case corresponds to a wave propagating at right angles to the magnetic field and hence no z variation of the variables. This case has already been mentioned as one for which equation (15) can be solved analytically. The equation becomes

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\varphi}{d\xi}\right) - \frac{\ell^2}{\xi^2}\varphi - 2\ell x \frac{\omega_p^2}{\omega|\omega_{ce}|}\varphi = 0 \ , \quad \ldots (21)$$

which will be recognised as Bessel's equation. There is no solution to equation (21) which satisfies either of the boundary conditions (16) or (20) for $\ell \geqslant 0$. Thus, choosing $\ell < 0$ in equation (21), the solution which is finite at the origin may be written

$$\varphi = A\, J_\ell(p\xi) \ ,$$

where $p^2 = 2\ell x\omega_p^2/\omega|\omega_{ce}|$ and J_ℓ is the ℓ^{th} order Bessel function of the first kind. If $z_{\ell s}$ is the s^{th} zero of the ℓ^{th} order Bessel function then the eigen-frequencies for conducting wall boundaries are given by:

$$\omega = \frac{2\ell x\ \omega_p^2}{z_{\ell s}^2\ |\omega_{ce}|} \ , \quad \ldots (23)$$

and are independent of the electron mass.

We notice immediately from (23) that the high frequency electrostatic wave propagating at right angles to $\underset{\sim}{B}_o$ is independent of the temperature. This is in contrast to the low frequency drift waves propagating perpendicularly to $\underset{\sim}{B}_o$ where the phase velocity is proportional to the electron temperature(2). The phase velocity ω/ℓ of the high frequency electrostatic wave is of the opposite sign to the diamagnetic drift velocity and the ratio of these two quantities is

$$|v_p/v_o| = \omega_p^2\, a^2/z_{\ell s}^2\, v_T^2 \,. \qquad \ldots (24)$$

We note that $v_{phase} \gg v_o$ for the range of parameters for which the electrostatic approximation is valid (see Appendix 2). This again contrasts with the low frequency drift wave where the wave propagates at the diamagnetic drift velocity. The much higher phase velocity of the high frequency wave compared with the low frequency drift wave is evidently due to the fact that a much larger space charge can build up in the former case because of the inability of the ions to respond to the electric field of the high frequency wave.

The discussion so far of the eigen-frequencies with $\underset{\sim}{k} \cdot \underset{\sim}{B}_o = 0$ has been for conducting wall boundaries. For a plasma bounded by a vacuum we obtain a similar result to (23) except that $z_{\ell s}$ is replaced by another quantity, say $y_{\ell s}$ which corresponds to the condition (20). The quantities $y_{\ell s}$ have been obtained graphically. In Fig.1 ω/ω' is plotted for various values of the mode number ℓ, where ω and ω' are the eigen-frequencies for vacuum and conducting wall boundary conditions respectively. It will be seen that the effect of the boundaries is quite marked, the frequency for the former case being approximately 5 times larger than for the latter. Fig.2 shows the dependence of ω and ω' on the plasma profile. ω' increases linearly with x since this corresponds to an increase of the density gradient. However, ω decreases with x which means that the effect of the wall charge is opposite to that of the density gradient and more than compensates for the reduction in the density gradient (N.B. For the form of profile chosen the density gradient is proportional to x and the wall charge to $1 - x$).

$\underset{\sim}{k} \cdot \underset{\sim}{B}_o \neq 0$

This condition corresponds to waves propagating obliquely to the magnetic field $\underset{\sim}{B}_o$. For this more general case the dispersion equation (15) can still be solved analytically but only for special density profiles and only for one frequency in each case. These solutions are useful for checking numerical solutions however.

The eigen-frequencies were obtained numerically using a programme written by McNamara[4]. The method employed was to express the differential equation and the boundary conditions as a homogeneous set of linear algebraic equations (using the method of finite differences). Setting the determinant of this set of equations equal to zero specifies a function whose complex zeros approximate the required eigen-values.

Figs.3 and 4 show the results of computations for the case when collisions are neglected and thus the frequencies are all real. Fig.3 corresponds to vacuum boundary conditions

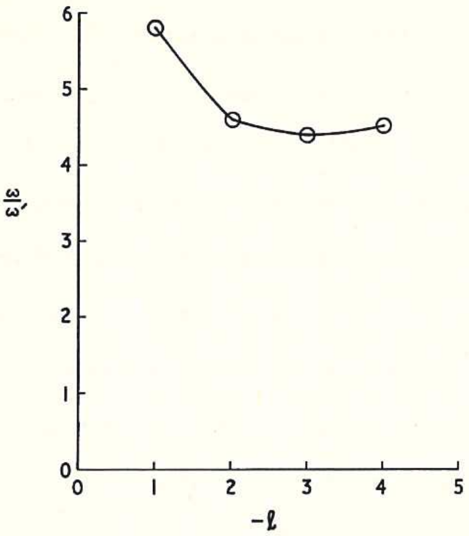

Fig.1 (CLM-R75)
Ratio of eigen frequencies of high frequency electrostatic wave for $\underline{k}.\,\underline{B}_o = 0$ v mode number where ω is the frequency for vacuum boundary conditions and ω' the frequency for conducting wall boundary conditions.

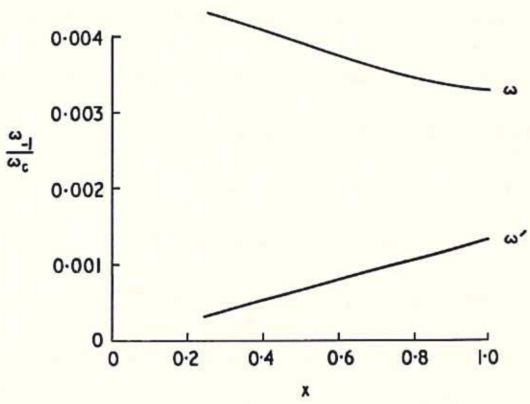

Fig.2 (CLM-R75)
Dependence of eigen frequencies of high frequency electrostatic wave on x for $\underline{k}.\,\underline{B}_o = 0$ and $\ell = -1$. ω and ω' have the same significance as for Fig.1 and ω_c is the electron cyclotron frequency. x is defined in equation (13) of the text.

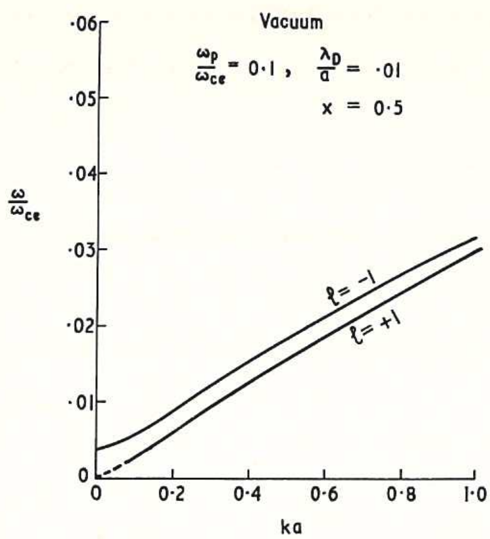

Fig.3 (CLM-R75)
Computed dispersion relation for high frequency electrostatic wave for $\underline{k}.\,\underline{B}_o \neq 0$ and vacuum boundary conditions.

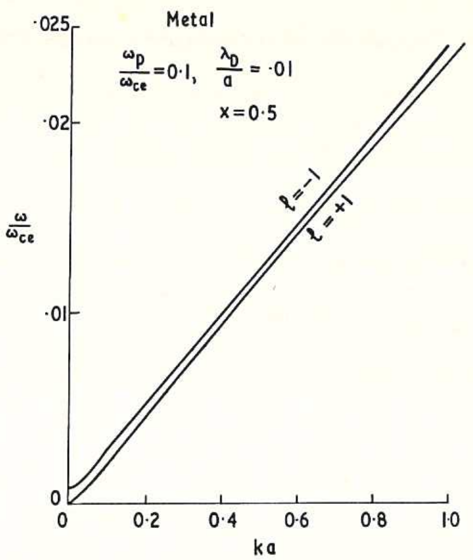

Fig.4 (CLM-R75)
Computed dispersion relation for high frequency electrostatic wave for $\underline{k}.\,\underline{B}_o \neq 0$ and conducting wall boundary conditions.

and $|\ell| = 1$. There are now two branches, the frequency of the lower one tending to zero with ka and that of the upper one to the value corresponding to the $\underset{\sim}{k} \cdot \underset{\sim}{B}_o = 0$ case. The appearance of two branches for finite ka is analogous to the low frequency drift wave case (for $c_S \ll c_A$) except that the phase velocities perpendicular to $\underset{\sim}{B}_o$ are reversed. This is seen by the fact that the lower branch corresponds to $\ell = +1$ and to the azimuthal phase velocity having the same direction of rotation as the diamagnetic drift velocity. For the low frequency drift waves it is the upper branch whose component of phase velocity in the direction perpendicular to $\underset{\sim}{B}_o$ has the same sign as the diamagnetic drift-velocity. (N.B. The results for the lower branch of Fig.3 are not valid for ka < 0·01 since we have assumed $\omega \gg \omega_{ci}$). Fig.4 is the corresponding curve for conducting wall boundary conditions. Both Figs.3 and 4 show that the phase velocity of high frequency electrostatic waves along the magnetic field is much greater than the mean electron thermal velocity ($\omega/k \gtrsim 20\, v_T$).

5. DAMPING OF THE HIGH FREQUENCY WAVE

Since the motions of the electrons parallel and perpendicular to the magnetic field are coupled, a damping of either of these motions should cause damping of the wave. As a first approximation only damping of the axial motion has been taken into account. The damping of the motion transverse to $\underset{\sim}{B}_o$ has been neglected. This is justified since the first is of order ν/ω and the second ν/ω_{ce}. Fig.5 shows the damping of the upper branch of the high frequency electrostatic waves for three different values of the collision frequency. For very small values of ka the damping depends inversely on the collision frequency, whereas for larger values of ka the damping increases with the collision frequency. For small values of ka, the effect of collisions is to decouple the electric field parallel to $\underset{\sim}{B}_o$ so that the wave still propagates at right angles to $\underset{\sim}{B}_o$ and the eigen-frequency corresponds to the $\underset{\sim}{k} \cdot \underset{\sim}{B}_o = 0$ case. Therefore, in this region the damping of the wave will be governed by terms $\sim \nu/\omega_{ce}$ which have so far been neglected. For small values of ν/ω_{ce} and ka, we may assume that the vacuum solution is close to the case considered in 4 giving

$$\frac{\omega}{|\omega_{ce}|} \simeq \frac{\omega_p^2}{\omega_{ce}^2} \frac{1}{y_{-11}^2}\left(1 - \frac{i\nu}{\omega_{ce}} \frac{9}{4}\right) \qquad \ldots (25)$$

for the $\ell = -1$ mode.

The damping of the lower branch of the high frequency electrostatic waves is shown in Fig.6. This branch is always more heavily damped. This is not surprising since the

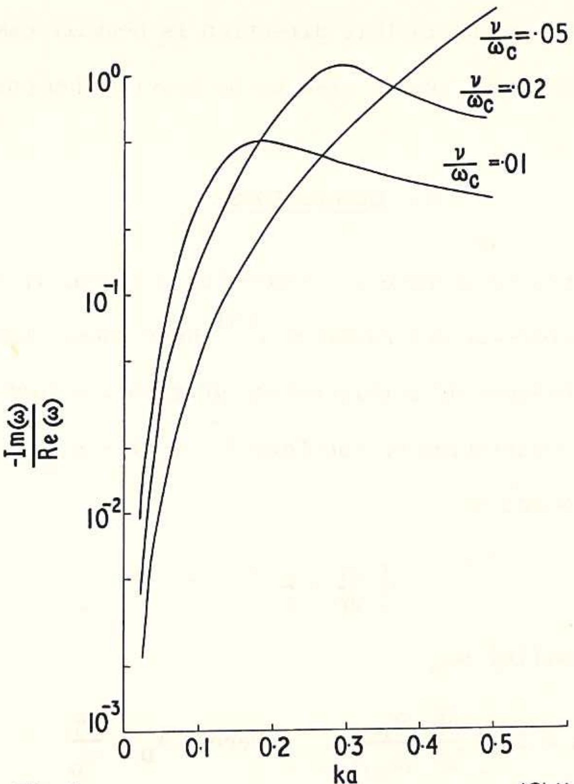

Fig. 5 (CLM-R75)

Computed damping rate of upper branch ($\ell = -1$) of high frequency electrostatic wave for vacuum boundary conditions. The parameters ω_p/ω_{ce}, λ_D/a and x have the same values as for Figs. 3 and 4.

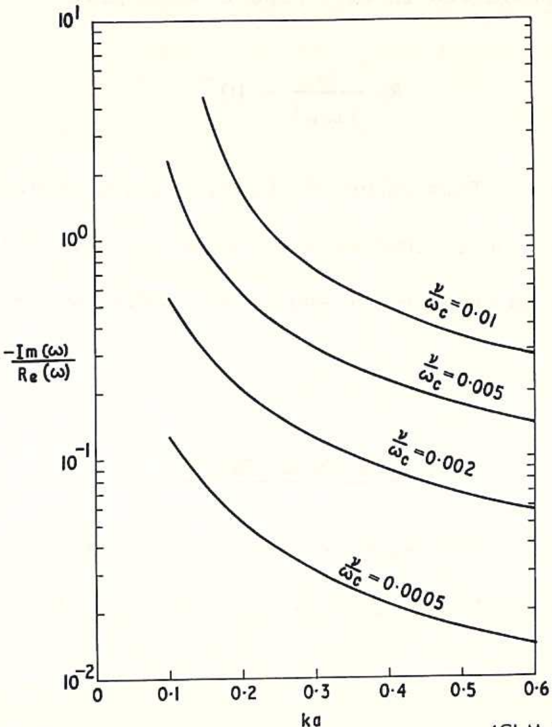

Fig. 6 (CLM-R75)

Computed damping rate of lower branch ($\ell = +1$) of high frequency electrostatic wave for vacuum boundary conditions. ω_p/ω_{ce}, λ_D/a and x as for Figs. 3, 4 and 5.

branch only appears when there are electric fields in the direction of the magnetic field. Thus, if the motion of the electrons in this direction is heavily damped one would expect the wave corresponding to the lower branch also to be heavily damped.

6. CONCLUSIONS

The high frequency electrostatic wave is stable for a plasma in which the temperature is constant. However, Mikhailovskii and Pashitskii[5] have shown that there is a high frequency instability in the presence of a temperature gradient. A further calculation[8] has shown that for a temperature variation of the form $T = T_o(1 - y \frac{r^2}{a^2})$ where y is a dimensionless parameter between 0 and 1

$$\frac{1}{T}\frac{dT}{dr} \sim \frac{y}{a}$$

and the condition for instability is

$$ka < 20\, y \frac{\lambda_D^2}{a^2} \frac{\omega_p}{|\omega_{ce}|}, \quad \text{where} \quad \lambda_D = \frac{v_T}{\omega_p}$$

This leads to the real part of the frequency having the value

$$\mathrm{Re}\, \frac{\omega}{|\omega_{ce}|} \sim \frac{1}{4} \frac{\omega_p}{|\omega_{ce}|}\, ka$$

which for the conditions considered in this report would give

$$\mathrm{Re}\, \frac{\omega}{|\omega_{ce}|} \sim 10^{-6}$$

where we have taken $y = 0{\cdot}1$. This value of $\mathrm{Re}(\omega)$ is well below the ion cyclotron frequency of even the heaviest ions. Therefore the high frequency electrostatic wave in the frequency range considered in this report should be stable even in the presence of a temperature gradient.

7. ACKNOWLEDGEMENTS

I am indebted to Dr. P.C. Thonemann for pointing out the existence of this wave and for helpful discussion. I would also like to thank Mr. G.N. Harding for his interest and encouragement.

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APPENDIX 1

THE FLUID APPROXIMATION

In the above analysis, the electron fluid equation has been used to describe the motion of the electrons. In order to justify its use we will now show that the Vlasov equation leads to the same equation except for terms of order $(kv_T/\omega)^2$ which is very much less than unity because of the large axial phase velocity.

Since we consider the frequency range $\omega_{ci} \ll \omega \ll |\omega_{ce}|$ we may use the Vlasov equation in electron drift space (6) assuming the ions to provide a background of stationary positive charge. Taking the distribution function to be,

$$f = f\,(\underset{\sim}{r}_G,\ \mu,\ u_z,\ t) \qquad \ldots (1.1)$$

where $\underset{\sim}{r}_G$ is the position of the guiding centre and $\mu(= \tfrac{1}{2}mu_\perp^2/B_o)$ the magnetic moment of the electron. In drift space Vlasov's equation becomes,

$$\frac{\partial f}{\partial t} + \frac{d\underset{\sim}{r}_G}{dt} \cdot \frac{\partial f}{\partial \underset{\sim}{r}_G} + \frac{d\mu}{dt} \cdot \frac{\partial f}{\partial \mu} + \frac{du_z}{dt}\frac{\partial f}{\partial u_z} = 0 \ . \qquad \ldots (1.2)$$

Splitting f into an equilibrium part and a perturbed part $f = f_o + f_1$ and using the fact that $\frac{d\mu}{dt} = 0$ to order $\omega/|\omega_{ce}|$, we obtain

$$\frac{\partial f_1}{\partial t} + u_z\frac{\partial f_1}{\partial z} = \frac{e}{m}E_z\frac{\partial f_o}{\partial u_z} - \frac{(\underset{\sim}{E} \times \underset{\sim}{B}_o)}{B_o^2} \cdot \frac{\partial f_o}{\partial \underset{\sim}{r}_G} \qquad \ldots (1.3)$$

where we have used the relations,

$$\frac{d\underset{\sim}{r}_G}{dt} = u_z + \frac{\underset{\sim}{E} \times \underset{\sim}{B}_o}{B_o^2} \qquad \ldots (1.4)$$

and

$$\frac{du_z}{dt} = \frac{-e}{m}E_z$$

Assuming an $\exp i(kz - \omega t)$ dependence, equation (1.3) gives,

$$f_1 = \frac{i}{(\omega - ku_z)}\left\{\frac{e}{m}E_z\frac{\partial f_o}{\partial u_z} - \frac{(\underset{\sim}{E} \times \underset{\sim}{B}_o)}{B_o^2} \cdot \frac{\partial f_o}{\partial \underset{\sim}{r}_G}\right\} \ . \qquad \ldots (1.5)$$

If we take

$$f_o = \frac{B_o e^{-u_\perp^2/2v_T^2}}{mv_T^2}\, f_o(\underset{\sim}{r}_G, u_z) \ ,$$

calculate the perturbed charge density from,

$$-n_1 e = -e \int f_1 \, du_z \, d\mu \qquad \ldots (1.6)$$

and substitute into Poisson's equation, we obtain,

$$\nabla^2\varphi - \frac{e^2 k}{\varepsilon_o m} \varphi \int_{-\infty}^{\infty} \frac{\partial f_o/\partial u_z}{(\omega - ku_z)} du_z$$

$$+ \frac{e^2}{\varepsilon_o m|\omega_{ce}|} \varphi \int \frac{1}{B_o} \frac{(\underset{\sim}{k}_\perp \times \underset{\sim}{B}_o)}{(\omega - ku_z)} \cdot \frac{\partial f_o}{\partial \underset{\sim}{r}_G} du_z = 0 \, . \qquad \ldots (1.7)$$

Specializing to cylindrical geometry, taking

$$f_o(\underset{\sim}{r}_G, u_z) = n_o(r) \; (2\pi v_T^2)^{-\frac{1}{2}} \; e^{-u_z^2/2v_T^2} \qquad \ldots (1.8)$$

and assuming an $e^{i\ell\theta}$ dependence as before, equation (1.7) becomes,

$$\nabla^2\varphi - \frac{e^2 k}{\varepsilon_o m} n_o(r) \; \varphi \; (2\pi v_T^2)^{-\frac{1}{2}} \int \frac{\frac{d}{du_z}(e^{-u_z^2/2v_T^2})}{(\omega - ku_z)} du_z$$

$$+ \frac{e^2 \ell}{m\varepsilon_o|\omega_{ce}|} \frac{1}{r} \frac{dn_o(r)}{dr} \varphi \int (2\pi v_T^2)^{-\frac{1}{2}} \frac{e^{(-u_z^2/2v_T^2)}}{(\omega - ku_z)} du_z = 0 \, . \qquad \ldots (1.9)$$

With the aid of a little algebra the two integrals in equation (1.9) may be expressed as follows,

$$\int_{-\infty}^{\infty} \frac{\frac{d}{du_z}\left(e^{-u_z^2/2v_T^2}\right)}{(\omega - ku_z)} du_z = \frac{\sqrt{2\pi}}{kv_T} \left[1 + \frac{1}{\sqrt{2}} \frac{\omega}{kv_T} \; W\left(\frac{\omega}{\sqrt{2}kv_T}\right) \right] \qquad \ldots (1.10)$$

and

$$\int_{-\infty}^{\infty} \frac{e^{-u_z^2/2v_T^2}}{(\omega - ku_z)} du_z = -\frac{\sqrt{\pi}}{k} W\left(\frac{\omega}{\sqrt{2}kv_T}\right) \qquad \ldots (1.11)$$

where

$$W(x) = \frac{+1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - x} dt \qquad \ldots (1.12)$$

the plasma dispersion function[7].

For $x \gg 1$, $W(x)$ has the following asymptotic expansion,

$$W(x) \sim 2i\sqrt{\pi} \exp(-x^2) - \frac{1}{x}\left[1 + \frac{1}{2x^2} + \frac{3}{4x^4} \right] \qquad \ldots (1.13)$$

Now we have already seen that $\omega/kv_T \gg 1$ so that we may use (1.13) to evaluate the expressions (1.10) and (1.11). Since $\omega/k \gg v_T$, the imaginary term in (1.13), which gives rise to Landau damping, is completely negligible and will be omitted. Making these substitutions, equation (1.9) becomes finally,

$$\nabla^2\varphi \quad + \frac{k^2e^2}{\varepsilon_o m}\frac{n_o(r)}{\omega^2}\left(1 + 3\,\frac{k^2v_T^2}{\omega^2}\right)\varphi$$

$$+ \frac{e^2\ell}{\omega m\varepsilon_o|\omega_{ce}|}\,\frac{1}{r}\,\frac{dn_o}{dr}\left(1 + \frac{k^2v_T^2}{\omega^2}\right)\varphi = 0 \qquad \ldots (1.14)$$

where we have neglected powers of $\frac{kv_T}{\omega}$ higher than the second. Comparing equation (1.14) with equation (12), obtained from the fluid approach, we see that for $\nu/\omega = 0$, the equations differ only in the coefficients of $\omega^2/k^2v_T^2$. Since we have already stressed the fact that this is very small the fluid approach is justified.

APPENDIX 2

THE ELECTROSTATIC APPROXIMATION

The analysis given above was greatly simplified by neglecting perturbations of the uniform magnetic field and assuming,

$$\underset{\sim}{E} = -\nabla\varphi \; .$$

At low frequencies it is the Alfvén mode which perturbs the magnetic field. However, at high frequencies, this mode can be neglected since the ion motion is negligible. Instead, we must show under what conditions the helicon mode can be neglected since it is this mode which will cause the greatest magnetic field perturbations. Starting from the complete set of Maxwell's equations,

$$\underset{\sim}{\nabla} \times \underset{\sim}{H_1} = \underset{\sim}{J_1} + \varepsilon_o \frac{\partial \underset{\sim}{E_1}}{\partial t} \qquad \ldots (2.1)$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E_1} = -\mu_o \frac{\partial \underset{\sim}{H_1}}{\partial t} \qquad \ldots (2.2)$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E_1} = \frac{-n_1 e}{\varepsilon_o} \qquad \ldots (2.3)$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{H_1} = 0 \qquad \ldots (2.4)$$

we express the fields in terms of the electromagnetic potentials $\underset{\sim}{A}$ and φ,

$$\underset{\sim}{E_1} = -\underset{\sim}{\nabla}\varphi - \frac{\partial A}{\partial t} \qquad \ldots (2.5)$$

$$\underset{\sim}{H_1} = \frac{1}{\mu_o} \underset{\sim}{\nabla} \times \underset{\sim}{A} \qquad \ldots (2.6)$$

For an $\exp(-i\omega t)$ time dependence and choosing the Lorentz gauge,

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{A} + \mu_o \varepsilon_o \frac{\partial \varphi}{\partial t} = 0 \qquad \ldots (2.7)$$

we obtain,

$$\underset{\sim}{A} = \frac{\mu_o \underset{\sim}{J_1}}{\left(k^2 + \frac{\omega^2}{c^2}\right)} \qquad \ldots (2.8)$$

where

$$k^2 = k_\perp^2 + k_z^2$$

(N.B. The spatial components of the gradient operator perpendicular to $\underset{\sim}{B_o}$ have been approximated by $i\underset{\sim}{k_\perp}$ since we are only interested in comparing magnitudes of various terms) To discover the importance of magnetic perturbations we simply compare the part of $\underset{\sim}{E_1}$ which arises from these perturbations with the total electric field. For the field perpendicular to $\underset{\sim}{B}_o$ we need to evaluate,

$$\left|\frac{-i\omega \underset{\sim}{A}_\perp}{\underset{\sim}{E}_\perp}\right|$$

From equation (2.8),

$$\underset{\sim}{A}_\perp = \frac{\mu_o \underset{\sim}{J}_{1\perp}}{\left(k^2 + \frac{\omega^2}{c^2}\right)} \qquad \text{... (2.9)}$$

We obtain $\underset{\sim}{J}_{1\perp}$ from equation (1) and the definition of current density $\underset{\sim}{J} = -ne\underset{\sim}{v}$,

$$\underset{\sim}{J}_{1\perp} = -en_o \frac{\underset{\sim}{E}_1 \times \underset{\sim}{B}_o}{B_o^2} - \frac{\kappa T}{B_o^2} \underset{\sim}{\nabla} n_1 \times \underset{\sim}{B}_o \qquad \text{... (2.10)}$$

where we have neglected collisions and terms of order $\omega/|\omega_{ce}|$. Substituting (2.10) into (2.9) and using equation (3) we obtain after a little manipulation,

$$\frac{\omega|\underset{\sim}{A}_\perp|}{|\underset{\sim}{E}_\perp|} = \frac{\omega_p^2}{c^2(k^2 + \omega^2/c^2)} \frac{\omega}{|\omega_{ce}|} + \frac{\beta}{2} \frac{\omega|\omega_{ce}|}{\omega_p^2} \frac{k^2}{k^2 + \frac{\omega^2}{c^2}} \qquad \text{... (2.11)}$$

where the second term was obtained assuming $\underset{\sim}{E} \approx -\underset{\sim}{\nabla}\varphi$. We see immediately from the first term on the RHS of (2.11) that, as expected, for helicon waves,

$$\frac{\omega|\underset{\sim}{A}_\perp|}{|\underset{\sim}{E}_\perp|} \approx 1$$

since for this case

$$k^2 \approx \frac{\omega_p^2}{c^2} \frac{\omega}{|\omega_{ce}|} \gg \frac{\omega^2}{c^2} \quad .$$

Note also that this term is independent of β. For the electrostatic wave we are considering the second term on the RHS of (2.11) is negligible provided $\beta \ll 1$. For the first term to be negligible, $\omega_p \ll |\omega_{ce}|$.

Finally, we compare the E_z fields. For this we need equations (2.8), (1) and (2) and we obtain,

$$\frac{\omega A_z}{E_z} = \frac{-i\mu_o/k^2\left(\frac{\omega^2}{k^2}\frac{n_oe^2}{\kappa T} + e\underset{\sim}{\nabla}n_o \cdot \frac{\underset{\sim}{E}_\perp \times \underset{\sim}{B}_o}{E_z B_o^2}\frac{\omega}{k_z}\right)}{\left[1 - \frac{\omega^2}{k_z^2}\frac{m}{\kappa T}\right]} \qquad \text{... (2.12)}$$

where we have again neglected collisions, and terms of order $\omega/|\omega_{ce}|$. Approximating

$$\underset{\sim}{E}_\perp \times \underset{\sim}{B}_o/E_z \quad \text{by} \quad (\underset{\sim}{k}_\perp \times \underset{\sim}{B}_o)\ \varphi/k_z\varphi \ ,$$

(2.12) may be written as,

$$\left|\frac{\omega\ A_z}{E_z}\right| = \frac{\dfrac{\omega_p^2}{c^2k^2}\left(\dfrac{\omega^2}{k_z^2v_T^2} + \dfrac{\omega}{|\omega_{ce}|}\,\dfrac{\underset{\sim}{K}_n \cdot \underset{\sim}{k}_\perp}{k_z^2}\right)}{\left(1 - \dfrac{\omega^2}{k_z^2v_T^2}\right)} \qquad \text{... (2.13)}$$

where $\underset{\sim}{K}_n = \frac{1}{n_o}\underset{\sim}{\nabla}\, n_o$. Since we are considering waves for which $\omega/k_z \gg v_T$ we require the following condition for the final justification of the electrostatic approximation,

$$\frac{\omega_p^2}{c^2k^2} + \frac{\omega_p^2}{\omega|\omega_{ce}|}\,\frac{\underset{\sim}{K}_n \cdot \underset{\sim}{k}_\perp}{k^2}\,\frac{v_T^2}{c^2} \ll 1\ . \qquad \text{... (2.14)}$$

The second term on the LHS is always $\ll 1$ except for relativistic plasmas since equation (2.11) requires $\omega_p \ll \omega_{ce}$. Thus, we must ensure that $\omega_p^2 \ll c^2k^2$.

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