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GRAVITATIONAL AND DRIFT WAVES IN A PERIODIC SLAB

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GRAVITATIONAL AND DRIFT WAVES IN A PERIODIC SLAB

by

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ABSTRACT

The low frequency electrostatic drift-acoustic mode and the gravitational flute mode in a periodic gravitational field in a collisionless plasma are examined. As a model, a one-dimensional plasma slab with straight field-lines, a small density gradient and a perpendicular gravity periodic along the field lines is considered. Only the long wavelength non-localised waves are investigated.

The periodicity along the field-lines causes the modes to combine producing unstable hybrid modes or standing waves. The energy transfer between the resonant electrons and the waves is also examined.

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C O N T E N T S

LIST OF SYMBOLS USED

ω	frequency
ω'	$\omega' \equiv \omega - \frac{k_x g_o}{\Omega_i}$
ω^*	the electron drift frequency $\omega^* \equiv \frac{k_x T_e}{a\, m_i\, \Omega_i} > 0$
γ	the growth rate ($\gamma > 0$ unstable)
$k_x, k_z^{(n)}$	components of the wave vector, $k_z^{(n)} = \frac{n}{\nu} k_1$, n ... mode number, ν ... number of the field periods
k_1	periodicity of the curvature $k_1 = \frac{\pi}{L}$, L ... connection length
$\frac{1}{a}$	density gradient in y-direction $\frac{1}{a} = \frac{1}{n}\frac{dn}{dy} > 0$
ρ_i	ion Larmor radius
b	$(k_x \rho_i)^2 = \frac{k_x^2\, T_i}{m_i\, \Omega_i^2}$
Q	$Q(b) \equiv 1 - e^{-b} I_o(b)$
g_o, g_1	first two coefficients of the Fourier expansion of the periodic gravity for ions ($g_o > 0$ for stability)
R_{co}, R_c	radii of curvature ($g_o = \frac{T_i}{m_i R_{co}}$)
m_i, m_e	particle masses
$e_i \equiv e$	elementary charge ($e > 0$)
T_i, T_e	temperatures in energy units
τ	$\tau \equiv \frac{T_e}{T_i}$
v_{Ti}, v_{Te}	the thermal velocities $v_{Ti}^2 = \frac{2T_i}{m_i}$
ψ	the perturbed electrostatic potential
$\hat{n}_i, \hat{n}_e$	the perturbed number densities
$n_o(y)$	the unperturbed density
ξ	$\xi \equiv \frac{k_1 z}{2}$
Ω_i	ion cyclotron frequency

1. INTRODUCTION

The aim of the present work is to summarise our studies on the stability of a collisionless shear-free plasma confined by a magnetic field possessing a periodic curvature. The theory is relevant only for the situations where the field-lines are self-closing. Our model is practically a one-dimensional plasma slab with the straight magnetic field in the z-direction and a small (positive) density gradient in the y-direction. The periodic curvature is simulated by a periodic gravitational g-field (0, g(z), 0)*

$$g(z) = g_o + g_1 \cos k_1 z \qquad \ldots (1)$$

where $g_1 > g_o \gtrsim 0$ (to simulate the $\int \frac{d\ell}{B}$ stability). We do not consider any effects of particle trapping, that is to say, we ignore any z-component of $\underset{\sim}{g}$ throughout our calculation.

To simplify the problem we make the electrostatic approximation and restrict ourselves to an investigation of the longwave non-localised modes as these are generally not Landau damped. We find that the modes of interest are the well known electrostatic flute mode(1) and the drift wave(2) which now, however, are coupled producing hybrid modes. So, for instance, the magnetohydrodynamically stabilised flute model is accompanied by a pair of long wavelength drift waves, which may be unstable due to resonant electrons. On the other hand the coupled drift waves can combine into a standing mode, that effectively behaves as a localised wave, even if the strong localisation condition(3) is not met.

In sections 2–7 we describe our model and develop the dispersion equation; the subsequent sections are devoted to its physical interpretation and solution. Throughout our calculations we assume that the parameters

$$\left(\frac{v_{Ti}}{\omega} \right)^2 \frac{\partial^2 \psi}{\partial z^2} \qquad \text{and} \qquad \tau \frac{k_x g_1}{\Omega_i \omega} \qquad \ldots (2)$$

are small quantities. The first condition allows us to neglect the ion Landau damping, the second one makes the use of our model of the gravity simulated curvature possible. To comply with the second condition (2) it is necessary to assume that finite Larmor radius terms are not too large, i.e.

$$(k_x \rho_i)^2 \lesssim 1 \qquad \ldots (3)$$

As the first parameter (2) may remain small (i.e. no Landau damping) even if the second one is not small we exclude the physically still interesting case of wavelengths very much

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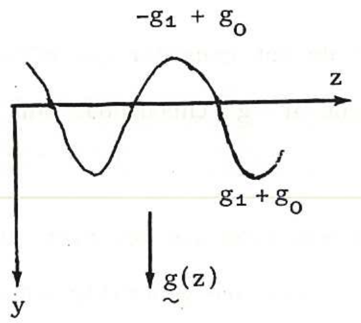
* This is a field acting on the ions, for the electrons we assume an effective field $\frac{m_i}{m_e} \frac{T_e}{T_i} g(z)$.

shorter than the ion Larmor radius. For further simplification we do not consider any propagation along the density gradient.

The concept of our calculations is rather similar to that of Ref.3 but we examine the shear-free case in somewhat more detail.

2. THE SITUATION IN OUR MODEL

We consider a plasma slab as follows:

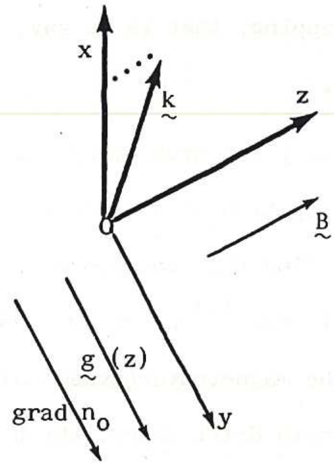


$\underset{\sim}{B} = (0,\ 0,\ B)$

$\text{grad}\ n_o = (0,\ n_o'(o),\ 0)$

$\underset{\sim}{g} = (0,\ g(z),\ 0)$

$\underset{\sim}{k} = (k_x,\ 0,\ k_z^{(n)})$



3. THE EQUILIBRIUM DISTRIBUTION FUNCTION

The equilibrium distribution function can be chosen in the following way:

$$F_{oi}(\underset{\sim}{v}) = f_{oi}(v)\,(1 + v_x\,\varphi_i)\ ,$$

$$f_{oi} = n_o(y)\,\frac{1}{\pi^{3/2}\,v_{Ti}^3}\exp\left(-\frac{v^2}{v_{Ti}^2}\right)\ , \qquad \ldots (4)$$

$$\varphi_i = \frac{1}{\Omega_i}\left(-\frac{1}{a} + \frac{2}{v_{Ti}^2}\,g(z)\right)\ .$$

The above form applies to the ions. For the electrons it is necessary to replace the ion quantities by the electron ones, including the changing of $g(z)$ to $\frac{T_e\,m_i}{T_i\,m_e}\,g(z)$.

4. THE EQUATIONS OF PARTICLE MOTION AND THEIR SOLUTION

The ion equations of motion are

$$m_i \ddot{X} = \frac{e_i}{c} \dot{Y} B \quad ,$$

$$m_i \ddot{Y} = - \frac{e_i}{c} \dot{X} B + m_i (g_o + g_1 \cos k_1 z) \quad , \qquad \ldots (5)$$

$$m_i \ddot{Z} = 0 \quad ,$$

with a solution to the order of accuracy $\frac{\rho_i}{R_c}$:

$$X(t) = x + \frac{v_\perp}{\Omega_i} \left[\sin(\Omega_i t - \chi) + \sin\chi \right]$$

$$+ \frac{g_o}{\Omega_i} t + \frac{g_1}{k_1 v_z \Omega_i} \left[\sin k_1(v_z t + z) - \sin k_1 z \right] , \qquad \ldots (6)$$

$$Z(t) = z + v_z t \quad .$$

$(v_\perp, \chi, v_z)$ are the polar coordinates in velocity space.

The z-dependent term in the expression for $X(t)$ represents the particle guiding centre wobble due to the varying curvature.

The analogous equations for electrons can be obtained again easily from the above expressions.

5. THE SOLUTION OF THE LINEARISED VLASOV EQUATION

The linearised Vlasov equation

$$\frac{\partial \hat{f}_i}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial \hat{f}_i}{\partial \underset{\sim}{r}} + \left[\frac{e_i}{m_i c} \underset{\sim}{v} \times \underset{\sim}{B} + \underset{\sim}{g}(z) \right] \cdot \frac{\partial \hat{f}_i}{\partial \underset{\sim}{v}} = \frac{e_i}{m_i} \operatorname{grad} \psi \cdot \frac{\partial F_{oi}}{\partial \underset{\sim}{v}} \qquad \ldots (7)$$

has a solution[1]

$$\hat{f}_i(\underset{\sim}{r}_1 \underset{\sim}{v}_1 t) = - \frac{e_i \psi}{T_i} F_{oi} + \frac{e_i}{T_i} \int_{-\infty}^{t} dt' \left[\frac{\partial \psi}{\partial t} F_{oi} - f_{oi} \frac{1}{\Omega_i} \left(\frac{v_{Ti}^2}{2a} - g(z) \right) \frac{\partial \psi}{\partial x} \right]_{\substack{\underset{\sim}{r} \to \underset{\sim}{R}(t'-t) \\ \underset{\sim}{v} \to \underset{\sim}{V}(t'-t) \\ t \to t'}} \qquad \ldots (8)$$

where $\underset{\sim}{R}(t)$, $\underset{\sim}{v}(t)$ are given by the equations (6).

To find the normal mode solution we assume that the perturbed electrostatic potential may be written in the form:

$$\psi(\underset{\sim}{r}, t) = \exp i(-\omega t + k_x x) \sum_{m=-\infty}^{\infty} \psi_m \exp iz(k_z^{(n)} + mk_1) \qquad \ldots (9)$$

Substituting (9) into (8) and integrating over t', an expression for the perturbed density may be readily obtained. There is a particle wobble term which is entirely analogous to the standard gyration term. The calculations give for the perturbed densities*:

$$\hat{n}_i(\underset{\sim}{r}, t) = \frac{en_o}{T_i} \sum_{r=-\infty}^{\infty} \left[-\psi_r + (1 - Q(b)) \int_{-\infty}^{\infty} \frac{dv_z}{\sqrt{\pi}\, v_{Ti}} e^{-\left(\frac{v_z}{v_{Ti}}\right)^2} \cdot \sum_{\substack{m \\ p}\}=-\infty}^{\infty} I_p\left(\frac{k_x g_1}{k_1 v_z \Omega_i}\right) \right.$$

$$\left. I_{p+m-r}\left(\frac{k_x g_1}{k_1 v_z \Omega_i}\right) \frac{\omega + \tau\omega^* - \frac{k_x g_o}{\Omega_i} - (p+m-r)k_1 v_z}{\omega - \frac{k_x g_1}{\Omega_i} - v_z[k_z^{(n)} + (p+r)k_1]} \psi_m \right] \exp i\left[-\omega t + k_x x + (k_z^{(n)} + rk_1)z\right],$$

$$\hat{n}_e(\underset{\sim}{r}, t) = -\frac{en_o}{T_e} \sum_{r=-\infty}^{\infty} \left[-\psi_n + \int_{-\infty}^{\infty} \frac{dv_z}{\sqrt{\pi}\, v_{Te}} e^{-\left(\frac{v_z}{v_{Te}}\right)^2} \sum_{\substack{m \\ p}\}=-\infty}^{\infty} I_p\left(-\tau \frac{k_x g_1}{k_1 v_z \Omega_i}\right) \cdot \right.$$

$$\left. I_{p+m-r}\left(-\tau \frac{k_x g_1}{k_1 v_z \Omega_i}\right) \frac{\omega - \omega^* + \tau \frac{k_x g_o}{\Omega_i} - (p+m-r)k\, v_z}{\omega + \tau \frac{k_x g_o}{\Omega_i} - v_z[k_z^{(n)} + (p+r)k_1]} \psi_m \right] \cdot \exp i\left[-\omega t + k_x x + (k_z^{(n)} + rk_1)\right],$$

... (10)

where

$$Q(b) = 1 - e^{-b} I_o(b)$$

$$b = (k_x \rho_i)^2$$

$$\tau = \frac{T_e}{T_i}$$

The quasi-neutrality condition

$$\hat{n}_i = \hat{n}_e \qquad \text{... (11)}$$

gives an infinite system of equations for the coefficients $\{\psi_m\}$ and if we equate the determinant of that system to zero we get the dispersion equation of the system.

6. THE SIMPLIFICATION OF THE EXPRESSIONS FOR THE DENSITIES

Using our original assumptions (1) and also the condition

$$\frac{\omega}{k_z^{(n)} v_{Te}} \ll 1 \qquad \text{... (12)}$$

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* The last term in the numerator of the fractions in (10) missing in[3] arises due to the z-dependence of the equilibrium in (4).

we can greatly simplify the expressions for the perturbed densities (10). In dealing with the resonant denominators we use the Landau prescription for each of the terms in the sums (10). So we obtain:

(a) The ion density

$$\hat{n}_i = \sum_{r=-\infty}^{\infty} \frac{n_o e}{T_i}\left[-\psi_r + \frac{1-Q}{\tau}\left\{ \frac{\tau\omega' + \omega^*}{\omega'}\left[1 + \frac{1}{2}\left(\frac{v_{T_i}\,(k_z^{(n)} + rk_1)}{\omega}\right)^2 \right]\psi_r + \frac{1}{2}\frac{\omega^*}{\omega^2}\frac{k_x g_1}{\Omega_i}(\psi_{r+1} + \psi_{r-1})\right\}\right]\cdot \exp i\left[-\omega t + k_x x + (k_z^{(n)} + rk_1)z\right],$$

where

$$\omega' = \omega - \frac{k_x\, g_o}{\Omega_i} \qquad \text{... (13)}$$

We neglect in our calculations the imaginary part of the ion density responsible for the ion Landau damping.

(b) The electron density

The simplification of the terms in the series for the electron density depends critically on the wave vector $k_z^{(n)}$. Namely, if the wave vector $k_z^{(n)}$ is an integral multiple of the curvature periodicity

$$k_z^{(n)} = -\, r_o\, k_1 \qquad \text{... (14)}$$

which implies that $\frac{n}{\nu} = -\, r_o$ is integer, then it is possible for the z-dependence in the denominator in (10) to disappear when

$$k_z^{(n)} + (p + r)\, k_1 = 0 \quad . \qquad \text{... (15)}$$

That particular term then assumes a form that corresponds to a flute type of wave. Examining the magnitudes of those terms that are exceptional in the sense of (15) we find that the most essential change occurs in the electron density series if

$$m = r = r_o \quad .$$

The above consideration should be applied to both real and imaginary parts of the electron density. If the condition (14) is not fulfilled a flute-like term cannot occur. Then the contributions of the terms in the electron series (10) are all the same and given essentially by the Boltzmann law and the standard resonant term, accounting for the universal drift instability. Thus the pure drift mode inherently cannot resonate with the flute.

Let us distinguish the two cases:

(i) The pure drift mode $(k_z^{(n)} \neq - r_o k_1$ for any $r_o)$:

$$\hat{n}_e = - \frac{en_o}{T_e} \sum_{r=-\infty}^{\infty} \left\{ - \psi_r - \frac{i\sqrt{\pi}}{v_{T_e} |k_z^{(n)} + rk_1|} \left[\left(\omega - \omega^* + \tau \frac{k_x g_o}{\Omega_i} \right) \psi_r + \right. \right.$$

$$\left. \left. + \frac{\tau}{2} \frac{k_x g_1}{\Omega_i} (\psi_{r+1} + \psi_{r-1}) \right] \right\} \exp i \left[- \omega t + k_x x + (k_z^{(n)} + rk_1) z \right] \quad \ldots (16)$$

and (ii) The flute mode $(k_n^{(z)} = - r_o k_1$ for some integer $r_o)$:

$$\hat{n}_e = \frac{en_o}{T_e} \left[\cdots + \left\{ \psi_{r_o-1} - \frac{i\sqrt{\pi}}{k_1 v_{T_e}} \left[(\omega - \omega^*) \psi_{r_o-1} + \right. \right. \right. \quad \ldots (17a)$$

$$\left. \left. + \tau \frac{k_x g_1}{2\Omega_i \omega} \left(\omega^* \psi_{r_o} + \omega \psi_{r_o-2} \right) \right] \right\} \exp i \left[- \omega t + k_x x - k_1 z \right] +$$

$$+ \left\{ - \psi_{r_o} + \frac{\omega - \omega^* + \tau \frac{k_x g_o}{\Omega_i}}{\omega + \tau \frac{k_x g_o}{\Omega_i}} \psi_{r_o} + \frac{\tau}{2} \frac{k_x g_1}{2\Omega_i \omega} (\psi_{r_o+1} + \psi_{r_o-1}) - \right.$$

$$\left. - \frac{i\sqrt{\pi}}{k_1 v_{T_e}} \left[\frac{1}{2} \left(\tau \frac{k_x g_1}{\Omega_i \omega} \right)^2 (\omega - \omega^*) \psi_{r_o} - \frac{\tau}{2} \frac{k_x g_1}{\Omega_i \omega} (\omega - \omega^*) (\psi_{r_o+1} + \psi_{r_o-1}) \right\} \quad \ldots (17b)$$

$$\exp i \left[- \omega t + k_x x \right] +$$

$$+ \left\{ - \psi_{r_o+1} - \frac{i\sqrt{\pi}}{k_1 v_{T_e}} \left[(\omega - \omega^*) \psi_{r_o+1} + \frac{\tau}{2} \frac{k_x g_1}{\Omega_i \omega} (\omega^* \psi_{r_o} + \omega \psi_{r_o+2}) \right] \right\} \cdot \quad \ldots (17c)$$

$$\left. \exp i \left[- \omega t + k_x x + k_1 z \right] + \cdots \right]$$

The remaining terms in the last series (17) are essentially of the same form as in (16). This exemplifies the difference between the pure drift (16) and flute type of mode. In the latter some of the terms are different, namely those where the relation (15) is valid.

7. THE PHYSICAL INTERPRETATION OF THE EXPRESSIONS FOR THE DENSITIES

(a) The real part of the densities:

The expression (13) and the real parts of (16) and (17) can be obtained in a more obvious way simply from the two fluid equations (including the FLR terms[4]) together with the isothermal gas laws. In the electrostatic approximation we obtain an equation connecting ψ and a perturbed density in the form:

$$\frac{v_{T_i}^2}{2} \frac{\partial}{\partial z} \frac{1}{\omega - \frac{k_x g(z)}{\Omega_i}} \frac{\partial}{\partial z} \left(\frac{\hat{n}_i}{n} + \frac{e}{T_i} \psi \right) + \frac{\hat{n}_i}{n} (1 + b) \left(\omega - \frac{k_x g(z)}{\Omega_i} \right) + \left[\frac{\omega^*}{\tau} + b \left(\omega - \frac{k_x g(z)}{\Omega_i} \right) \right] \frac{e}{T_i} \psi = 0. \quad \ldots (18)$$

A similar expression is valid for electrons.

Substituting into (18) the series (9) we recover once again the real parts of (13) and either (16) or (17) depending on the condition (15). It is also easy to demonstrate from (18) that for a pure drift mode (condition (15) is not valid) (18) yields a Mathieu equation for the real part of the perturbed potential:

$$\frac{d^2\psi}{d\xi^2} + (A(\omega) - 2q \cos 2\xi)\, \psi = 0 \qquad \ldots (19)$$

where

$$A(\omega) = 2\left(\frac{2\omega}{k_1 v_{T_i}}\right)^2 \left[-1 + \frac{1+\tau}{1 - Q(b)} \; \frac{\omega'}{\tau\omega' + \omega^*} \right]$$

$$q = \left(\frac{2\omega}{k_1 v_{T_i}}\right)^2 \frac{\omega^*}{\omega'(\tau\omega + \omega^*)} \; \frac{k_x g_1}{\Omega_i}$$

and

$$\xi = \frac{k_1 z}{2}$$

The above equation is invalid for the flute mode.

(b) <u>The imaginary part of the electron density</u>:

The most convenient way to interpret the imaginary terms in (16) and (17) is to examine the power transfer in the electron gas. To make sure that we correctly include the y-variations we start from the equation of the charge continuity multiplied by ψ^* :

$$\frac{\partial \hat{\rho}}{\partial t}\, \psi^* = -\psi^* \operatorname{div} \hat{\underset{\sim}{j}} = -\operatorname{div}(\hat{\underset{\sim}{j}}\, \psi^*) + \hat{\underset{\sim}{j}}\, \hat{\underset{\sim}{E}}^* \qquad \ldots (20)$$

If we integrate over a large volume, then, apart from the surface term, the right hand side of (20) gives an increase in the electrostatic energy of the wave in terms of the contributions from the current components. Clearly, if we examine the structure of the imaginary part of $-e \frac{\partial \hat{n}_e}{\partial t}$, composed of the imaginary parts of the electron currents in the expression $-\operatorname{div} \hat{\underset{\sim}{j}}_e$, we shall find which way the energy flows.

We calculate the electron current again from (4), (6) and (8). We obtain

$$i\, k_x \hat{j}_{xe} = i\, \frac{n_o e^2}{T_e} \sum_{r=-\infty}^{\infty} \Bigg[\left(\omega^* - \tau \frac{k_x g_o}{\Omega_i} \right) \psi_r - \frac{\tau}{2} \frac{k_x g_1}{\Omega_i} (\psi_{r+1} + \psi_{r-1}) +$$

$$+ \int_{-\infty}^{\infty} \frac{dv_z}{\sqrt{\pi} v_{T_e}}\, e^{-\left(\frac{v_z}{v_{T_e}}\right)^2} \sum_{m,p=-\infty}^{\infty} I_p\left(-\tau \frac{k_x g_1}{k_1 v_z \Omega_i}\right) I_{p+m-r}\left(-\tau \frac{k_x g_1}{k_1 v_z \Omega_i}\right) \cdot$$

$$\cdot \frac{\omega - \omega^* + \tau \frac{k_x g_o}{\Omega_i} - (p+m-r) k_1 v_z}{\omega + \tau \frac{k_x g_o}{\Omega_i} - v_z [k_z^{(n)} + (p+r) k_1]} \left(\omega^* - \tau \frac{k_x g_o}{\Omega_i} + p k_1 v_z \right) \psi_m \Bigg] \cdot \exp i \left[-\omega t + k_x x + (k_z^{(n)} + r k_1) z \right]$$

$$\ldots (20)$$

$$\frac{1}{a}\hat{j}_y = -i\frac{n_o e^2}{T_e}\omega^* \sum_{r=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dv_z}{\sqrt{\pi} v_{T_e}} e^{-\left(\frac{v_z}{v_{T_e}}\right)^2} \sum_{\left.\begin{matrix}m\\p\end{matrix}\right\}=-\infty}^{\infty} I_p\left(-\tau\frac{k_x g_1}{k_1 v_z \Omega_i}\right) I_{p+m-r}\left(-\tau\frac{k_x g_1}{k_1 v_z \Omega_i}\right)$$

... (21)

$$\frac{\omega-\omega^*-\tau\frac{k_x g_o}{\Omega_i}-(p+m-r)k_1 v_z}{\omega+\tau\frac{k_x g_o}{\Omega_i}-v_z[k_z^{(n)}+(p+r)k_1]}\psi_m \cdot \exp i\left[-\omega t+k_x x+(k_z^{(n)}+rk_1)z\right]$$

$$\frac{\partial\hat{j}_{ze}}{\partial z} = i\frac{n_o e^2}{T_e}\sum_{r=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dv_z}{\sqrt{\pi}v_{T_e}} e^{-\left(\frac{v_z}{v_{T_e}}\right)^2}\sum_{\left.\begin{matrix}m\\p\end{matrix}\right\}=-\infty}^{\infty} \cdot I_p\left(-\tau\frac{k_x g_1}{k_z v_z \Omega_i}\right) I_{p+m-r}\left(-\tau\frac{k_x g_1}{k_1 v_z \Omega_i}\right)$$

... (22)

$$\frac{\omega+\tau\frac{k_x g_o}{\Omega_i}-\omega^*-(p+m-r)k_1 v_z}{\omega+\tau\frac{k_x g_o}{\Omega_i}-v_z[k_z^{(n)}+(p+r)k_1]}(k_z^{(n)}+rk_1)\, v_z\psi_m \cdot \exp i\left[-\omega t+k_x x+(k_z^{(n)}+rk_1)z\right]$$

Combining the expressions (20), (21) and (22) we obtain:

$$-\operatorname{div}\hat{\underset{\sim}{j}}_e = -i\frac{n_o e^2}{T_e}\sum_{r=-\infty}^{\infty}\left\{\left(\omega^*-\tau\frac{k_x g_o}{\Omega_i}\right)\psi_r-\frac{\tau}{2}\frac{k_x g_1}{\Omega_i}(\psi_{r+1}+\psi_{r-1})+\right.$$

$$+\int_{-\infty}^{\infty}\frac{dv_z}{\sqrt{\pi}v_{T_e}} e^{-\left(\frac{v_z}{v_{T_e}}\right)^2}\sum_{\left.\begin{matrix}m\\p\end{matrix}\right\}=-\infty}^{\infty} I_p\left(-\tau\frac{k_x g_1}{k_1 v_z \Omega_i}\right)\cdot I_{p+m-r}\left(-\tau\frac{k_x g_1}{k_1 v_x \Omega_i}\right)\cdot$$

$$\cdot\frac{\omega+\tau\frac{k_x g_o}{\Omega_i}-\omega^*-(p+m-r)k_1 v_z}{\omega+\tau\frac{k_x g_o}{\Omega_i}-v_z[k_z^{(n)}+(p+r)k_1]}\cdot$$

$$\left.\cdot\left[-\tau\frac{k_x g_o}{\Omega_i}+p\,k_1\,v_z+(k_z^{(n)}+rk_1)v_z\right]\psi_m\right\}\cdot$$

$$\cdot\exp i\left[-\omega t+k_x x+(k_z^{(n)}+rk_1)z\right] = i\,\omega\, e\,\hat{n}_e$$

... (23)

Only the integral expression in (23) is responsible for the imaginary part of $\hat{n}_e$. Inside the square bracket there we recognise the terms

$$-\tau\frac{k_x g_o}{\Omega_i}+p\,k_1\,v_z$$

originating from the perpendicular g-drifts and the longitudinal part

$$\left(k_z^{(n)} + rk_\perp \right) v_z \quad .$$

From here it follows that energy is tranferred in two different ways:

(a) By the longitudinal motion of electrons along the field lines resonating with the $\hat{E}_z$ component of the perturbed electric field,

(b) by the perpendicular resonance between the particle wobble due to the g-field and the $\hat{E}_x$ component of the field.

Both of these cases can be conveniently distinguished by the indices r and p.

So in the equation (16) just mechanism (a) produces the imaginary part, however, the mechanism (b) is solely responsible for all the imaginary terms in (17b) and combines with (a) in producing the terms in (17a) and (17b).

8. THE GRAVITATIONAL FLUTE MODE

The further procedure is now obvious. We substitute the expressions (15) and (17) into the quasi-neutrality condition (11) and calculate the frequencies and growth rates from the condition for the determinant of the system obtained to vanish. In the gravitational flute case $k_z^{(n)} = 0$ (including those of the drift modes with $k_z^{(n)} = - r_o k_\perp$, which can resonate with the flute) we obtain an equation governing the frequency and growth rate:

$$\left\{ (\omega')^2 + \omega' \frac{\omega^*}{\tau} - \frac{\omega^*}{\tau} \frac{k_x g_o}{\Omega_i} \frac{1-Q}{Q} + i \frac{\sqrt{\pi}}{k_\perp v_{T_e}} \frac{\tau}{2Q} \left(\frac{k_x g_\perp}{\Omega_i} \right)^2 (\omega - \omega^*) \right\} \cdot$$

$$\left\{ \omega' - \omega^* \frac{(1-Q)\left[1 + \frac{1}{2}\left(\frac{k_\perp v_{T_i}}{\omega} \right)^2 \right]}{1 + \tau \left[Q - \frac{1}{2} \left(\frac{k_\perp v_{T_e}}{\omega} \right)^2 (1-Q) \right]} + i \frac{\sqrt{\pi}}{k_\perp v_{T_e}} \frac{\omega' \left(\omega - \omega^* + \tau \frac{k_x g_o}{\Omega_i} \right)}{1 + \tau Q} \right\} =$$

$$= \frac{1}{2Q} \left(\frac{k_x g_\perp}{\Omega_i} \right)^2 \frac{\omega^*}{\tau (1 + \tau Q) \omega} \left[\tau \omega - \omega^* (1-Q) + i \frac{\sqrt{\pi}}{k_\perp v_{T_e}} \tau \omega (\omega - \omega^*) \right] \cdot \left(1 - Q - i \frac{\sqrt{\pi}}{k_\perp v_{T_e}} \omega \right) \quad . \qquad \ldots (24)$$

The last equation has been derived on the assumption that only ψ_o and $\psi_{\pm 1}$ are the non-zero terms in the expressions (10). The first curled bracket represents the well known electrostatic flute mode of Ref.1, the second one the drift mode of Ref.2. The right hand side is an interaction term, specific for our model. By the analysis of the complete determinant it can be found that out of the two possible drift wave solutions with the

$k_z^{(n)} = k_1$ it is the even one that can resonate with the flute and appears in (24), the odd solution exists quite independently of the flute.

Denoting now:

$$\omega_{1,2} = -\frac{\omega^*}{2\tau} \pm \sqrt{\left(\frac{\omega^*}{2\tau}\right)^2 + \frac{\omega^*}{\tau}\frac{k_x g_o}{\Omega_i}\frac{1-Q}{Q}} \qquad \text{... (25)}$$

and assuming that the ω_1 (unstable root) and the drift frequency are sufficiently distant from each other, i.e.

$$\left|\omega_1 - \omega^*\frac{1-Q}{1+\tau Q}\right| > 2\sqrt{\left(\frac{k_x g_1}{\Omega_i}\right)^2 \frac{\omega^*(\tau\omega+\omega^*)}{2Q\,(\omega-\omega_1)\tau\omega}} \qquad \text{... (26)}$$

it is possible to calculate the growth rates both of the gravitational and the drift wave in the non-resonant case. We obtain

(a) Non-resonant flute growth-rate (for $(k_x\rho_i)^2 \lesssim 1$):

$$\gamma = -\frac{\sqrt{\pi}}{2Qk_1 v_{T_e}}\left(\tau\frac{k_x g_1}{\Omega_i}\right)^2 \frac{1}{2\omega_1\tau+\omega^*}\;\frac{(\omega_1-\omega^*)^2+\omega^*(\omega_1-\omega^*)+(\omega^*\frac{1+\tau}{\tau})^2}{\omega_1-\omega^*} \qquad \text{... (27)}$$

The last expression says that the instability condition for the non-resonant flute is now the same as for a usual drift-wave:

$$\omega_1 < \omega^* \Rightarrow \tau b > \frac{a}{R_{co}} \qquad \text{... (28)}$$

in agreement with Ref.3.

The instability is now caused entirely by the presence of the resonant electrons.

(b) The non-resonant drift wave:

If the parameter q of (19) is moderate $q \lesssim 1$ the drift wave associated with the flute mode $(k_z^{(n)} = k_1)$ is non-localised. Its frequency is slightly shifted, but not enough to influence the growth rate. For the stability criterion it is possible to accept the usual condition for the drift wave stability:

$$\tau b\left(\tau b + \frac{a}{R_{co}}\right) < (k_1\, a)^2 \qquad \text{... (29)}$$

(c) The resonant case

In this case when the condition (26) is not fulfilled the flute mode strongly interacts with the drift wave and the growth rate is enhanced. It can be visualised from the equation (27) letting ω_1 approach ω^*. To find the growth rate we have to go back to the starting equation (24) and solve it for the case $\omega_1 = \omega^*$. The growth rate obtained depends strongly on the way we order our parameters. If we assume that:

$$\left[Q - \tfrac{1}{2}\left(\frac{k_1 v_{T_e}}{2\omega}\right)^2 - \frac{k_x g_o}{\Omega_i \omega^*}\right]^2 < \frac{1}{2Q}\left(\frac{k_x\, g_1}{\Omega_i}\right)^2 \frac{1}{\tau(1+\tau)\omega^*(\omega^* - \omega_2)} \qquad \dots (30)$$

we find that

$$\gamma = \mp \sqrt{\frac{a}{2Q}} \left| \frac{k_x g_1}{\Omega_i} \right| \frac{\omega^*}{k_1\, v_{T_e}} \left(\frac{1+\tau}{1+2\tau}\right)^{\frac{1}{2}} \qquad \dots (31)$$

The opposite inequality in (30) has been examined in Ref.3.

In conclusion we shall justify the reduction of the full determinant of the system to the 3×3 determinant. This is possible when the off diagonal terms are small with respect to the diagonal one. We limit our investigation to the resonant case as there the convergence is hardest to attain. We find that a typical ratio of an off-diagonal and a diagonal term is

$$\sim k_x \rho_i \sqrt{\frac{1+\tau}{1+2\tau}}$$

so that we are again limited to the long wave case in our investigations.

The order of magnitude of the growth rate of the non-resonant flute is roughly the same as that of a conventional drift-wave and it is approximately $\frac{1}{|k_x \rho_i|}$ higher in the case of the resonance (which cannot be stabilised). But the resonance band width around $\omega^* \sim \omega_1$ is very narrow, given for $(k_x \rho_i)^2$ small by the equation (26).

9. THE DRIFT WAVE

In the case of a pure drift wave the real quantities are now governed by equation (19). If the resonant effects are included it is unfortunately impossible to replace A and q of (19) just by the corresponding complex quantities without changing the equation itself. The reason is that the corresponding perturbation term in (19) would have an integral form affecting each of the harmonics composing the zero order solution of (19) (the appropriate Mathieu function) in a different way. It is therefore more convenient to use the relevant Hill determinant rather than the equation (19) itself. The corresponding system is ($r = \dots, -1, 0, 1, \dots$):

$$\psi_r \left[A(\omega) - \left(2\frac{n}{\nu} + 2r\right)^2 + \frac{i}{\left|\frac{n}{\nu} + r\right|} 2 \left(\frac{2\omega}{k_1 v_{T_i}}\right)^2 \frac{\sqrt{\pi}}{k_1 v_{T_e}} \frac{\omega'}{\tau\omega' + \omega^*} \left(\omega - \omega^* + \tau \frac{k_x g_o}{\Omega_i}\right)\right] +$$

$$+ \left[- q(\omega) + \frac{i}{\left|\frac{n}{\nu} + r\right|} \left(\frac{2\omega}{k_1 v_{T_i}}\right)^2 \frac{\sqrt{\pi}}{k_1 v_{T_i}} \tau \frac{k_x g_1}{\Omega_i} \frac{\omega'}{\tau\omega' + \omega^*}\right] (\psi_{r+1} + \psi_{r-1}) = 0 \qquad \dots (32)$$

where $A(\omega)$ and q are given by (19).

The system (32) may be solved conveniently by regarding the imaginary terms The new order solutions are the well known Mathieu functions[5] $ce_{\frac{2n}{\nu}}(\xi)$ or $se_{\frac{2n}{\nu}}(\xi)$. The first order system is

$$\left[A^R - \left(2\frac{n}{\nu} + 2r \right)^2 \right] \psi_r^I - q^R \left(\psi_{r+1}^I + \psi_{r-1}^I \right) =$$

$$= - \psi_r^R \left\{ A^I + \frac{i}{|\frac{n}{\nu} + r|} \, 2 \left(\frac{2\omega}{k_1 v T_e} \right)^2 \frac{\sqrt{\pi}}{k_1 v T_e} \frac{\omega'}{\tau\omega' + \omega^*} \left(\omega - \omega^* + \tau \frac{k_x g_o}{\Omega_i} \right) + \right. \qquad \ldots (33)$$

$$\left. + \left[- q^I + \frac{1}{|\frac{n}{\nu} + r|} \left(\frac{2\omega}{k_1 v T_e} \right)^2 \frac{\sqrt{\pi}}{k_1 v T_e} \, \tau \frac{k_x g_1}{\Omega_i} \frac{\omega'}{\tau\omega' + \omega^*} \right] \frac{A^R - (2\frac{n}{\nu} + 2r)^2}{q^R} \right\}$$

Here A^R, A^I etc. means the real and imaginary parts of the eigen-value etc. On multiplying (33) by ψ_r^R and summing, the left-hand side subsequently cancels out and the right-hand side gives an equation for the growth rate (neglecting q^I as inessential):

$$\gamma = \frac{\sqrt{\pi}}{k_1 v_r} \frac{(\omega')^2}{1 - Q} \sum_{r=-\infty}^{\infty} \frac{(\psi_r^R)^2}{|\frac{n}{\nu} + r|} \left\{ \frac{(1+\tau)\left[Q - \frac{1}{2} \left(\frac{k_1 v T_i}{2\omega} \right)^2 A^R (1-Q) \right]}{1+\tau \left[Q - \frac{1}{2} \left(\frac{k_1 v T_i}{2\omega} \right)^2 A^R (1-Q) \right]} + (1+\tau) \frac{k_x g_o}{\omega^* \Omega_i} - \frac{\tau}{2} \frac{k_x g_1}{\omega^* \Omega_i} \frac{A^R - (2\frac{n}{\nu} + 2r)}{q^R} \right\} ,$$

$$\ldots (34)$$

with

$$\sum_{r=-\infty}^{\infty} (\psi_r^R)^2 = 1 \quad .$$

Now for q moderate there is a whole class of the standing wave solutions to (19), namely those where $\frac{2n}{\nu}$ is an integer. The most interesting one is the solution $se_1(\xi)$ as it has its antinodes in the unfavourably curved regions and samples the average of the adverse curvature. Then the second term in the curly bracket in (34) becomes important causing the growth rate of the $se_1(\xi)$ mode to predominate:

$$\frac{A^R - 1}{q^R} = - 1 - \frac{q}{8} + \ldots \qquad \ldots (35)$$

For the fractional order long wave modes with $\frac{2n}{\nu} < 1$ we get

$$\frac{A^R_{\frac{2n}{\nu}} - (\frac{2n}{\nu})^2}{q^R} \doteq \frac{1}{2\left[(\frac{2n}{\nu})^2 - 1 \right]} q \qquad \ldots (36)$$

and the last term in (34) is again positive (amplifying) proportional to q. Here, however, we do not get the standing waves as the odd and even modes are not distinguished by the growth rate.

10. CONCLUSION

Both the flute and long wavelength drift wave can be unstable in the model with periodic gravity. A principal flute mode interacts with the periodic geometry producing secondary oblique waves. It also can absorb energy from the perpendicular electron wobble motion. The result is a resonant electron instability which is strongest when the gravitational flute and electron drift mode have frequencies close to each other and both waves resonate. In the case of a pure drift mode the harmonics induced by the periodic curvature can combine into a standing wave, which effectively behaves as a localised wave modified by the local properties of the periodic geometry.

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