

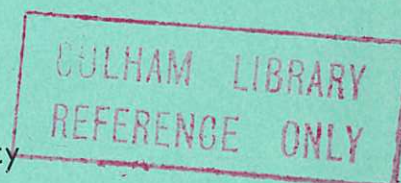
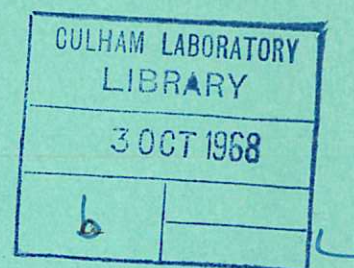
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Report



THE PROPERTIES OF LINEAR FILAMENTARY MULTIPOLE MAGNETIC FIELDS

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THE PROPERTIES OF LINEAR FILAMENTARY MULTIPOLE MAGNETIC FIELDS

by

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A B S T R A C T

This report is a catalogue of the properties of the vacuum magnetic fields generated by a class of linear multipoles.

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1. INTRODUCTION

Within the context of plasma confinement a multipole magnetic field is defined as that generated by a set of parallel conductors, all carrying currents in the same direction, the return current being carried either in a conducting outer wall or in external conductors^(1,2). Ideally the conductors are coaxial axisymmetric rings but linear systems are also of interest provided they are of sufficient length for end effects to be negligible.

Multipoles are of importance as plasma containment devices for the following reasons. Firstly, the field lines, which lie in planes perpendicular to the conductors, are closed. As a consequence the connection length, that is the distance over which a particle averages favourable and unfavourable stability properties, is short. This should be contrasted with the open line systems possessing magnetic surfaces, where particles may have to travel much larger distances in order to detect, or produce, the average surface properties on which the stability depends. Secondly, the magnetic fields are such that there exist closed regions of space in which the conditions $\nabla p \times \nabla U = 0$ and $\nabla p \cdot \nabla U > 0$ can be satisfied simultaneously, where p is the (scalar) plasma pressure and $U = \oint d\ell/|B|$, the integral being taken once around a closed field line⁽³⁾. The former ensures low pressure MHD equilibrium while the latter is theoretically sufficient for the stability of (low pressure) interchange modes. Thirdly, provided the fields are both invariant in the direction parallel to the conductors and independent of time, and collisions are negligible, particles of a given energy and 'axial' momentum are absolutely confined within a finite region of space⁽⁴⁾.

The major disadvantage of the multipole, at least from a fusion reactor standpoint, is that the regions of stability and absolute confinement both completely encircle the conductors, thereby making their support, cooling and current supply technologically difficult.

The particular class of multipole fields whose properties are listed in this report is that generated by an even number of infinitely long, linear, current filaments placed symmetrically, in vacuo, within a perfectly conducting (flux conserving) cylindrical wall.

The justification for studying this model is firstly that it is particularly amenable to analysis, thus enabling a catalogue of its properties to be drawn up in considerable detail. Secondly, the use of infinitesimally thin filaments is not particularly restrictive since solid conductors can always be placed on the almost circular field lines close to the singularities without disturbing the magnetic fields elsewhere. Thirdly, in closed toroidal

multipoles the purely toroidal effects are in general so small or lie sufficiently close to a $1/r$ dependence that they can be neglected. Finally the results derived from the model are sufficiently simple in form for them to be of considerable value when studying the behaviour of the plasma itself, particularly those aspects which do not depend on the very detailed nature of the field configuration; overall stability characteristics⁽⁵⁾, diffusion⁽⁶⁾ and single particle motion⁽⁷⁾, for example.

The following sections establish the model in detail and describe the properties of the resultant vacuum fields. The last section deals briefly with the effects of adding an axial field component.

No attempt is made in this report to discuss the physical significance of any of the properties derived.

2. THE COMPLEX POTENTIAL FUNCTION

Consider a vacuum magnetic field $\vec{B}(x,y) = (B_x(x,y), B_y(x,y), 0)$ generated by currents flowing perpendicular to the x - y plane. This field can be described either in terms of a magnetic potential $\phi(x,y)$ defined by

$$\vec{B} = -\nabla\phi$$

or by a magnetic stream function $\chi(x,y)$ defined by

$$\vec{B} = \nabla \times (\hat{k}\chi),$$

where \hat{k} is a unit normal to the x - y plane. The lines $\chi(x,y) = \text{const.}$ and $\phi(x,y) = \text{const.}$ are mutually orthogonal, the former being the magnetic field lines. Thus each field line is labelled by a particular value of χ while each point on a line is labelled by ϕ .

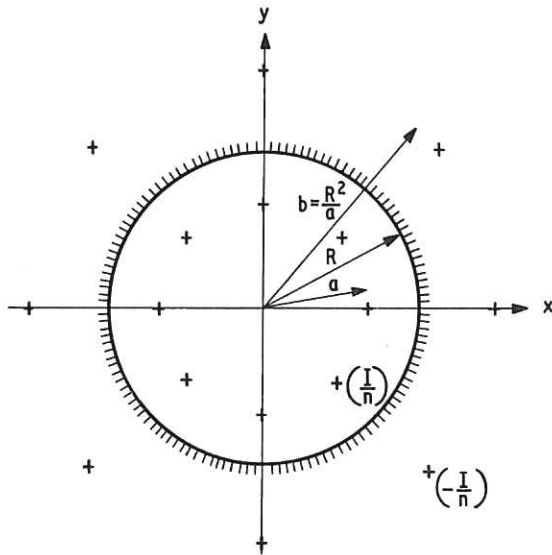


Fig.1 Linear multipole geometry (CLM-R 95)

Since both ϕ and χ satisfy Laplace's equation, a complex potential $\Psi(z)$, an analytic function of $z = x + iy$, can be defined by

$$\Psi(z) = \chi(x,y) + i\phi(x,y).$$

Direct differentiation shows

$$|B| = |\nabla\phi| = |\nabla\chi| = |d\Psi/dz|.$$

The complex potential function due to a set of n filamentary conductors (Fig.1), each carrying a current I/n , spaced equally and symmetrically around a circle of radius a

inside a perfectly conducting cylinder of radius R can be written

$$\Psi(z) = -\frac{2I}{n} \ln \left\{ \left(\frac{b}{a} \right)^n \cdot \frac{z^n - a^n}{z^n - b^n} \right\}, \quad \dots (1)$$

where the cylinder has been replaced by n image currents $-I/n$ on a circle of radius $b = R^2/a$. The factor $(b/a)^n$ has been introduced to give a convenient normalisation ($\chi = 0$) at the origin. Defining $\delta = (a/R)^{2n}$, equation (1) then becomes

$$\Psi(z) = \frac{2I}{n} \ln \frac{1 - \left(\frac{z}{a} \right)^n}{1 - \delta \left(\frac{z}{a} \right)^n}. \quad \dots (2)$$

3. FIELD LINE GEOMETRY

It is convenient to rewrite equation (2) in the form

$$qe^{i\Phi} = \frac{1 - \left(\frac{z}{a} \right)^n}{1 - \delta \left(\frac{z}{a} \right)^n} \quad \dots (3)$$

where now $q \equiv e^{-n\chi/2I}$ labels a field line and $\Phi \equiv -\frac{n\phi}{2I}$ labels a point on it.

The modulus of this equation, with $z = re^{i\theta}$, then immediately generates the polar equation of a field line ($q = \text{const}$):

$$\left(\frac{r}{a} \right)^n = \frac{1 - \delta q^2}{1 - \delta^2 q^2} \left[\cos n\theta \pm \sqrt{q_1^2 - \sin^2 n\theta} \right], \quad \dots (4)$$

where $q_1 = q(1-\delta)/(1-\delta q^2)$ and the negative square root applies only for $q < 1$.

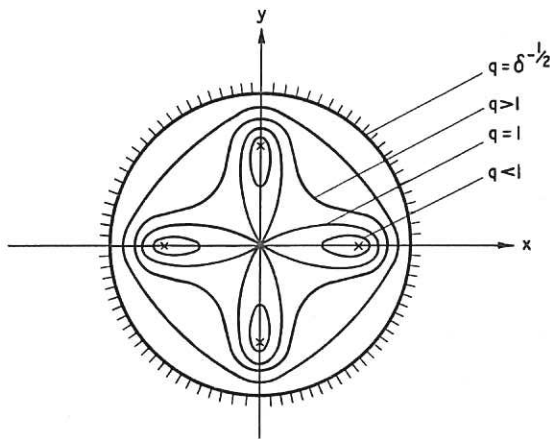


Fig. 2 (CLM-R 95)
Typical octopole ($n = 4$) field configuration

All field lines are closed, c.f. Fig.2, those ($0 < q < 1$) encircling only one filament being separated from those ($1 < q \leq \delta^{-1/2}$) encircling all the filaments by the separatrix ($q = 1$, $\chi = 0$). The separatrix, the equation of which is

$$\left(\frac{r}{a} \right)^n = \frac{2 \cos \theta}{1 + \delta},$$

consists of n 'petals' radiating from the centre, each subtending an angle $\theta = \pi/n$ at the origin and extending to a maximum radius $2^{1/n}(1 + \delta)a$, c.f. Fig.3.

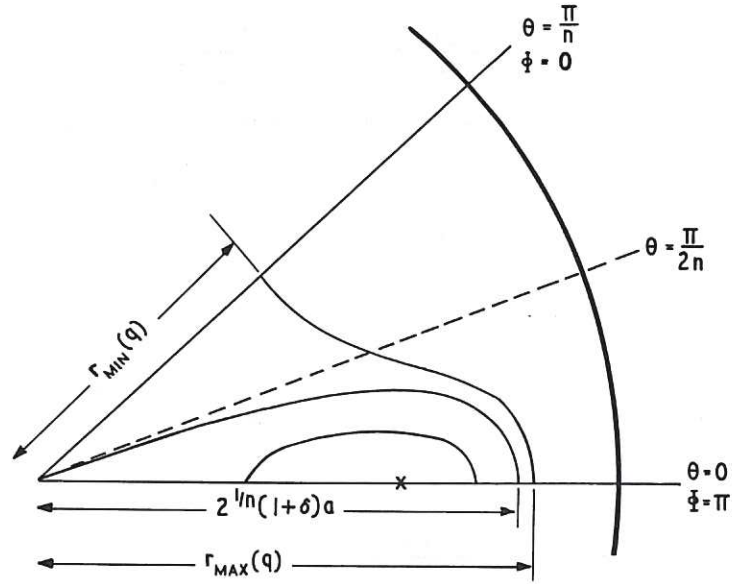


Fig. 3 (CLM-R 95)
A segment of the general field configuration

All lines oscillate between the radii

$$r_{\max}(q) = \left(\frac{1+q}{1+\delta q} \right)^{1/n} a$$

and

$$r_{\min}(q) = \left| \frac{1-q}{1-\delta q} \right|^{1/n} a.$$

Thus the effect of the cylindrical wall, i.e. $\delta > 0$, is to push the field lines closer together and to the filament along the radii $\theta = 0, \pm 2\pi/n, \pm 4\pi/n$, etc. and to pull them apart and away from the origin along the radii $\theta = \pm \pi/n, \pm 3\pi/n$, etc..

In the evaluation of several functions of position it is convenient to be able to make transformations between the polar coordinates (r, θ) and the magnetic coordinates (q, Φ) . This is facilitated by making use of the four functions

$$\begin{aligned} F_1(q, \Phi) &= \left| 1 - qe^{i\Phi} \right|^2 = 1 - 2q \cos \Phi + q^2 \\ F_2(q, \Phi) &= \left| 1 - \delta qe^{i\Phi} \right|^2 = 1 - 2\delta q \cos \Phi + \delta^2 q^2 \\ G_1(r, \theta) &= \left| 1 - \left(\frac{r}{a} \right)^n \right|^2 = 1 - 2 \left(\frac{r}{a} \right)^n \cos n\theta + \left(\frac{r}{a} \right)^{2n} \\ G_2(r, \theta) &= \left| 1 - \delta \left(\frac{r}{a} \right)^n \right|^2 = 1 - 2\delta \left(\frac{r}{a} \right)^n \cos n\theta + \delta^2 \left(\frac{r}{a} \right)^{2n} \end{aligned}$$

which are related by

$$F_1 = \frac{(1-\delta)^2}{G_2} \left(\frac{r}{a} \right)^{2n},$$

$$F_2 = \frac{(1-\delta)^2}{G_2}$$

and have the property

$$\frac{F_1}{F_2} = \left(\frac{r}{a}\right)^{2n}$$

$$\frac{G_1}{G_2} = q^2.$$

Thus at the point (r, θ) , q and Φ are given by

$$q = \left(\frac{G_1}{G_2}\right)^{1/2},$$

$$\cos \Phi = \frac{G_1 + G_2 - (1-\delta)^2 \left(\frac{r}{a}\right)^{2n}}{2(G_1 G_2)^{1/2}}$$

and at the point (q, Φ) , r and θ are given by

$$\frac{r}{a} = \left(\frac{F_1}{F_2}\right)^{1/2n},$$

$$\cos n\theta = \frac{F_1 + F_2 - (1-\delta)^2 q^2}{2(F_1 F_2)^{1/2}}.$$

4. LINE CURVATURE

The radius of curvature R_C at any point on a field line may be obtained by evaluating either

$$R_C = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left[r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}\right]}$$

or

$$\frac{1}{R_C} = \left(\frac{\partial |B|}{\partial \chi}\right)_\Phi$$

The result is, for $q \geq 1$

$$\frac{R_C}{a} = q_1 \left(\frac{1-\delta q^2}{1-\delta^2 q^2}\right)^{1/n} \frac{\left[\cos n\theta + \sqrt{q_1^2 - \sin^2 n\theta}\right]^{1/n}}{\left[n \cos n\theta + \sqrt{q_1^2 - \sin^2 n\theta}\right]}$$

... (5)

$$= \frac{q_1}{(1+\delta q^2)} \frac{F_1^{\frac{n+1}{2n}} F_2^{\frac{n-1}{2n}}}{n(1-q_2 \cos \Phi) + q_1(q_2 - \cos \Phi)},$$

where $q_2 = q(1+\delta)/(1+\delta q^2)$ and R_C is positive inwards.

On any particular line therefore the curvature lies between the limits

$$-\frac{(n-q_1)}{q_1} \left(\frac{1-\delta q}{q-1} \right)^{1/n} \leq \frac{a}{R_c} \leq \frac{(n+q_1)}{q_1} \left(\frac{1+\delta q}{q+1} \right)^{1/n} . \quad \dots (6)$$

When $q_1 < n$, the curvature oscillates in sign, the zero occurring at

$$n\theta = \cos^{-1} \left[-\sqrt{\frac{q_1^2-1}{n^2-1}} \right] ,$$

$$\Phi = \cos^{-1} \left[\frac{n+q_1q_2}{q_1+nq_2} \right] ,$$

where $n\theta$ lies in the quadrant $\pi/2 < n\theta < \pi$. When $q_1 > n$ the curvature is everywhere positive. The extreme curvatures given by the inequality (6) are shown in Figs.4 and 5.

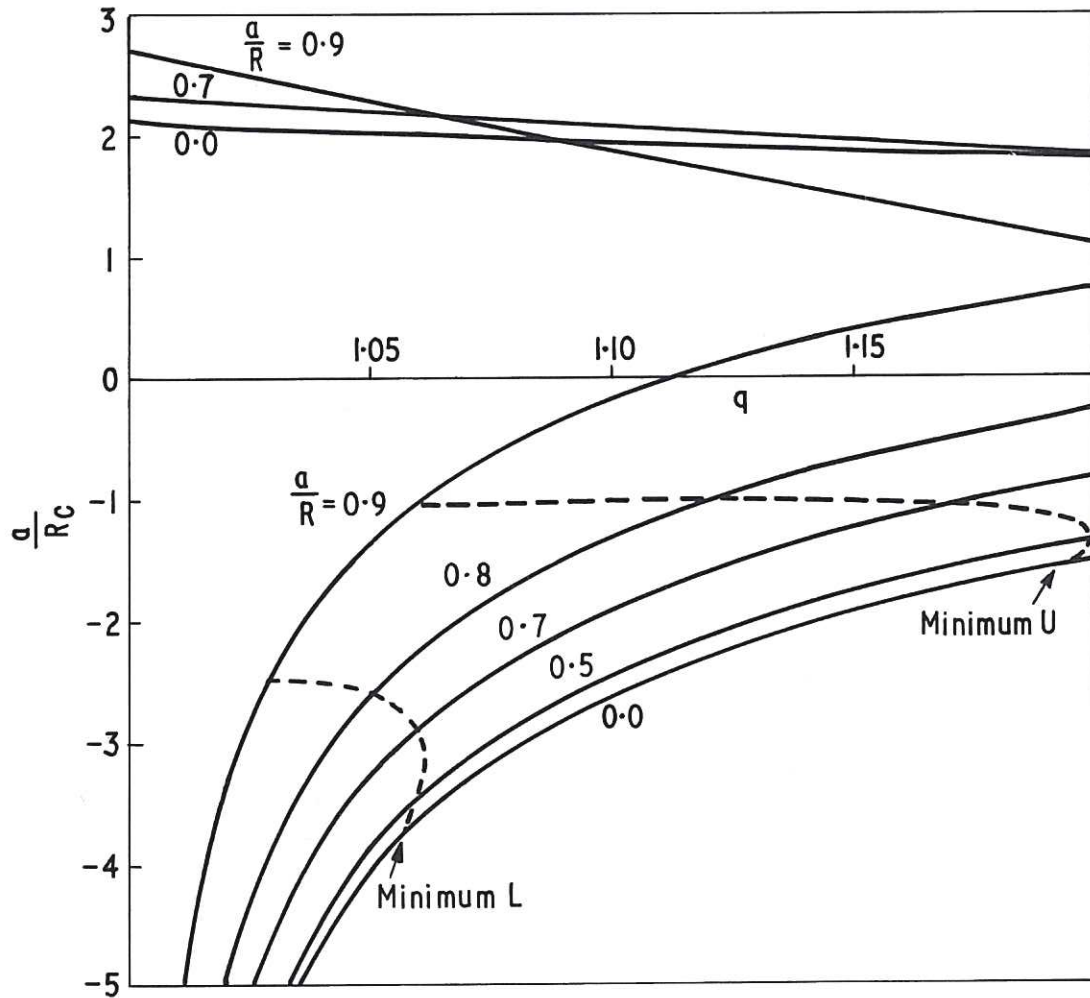


Fig. 4 (CLM-R 95)
Extreme curvatures for a quadrupole

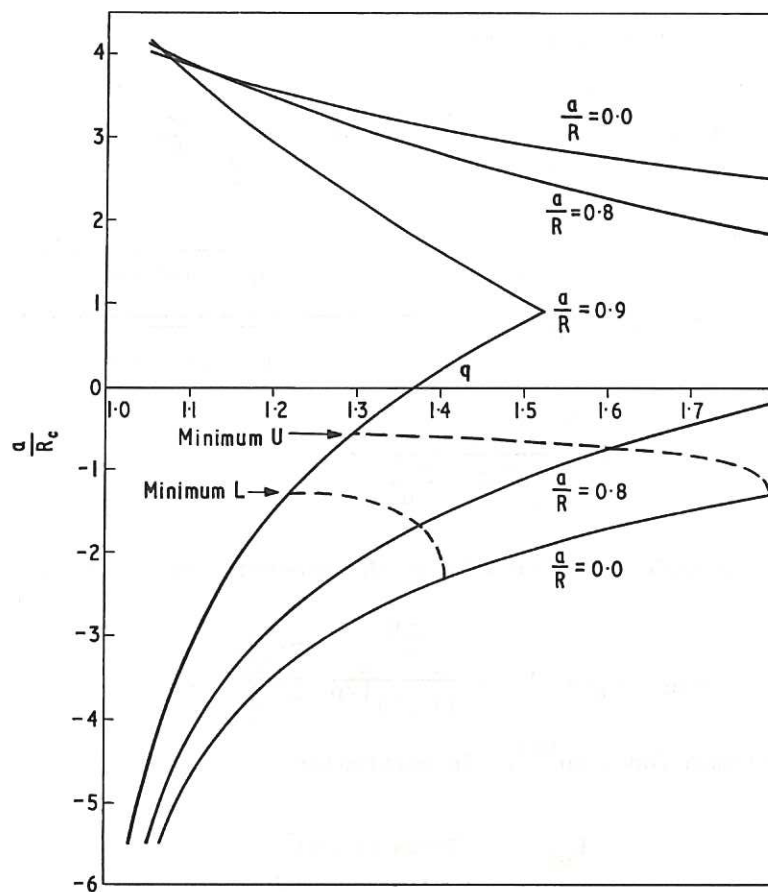


Fig. 5 (CLM-R 95)
Extreme curvatures for an octopole

The values of $q(=q_m)$ corresponding to $q_1 = n$, the last line with any negative curvature, are given in Table I (see also Section 7).

TABLE I

a/R	q_m	
	n = 2	n = 4
0.0	2.00	4.00
0.1	2.00	4.00
0.2	2.00	4.00
0.3	1.96	4.00
0.4	1.88	4.00
0.5	1.77	3.80
0.6	1.57	3.35
0.7	1.40	2.61
0.8	1.25	1.90
0.9	1.12	1.37

5. LINE LENGTH

The length of a field line is given by

$$\begin{aligned}
 L_n(q) &= \oint d\ell = \int_0^{2\pi} \left[r^2 + \left(\frac{dr}{d\theta} \right)^2 \right]^{1/2} d\theta = 4\Gamma \int_0^\pi \frac{d\Phi}{|B|} \\
 &= 2aq_1 \left(\frac{1 - \delta q^2}{1 - \delta^2 q^2} \right)^{1/n} \int_0^\pi \frac{\left[\cos n\theta + \sqrt{q_1^2 - \sin^2 n\theta} \right]^{1/n}}{\sqrt{q_1^2 - \sin^2 n\theta}} d(n\theta) \\
 &= 2aq(1 - \delta) \int_0^\pi \frac{d\Phi}{F_1^{\frac{n-1}{2n}} F_2^{\frac{n+1}{2n}}} \quad \dots (7)
 \end{aligned}$$

This cannot be integrated analytically in the general case. For $q = 1$, however

$$L_{sn} = L_n(q=1) = \frac{2^{\frac{2-n}{n}} a}{(1+\delta)^{1/n}} \frac{\Gamma^2\left(\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{n}\right)},$$

where $\Gamma(x)$ is the Gamma function⁽⁸⁾. In particular

$$\begin{aligned}
 L_{s2} &= 7.42a (1+\delta)^{-1/2} \\
 L_{s4} &= 11.04a (1+\delta)^{-1/4}.
 \end{aligned}$$

The length $L_n(q)$ is not a monotonic function of q , but has a minimum at $q = q_\ell$. The variation of q_ℓ with a/R is shown in Figs.10 and 11.

The distance between the minimum L surface and the separatrix varies between the limits

$$\begin{aligned}
 \Delta_{Lmax} &= r_{\min}(q_\ell) \\
 \Delta_{Lmin} &= r_{\max}(q_\ell) - r_{\max}(1).
 \end{aligned}$$

The dependence of these limits on a/R is shown in Fig.12.

6. AREA

The area, or alternatively volume/unit length, enclosed by a field line is given by

$$\begin{aligned}
 A_n &= \frac{1}{2} \int_0^{2\pi} r^2 d\theta \\
 &= \left(\frac{1 - \delta q^2}{1 - \delta^2 q^2} \right)^{\frac{2}{n}} a^2 \int_0^\pi \left[\cos n\theta + \sqrt{q_1^2 - \sin^2 n\theta} \right]^{2/n} d(n\theta). \quad \dots (8)
 \end{aligned}$$

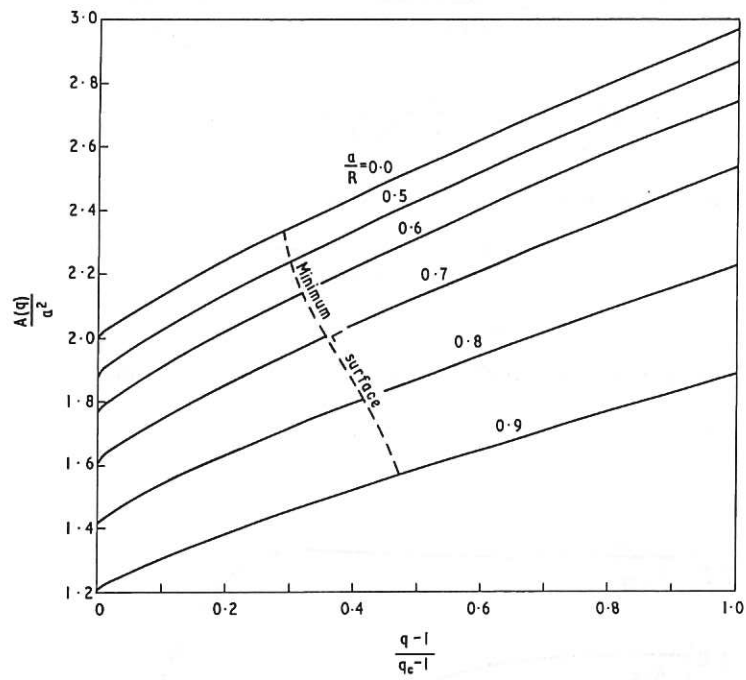


Fig. 6 (CLM-R 95)
The area within a field line q for a quadrupole

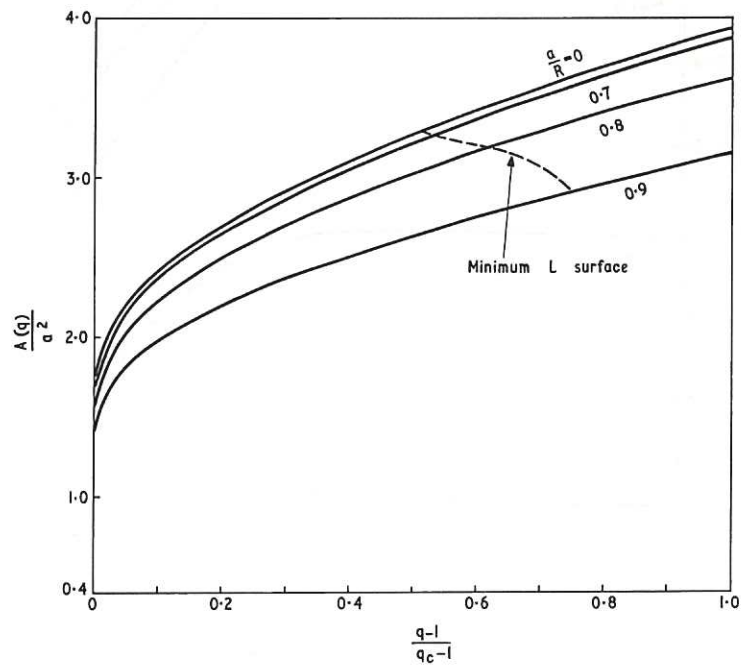


Fig. 7 (CLM-R 95)
The area within a field line q for an octopole

This is not in general integrable, but two useful particular results can be obtained, namely

$$A_{ns} = A_n(q=1) = \frac{a^2}{(1+\delta)^{2/n}} 2^{\frac{2-n}{n}} \sqrt{\pi} \frac{\Gamma(\frac{1}{n} + \frac{1}{2})}{\Gamma(\frac{1}{n} + 1)},$$

$$A_2(q) = 2a^2 \frac{(1-\delta^2)}{(1-\delta^4 q^2)} q E\left(\frac{1}{q^2}\right),$$

where the complete elliptic integral $E(k^2)$ is given by⁽⁸⁾

$$E(k^2) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t} \, dt.$$

Equation (8) has been evaluated numerically for $n = 2$ and 4 , and the results are displayed in Figs.6,7 and 8.

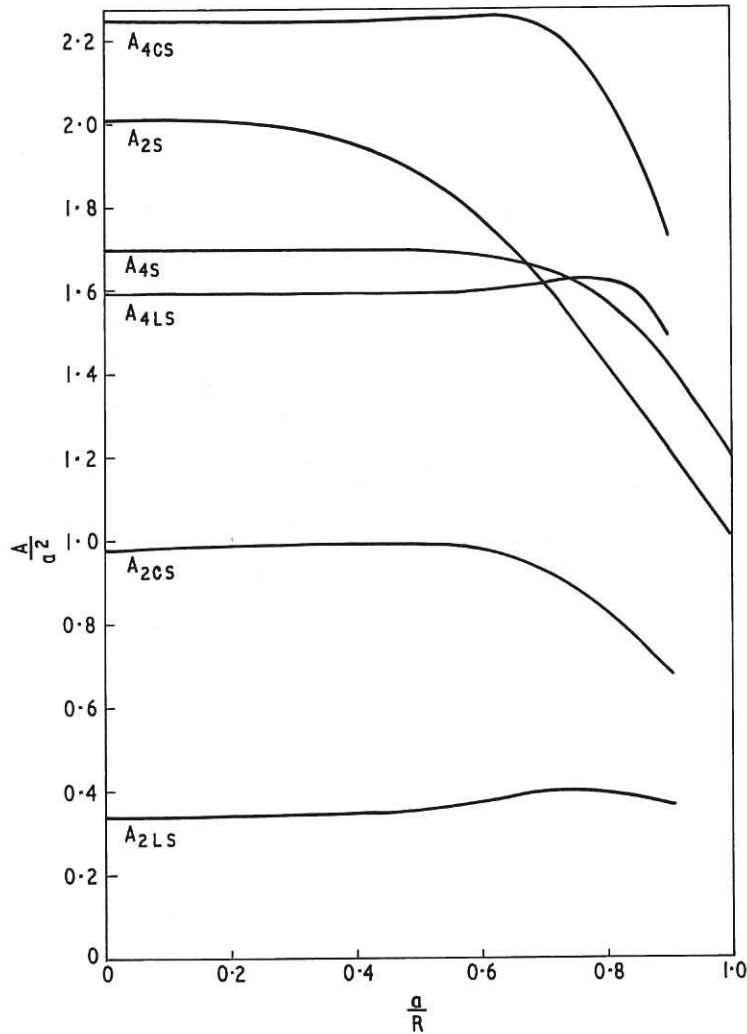


Fig. 8

(CLM-R 95)

The areas within the separatrix (A_{ns}), between the critical surface and separatrix (A_{nCS}), and between the minimum L surface and separatrix (A_{nLS})

7. MAGNETIC FIELD

The magnitude of the magnetic field can be calculated directly from

$$|B| = \left| \frac{d\psi}{dz} \right| = \frac{2I}{a} \frac{(1-\delta) \left| \frac{z}{a} \right|^{n-1}}{\left| 1 - \left(\frac{z}{a} \right)^n \right| \left| 1 - \delta \left(\frac{z}{a} \right)^n \right|}$$

and is most conveniently expressed as a function of Φ ,

$$|B| = \frac{2I}{a} \frac{F_1^{\frac{n-1}{2n}} F_2^{\frac{n+1}{2n}}}{q(1-\delta)} \quad \dots (9)$$

The magnetic field varies around a field line between the limits $|B|_{\min}$, at $\theta = \pi/n$, $\Phi = 0$ etc. and $|B|_{\max}$, at $\theta = 0$, $\Phi = \pi$, etc., where

$$|B|_{\min} = \frac{2I}{a} \frac{|q-1|}{q}^{\frac{n-1}{n}} \frac{(1-\delta q)}{(1-\delta)}^{\frac{n+1}{n}}$$

and

$$|B|_{\max} = \frac{2I}{a} \frac{(q+1)}{q}^{\frac{n-1}{n}} \frac{(1+\delta q)}{(1-\delta)}^{\frac{n+1}{n}}.$$

$|B|_{\max}$ is a monotonically decreasing function of q , falling from $(2I/a) 2^{\frac{n-1}{n}} (1+\delta)^{\frac{n+1}{n}} / (1-\delta)$ on the separatrix to $(2I/R) (1+\sqrt{\delta}) / (1-\sqrt{\delta})$ at the wall. $|B|_{\min}$ rises from zero on the separatrix to a maximum, or more precisely a saddle point, B_m on the line $q=q_m$ (Section 4), falling to $(2I/R)(1-\sqrt{\delta}) / (1+\sqrt{\delta})$ at the wall. The contour $|B| = B_m$ therefore bounds the true minimum B region and all $|B|$ contours with $|B| > B_m$ intersect the wall (Fig.9). The dependence of B_m on a/R is shown in Table II.

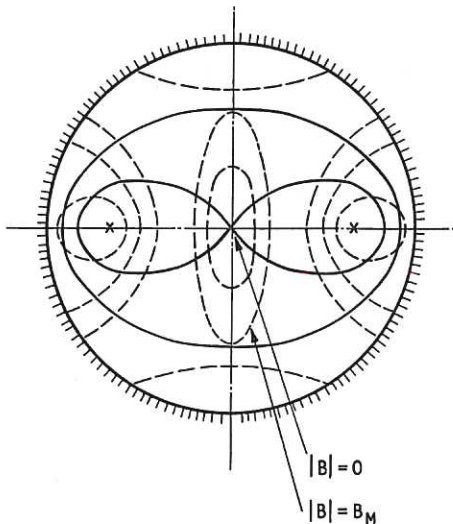


Fig. 9 (CLM-R 95)
Typical $|B|$ contours (dashed) for a quadrupole

TABLE II

a/R	$\frac{a}{I} B_m$	
	n = 2	n = 4
0.0	1.000	1.140
0.1	0.999	1.140
0.2	0.997	1.140
0.3	0.984	1.140
0.4	0.951	1.140
0.5	0.887	1.122
0.6	0.785	1.072
0.7	0.643	0.948
0.8	0.462	0.723
0.9	0.245	0.400

The mirror ratio on the line q is given by

$$M_n(q) = \left(\frac{q+1}{q-1} \right)^{\frac{n-1}{n}} \left(\frac{1+\delta q}{1-\delta q} \right)^{\frac{n+1}{n}}, \quad \dots (10)$$

the value at the wall being $(1+\sqrt{\delta})^2/(1-\sqrt{\delta})^2$. It does not fall monotonically with increasing q but has a minimum on the line

$$q = \left[\frac{1}{\delta} \frac{n(1+\delta) - (1-\delta)}{n(1+\delta) + (1-\delta)} \right]^{\frac{1}{2}}.$$

This corresponds to a line lying between the last line with negative curvature (q_m) and the wall ($q = \delta^{-1/2}$).

8. VOLUME OF A UNIT FLUX TUBE

The volume of a unit flux tube U can be evaluated from either

$$U(q) = - \frac{dA}{d\chi} = + \frac{nq}{2I} \frac{dA}{dq}$$

or

$$U(q) = \oint \frac{d\ell}{|B|} = 4I \int_0^\pi \frac{d\Phi}{|B|^2},$$

giving

$$U_n(q, \delta) = \frac{a^2}{I} q^2 (1-\delta)^2 \int_0^\pi \frac{d\Phi}{F_1^{\frac{n-1}{n}} F_2^{\frac{n+1}{n}}}.$$

This is integrable in terms of complete elliptic integrals⁽⁸⁾ for two special cases: when $n = 2$ and when $n = 4$, $\delta = 0$.

$$U_2(q, \delta) = \frac{2a^2}{I} \frac{q_1}{(1-\delta^2 q^2)} \left[(1-\delta q^2) K\left(\frac{1}{q_1}\right) - \frac{2\delta(1-\delta)q^2}{(1-\delta^2 q^2)} E\left(\frac{1}{q_1}\right) \right],$$

$$U_4(q, 0) = \frac{2a^2}{I} \frac{q^{3/2}}{(q^2-1)^{1/2}} K\left(\frac{q-(q^2-1)^{1/2}}{2q}\right),$$

where

$$K(k^2) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2 \sin^2 t}}$$

and $E(k^2)$ is defined in Section 6.

The most important property of a multipole is the location of the critical surface, denoted by a subscript c , on which $dU/d\chi = 0$. The dependencies of q_c , U_c and $|\chi_c|$ (the flux between the separatrix and the critical surface) on a/R are shown in Figs. 10 and 11.

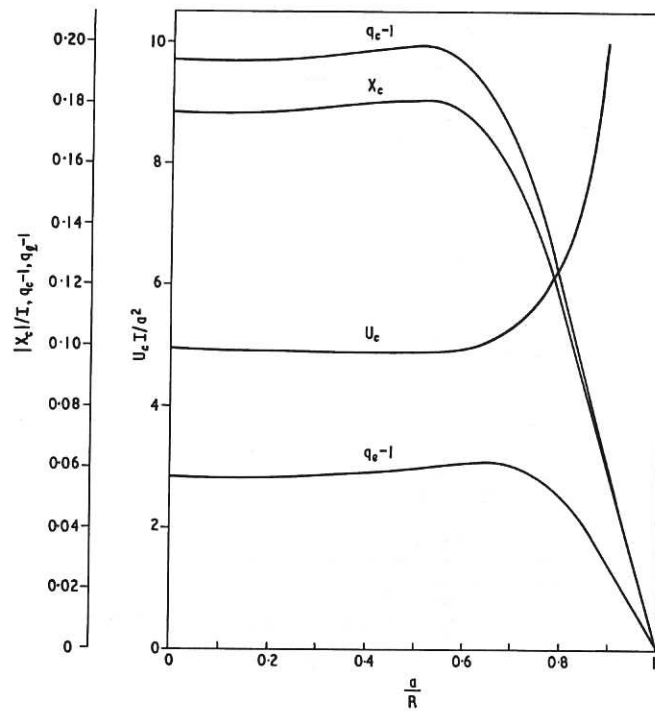


Fig. 10 (CLM-R 95)
Minimum U and minimum L properties for a quadrupole

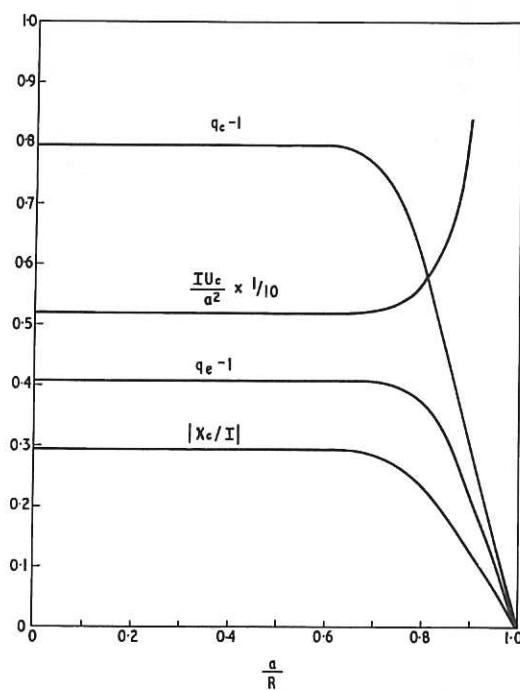


Fig. 11 (CLM-R 95)
Minimum U and minimum L properties for an octopole

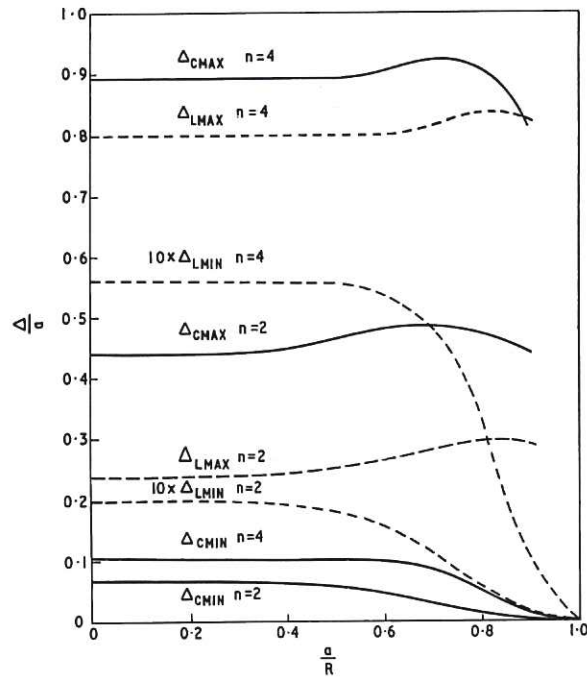


Fig. 12 (CLM-R 95)
Separation of the critical and minimum L surfaces from the separatrix

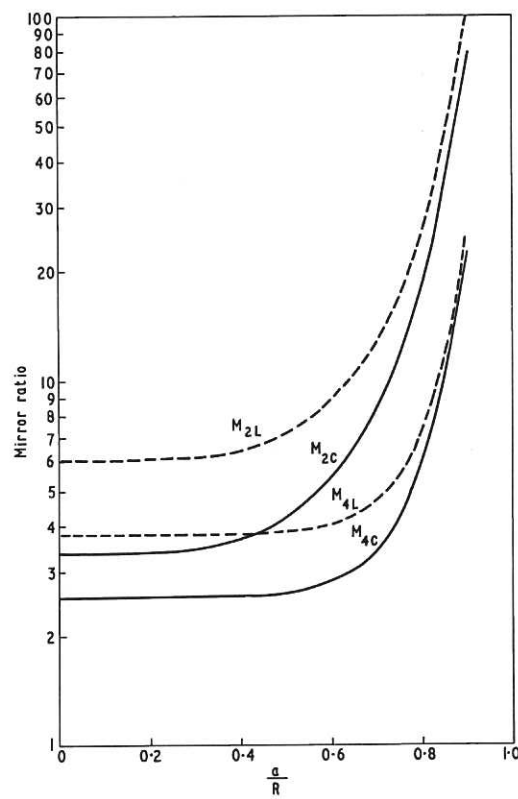


Fig. 13 (CLM-R 95)
Mirror ratios on the critical and minimum L surfaces

A measure of the 'efficiency' with which the magnetic flux is being used can be obtained by comparing χ_c with the flux χ_ω between the separatrix and the wall, where

$$\chi_\omega = 2I \ln(a/R) .$$

For $a/R = 0.7$, $\chi_c/\chi_\omega = 0.22$ for a quadrupole and 0.4 for an octopole.

The distance between the critical surface and the separatrix varies between the limits

$$\Delta_{c \max} = r_{\min}(q_c)$$

and

$$\Delta_{c \min} = r_{\max}(q_c) - r_{\max}(1) .$$

The dependence of these limits on a/R is shown in Fig.12.

The variation of the mirror ratio on the critical surface is shown (with the equivalent quantity for the minimum L surface) in Fig.13. (Note that the scale is logarithmic).

9. THE INTEGRAL $\oint d\ell/|B|^3$

An integral of interest from the point of view of Universal stability is

$$W(q) = \oint \frac{d\ell}{|B|^3} = 4I \int_0^\pi \frac{d\Phi}{|B|^4} = 2a \left(\frac{a}{2I}\right)^3 q^4 (1-\delta)^4 \int_0^\pi \frac{d\Phi}{F_1^n F_2^n} .$$

This has been evaluated for the particular cases: $n = 2$ and $n = 4$, $\delta = 0$.

$$W_2(q, \delta) = 2\pi a \left(\frac{a}{2I}\right)^3 \frac{(1-\delta)^4 q^4}{q^2-1} \frac{(1+4\delta^2 q^2 + \delta^4 q^4)(1+\delta^3 q^3) + 3\delta q^2 [(1-\delta q^2)(1-\delta^4 q^4) - 2\delta^2 q^2(1+\delta q^2)]}{(1-\delta q^2)^3 (1-\delta^2 q^2)^5}$$

$$W_4(q, 0) = 4a \left(\frac{a}{2I}\right)^3 \frac{q^3}{q^2-1} \left[\frac{2q^2}{q^2-1} E\left(\frac{1}{q^2}\right) - K\left(\frac{1}{q^2}\right) \right] .$$

These integrals have been plotted in Figs.14 and 15 in the way which is probably most useful from the point of view of stability theory.

10. ROTATIONAL TRANSFORM

The addition of a uniform axial field B_0 causes the magnetic field lines to break open into helices. They still lie on the original surfaces, however, and their projection onto planes perpendicular to the filaments remain unchanged. The helical pitch λ_p and

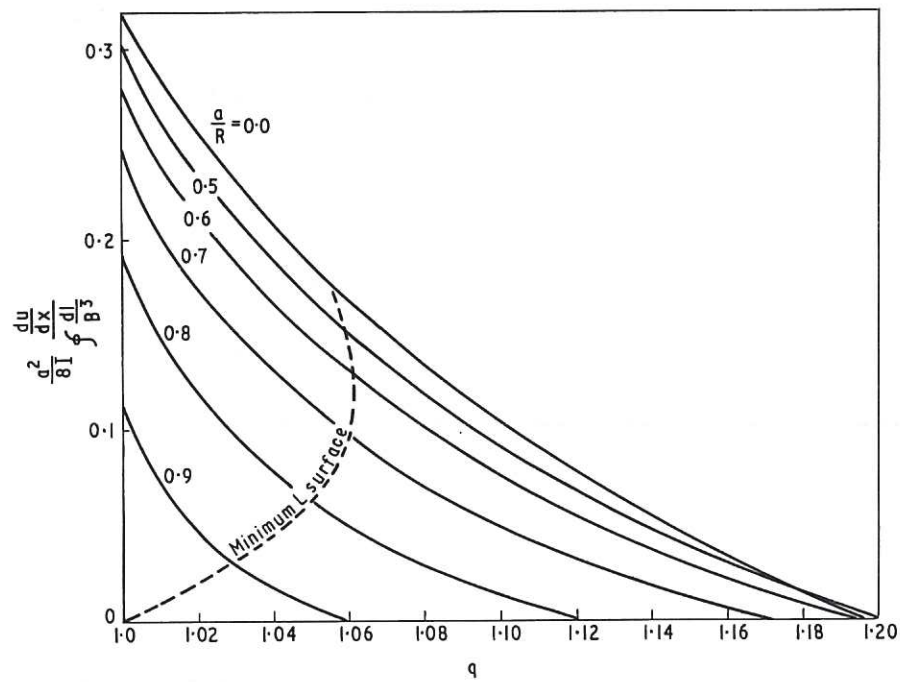


Fig. 14 (CLM-R 95)
Variation of $(dU/dx)/\int dI/B^3$ with q for a quadrupole

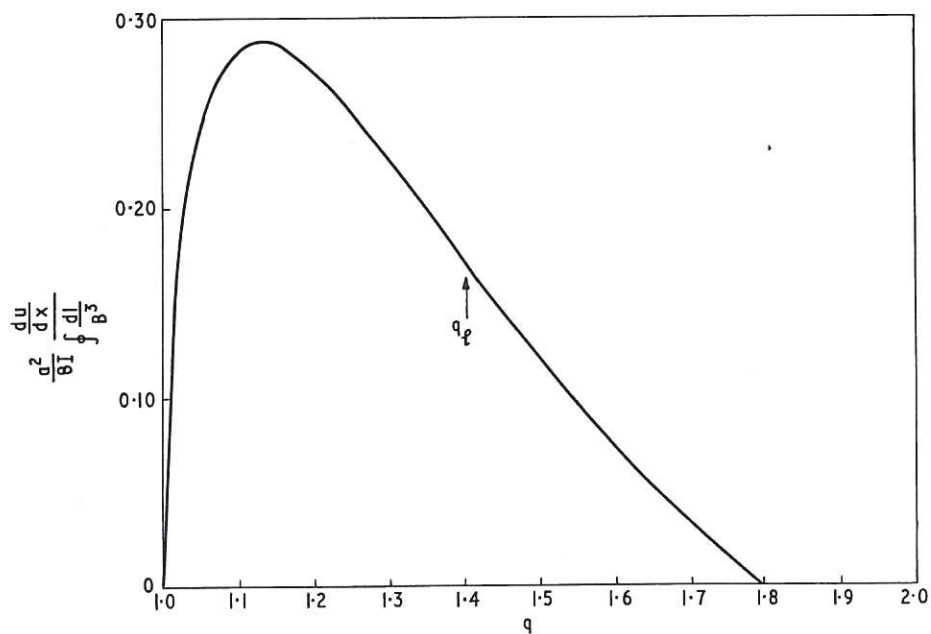


Fig. 15 (CLM-R 95)
Variation of $(dU/dx)/\int dI/B^3$ with q for an 'unscreened' ($\frac{a}{R} \leq 0.6$) octupole

the corresponding rotational transform ι , that is the mean angle through which a line rotates about the axis in moving unit axial distance, are related by

$$\iota = \frac{2\pi}{\lambda_p} = \frac{2\pi}{B_0 U},$$

where U is evaluated as if the axial field were absent.

Thus ι has a maximum value ι_c corresponding to the minimum in U_c . Since shear is proportional to $d\iota/dx$, the edge of the MHD stable region must by definition be in a shearless field. The rotational transform is zero on the separatrix and since

$$U_n(\text{wall}) = \frac{\pi R^2}{I} \frac{1+\delta}{1-\delta}$$

it falls to ι_ω where,

$$\iota_\omega = \frac{2I}{B_0 R^2} \frac{1-\delta}{1+\delta}$$

The variation of ι_c and ι_ω is shown in Fig.16.

11. REFERENCES

1. OHKAWA, T. and KERST, D.W. Stable plasma confinement by multipole fields. Phys. Rev. Lett., vol.7, no.2, July 15, 1961, pp.41-42.
2. BRAGINSKII, S.I. and KADOMTSEV, B.B. Stabilisation of a plasma by the use of 'guard conductors'. In Plasma Physics and Problems of Controlled Thermonuclear Reactions (ed. Leontovich, M.A.), vol.III, pp.356-386. Oxford, Pergamon Press, 1959.
3. KADOMTSEV, B.B. Hydrodynamics of a low pressure plasma. In Plasma Physics and Problems of Controlled Thermonuclear Reactions (ed. Leontovich, M.A.), vol. IV, pp.17-25. Oxford, Pergamon Press, 1960.
4. ARTSIMOVICH, L.A. Controlled Thermonuclear Reactions. Chapter VII, pp.236-285, Magnetic traps : general principles. Edinburgh, Oliver & Boyd, 1964.
5. FISHER, D.L., HASTIE, R.J., McNAMARA, B. and TAYLOR, J.B. (Culham Laboratory). Paper in preparation.
6. HOBBS, G.D. and TAYLOR, J.B. Plasma diffusion in multipoles. Plasma Phys., vol.10, no.3, March 1968, pp.207-212.
7. HASTIE, R.J., HOBBS, G.D. and TAYLOR, J.B. (Culham Laboratory). Paper in preparation.
8. ABRAMOWITZ, M. and STEGUN, I.A. (eds.) Handbook of Mathematical Functions. New York, Dover Publications, 1965.

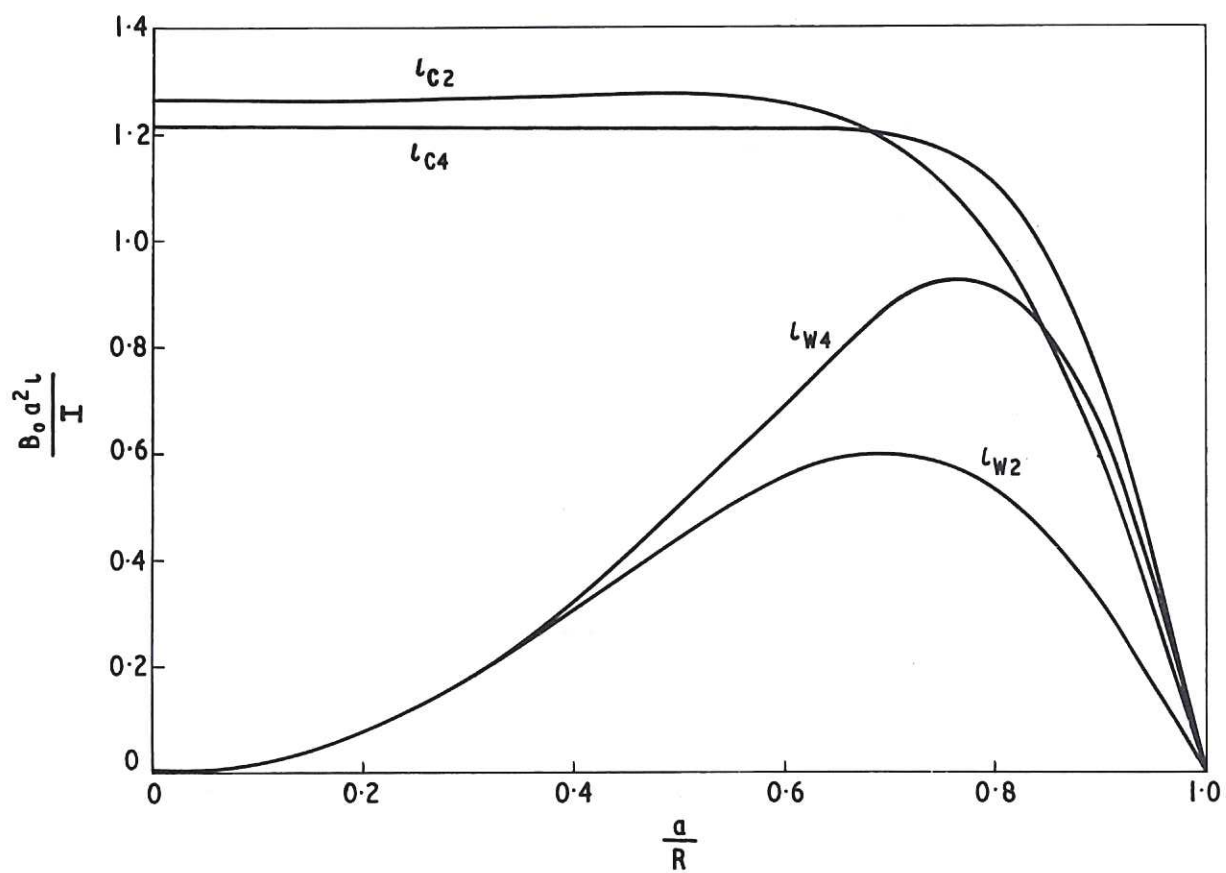


Fig. 16 (CLM-R 95)
The rotational transform on the critical surface and at the wall



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