CLM-R 97

CLM-R 97

United Kingdom Atomic Energy Authority

RESEARCH GROUP

Report

DYNAMIC STABILISATION OF UNSTABLE MHD EQUILIBRIA

A GENERAL APPROACH

G. BERGE

Culham Laboratory
Abingdon Berkshire

1969

Available from H. M. Stationery Office
EIGHT SHILLINGS NET

C.18

U.D.C.
533.951.8

DYNAMIC STABILISATION OF UNSTABLE MHD EQUILIBRIA

A GENERAL APPROACH

by

G. BERGE*

ABSTRACT

The basic equations used in this work are the ideal magnetohydrodynamic equations. A general theory is developed for investigating the stability of a plasma with an oscillatory motion. The oscillatory motion is assumed to be forced on the system and controlled from outside. It takes place around some equilibrium of the system. This equilibrium may be 'weakly' unstable or only marginally stable. 'Weakly' unstable in this context means that the characteristic frequency of any instability present in the static system must be much less than the frequency of the forced oscillations. The aim of the investigation is to determine whether the dynamic effect of such oscillations can improve the stability of the system.

First a vector equation is derived for the Lagrangian displacement vector of an arbitrary fluid element. This equation determines the stability of a hydromagnetic fluid in arbitrary motion. Then the assumption of a small amplitude periodic motion around some equilibrium is made. In order to deal with the explicit time dependence in the problem the multiple timescale method is used. An expansion is made in terms of a small parameter which is associated with the applied frequency and the amplitude of the oscillations. The first relevant equations obtained, result in stability conditions related to the energy principle for the static case. The results are not restricted to any particular geometry. The special case of a plasma separated from the vacuum region by a discontinuity surface is worked out. The problem of parametric resonance and associated instability is briefly discussed.

* On leave of absence from University of Bergen, Norway

U.K.A.E.A. Research Group,
Culham Laboratory,
Abingdon,
Berks.

May, 1968 (C/18 IMG)

C O N T E N T S

APPENDICES

PREFACE

Recently a discussion of the general problem of dynamic stabilisation of MHD equilibria and general stability criteria was published*. The theory was applied to axisymmetric $\beta = 1$ plasmas. This paper was, however, very condensed because of the strict limitations on the length of the papers being accepted on this occasion.

For this reason it is of interest to give a more complete and detailed discussion of the general theory than that presented in the paper referred to above. In the meantime it has also turned out to be possible to derive some of the results in a more straight-forward manner. There are, however, no essential differences, but for reasons of convenience ε^2 is replaced by ε as the expansion parameter. Moreover the basic equation governing the perturbed motion (second order in the time derivative) is expanded directly without first transforming it to two first order equations. The detailed calculations are, however, kept separate from the main text and are given in appendices.

* BERGE, G. IAEA, Third Conference on Plasma Physics and Controlled Nuclear Fusion research, Novosibirsk, U.S.S.R. (1968) paper CN-24/J-11.

1. INTRODUCTION

The stabilisation of a plasma in thermonuclear conditions seems to be one of the most difficult problems in the field of thermonuclear research. One possible way of improving the stability of a plasma contained by a magnetic field is by means of dynamic stabilisation.

The purpose of this report is to formulate a general approach to the problem of dynamic stablisation. The ideal magnetohydrodynamic (MHD) equations are used throughout. The idea of dynamic stabilisation is not new in plasma physics[1,2,3]. However, as far as we know, no general approach to this problem has previously been made.

The list of references given is to be regarded as representative rather than complete. There also exists an extensive literature, especially Russian, in a related topic commonly referred to as RF (radio-frequency) or HF (high-frequency) stabilisation (and confinement). This type of problem will, however, not be discussed here.

Section 2 deal with a generalisation of results given by Frieman and Rotenberg[4]. An equation is derived for determining the stability of a fluid in an arbitrary state of motion, and in Section 3 a Lagrangian and Hamiltonian formulation is presented.

In Section 4 a systematic way of solving this equation for systems with periodic motion is outlined. A general necessary condition for stability is derived. The results obtained are related to the energy principle[5] for the static case. The stability theory is in the linear regime. The MHD equations are, however, highly nonlinear. Therefore even an exact solution to the periodic motion is extremely difficult. However, the method used to solve the problem here does not depend on the knowledge of such a solution. The periodic motion is assumed to have a small amplitude. This is the basis for an asymptotic expansion, and the problem is solved successively order by order.

In Section 5 the general results are formulated as an energy principle. The problem of parametric resonance is briefly discussed in Section 6. Section 7 deals with the problem of a plasma separated from the vacuum region by a surface of discontinuity and in Section 8 the conclusions are given.

Dyadic notation is used mainly in agreement with Brand[6]. The 'double dot' symbol is used according to the definition:

$$\underline{a}\ \underline{b} : \underline{c}\ \underline{d} \equiv a_i d_i\ b_j c_j \qquad \ldots (1.1)$$

where $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ are vectors represented in an orthogonal system and the double suffix summation convention is used. The operator ∇ is defined by :

$$\nabla = \underline{e}_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \underline{r}} \quad . \quad \underline{r} = \underline{e}_i x_i \ , \qquad \ldots (1.2)$$

where $\underline{r}$ is the position vector. We shall use the convention that this operates only on the first quantity to the right, and a parenthesis counts as one quantity in this respect.

2. EQUATIONS OF MOTION

We list the equations of motion governing an ideal hydromagnetic fluid :

$$\rho \frac{d\underline{v}}{dt} = - \nabla p + \underline{J} \times \underline{B} - \rho \nabla \psi \ , \qquad \ldots (2.1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \ , \qquad \ldots (2.2)$$

$$\frac{d}{dt} (p \rho^{-\gamma}) = 0 \ , \qquad \ldots (2.3)$$

$$\underline{E} + \underline{v} \times \underline{B} = 0 \ , \qquad \ldots (2.4)$$

$$\nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t} \ , \qquad \ldots (2.5)$$

$$\nabla \times \underline{B} = \underline{J} \ , \qquad \ldots (2.6)$$

$$\nabla \cdot \underline{B} = 0 \ . \qquad \ldots (2.7)$$

All the symbols have their usual meaning; ρ, the mass density; t the time ; $\underline{v}$ the fluid velocity; $\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla$; p the kinetic pressure; $\underline{J}$ the current density; $\underline{B}$ the magnetic field; γ the ratio between the specific heats; $\underline{E}$ the electric field and $\nabla \psi$ an external force field.

We shall be concerned with systems where the following boundary conditions apply.

(i) At a fluid vacuum interface:

$$\langle p + \tfrac{1}{2} B^2 \rangle = 0 \ , \qquad \ldots (2.8)$$

$$\underline{n} \times \langle \underline{E} \rangle = \underline{n} \cdot \underline{v} \langle \underline{B} \rangle \ , \qquad \ldots (2.9)$$

$$\underline{n} \cdot \underline{B} = 0 \ , \qquad \ldots (2.10)$$

$$\underline{n} \times \langle \underline{B} \rangle = \underline{K} \ . \qquad \ldots (2.11)$$

Here $\langle X \rangle = X_{vacuum} - X_{plasma}$ is the jump in X across the fluid vacuum interface, $\underline{n}$ is the unit normal pointing outward from the plasma surface, and $\underline{K}$ is the surface current. The surfaces limiting the vacuum region are assumed to be rigid and to have either infinite conductivity or infinite resistivity. At a surface having infinite conductivity we must have

$$\underline{n} \cdot \underline{B} = 0 \qquad \ldots (2.12)$$

(ii) If the plasma extends to a rigid infinitely conducting wall we must have

$$\underline{n} \cdot \underline{v} = 0 \qquad \ldots (2.13)$$

at the wall in addition to eq.(2.12).

These boundary conditions have been discussed previously[4,5]. We have, however, rederived eqs.(2.8)-(2.11) in order to assure that they apply to the present situation, which is different from the situations previously considered. Details are given in Appendix B.

In all systems of practical interest from a fusion point of view, the plasma has to be kept away from material walls by a vacuum magnetic field. The boundary condition $\underline{n}\cdot\underline{v} = 0$ is therefore of little interest. When the plasma is separated from the walls by a vacuum magnetic field, it is often useful to deal with a sharp boundary as a first approximation to the solution of a particular problem. In this case the boundary conditions given by eqs.(2.8)-(2.11) apply.

The fluid motion is now perturbed and the perturbation is described in terms of a Lagrangian displacement $\underline{\xi}(r^o,t)$ of each fluid element. We take $\underline{r}^o$ to be the position vector the fluid element would have had at the particular time t if it were moving along its unperturbed trajectory. This is an extension of the technique developed by Frieman and Rotenberg[4] for dealing with stationary fluid motion. We have derived expressions for the perturbed quantities which are similar to those derived by Frieman and Rotenberg. Formally the results are the same, the differences enter only in the explicit time-dependence appearing in the unperturbed quantities in our case. The results are,

$$\rho(\underline{r}^o + \underline{\xi},\ t) = \rho(1 - \nabla\cdot\underline{\xi})\ , \qquad \ldots (2.14)$$

$$p(\underline{r}^o + \underline{\xi},\ t) = p(1 - \gamma\ \nabla\cdot\underline{\xi})\ , \qquad \ldots (2.15)$$

$$\underline{B}(\underline{r}^o + \underline{\xi},\ t) = \underline{B} - \underline{B}\ \nabla\cdot\underline{\xi} + \underline{B}\cdot\nabla\underline{\xi}\ , \qquad \ldots (2.16)$$

$$\underline{v}(\underline{r}^o + \underline{\xi},\ t) = \underline{v} + \underline{v}\cdot\nabla\underline{\xi} + \frac{\partial\underline{\xi}}{\partial t}\ , \qquad \ldots (2.17)$$

$$n(\underline{r}^o + \underline{\xi},\ t) = \underline{n} + \underline{n}\ \underline{n}\cdot\nabla\ \underline{\xi}\cdot\underline{n} - \nabla\underline{\xi}\cdot\underline{n}\ , \qquad \ldots (2.18)$$

where $\nabla = \frac{\partial}{\partial \underline{r}^o}$. The quantities ρ, p, $\underline{B}$, $\underline{v}$ and $\underline{n}$ are associated with the unperturbed motion, and they are all of the general form $q = q(\underline{r}^o,\ t)$. Furthermore we have the operator equation

$$\nabla' = \nabla - \nabla\underline{\xi}\cdot\nabla \qquad \ldots (2.19)$$

where $\nabla' = \frac{\partial}{\partial \underline{r}'}$, $\underline{r}' = \underline{r}^o + \underline{\xi}$, which has been used in obtaining eqs.(2.14)-(2.18). All these expressions are correct only to the first order in $\underline{\xi}$. Details of the derivation of eqs.(2.14)-(2.19) are given in Appendix A, where the complete expressions for the perturbed quantities correct to all orders in $\underline{\xi}$ are also obtained.

In the vacuum it is convenient to introduce the first order (with respect to $\underline{\xi}$)

scalar- and vector-potentials,

$$\underline{E}_{vac.} (\underline{r}^o + \underline{\xi}, t) = \underline{E}_{vac.} + \underline{\xi}\cdot\nabla \underline{E}_{vac.} - \nabla\varphi - \frac{\partial \underline{A}}{\partial t} , \qquad \ldots (2.20)$$

$$\underline{B}_{vac.} (\underline{r}^o + \underline{\xi}, t) = \underline{B}_{vac.} + \underline{\xi}\cdot\nabla \underline{B}_{vac.} + \nabla \times \underline{A} . \qquad \ldots (2.21)$$

By using the boundary conditions, eqs.(2.8) and (2.9), we can derive the following relationships between $\underline{\xi}$ and $\underline{A}$ which must be satisfied at a vacuum plasma interface

$$- \gamma\, p\, \nabla\cdot\underline{\xi} - \underline{\xi}\cdot\nabla p + \underline{B}\cdot\underline{Q} = \underline{B}_{vac.}\cdot \nabla \times \underline{A} + \underline{n}\cdot\underline{\xi}\, \underline{n}\cdot \langle\nabla(p + \frac{B^2}{2})\rangle \qquad \ldots (2.22)$$

and

$$\underline{n} \times \underline{A} = - \underline{n}\cdot\underline{\xi}\, \underline{B}_{vac.} , \qquad \ldots (2.23)$$

where

$$\underline{Q} = \nabla \times (\underline{\xi} \times \underline{B}). \qquad \ldots (2.24)$$

For details see Appendix B.

In the unperturbed state an arbitrary fluid element moves along the trajectory $\underline{r}^o = \underline{r}^o (\underline{r}_o, t)$, where $\underline{r}_o = \underline{r}^o (\underline{r}_o, 0)$ is the position vector of the fluid element at $t = 0$. In the perturbed state the same fluid element is following a different trajectory given by

$$\underline{r}' = \underline{r}^o(\underline{r}_o, t) + \underline{\xi}(\underline{r}^o, t). \qquad \ldots (2.25)$$

The unperturbed trajectories do not form stream-lines in the general case because the whole of the fluid motion is changing with time. It is, however, still meaningful to pose the question of the stability of such a system. The question is, will a small perturbation of the fluid motion cause an arbitrary fluid element to move along a trajectory which is arbitrarily close to the unperturbed trajectory for all t or not*. The Lagrangian displacement vector $\underline{\xi}$ is an adequate measure for this property of the system. Accordingly we take as a definition of stability:

The system is stable or unstable dependent on whether the equation determining $\underline{\xi}$ allows only solutions bounded in time or not.

This is essentially Lyapunov's definition of stability[7].

---

*However, one would in this connection count as unstable a situation where the trajectories for the perturbed and unperturbed state stay arbitrarily close for all time, but where the perturbed fluid element still moves away from the unperturbed reference fluid element. This distinction is valuable in classical mechanics when dealing with the problem of orbital stability. It is also difficult to rule out the possibility of similar situations arising in a continuum system. However, it is hardly of the same importance here. The boundaries will never be moved away by such perturbations for example.

The equation determining $\underline{\xi}$ is now obtained by substituting for the quantities listed in equations (2.14) to (2.17) into equation (2.1). After some manipulation where equations (2.1) and (2.2) governing the motion of the unperturbed system have been used, we obtain the equation

$$\rho \frac{\partial^2 \underline{\xi}}{\partial t^2} + 2\, \rho \underline{v} \cdot \nabla \frac{\partial \underline{\xi}}{\partial t} + \frac{\partial}{\partial t} (\rho \underline{v}) \cdot \nabla \underline{\xi} = \underline{F}(\underline{\xi}) \ , \qquad \ldots (2.26)$$

where

$$\underline{F}(\underline{\xi}) = \nabla(\gamma p\, \nabla \cdot \underline{\xi} + \underline{\xi} \cdot \nabla p) - \underline{B} \times (\nabla \times \underline{Q}) - \underline{Q} \times (\nabla \times \underline{B}) + \nabla \cdot (\underline{\xi} \rho \frac{d\underline{v}}{dt} - \rho \underline{v}\, \underline{v} \cdot \nabla\, \underline{\xi}) + \nabla \cdot (\rho \underline{\xi})\, \nabla \psi \ . \qquad \ldots (2.27)$$

The details of the derivation of these equations are given in Appendix C. The operator $\underline{F}$ can be proved to be self-adjoining for the boundary conditions under consideration. It is, however, a lengthy procedure which involves many integrations by parts and application of the equations governing the unperturbed motion. It should also be noticed that in the case of a vacuum region surrounding the plasma it is assumed that all material objects (conductors, walls) present in this region are rigid and have either infinite conductivity or infinite resistivity. If this is not true $\underline{F}$ will no longer be self-adjoining in general. The details of this proof are given in Appendix D.

3. THE LAGRANGIAN AND HAMILTONIAN FORMULATION

It is of some general interest to construct Lagrangian and Hamiltonian functionals from which equation (2.26) can be derived. So far we have not been able to make use of this more general formulation, but for the sake of completeness we shall briefly list the results here. A more detailed discussion, including some derivations is given in Appendix E.

The following quantity can be taken to be a Lagrangian density

$$\mathcal{L} = \frac{1}{2} \rho \left(\frac{\partial \underline{\xi}}{\partial t}\right)^2 + \rho \frac{\partial \underline{\xi}}{\partial t} \underline{v} : \nabla \underline{\xi} - \frac{1}{2}\, \underline{\xi} \cdot \underline{F}(\underline{\xi}) \ . \qquad \ldots (3.1)$$

By taking the generalised co-ordinates to be the components of the vector $\underline{\xi}$, the canonical conjugate momentum density defined in the usual way is given by :

$$\underline{\pi} = \rho \frac{\partial \underline{\xi}}{\partial t} + \rho\, \underline{v} \cdot \nabla \underline{\xi} \ . \qquad \ldots (3.2)$$

Moreover the resulting Hamiltonian density is given by :

$$\mathcal{H} = \frac{1}{2\rho} (\underline{\pi} - \rho \underline{v} \cdot \nabla \underline{\xi})^2 - \frac{1}{2}\, \underline{\xi} \cdot \underline{F}(\underline{\xi}) \ . \qquad \ldots (3.3)$$

From the Lagrangian functional

$$L = \int_{V_p} \mathcal{L}\, d\underline{r} \ , \qquad \ldots (3.4)$$

or the Hamiltonian functional

$$H = \int_{V_p} \mathcal{H}\, d\underline{r} \ , \qquad \ldots (3.5)$$

the equation of motion, e.g. (2.26), can be obtained variationally in the usual way. The integration is performed over the plasma volume V_p.

In the formulation given here there appears to be no formal difference between a stationary system and a system where the fluid is in an arbitrary state of motion. There is, however, a real difference since in the stationary case all quantities referring to the unperturbed state are not explicitly dependent on time. In particular the Hamiltonian H, equation (3.5), is a constant of the motion in the stationary case, as we have under quite general assumptions, see Appendix E, that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \; . \qquad \ldots (3.6)$$

From equations (3.3) and (3.5) it follows that if δW is positive definite, where

$$\delta W = -\frac{1}{2}\int_{V_p} \underline{\xi}\cdot \underline{F}(\underline{\xi})\; d\underline{r} \; ,$$

then H is positive definite. Moreover, since $\frac{dH}{dt} = 0$ we have that H is a Lyaponov functional for the system[6]. Thus the most restrictive sufficient condition derived by Frieman and Rotenberg[4] is an obvious consequence of this fact. In the case of a rapid oscillatory system it might be possible to derive some similar results for an averaged motion, but no specific results have been obtained so far.

4. THE STABILITY OF PERIODIC SYSTEMS

In this section we shall restrict the discussion to the stability problem of periodic systems, which is the main purpose of this work. We assume a periodic motion with a basic frequency ω_s. Further the periodic motion takes place around some equilibrium characterised by $\rho = \rho_o$, $p = p_o$, $\underline{v} = 0$, $\underline{B} = \underline{B}_o$. This motion is assumed to be described by the following expansion in ε,

$$\begin{aligned} \rho &= \rho_o + \varepsilon\,\rho_1 + \varepsilon^2\rho_2 + \ldots \; , \\ p &= p_o + \varepsilon\, p_1 + \varepsilon^2 p_2 + \ldots \; , \\ \frac{\underline{v}}{\omega_s} &= 0 + \varepsilon\,\underline{v}_1 + \varepsilon^2\underline{v}_2 + \ldots \; , \\ \underline{B} &= \underline{B}_o + \varepsilon\,\underline{B}_1 + \varepsilon^2\underline{B}_2 + \ldots \; . \end{aligned} \qquad \ldots (4.1)$$

It is convenient to expand $\underline{v}/\omega_s$ rather than $\underline{v}$, ω_s is of course constant and $\underline{v}$ is proportional to ω_s. Thus $\underline{v}_1, \underline{v}_2, \ldots$ have the dimensions of length and $\underline{v}_n\cdot\nabla$ is a non-dimensional quantity. In some general sense ε represents the amplitude of the oscillatory motion. By introducing the non-dimensional time $\tau = \omega_s t$, equation (2.26) can be rewritten as:

$$\rho \frac{\partial^2 \underline{\xi}}{\partial \tau^2} + 2 \frac{\rho \underline{v}}{\omega_s} \cdot \nabla \frac{\partial \underline{\xi}}{\partial \tau} + \frac{\partial}{\partial \tau}\left(\frac{\rho \underline{v}}{\omega_s}\right) \cdot \nabla \underline{\xi} = \frac{1}{\omega_s^2} \underline{F} (\underline{\xi}) \; . \qquad \ldots (4.2)$$

It is now convenient to write :

$$\underline{F} (\underline{\xi}) = F_e (\underline{\xi}) + \varepsilon \underline{\tilde{F}}_1 (\underline{\xi}) \quad \varepsilon^2 \underline{\tilde{F}}_2 (\underline{\xi}) + \ldots \; , \qquad \ldots (4.3)$$

where according to equation (2.27) and equations (4.1) we have :

$$\underline{F}_e(\underline{\xi}) = \nabla(\gamma p_o \nabla \cdot \underline{\xi} + \underline{\xi} \cdot \nabla p_o) - \underline{B}_o \times (\nabla \times \underline{Q}_o) - \underline{Q}_o \times (\nabla \times \underline{B}_o) + \nabla \cdot (\rho_o \underline{\xi}) \nabla \psi \; , \qquad \ldots (4.4)$$

$$\underline{\tilde{F}}_1 (\underline{\xi}) = \nabla(\gamma p_1 \; \nabla \cdot \underline{\xi} + \underline{\xi} \cdot \nabla p_1) - \Big[\underline{B} \times (\nabla \times \underline{Q}) + \underline{Q} \times (\nabla \times \underline{B}) \Big]_1 + \nabla \cdot \left(\underline{\xi} \rho_o \omega_s^2 \frac{\partial \underline{v}_1}{\partial \tau_o} \right) + \nabla \cdot (\rho_1 \underline{\xi}) \nabla \psi \; , \qquad \ldots (4.5)$$

$$\underline{\tilde{F}}_2 (\underline{\xi}) = \nabla(\gamma p_2 \; \nabla \cdot \underline{\xi} + \underline{\xi} \cdot \nabla p_2) - \Big[\underline{B} \times (\nabla \times \underline{Q}) + Q \times (\nabla \times \underline{B}) \Big]_2$$

$$+ \nabla \cdot \left[\underline{\xi} \left(\rho \frac{d\underline{v}}{dt} \right)_2 - \rho_o \omega_s^2 \underline{v}_1 \underline{v}_1 \cdot \nabla \underline{\xi} \right] + \nabla \cdot (\rho_2 \underline{\xi}) \nabla \psi \; , \qquad \ldots (4.6)$$

$$\underline{\tilde{F}}_3 (\underline{\xi}) = \ldots \; ,$$

$$\vdots$$

Here $\underline{F}_e(\underline{\xi})$ is the forcing term which determines the stability of the equilibrium state, i.e., in the absence of the oscillatory motion. The question of stability of the equilibrium state is in general answered by the usual energy principle[5].

We now make the assumption that for the unstable modes under consideration the driving force for the instability is small. To be more precise we shall assume that:

$$\omega_s^{-2} \; \underline{F}_e(\underline{\xi}_o) = O(\varepsilon^2 \rho_o \underline{\xi}_o) \; . \qquad \ldots (4.7)$$

In principle this is always possible by making ω_s sufficiently large, provided the growth rate of any instability present in the static system is limited. (We shall return to the discussion of this assumption in Section 5.) The validity of the M H D equations is of course restricted by an upper limit for the frequency under investigation[5].

One can in fact show that the ordering given in equation (4.7) is the only proper ordering, see Appendix F, Section (a). This also applies to another simple dynamic system, namely the inverted pendulum with oscillating support. For details see Appendix F, Section (c), and also Bogoliubov and Mitropolsky[8]. Landau and Lifshitz[9] also briefly discuss this problem and more general similar problems in ordinary mechanics.

In general the ordering procedure seems to be: If the small amplitude oscillatory force acting is of order ε, then the driving force of any instability to be stabilised must not be bigger than of order ε^2. The reason for this is simple enough. The effect of the applied oscillatory force is non-linear in character and can be regarded as an averaged effect over one period. Consequently the significant contribution is of order ε^2, and in order to be effective this contribution has to be comparable to the driving force of the instability.

Returning to equation (4.2) we notice that all terms except $\rho \frac{\partial^2 \underline{\xi}}{\partial \tau^2}$ are small, of order ε. Thus this term also must be small, of order ε. This means that $\underline{\xi}$ must be a slowly varying function of time, or to be more precise $\underline{\xi}$ changes only by a small amount of order ε over a time interval of the order $\frac{2\pi}{\omega_s}$. Equation (4.2) is therefore in a form which is suitable for an asymptotic expansion in time in terms of ε.

When doing the expansion all coefficients of powers of ε are treated as numbers of order unity. This will, however, put a practical limit to the magnitude of any element $\frac{\partial \xi_i}{\partial x_j}$ of the dyadic $\nabla \underline{\xi}$. Let L be a characteristic length in the proper direction of the system, then we must have

$$L \frac{\partial \xi_i}{\partial x_j} < \frac{1}{\varepsilon} \xi_i .$$

If this is not true we must expect our expansion to break down. In practice this means that we must limit ourselves to long wavelength perturbations. In this connection we would, however, like to point out that when short wavelength perturbations are investigated the MHD equations are not a valid description of most real plasmas of practical interest. Under such circumstances finite Larmor radius effects for example, should be incorporated in the equations.

We shall make use of the multiple time-scale (MTS) method[10,11,12] in our expansion scheme. According to this method we introduce a space of times $\tau_0, \tau_1, \tau_2 \ldots$ replacing the single time τ. We take

$$\tau_n = \varepsilon^n \tau_0 . \qquad \ldots (4.8)$$

The only time appearing explicitly in equation (4.2) is the time associated with the forced oscillations in the system. This is the fastest time-scale in our problem and by definition we take this time to be τ_0.

The expansion of equation (4.2) is straightforward but tedious. Each of the quantities appearing in the equation are expanded in ε, including $\underline{\xi}$ and $\frac{\partial}{\partial \tau}$. The solution of the problem is determined by the extra conditions necessary to remove secular terms. The motivation for removing these terms is to preserve the asymptotic character of the solution for large τ.

The zeroth order equations just leave us with the information that $\underline{\xi}_0$ is independent of τ_0. This is the more precise way of expressing the fact that $\underline{\xi}$ is a slowly varying function of τ. We now obtain the first and second order equations. In the first order equation $\underline{\xi}_1$ is determined in terms of $\underline{\xi}_0$ and unperturbed quantities. After having assumed that the explicit time dependence on τ_0 is sinusoidal in leading order in the expansion, this equation reduces to equation (4.15) listed below. One main difficulty concerning this equation is that $\underline{\xi}_1$ is determined implicitly and not explicitly in terms of $\underline{\xi}_0$. We shall return to this problem later on. One important feature of these equations is that they are not explicitly dependent on τ_1, that is, they are dependent on τ_1 only through $\underline{\xi}_0$. We can therefore seek normal mode solutions of the form :

$$\underline{\xi}_0(\tau_1) = \underline{\xi}_0 \, e^{i\omega\tau_1} \ . \qquad \text{... (4.9)}$$

By doing so we have of course limited ourselves to the study of exponential stability.

We now proceed by obtaining two equations in the following way. We get the first equation by multiplying the second order equation by $\underline{\xi}_0^*$ and integrating it over the plasma volume as well as averaging over the fast timescale. The time averaging process is defined by

$$\bar{\int} x(\underline{r},\tau_0)\, d\underline{r} = \frac{1}{2\pi}\int_0^{2\pi}\left\{\int x(\underline{r},\tau_0)\,d\underline{r}\right\} d\tau_0 \ . \qquad \text{... (4.10)}$$

We also make use of the formulae:

$$\frac{1}{2\pi}\int_0^{2\pi}\left\{\int_{V(t)} \frac{\partial}{\partial t} f(\underline{r},t)\,d\underline{r}\right\} dt = 0 \ , \qquad \text{... (4.11)}$$

which is correct provided the following conditions are met: (a) $f(\underline{r},t)$ is a periodic function of t with periods 2π, and (b) either the normal velocity of the surface S bounding the volume V vanishes, or f vanishes over S. See Appendix F, equation (F.31).

The second equation is obtained by taking the complex conjugate of the first order equation, multiplying it by $\underline{\xi}_0$, integrating it over the plasma volume and averaging over time.

The two equations obtained in this way can be combined to give the following result:

$$\omega^2 = \frac{\delta\bar{W}(\underline{\xi}_0^*, \underline{\xi}_0)}{\bar{K}(\underline{\xi}_0^*, \underline{\xi}_0)} \ . \qquad \text{... (4.12)}$$

We have

$$\delta\bar{W}(\underline{\xi}_0^*, \underline{\xi}_0) = \delta\bar{W}_0 + \delta\bar{W}_D + \delta\bar{W}_E \qquad \text{... (4.13)}$$

and

$$\bar{K}(\underline{\xi}_0^*, \underline{\xi}_0) = \bar{\int} \rho_0 |\underline{\xi}_0|^2 \, d\underline{r} \ , \qquad \text{... (4.14)}$$

where

$$\delta\bar{W}_0 = -\omega_s^{-2} \bar{\int} \underline{\xi}_0^* \cdot \left(\varepsilon^{-2}\, \underline{F}_e + \underline{\tilde{F}}_2\right)(\underline{\xi}_0)\, d\underline{r} \ , \qquad \text{... (4.15)}$$

$$\delta\bar{W}_D = \bar{\int} \rho_0^{-1} \left|\underline{F}_1'(\underline{\xi}_0)\right|^2 d\underline{r} \ , \qquad \text{... (4.16)}$$

$$\delta\bar{W}_E = \omega_s^{-2} \bar{\int}\left\{-\underline{\xi}_1^*\, \underline{F}_e(\underline{\xi}_1) - \rho_0^{-1}\omega_s^{-2}\left|\underline{F}_e(\underline{\xi}_1)\right|^2\right\} d\underline{r} \ , \qquad \text{... (4.17)}$$

and

$$\underline{F}_1'(\underline{\xi}_0) = \omega_s^{-2} \widetilde{\underline{F}}_1(\underline{\xi}_0) - \rho_0 \frac{\partial \underline{v}_1}{\partial \tau_0} \cdot \nabla \underline{\xi}_0 , \quad \ldots (4.18)$$

$$\rho_0 \underline{\xi}_1 = - \underline{F}_1'(\underline{\xi}_0) - \omega_s^{-2} \underline{F}_e (\underline{\xi}_1) . \quad \ldots (4.19)$$

Details of this derivation are given in Appendix F, Section b.

5. AN ENERGY PRINCIPLE

We now turn to the discussion of the results obtained in equations (4.12) - (4.19). It is apparent from the symmetric form of the expression for ω^2 , equation (4.12), that ω^2 is real. Furthermore if we look at the eigenvalue problem where ω^2 constitutes the eigenvalues then to each eigenvalue, say ω_n^2, there corresponds at least one eigenfunction, say $\underline{\xi}_{on}$. It is fairly easy to show, see Appendix G, that the eigenfunctions belonging to different eigenvalues are orthogonal with the weight function ρ_0 in the time averaged sense.

Moreover, if the set of eigenfunctions $\underline{\xi}_{on}$ forms a complete set the following two related theorems can be proved:

<u>Theorem 1</u>: The system is exponentially unstable on the τ_1 timescale if and only if there exists an admissible displacement $\underline{\xi}_0$ which makes $\delta\bar{W}$ negative.

<u>Theorem 2</u>: A necessary condition for stability of the system is that $\delta\bar{W} > 0$ for all admissible displacements $\underline{\xi}_0$.

Details of the proofs are given in Appendix H.

All other quantities appearing in the theory except $\underline{\xi}_0$ (and $\underline{\xi}_1$) are real, therefore $\underline{\xi}_0$ (and $\underline{\xi}_1$) can also be taken to be real, since ω^2 is real. It is, however, sometimes convenient to take $\underline{\xi}_0$ to be complex rather than real. However, if there exists a complex $\underline{\xi}_0$ for which $\delta\bar{W}$ is negative, there also exists a real $\underline{\xi}_0$ for which it is negative.

The energy principle for the static case [5] was originally proved under the same assumptions of completeness of the eigenfunctions as Theorems 1 and 2. This energy principle has, however, been proved without this assumption by Laval, Mercier and Pellat [13]. We suggest that it might also be possible to prove Theorems 1 and 2 without this assumption.

We now give some interpretations of the results in terms of energy. First, the term $\delta\bar{W}_0$ is easily interpreted as twice the averaged potential energy divided by ω_s^2 . It is closely related to the potential energy in the energy principle for the static case.

Secondly, the term $\delta \bar{W}_D$ is a contribution essential for dynamic stabilisation. It is really a form of kinetic energy which appears as potential energy in the present context. This term originates from the expression:

$$-\int \rho_o \underline{\xi}_1^* \cdot \frac{\partial^2}{\partial \tau_o^2} \underline{\xi}_1 \, d\underline{r} = \int \rho_o \left| \frac{\partial \underline{\xi}_1}{\partial \tau_o} \right|^2 d\underline{r} \; . \qquad \ldots (5.1)$$

Written in this way the connection with a form of kinetic energy is clearly seen. At this point it is also interesting to look at the similarity between this problem and that of the inverted pendulum with oscillating support. One can easily identify the quantities corresponding to $\delta \bar{W}_O$ and $\delta \bar{W}_D$ for this simple mechanical system. A discussion of the inverted pendulum is given in Appendix F, Section c.

Finally we have the extra term $\delta \bar{W}_E$ which contains some extra contributions with no analogy in a simple mechanical system like the inverted pendulum. This contribution is something special to systems having more than one degree of freedom. It is connected with the fact that an unstable static system of this kind may have two classes of perturbations, related to the stable and the unstable eigenmodes.

Definition: A certain eigenfunction $\underline{\xi}_{on}$ belongs to the unstable eigenmodes provided:

$$\delta \bar{\bar{W}}_O (\underline{\xi}_{on}^*, \underline{\xi}_{on}) < 0 \; , \qquad \ldots (5.2)$$

and to the stable eigenmodes if

$$\delta \bar{\bar{W}}_O (\underline{\xi}_{on}^*, \underline{\xi}_{on}) > 0 \; , \qquad \ldots (5.3)$$

where $\delta \bar{W}_O = \delta \bar{\bar{W}}_O = \delta \bar{\bar{\bar{W}}}_O$, see equation (4.15), and $\delta \bar{\bar{W}}_O$ is associated with the static system

Now we can take the eigenfunctions to be normalised, that is we have:

$$\bar{K}(\underline{\xi}_{on}^*, \underline{\xi}_{on}) = 1 \; , \qquad \ldots (5.4)$$

where $\bar{K}$ is given by equation (4.14). Consequently we can write:

$$\delta \bar{\bar{W}}_O (\underline{\xi}_{on}^*, \underline{\xi}_{on}) = \frac{\omega_n^2}{\omega_s^2} \; . \qquad \ldots (5.5)$$

Looking at the problem this way, the basic assumption equation (4.7) means that for all n, where $\omega_n^2 < 0$ we have

$$\frac{\omega_n^2}{\omega_s^2} = O(\varepsilon^2) \; . \qquad \ldots (5.6)$$

However, we have not made any assumption about the order of magnitude of this quantity associated with the stable eigenmodes. This means that we allow for stable oscillations, wave-modes, with frequencies of the same order as the applied frequency. As far as we can see, these modes are of most interest from a stability point of view in connection with parametric resonances, a problem which will be discussed in Section 6. However, these modes also enter in connection with $\delta \bar{W}_E$ because unless $\underline{\xi}_1$ contains stable eigenmodes for which $(\omega_n/\omega_s)^2$ is of zeroth order this term will not give any contribution.

This discussion is only concerned with perturbations for which $\delta\bar{W}_O(\underline{\xi}_o^*, \underline{\xi}_o)$ can be made negative (or zero). Consequently high frequency stable modes do not enter in connection with $\underline{\xi}_o$. However, we must allow for the possibility that $\underline{\xi}_1$ contains such eigenmodes. Accordingly, $\delta\bar{W}_O(\underline{\xi}_1^*, \underline{\xi}_1)$ must be treated as a zeroth order quantity and so must $\delta\bar{w}_E$. Put in another way we may have

$$\omega_s^{-2}\, \underline{F}_e(\underline{\xi}_1) = O(\rho_o \underline{\xi}_1) \quad \text{whereas} \quad \omega_s^{-2} \underline{F}_e(\underline{\xi}_o) = O(\varepsilon^2\, \rho_o\, \underline{\xi}_o)\;. \qquad \text{... (5.7)}$$

Notice that for an unstable system, $\underline{\xi}_o$ can be chosen such as to make $\delta\bar{\bar{W}}_0(\underline{\xi}_o^*, \underline{\xi}_o)$ negative, but when this choice is made $\underline{\xi}_1$ is determined in terms of $\underline{\xi}_o$ and the unperturbed quantities by equation (4.19). For the special cases which we are going to discuss later this particular feature of the problem will become clearer.

Finally, what really matters, from a stability point of view, is the sign of $\delta\bar{W}_E$. This problem is discussed in Appendix H, Section b. On the basis of this discussion we can conclude that it is very likely that when $\delta\bar{W}_E$ makes a significant contribution then $\delta\bar{W}_E$ has got a definite sign. Moreover, this sign is favourable for stability, that is:

$$\delta\bar{W}_E > 0\;. \qquad \text{... (5.8)}$$

This result is also reasonable from a physical point of view, since by the $\underline{\xi}_1$ part of the solution the energy is fed into stable oscillations. Thus it is hard to see how this can be the source of instabilities apart from instabilities of the parametric resonance type which is not included in this discussion.

6. PARAMETRIC RESONANCE (INSTABILITY)

So far we have only discussed the problem of stabilisation of the unstable modes of motion. It is likely, however, that the forced oscillations introduce completely new modes of motion, which arise through the resonance between the forced oscillations and the natural stable modes of the system.

The expansion used in solving equation (2.26) is based on the assumption that $\omega_s^{-2}\, \underline{F}_e(\underline{\xi})$ is small for the unstable modes. Normally, however, there exist stable (oscillatory) modes, as pointed out in Section 5, for which $\omega_s^{-2}\, \underline{F}_e(\underline{\xi})$ is of order unity. These modes are in general associated with bending of the field-lines and compression of the fluid. They are therefore related to the hydromagnetic waves in an infinite homogeneous medium.

We shall now briefly discuss the general problem, but because of the complexity the discussion will be more qualitative than quantitative. By multiplying equation (2.26) by:

$$\frac{\rho_0}{\rho} = 1 - \frac{\rho - \rho_0}{\rho}$$

we obtain

$$\rho_0 \frac{\partial^2 \underline{\xi}}{\partial \tau^2} + 2\,\varepsilon \rho_0 \underline{\tilde{v}}_1 \cdot \nabla \frac{\partial \underline{\xi}}{\partial \tau} = \frac{1}{\omega_s^2}\left(\underline{F}_e(\underline{\xi}) + \frac{\rho_0}{\rho}\,\underline{\tilde{F}}(\underline{\xi}) - \frac{\rho - \rho_0}{\rho}\,\underline{F}_e(\underline{\xi})\right) - \varepsilon\,\frac{\rho_0}{\rho}\,\frac{\partial}{\partial \tau}(\rho \underline{\tilde{v}}_1) \cdot \nabla \underline{\xi} \,. \quad \ldots (6.1)$$

Here we have written $\underline{F} = \underline{F}_e + \underline{\tilde{F}}$ where $\underline{F}_e = \lim_{\varepsilon \to 0} \underline{F}$ and $\underline{v} = \varepsilon\, \omega_s\, \underline{\tilde{v}}_1$. Notice that $\tilde{F}$ and $(\rho - \rho_0)/\rho$ are of order ε.

We assume that we can write :

$$\underline{\xi}(\underline{r},\tau) = \underline{\xi}(\underline{r})\; T(\tau)\,.$$

By multiplying equation (6.1) by $\underline{\xi}(\underline{r})$ and integrating over the volume occupied by the plasma we obtain

$$\ddot{T} + \varepsilon\, a\dot{T} + \sigma^2 T + \varepsilon\, bT = 0\,, \quad \ldots (6.2)$$

where

$$a = \frac{2}{\bar{K}} \int \underline{\xi} \rho_0\, \underline{\tilde{v}}_1 \;:\; \nabla \underline{\xi}\;\; d\underline{r}\,, \quad \ldots (6.3)$$

$$\sigma^2 = -\frac{1}{\omega_s^2 \bar{K}} \bar{\int} \underline{\xi} \cdot \underline{F}_e(\underline{\xi})\; d\underline{r}\,, \quad \ldots (6.4)$$

$$b = \frac{1}{\omega_s^2 K} \int \underline{\xi} \cdot \left\{ \frac{\rho_0}{\varepsilon\rho}\,\underline{\tilde{F}}(\underline{\xi}) - \frac{\rho - \rho_0}{\varepsilon\rho}\,\underline{F}_e(\underline{\xi}) - \frac{\omega_s^2 \rho_0}{\rho}\,\frac{\partial}{\partial \tau}(\rho \underline{v}_1) \cdot \nabla \underline{\xi} \right\} d\underline{r} + c\,, \quad \ldots (6.5)$$

$$c = \frac{1}{\varepsilon}\left\{ \frac{1}{\omega_s^2 K} \int \underline{\xi} \cdot \underline{F}_e(\underline{\xi})\, d\underline{r} + \sigma^2 \right\}, \quad \ldots (6.6)$$

$$K = \int \rho_0\, \underline{\xi}^2\, d\underline{r}\,, \quad \bar{K} = \bar{\int} \rho_0\, \underline{\xi}^2\, d\underline{r}\,. \quad \ldots (6.7)$$

Here $\bar{\int}$ means the integral over the volume in the limit $\varepsilon \to 0$, and we have written $\underline{\xi}$ for $\underline{\xi}(\underline{r})$. Thus a and b are periodic functions of τ with period 2π. By introducing the new variables

$$x_1 = T\,, \quad x_2 = \dot{T}\,, \quad (\dot{T} = \frac{d}{d\tau} T)\,, \quad \underline{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

we can write equation (6.2) as :

$$\underline{\dot{x}} = \{A + \varepsilon\, B(\tau)\} \cdot \underline{x}\,, \quad \ldots (6.8)$$

and the matrices A and B are given by:

$$A = \begin{Bmatrix} 0 & 1 \\ -\sigma^2 & 0 \end{Bmatrix}, \quad B(\tau) = \begin{Bmatrix} 0 & 0 \\ -b & -a \end{Bmatrix}.$$

Equation (6.8) is now in the standard form of a generalised Hill equation which is discussed in the literature, see for instance L. Cesari [14]. For ε small, 'resonance' or 'parametric instability', may occur only at the lattice points :

$$2\sigma = n\,, \quad n = 1\,,\, 2\,,\, 3\,,\, \ldots\,. \quad \ldots (6.9)$$

Notice that the eigenvalues of A are $\pm\, i\sigma$. If, however, the modes associated with the eigenvalues of A are damped, that is if the eigenvalues of A have got a small negative real part, then the solutions of equation (6.8) are bounded in time for ε sufficiently small. In the present theory this cannot happen because there is no dissipative effects included in the equations. Such effects will, however, be present in all real physical systems.

Furthermore, if the system is finite then the eigenmodes will be restricted to satisfy the boundary conditions. In general the boundary conditions and the resonance conditions, equation (6.9), will be incompatible. However, if there exist eigenmodes having a frequency close enough to one of the resonance frequencies, one would still expect parametric resonance. Close enough in this context means inside some definite interval symmetric about the resonance frequency in question. The width of the interval is dependent on ε and approaches zero as ε approaches zero. The intervals corresponding to different resonance frequencies become more and more narrow for higher resonance frequencies, (for further details see L. Cesari[14] and also Landau and Lifshitz[9], p.80). The effect of damping is also discussed in the latter reference. When there is damping present the width of these intervals can be made 'non zero' only for sufficiently <u>large</u> ε.

More detailed discussion of this subject can hardly be carried out without discussing specific problems. However, we can in general conclude that :

(a) Dissipative effects can prevent this sort of instability;

(b) Finite geometry will normally restrict the number of possible modes and can therefore make it more difficult to fulfil the resonance conditions;

(c) A combination of the two effects given under (a) and (b) may also act in such a way as to prevent this kind of instability.

There also exists the possibility of making use of the resonance effects in creating the oscillatory motion in the first place*. From a physical point of view it should be easy to feed energy into some natural mode of the plasma. The oscillatory motion and the resulting dynamic stabilisation will probably also be cheaper to generate this way.

From this point of view one could say that the problem is not so much to avoid resonance effects, but to control them, and avoid the associated instabilities. However, if this should turn out to be difficult it might be possible to make the stabilizing frequency ω_s change with time. One would expect the characteristic time T_R for build up of resonances to be long compared to one period $T_s = \frac{2\pi}{\omega_s}$. Thus, taking ω_s to change with time such that :

---

* Dr K.V. Roberts. Private communication

$$T_S \ll \left| \omega_S \Big/ \frac{\partial \omega_S}{\partial t} \right| \leq T_R ,$$

it may be possible to avoid parametric instability and still have the stabilising effect of the frequency modulated dynamic stabilisation.

7. A SHARP BOUNDARY

It is of interest to be able to deal with systems where the plasma is separated from the vacuum region by a sharp boundary. The interface between plasma and vacuum is in general a surface of discontinuity carrying a sheet current. In this connection we notice that the first term $\delta\bar{W}_O$ in the expression for $\delta\bar{W}$ equation (4.15), is composed of the zero order terms in the expansion of

$$G(\underline{\xi}_o^*, \underline{\xi}_o) = \frac{-1}{\varepsilon^2 \omega_s^2} \bar{\int_{V_p}} \left\{ \underline{\xi}_o^* \cdot \underline{F}(\underline{\xi}_o) - \varepsilon\, \underline{\xi}_o^* \cdot \tilde{\underline{F}}_1(\underline{\xi}_o) \right\} d\underline{r} \qquad \ldots (7.1)$$

Further we can derive the following result ,

$$\begin{aligned} - \int_{V_p} \underline{\xi} \cdot \underline{F}(\underline{\xi})\, d\underline{r} = & \int_{V_v} (\nabla \times \underline{A})^2\, d\underline{r} + \int_S (\underline{n} \cdot \underline{\xi})^2\, \underline{n} \cdot \left\langle \nabla \left(p + \frac{B^2}{2}\right) \right\rangle d\sigma \\ & + \int_{V_p} \left\{ (\underline{B} \cdot \nabla \underline{\xi})^2 - \rho\, (\underline{v} \cdot \nabla \underline{\xi})^2 + \underline{\xi}\underline{\xi} : \nabla\nabla \left(p + \frac{B^2}{2}\right) + (\nabla \cdot \underline{\xi})^2 \left(\gamma p + \frac{B^2}{2}\right) \right. \\ & \left. + \; 2 \nabla \cdot \underline{\xi} \left[\underline{\xi} \cdot \nabla \left(p + \frac{B^2}{2}\right) - \underline{B}\underline{B} : \nabla \underline{\xi} \right] \right\} d\underline{r} \, . \end{aligned} \qquad \ldots (7.2)$$

Here V_V is the volume of the vacuum-region and S is the surface bounding the plasma. We observe that in leading order the destabilising term $\rho(\underline{v} \cdot \nabla \underline{\xi}_o)^2$ will cancel the term $\rho_o \left(\frac{\partial \underline{v}}{\partial \tau_o} \cdot \nabla \underline{\xi}_o \right)^2$ coming from the expansion of the second term $\delta\bar{W}_D$ in the expression for $\delta\bar{W}$, equation (4.13). When deriving equation (7.2) we have assumed $\underline{\xi}$ to be real; one can, however, easily derive a similar expression if it is convenient to work with a complex $\underline{\xi}$. A detailed discussion leading to equation (7.2) is given in Appendix H, Section c. The expression in the case of a complex $\underline{\xi}$ is also given there.

8. CONCLUSIONS

The treatment given here represents the first steps towards a solution of the general problem of dynamic stabilisation in the framework of the ideal MHD equations. It is demonstrated that if $\delta\bar{W}$ given by equation (4.13) is positive for all possible $\underline{\xi}$, then the system is stabilised in leading order in the expansion parameters. They are essentially the normalised amplitude of the forced oscillations and the ratio between the characteristic growth rate of any instability and the applied frequency. The problem is therefore reduced to that of determining the sign of $\delta\bar{W}$, and the procedure is to minimise $\delta\bar{W}$ with respect to $\underline{\xi}$.

The results obtained are not necessarily valid for short wavelength perturbations. Generally speaking the effect of a forced oscillatory motion, with sufficiently high frequency, imposed on an unstable equilibrium will always be a stabilising one. Some unstable modes may, however, not be affected, and it is a design problem to make dynamic stabilisation work on the unstable modes of interest. This will become more clear in connection with the application of the present theory in particular cases, but this is outside the scope of this report.

However, one has to worry about entirely new modes of motion being introduced into the system by dynamic stabilisation. This is the problem of parametric resonance and associated instability. Only a brief account of this very complex problem is given, where the main characteristics of the problem are listed and a possible way of avoiding serious effects of this sort is suggested. It might even be possible to take advantage of the resonance effects when solving the difficult technical problem of generating the oscillatory motion at sufficiently high frequency and sufficiently large amplitude in the first place. The stabilising frequency must of course be much greater than the growth rate of any instability to be stabilised.

ACKNOWLEDGEMENTS

The author wishes to thank the members of the Theory Division of the Culham Laboratory for many helpful discussions during the course of this work, and wishes in this connection specially to mention Dr J. Wesson.

This work has been performed at the U.K.A.E.A. Culham Laboratory and was supported by a grant from the Royal Norwegian Council for Scientific and Industrial Research.

REFERENCES

1. WEIBEL, E.S., Phys. Fluids, vol.3, p.946, 1960, and
   WEIBEL, E.S., Second Conf. on Plasma Phys., Salzburg, 1961. Nucl. Fusion Suppl., p.1137, 1962.
2. VANDAKUROV, Yu.V., Nucl. Fusion, vol.6, p.188, 1966
3. HAAS, F.A. and WESSON, J.A. Phys. Rev. Letters, vol.19, p.833, 1967.
4. FRIEMAN, E. and ROTENBERG, M. Rev. Mod. Phys., vol.32, p.898, 1960
5. BERNSTEIN, I.B., FRIEMAN, E.A., KRUSKAL, M.D. and KULSRUD, R.M. Proc. Roy. Soc. (London), vol.A244, p.17, 1958
6. BRAND, L., Vector and Tensor Analysis. Wiley, New York, 1958
7. ZUBOV, V.I. Methods of A.M. Lyapunov and their Application. P. Noordhoff Ltd., Groningen, The Netherlands. 1964. p.1
8. BOGOLIUBOV, N.N. and MITROPOLSKY, Y.A., Asymptotic Methods in the Theory of Non-Linear Oscillations. Hindustan Publishing Corp (India), Delhi 6, 1961. p.404
9. LANDAU, L.D. and LIFSHITZ, E.M. Mechanics. Pergamon Press, Oxford, 1960. p.94.
10. FRIEMAN, E.A. J. Math. Phys., vol.4, p.410, 1963.
11. SU, C.H., FRIEMAN, E.A. and KRUSKAL, M.D. Report Matt-238, Princeton University, 1964.
12. SANDRI, J.G. Nuovo Cimento, vol.XXXVI, no.1, p.67, 1965
13. LAVAL, G., MERCIER, C. and PELLAT, R. Nucl. Fusion, vol.5, p.156, 1965.
14. CESARI, L., Asymptotic behaviour and stability problems in ordinary differential equations. 2nd ed. Springer-Verlag, Berlin, 1963, p.72.
15. TRUESDELL, C. The kinematics of vorticity. I.U.P.S.S. No.19. Indiana University Press, Bloomington, 1954. pp.30and 53.
16. LUNDQUIST, S. Ark. Mat. Ast. Fys., vol.5, p.297, 1952.
17. Proceedings of the International School of Physics 'Enrico Fermi'; Advanced Plasma Theory. Academic Press, New York, 1964. p.69.

APPENDIX A

INTEGRATION OF THE CONSTRAINT EQUATIONS

(a) INTRODUCTORY REMARKS

The description of fluid motion goes back to the classic works by Lagrange and Euler. There exist two basic descriptions commonly referred to as the Lagrangian and Eulerian description. The main content of the Lagrangian description is that one assumes the initial position $\underline{r}_0$ of each fluid particle (- element) to be known. The trajectories $\underline{r} = \underline{r}(\underline{r}_0, t)$ of the fluid particles are then calculated by integrating Newton's second law of motion. The forces acting upon the fluid particles are given as function of position $\underline{r}$ and time t.

In the Eulerian description the fluid motion is described in terms of a velocity vector field $\underline{v}(\underline{r}, t)$. The velocity field is defined in such a way that a fluid particle which happens to be located at the point $\underline{r}_0$ in space at the time t_0 has the velocity $\underline{v}(\underline{r}_0, t_0)$. The aim of this description is to determine the velocity field $\underline{v}(\underline{r}, t)$. An excellent discussion of these problems with reference to the original works is given by Truesdell[15] the problem is also discussed by Brand[6].

In the following analysis we shall find it convenient to use a mixture of the Eulerian and Lagrangian description. It is therefore important to be aware of the two descriptions and the way in which they are connected.

A Lagrangian displacement $\underline{\xi}$ is introduced as illustrated in Fig.1. This displacement is, however, to be regarded as a vector field in terms of Eulerian variables. We regard equation (2.1) as the basic equation of motion and all the other equations as constraint equations which have to be satisfied by any solution of equation (2.1). The advantage of the present approach is that the introduction of the displacement $\underline{\xi}$ enables us to integrate all the constraint equations in terms of $\underline{\xi}$ for the perturbed motion.

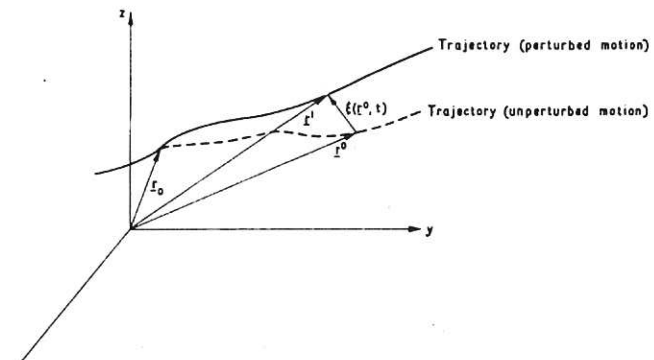

Fig. 1 Kinematics of the fluid motion (CLM-R 97)

(b) THE PERTURBED VELOCITY AND ACCELERATION

The following definitions and notations are used throughout this Appendix :

$$\nabla^o \overset{def}{=} \frac{\partial}{\partial \underline{r}^o} \quad , \quad \nabla' \overset{def}{=} \frac{\partial}{\partial \underline{r}'} \quad , \quad \nabla_o \overset{def}{=} \frac{\partial}{\partial \underline{r}_o} \quad , \qquad \text{... (A.1)}$$

$$\underline{r}' = r'\{\underline{r}^o(\underline{r}_o,t),t\} \overset{def}{=} \underline{r}^o(\underline{r}_o,t) + \underline{\xi}\{\underline{r}^o(\underline{r}_o , t) , t\} \ , \qquad \text{... (A.2)}$$

$$\underline{v} \overset{def}{=} \frac{d}{dt}\underline{r}^o(\underline{r}_o , t) \quad , \quad \frac{d}{dt} \overset{def}{=} \frac{\partial}{\partial t} + \underline{v}\{\underline{r}^o(\underline{r}_o , t) , t\}\cdot\nabla^o \qquad \text{... (A.3)}$$

In equation (A.2) the functional form to be used is indicated. The physical meaning of the symbols appearing in this equation is obvious from Fig.1.

The perturbed velocity and acceleration are readily obtained by taking the time derivative of $\underline{r}'$, thus :

$$\underline{v}(\underline{r}^o + \underline{\xi}) = v + \underline{v}\cdot\nabla^o\,\underline{\xi} + \frac{\partial\underline{\xi}}{\partial t} \quad , \quad \underline{v} = \underline{v}\,(\underline{r}^o , t) \ , \qquad \text{... (A.4)}$$

$$\underline{a}(\underline{r}^o + \underline{\xi}) = \frac{d}{dt}\,\underline{v}(\underline{r}^o + \underline{\xi}) = \frac{\partial\underline{v}}{\partial t} + \underline{v}\,.\,\nabla^o\,\underline{v}$$

$$+ 2\underline{v}\cdot\nabla^o\frac{\partial\underline{\xi}}{\partial t} + \underline{v}\cdot\nabla^o\,(\underline{v}\cdot\nabla^o\,\underline{\xi}) + \frac{\partial\underline{v}}{\partial t}\cdot\nabla^o\underline{\xi} + \frac{\partial^2\underline{\xi}}{\partial t^2} \qquad \text{... (A.5)}$$

Comments on equations (A.4) and (A.5). Since we always use $\frac{d}{dt}$ in the way it is defined by equation (A.3), then this is one possible way of expressing the time derivative along with the motion in the perturbed state. Notice also that the 'local' time derivative $\frac{\partial}{\partial t}$ in the operation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v}\cdot\nabla^o,$$

is not the rate of change with time for an observer being at rest at the point r', but at $\underline{r}^o$. If we take the local time derivative at $\underline{r}'$ to be $\frac{\partial}{\partial t'}$, we also have:

$$\frac{d}{dt}\,\underline{v}(\underline{r}^o + \underline{\xi}) = \left\{\frac{\partial}{\partial t'} + \underline{v}(\underline{r}^o + \underline{\xi})\cdot\nabla'\right\}\left\{\underline{v}\,(\underline{r}^o) + \frac{d\underline{\xi}}{dt}\right\}.$$

By identifying this expression by that given in equation (A.5), it is easy to derive the relationship between $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial t'}$. To first order in $\underline{\xi}$ it is given by

$$\frac{\partial}{\partial t} = \frac{\partial\underline{\xi}}{\partial t}\cdot\nabla^o + \frac{\partial}{\partial t'} \ ,$$

a relation which is easy to interpret kinematically. We can also look at the problem in the following way. Let the velocity and the acceleration of a fluid element in the perturbed state at the point $\underline{r}^o$ be given by $\underline{v}_E$ and $\underline{a}_E$ respectively. Then to first order in $\underline{\xi}$ we obtain:

$$\underline{v}_E = \underline{v} + \frac{d\underline{\xi}}{dt} - \underline{\xi}\cdot\nabla^o\underline{v} \ ,$$

$$\underline{a}_E = \frac{\partial}{\partial t}\,\underline{v}_E + \underline{v}_E\cdot\nabla^o\,\underline{v}_E \ .$$

Furthermore we introduce:

$$\underline{a}'_E = \underline{a}(\underline{r}^o + \underline{\xi}) - \underline{\xi} \cdot \nabla^o \frac{d\underline{v}}{dt} \; .$$

By using equation (A.5) it is straightforward to prove that $\underline{a}_E \equiv \underline{a}'_E$ to first order in $\underline{\xi}$ (the result must of course be true to all orders in $\underline{\xi}$). The interpretation is obvious.

(c) THE PERTURBED DENSITY

Equation (2 .2) can be re-written as:

$$\frac{1}{\rho(\underline{r}^o + \underline{\xi})} \frac{d}{dt} \rho(\underline{r}^o + \underline{\xi}) = - \nabla' \cdot \left\{ \underline{v} + \frac{d\underline{\xi}}{dt} \right\} , \qquad \text{... (A.6)}$$

$$- \frac{1}{\rho(\underline{r}^o)} \frac{d}{dt} \rho(\underline{r}^o) = \nabla^o \cdot \underline{v} , \qquad \text{... (A.7)}$$

where equation (A.6) refers to the perturbed state and equation (A.7) to the unperturbed state. Furthermore the following operator equation follows from straightforward differentiation and application of the chain rule :

$$\nabla' = \frac{\partial}{\partial \underline{r}'} = \nabla^o - \nabla' \underline{\xi} \cdot \nabla^o \; . \qquad \text{... (A.8)}$$

By adding equation (A.6) and equation (A.7) we obtain:

$$\frac{d}{dt} \left\{ \ell n \frac{\rho(\underline{r}^o + \underline{\xi})}{\rho(\underline{r}^o)} \right\} = - \nabla' \cdot \left(\underline{v} + \frac{d\underline{\xi}}{dt} \right) + \nabla^o \cdot \underline{v} \; . \qquad \text{... (A.9)}$$

It can be proved that the right-hand side of equation (A.9) is equal to:

$$\nabla' \underline{\xi} : \nabla^o \underline{v} - \nabla' \underline{v} : \nabla^o \underline{\xi} - \frac{d}{dt} (\nabla' \cdot \underline{\xi}) \; .$$

The first two terms in this expression cancel to all orders in $\underline{\xi}$ when use is made of the recurrence relation given by equation (A.8). Thus we obtain:

$$\frac{d}{dt} \left\{ \ell n \frac{\rho(\underline{r}^o + \underline{\xi})}{\rho(\underline{r}^o)} + \nabla' \cdot \underline{\xi} \right\} = 0 ,$$

or in integrated form:

$$\rho(\underline{r}^o + \underline{\xi}) = \rho(\underline{r}^o) \exp [- \nabla' \cdot \underline{\xi}] \; . \qquad \text{... (A.10)}$$

By expansion of the exponential we obtain :

$$\rho(\underline{r}^o + \underline{\xi}) = \rho(\underline{r}^o) \left\{ 1 - \nabla' \cdot \underline{\xi} + \frac{1}{2} (\nabla' \cdot \underline{\xi})^2 + \dots \right\} . \qquad \text{... (A.11)}$$

Moreover by an iterative procedure using equation (A.8) we can obtain :

$$\nabla' \cdot \underline{\xi} = \sum_{n=o}^{\infty} (-1)^n (\nabla^o \underline{\xi})^n : \nabla^o \underline{\xi} , \qquad \text{... (A.12)}$$

where $(\nabla^o \underline{\xi})^n = \nabla^o \underline{\xi} \cdot \nabla^o \underline{\xi} \cdot \nabla^o \underline{\xi} \cdots$ (up to n factors) and $(\nabla^o \underline{\xi})^o = \underset{\sim}{I}$, the unit dyadic.

(When obtaining equation (A.12) it is assumed that the process converges. This will always be true if $|\underline{\xi}|$ is sufficiently small compared with the scale lengths in the problem.) Thus equations (A.11) and (A.12) determine $\rho(\underline{r}^o + \underline{\xi})$ to all orders in $\underline{\xi}$. When non-linear terms in $\underline{\xi}$ are left out we obtain:

$$\rho(\underline{r}^o + \underline{\xi}) = \rho(\underline{r}^o)\left\{1 - \nabla^o \cdot \underline{\xi}\right\}. \qquad \ldots (A.13)$$

(Notice that even if it is not explicitly stated all quantities $\underline{q}$ associated with the unperturbed motion can be explicitly dependent on time, i.e. $\frac{\partial \underline{q}}{\partial t} \neq 0$ in general.)

(d) THE PERTURBED PRESSURE

From equation (2.3), which can be integrated at once, we obtain:

$$\frac{p(\underline{r}^o + \underline{\xi}_o)}{p(\underline{r}^o)} = \left\{\frac{\rho(\underline{r}^o + \underline{\xi})}{\rho(\underline{r}^o)}\right\}^\gamma = e^{-\gamma\, \nabla' \cdot \underline{\xi}}, \qquad \ldots (A.14)$$

where equation (A.10) is used. Thus by making use of equation (A.12) the perturbed pressure is readily obtained to all orders in $\underline{\xi}$. To first order it is given by

$$p(\underline{r}^o + \underline{\xi}) = p(r^o)\{1 - \gamma \nabla^o \cdot \underline{\xi}\}. \qquad \ldots (A.15)$$

(e) THE PERTURBED MAGNETIC FIELD

From equations (2.2), (2.4) and (2.5) the following well-known relation[16] can be derived:

$$\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) = \frac{\underline{B}}{\rho} \cdot \nabla \underline{v}. \qquad \ldots (A.16)$$

Derivation: The left-hand side of equation (A.16) can be re-written as:

$$\frac{d}{dt}\left(\frac{\underline{\underline{B}}}{\rho}\right) = \frac{1}{\rho}\left(\frac{\partial \underline{\underline{B}}}{\partial t} + \underline{v} \cdot \nabla \underline{B} - \underline{B}\ \nabla \cdot \underline{v}\right).$$

Furthermore, from:

$$\nabla \times (\underline{E} + \underline{v} \times \underline{B}) = 0 \quad \text{and} \quad \nabla \times \underline{E} = -\frac{\partial \underline{\underline{B}}}{\partial t},$$

it follows that:

$$\frac{\partial}{\partial t}\underline{B} + \underline{v} \cdot \nabla \underline{B} - \underline{B}\, \nabla \cdot \underline{v} = \underline{B} \cdot \nabla \underline{v}.$$

This completes the derivation of equation (A.16).

From equation (A.16) referring to the perturbed state we obtain:

$$\frac{d}{dt}\underline{\Omega}(\underline{r}') = \underline{\Omega}(\underline{r}') \cdot \nabla' \underline{v}', \qquad \ldots (A.17)$$

where:

$$\underline{\Omega}(\underline{r}') = \frac{\underline{B}(\underline{r}')}{\rho(\underline{r}')} = \frac{\underline{B}(\underline{r}^o + \underline{\xi})}{\rho(\underline{r}^o + \underline{\xi})}. \qquad \ldots (A.18)$$

The following identities are easily obtained by using the chain rule of differentiation:

$$\nabla' f = \nabla' \underline{r}_o \cdot \nabla_o f \ , \qquad \text{... (A.19a)}$$

$$\nabla_o f = \nabla_o \underline{r}' \cdot \nabla' f \ . \qquad \text{... (A.19b)}$$

Here f can be any tensor point function. Notice that the determinant or scalar associated with the dyadic $\nabla' \underline{r}_o$ is the Jacobian:

$$\frac{\partial(x_o\, , y_o\, , z_o)}{\partial(x'\, , y'\, , z')} \ ,$$

where we have made the special choice of letting $\underline{r}'$ and $\underline{r}_o$ be represented by the Cartesian co-ordinates $\underline{r}' = \{x'\, , y'\, , z'\}$ and $\underline{r}_o = \{x_o\, , y_o\, , z_o\}$.

By taking f to be $\underline{r}'$ and $\underline{r}_o$ respectively in equations (A.19a) and (A.19b) we obtain:

$$\underset{\sim}{I} = \nabla' \underline{r}_o \cdot \nabla_o \underline{r}' \ , \qquad \text{... (A.20)}$$

$$\underset{\sim}{I} = \nabla_o \underline{r}' \cdot \nabla' \underline{r}_o \ , \qquad \text{... (A.21)}$$

where $\underset{\sim}{I}$ is the unit dyadic as before. By taking the derivative of equation (A.20) we obtain:

$$\frac{d}{dt}(\nabla' \underline{r}_o) \cdot \nabla_o r' = - \nabla' \underline{r}_o \cdot \frac{d}{dt}(\nabla_o r') \ . \qquad \text{... (A.22)}$$

Moreover we have:

$$\nabla' \underline{v}' = \nabla' \frac{d}{dt} \underline{r}' = \nabla' \underline{r}_o \cdot \nabla_o \left(\frac{d}{dt} \underline{r}' \right)$$

$$= \nabla' \underline{r}_o \cdot \frac{d}{dt}(\nabla_o \underline{r}') = - \frac{d}{dt}(\nabla' \underline{r}_o) \cdot \nabla_o r' \ , \qquad \text{... (A.23)}$$

where equation (A.22) has been used in the final step. Notice also that the operators $\frac{d}{dt}$ and ∇_o commute, since $\underline{r}' = \underline{r}'\{\underline{r}^o(\underline{r}_o,t),t\}$, and $\frac{d}{dt}$ is evaluated at fixed $\underline{r}_o$. By making use of this result equation (A.17) can be rewritten as:

$$\frac{d}{dt} \underline{\Omega}(\underline{r}') + \underline{\Omega}(r') \cdot \frac{d}{dt} (\nabla' r_o) \cdot \nabla_o \underline{r}' = 0 \ . \qquad \text{... (A.24)}$$

By multiplying equation (A.24) by $\nabla' \underline{r}_o$ as a post-factor and using equation (A.21), we obtain:

$$\frac{d}{dt} \{ \underline{\Omega}(\underline{r}') \cdot \nabla' \underline{r}_o \} = 0 \ . \qquad \text{... (A.25)}$$

Equation (A.25) can now be integrated to give:

$$\underline{\Omega}(\underline{r}') \cdot \nabla' \underline{r}_o = \underline{\Omega}(\underline{r}_o) \ . \qquad \text{... (A.26)}$$

Equation (A.26) is also true with $\underline{r}'$ replaced by $\underline{r}^o$, thus:

$$\underline{\Omega}(\underline{r}^o) \cdot \underline{\nabla}^o \underline{r}_o = \underline{\Omega}(\underline{r}_o) \ . \qquad \text{... (A.27)}$$

Notice that equation (A.26) refers to the perturbed motion and equation (A.27) to the unperturbed motion. From equations (A.26) and (A.27) we readily obtain:

$$\underline{\Omega}(r') \cdot \nabla' \underline{r}_o = \underline{\Omega}(\underline{r}^o) \cdot \nabla^o \underline{r}_o \ . \qquad \text{... (A.28)}$$

By multiplying equation (A.28) by $\nabla_o \underline{r}'$ and making use of equation (A.20) we obtain:

$$\underline{\Omega}(\underline{r}') = \underline{\Omega}(\underline{r}^o) \cdot \nabla^o \underline{r}_o \cdot \nabla_o \underline{r}' \quad . \qquad \dots (A.29)$$

Moreover, we have :

$$\nabla^o \underline{r}_o \cdot \nabla_o r' = \nabla^o \underline{r}' = \nabla^o(\underline{r}^o + \underline{\xi}) = \underset{\sim}{I} + \nabla^o \underline{\xi} \ ,$$

and finally we obtain :

$$\underline{\Omega}(\underline{r}') = \underline{\Omega}(\underline{r}^o) + \underline{\Omega}(\underline{r}^o) \cdot \nabla^o \underline{\xi} \quad ,$$

or by equation (A.18) :

$$\underline{B}(\underline{r}^o + \underline{\xi}) = \frac{\rho(\underline{r}^o + \underline{\xi})}{\rho(\underline{r}^o)} \left\{ \underline{B}(\underline{r}^o) + \underline{B}(\underline{r}^o) \cdot \nabla^o \underline{\xi} \right\} . \qquad \dots (A.30)$$

This result, as the previous results, is general. Since $\rho(\underline{r}^o + \underline{\xi})$ is already obtained in equations (A.10) and (A.12) this determines the perturbed magnetic field at $\underline{r}^o + \underline{\xi}$ in terms of $\underline{\xi}$. If we neglect all non-linear terms, equation (A.30) gives :

$$\underline{B}(\underline{r}^o + \underline{\xi}) = \underline{B}(\underline{r}^o) + \underline{B}(\underline{r}^o) \cdot \nabla^o \underline{\xi} - \underline{B}(\underline{r}^o) \nabla^o \cdot \underline{\xi} \ . \qquad \dots (A.31)$$

(f) THE PERTURBED SURFACE UNIT NORMAL

Let the surface of discontinuity in the unperturbed state be given by the parametric representation :

$$\underline{r}^o = \underline{r}^o \, (u,v) \ . \qquad \dots (A.32)$$

Moreover, let the subscripts u and v denote differentiation with respect to u and v respectively.

The unit normal $\underline{n} = \underline{n}(\underline{r}^o)$ at the point $\underline{r}^o$ of the unperturbed surface is then given by:

$$\underline{n} = \frac{\underline{r}^o_u \times \underline{r}^o_v}{H} \quad , \quad H = \left[\underline{r}^o_u \ \underline{r}^o_v \ \underline{n} \right] . \qquad \dots (A.33)$$

The unit normal $\underline{n}' = \underline{n}(\underline{r}^o + \underline{\xi})$ at the corresponding point $\underline{r}^o + \underline{\xi}(\underline{r}^o, t)$ of the perturbed surface is given by :

$$H' \underline{n}' = H \underline{n} + \underline{r}^o_u \times \underline{\xi}_v - \underline{r}^o_v \times \underline{\xi}_u \ , \qquad \dots (A.34)$$

where H′ is a scalar function to be determined by the condition $\underline{n}' \cdot \underline{n}' = 1$.

If $\underline{e}$ is a unit vector then any vector $\underline{A}$ can be decomposed as:

$$\underline{A} = \underline{e}\,\underline{e} \cdot \underline{A} - \underline{e} \times (\underline{e} \times \underline{A}) \ . \qquad \dots (A.35)$$

If we take :

$$\underline{A} \overset{\text{def}}{=} \underline{r}^o_u \times \underline{\xi}_v - \underline{r}^o_v \times \underline{\xi}_u \ ,$$

we obtain :

$$\underline{n} \cdot \underline{A} = \underline{n} \times \underline{r}^o_u \cdot \underline{\xi}_v + \underline{r}^o_v \times \underline{n} \cdot \underline{\xi}_u \ . \qquad \dots (A.36)$$

Furthermore, denoting the reciprocal set to $\underline{r}^o_u, \underline{r}^o_v, \underline{n}$ by $\underline{a}, \underline{b}, \underline{c}$, we have

$$\underline{a} = \frac{\underline{r}^o_v \times \underline{n}}{H} \quad , \quad \underline{b} = \frac{\underline{n} \times \underline{r}^o_u}{H} \quad , \quad \underline{c} = \frac{\underline{r}^o_u \times \underline{r}^o_v}{H} = \underline{n} \ , \qquad \dots (A.37)$$

and we can write :

$$\nabla^o = \underline{a}\frac{\partial}{\partial u} + \underline{b}\frac{\partial}{\partial v} + \underline{n}\,\underline{n}\cdot\nabla^o \,. \qquad \text{... (A.38)}$$

Thus equations (A.36), (A.37) and (A.38) yield :

$$\underline{n}\cdot\underline{A} = \left\{\nabla^o\cdot\underline{\xi} - \underline{n}\,\underline{n} : \nabla^o\underline{\xi}\right\} H \,. \qquad \text{... (A.39)}$$

Furthermore, we have :

$$\begin{aligned}\underline{n}\times(\underline{n}\times\underline{A}) &= \underline{n}\times\left\{\underline{n}\times\left(\underline{r}^o_u\times\underline{\xi}_v - \underline{r}^o_v\times\underline{\xi}_u\right)\right\}\\ &= \left(\underline{n}\times\underline{r}^o_u\,\underline{\xi}_v + \underline{r}_v\times\underline{n}\,\underline{\xi}_u\right)\cdot\underline{n}\\ &= \left(\nabla^o\underline{\xi}\cdot\underline{n} - \underline{n}\,\underline{n}\,\underline{n} : \nabla^o\underline{\xi}\right) H \,. \qquad \text{... (A.40)}\end{aligned}$$

In the first step here we have expanded the vector product inside the braces { } and used the fact that $\underline{r}^o_u\cdot\underline{n} = \underline{r}^o_v\cdot\underline{n} = 0$. In the final step equations (A.37) and (A.38) are used again.

Now in equation (A.35) we take $\underline{e}$ to be $\underline{n}$ and substitute for $\underline{n}\cdot\underline{A}$ and $\underline{n}\times(\underline{n}\times\underline{A})$ from equations (A.39) and (A.40) to obtain :

$$\underline{r}^o_u\times\underline{\xi}_v - \underline{r}^o_v\times\underline{\xi}_u = \left\{\underline{n}\,\nabla^o\cdot\underline{\xi} - \nabla^o\underline{\xi}\cdot\underline{n}\right\} H \,. \qquad \text{... (A.41)}$$

By using this result equation (A.34) can be re-written as:

$$H'\underline{n}' = H(\underline{n} + \underline{n}\,\nabla^o\cdot\underline{\xi} - \nabla^o\underline{\xi}\cdot\underline{n}) \;, \qquad \text{... (A.42)}$$

and it follows that

$$\left(\frac{H'}{H}\right)^2 = 1 + 2\nabla^o\cdot\underline{\xi} - 2\,\underline{n}\,\underline{n} : \nabla^o\underline{\xi} - 2\,\nabla^o\cdot\underline{\xi}\,\underline{n}\,\underline{n} : \nabla^o\underline{\xi} + (\nabla^o\cdot\underline{\xi})^2 + (\nabla^o\underline{\xi}\cdot n)^2 \,. \qquad \text{... (A.43)}$$

Now from equations (A.42) and (A.43) $\underline{n}'$ can readily be obtained to all orders in $\underline{\xi}$. To first order in $\underline{\xi}$ we have :

$$\underline{n}' = \underline{n}(\underline{r}^o + \underline{\xi}) = \underline{n}(\underline{r}^o) - \nabla^o\underline{\xi}\cdot\underline{n} + \underline{n}\,\underline{n} : \nabla^o\underline{\xi} \,. \qquad \text{... (A.44)}$$

APPENDIX B

BOUNDARY CONDITIONS

(a) THE BOUNDARY CONDITIONS IN GENERAL

(1) We first discuss equation (2.8). Notice that equation (2.1) can be re-written as:

$$\rho \frac{d\underline{v}}{dt} = - \nabla \cdot \left\{ \underset{\sim}{I} \left(p + \frac{B^2}{2} \right) - \underline{B}\,\underline{B} \right\} , \qquad \text{... (B.1)}$$

where $\underset{\sim}{I}$ is the unit dyadic (tensor). We consider a small volume :

$$\Delta\tau = \delta \Delta S ,$$

of a thin sheath across which the changes occur (see Fig.2). Here δ is the thickness of the sheath and $\Delta\tau$ the volume. We integrate equation (B.2) over the volume $\Delta\tau$. By using Gauss' theorem we obtain:

$$\int_{\Delta\tau} \rho \frac{d\underline{v}}{dt} d\tau = \int_{S} \left\{ \underset{\sim}{I} \left(p + \frac{B^2}{2} \right) - \underline{B}\,\underline{B} \right\} \cdot \underline{n}\, dS \qquad \text{... (B.2)}$$

where S is the surface bounding the volume $\Delta\tau$ and $\underline{n}$ is the surface unit normal. We now assume that the surface has no sheet mass, i.e.

$$\lim_{\delta \to 0} \int_{\Delta\tau} \rho \, d\tau = 0 \qquad \text{... (B.3)}$$

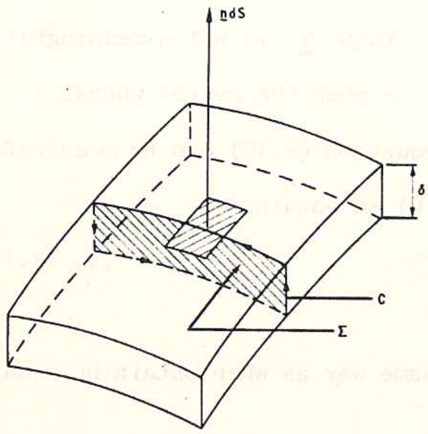

Fig. 2 (CLM-R 97)
Slab of the plasma sheath which in the limit $\delta \to 0$ shrinks to the surface of discontinuity

Moreover $\frac{d\underline{v}}{dt}$ which is the acceleration of a fluid element must always be finite for physical reasons. Thus by passing to the limit $\delta \to 0$, we obtain from equation (B.2)

$$\underline{n} \left\langle p + \frac{B^2}{2} \right\rangle - \left\langle \underline{n} \cdot \underline{B}\,\underline{B} \right\rangle = 0 . \qquad \text{... (B.4)}$$

Notice that $\underline{n} \cdot \underset{\sim}{I} = \underline{n}$ and that :

$$\underline{n} \cdot \langle \underline{B} \rangle = 0 , \qquad \text{... (B.5)}$$

where equation (B.5) follows immediately from equation (2.7). Multiplying equation (B.4) by $\underline{n}\cdot$ and using equation (B.5) yields :

$$\left\langle p + \frac{B^2}{2} \right\rangle = 0 . \qquad \text{... (B.6)}$$

Multiplying equation (B.4) by $\underline{n}\times$ and using equation (B.5) yields:

$$\underline{n} \cdot \underline{B}\; \underline{n} \times \langle \underline{B} \rangle = 0 . \qquad \text{... (B.7)}$$

It is clear that this result is not dependent on the dynamic state of motion of the surface.

(2) The boundary condition, equation (2.9), can be obtained from equation (2.4). We first take the curl of equation (2.4). By integrating the resulting equation over a surface Σ, bounded by the curve C, (see Fig.2) and transforming this surface integral into a line integral along C using Stokes' theorem, we obtain in the limit as $\delta \to 0$:

$$\underline{n} \times \langle \underline{E} \rangle = \langle \underline{n} \cdot \underline{v} \underline{B} - \underline{n} \cdot \underline{B} \underline{v} \rangle . \qquad \text{... (B.8)}$$

From equation (2.2) we obtain :

$$\lim_{\delta \to 0} \int_{\Delta\tau} \frac{1}{\rho} \frac{d}{dt} \rho \, d\tau = - \underline{n} \cdot \langle \underline{v} \rangle . \qquad \text{... (B.9)}$$

The motion of the sheath is now assumed to be incompressible, and therefore that $\delta \to 0$ is preserved during the course of the motion. It then follows from equation (B.9) that :

$$\underline{n} \cdot \langle \underline{v} \rangle = 0 . \qquad \text{... (B.10)}$$

If there is vacuum on one side of the discontinuity surface then $\underline{v}$ is not a meaningful quantity on the vacuum side. However, we can take $\underline{n} \cdot \underline{v}$ to mean the motion normal to the surface of the magnetic lines of force. Therefore, equation (B.10) can be meaningful also in this case. By combining equations (B.8) and (B.10) we obtain :

$$\underline{n} \times \langle \underline{E} \rangle = \underline{n} \cdot \underline{v} \langle \underline{B} \rangle - \langle \underline{n} \cdot \underline{B} \underline{v} \rangle . \qquad \text{... (B.11)}$$

(3) Starting from equation (2.6) and proceeding in the same way as when obtaining equation (B.8), we obtain :

$$\underline{n} \times \langle \underline{B} \rangle = \underline{k} , \qquad \text{... (B.12)}$$

where :

$$\underline{k} = \lim_{\delta \to 0} \int_{\Sigma} \underline{J} \cdot d\underline{\Sigma} , \qquad \text{... (B.13)}$$

is the sheet current flowing in the discontinuity surface. By hypothesis we have $\underline{k} \neq 0$. Thus from equation (B.7) we must have :

$$\underline{n} \cdot \underline{B} = 0 , \qquad \text{... (B.14)}$$

otherwise the boundary conditions would not be consistent. Equation (B.11) now reduces to equation (2.9)

(b) THE BOUNDARY CONDITIONS IN THE PERTURBED STATE

(1) In the following relations only first order contributions in $\underline{\xi}$ are retained. By taking :

$$\delta p = p(\underline{r}^0 + \underline{\xi}) - p , \qquad \text{(eq. (2.15))}, \qquad \text{... (B.15)}$$

$$\delta \underline{B}_p = \underline{B}(\underline{r}^0 + \underline{\xi}) - \underline{B} , \qquad \text{(eq. (2.16))}, \qquad \text{... (B.16)}$$

$$\delta \underline{B}_v = \nabla \times \underline{A} + \underline{\xi} \cdot \nabla \underline{B}_v , \qquad \text{(eq. (2.21))}, \qquad \text{... (B.17)}$$

condition (2.8) yields:

$$\langle \delta p + \underline{B} \cdot \delta \underline{B} \rangle = 0 . \qquad \text{... (B.18)}$$

Explicitly in terms of $\underline{\xi}$ equation (B.18) reads:

$$- \gamma p \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla p + \underline{B}_p \cdot \underline{Q} = \underline{B}_v \cdot \nabla \times \underline{A} + \underline{\xi} \cdot \left\langle \nabla \left(p + \frac{B^2}{2} \right) \right\rangle . \qquad \text{... (B.19)}$$

Note that $p + \frac{B^2}{2}$ is a continuously differentiable quantity on each side of the discontinuity surface. Thus it follows from equation (2.8) (by applying the operator $\underline{n} \times \nabla$ which involves only differentiation along the discontinuity surface) that

$$\underline{n} \times \left\langle \nabla \left(p + \frac{B^2}{2} \right) \right\rangle = 0 . \qquad \text{... (B.20)}$$

(Note also that it is meaningless to apply the operator $\underline{n} \cdot \nabla$ to the same equation because the quantities involved are not differentiable across the discontinuity surface.)

By using equation (A.35) with $\underline{A} = \underline{\xi}$ and $\underline{e} = \underline{n}$ we obtain :

$$\underline{\xi} \cdot \left\langle \nabla \left(p + \frac{B^2}{2} \right) \right\rangle = \underline{n} \cdot \underline{\xi}\, \underline{n} \cdot \left\langle \nabla \left(p + \frac{B^2}{2} \right) \right\rangle , \qquad \text{... (B.21)}$$

since by equation (B.20) we have

$$- \underline{n} \times (\underline{n} \times \underline{\xi}) \cdot \left\langle \nabla \left(p + \frac{B^2}{2} \right) \right\rangle = (\underline{n} \times \underline{\xi}) \cdot \underline{n} \times \left\langle \nabla \left(p + \frac{B^2}{2} \right) \right\rangle = 0 .$$

A combination of equations (B.19) and (B.21) immediately gives equation (2.22).

(2) We take $\delta \underline{n}$ to be

$$\delta \underline{n} = \underline{n}(\underline{r}^0 + \underline{\xi}) - \underline{n}(\underline{r}^0) \quad , \quad \text{(eq. (2.18))} . \qquad \text{... (B.22)}$$

From equation (2.10) we obtain :

$$\delta \underline{n} \cdot \underline{B}_v + \underline{n} \cdot \nabla \times \underline{A} + \underline{n}\underline{\xi} : \nabla \underline{B}_v = 0 , \qquad \text{... (B.23)}$$

where equation (2.21) has been used. Now we have

$$\delta \underline{n} = \underline{n}\,\underline{n}\,\underline{n} : \nabla \underline{\xi} - \nabla \underline{\xi} \cdot \underline{n} , \qquad \text{... (B.24)}$$

and it follows that :

$$\underline{B}_v \cdot \delta \underline{n} = - \underline{n}\underline{B}_v : \nabla \underline{\xi} \quad , \quad (\underline{n} \cdot \underline{B}_v = 0) . \qquad \text{... (B.25)}$$

By using this result, equation (B.23) yields:

$$(\underline{\xi} \cdot \nabla \underline{B}_v - \underline{B}_v \cdot \nabla \underline{\xi}) \cdot \underline{n} + \underline{n} \cdot \nabla \times \underline{A} = 0 . \qquad \text{... (B.26)}$$

Furthermore, we have that :

$$(\underline{\xi} \cdot \nabla \underline{B}_v - \underline{B}_v \cdot \nabla \underline{\xi}) \cdot \underline{n} = - \underline{n} \cdot \nabla \times (\underline{\xi} \times \underline{B}_v) . \qquad \text{... (B.27)}$$

Therefore, equation (B.26) can be re-written as :

$$\underline{n} \cdot \nabla \times (\underline{A} - \underline{\xi} \times \underline{B}_v) = 0 . \qquad \text{... (B.28)}$$

In discussing this equation it is convenient to introduce the surface gradient :

$$\nabla_s \overset{\text{def}}{=} \nabla - \underline{n}\,\underline{n} \cdot \nabla . \qquad \text{... (B.29)}$$

Let $\underline{v}$ be any vector field defined and continuously differentiable over the surface S. Let $\underline{n}$ as before denote the surface unit normal, then the following vector identities exists, see Brand[6] :

$$\underline{n} \cdot \nabla \times \underline{v} = \underline{n} \cdot \nabla_s \times \underline{v} \ , \qquad \ldots (B.30)$$

$$\underline{n} \cdot \nabla_s \times \underline{v} = - \nabla_s \cdot (\underline{n} \times \underline{v}) \ , \qquad \ldots (B.31)$$

$$\nabla_s \times \underline{n} = 0 \ . \qquad \ldots (B.32)$$

Making use of equations (B.30) and (B.31) we can re-write equation (B.28) as :

$$\nabla_s \cdot \{ \underline{n} \times (\underline{A} - \underline{\xi} \times \underline{B}_V) \} = 0 \ . \qquad \ldots (B.33)$$

Now an obvious solution to equation (B.33) is:

$$\underline{n} \times (\underline{A} - \underline{\xi} \times \underline{B}_V) = 0 \ . \qquad \ldots (B.34)$$

There is, however, no loss in generality by taking the solution of equation (B.33) to be given by equation (B.34). To prove this let us assume on the contrary to equation (B.34) that:

$$\underline{n} \times (\underline{A} - \underline{\xi} \times \underline{B}_V) = \underline{n} \times \underline{b} \neq 0 \ , \qquad \ldots (B.35)$$

where $\underline{b}$ is an arbitrary solution of the equation:

$$\nabla_s \cdot (\underline{n} \times \underline{b}) = 0 \ , \qquad \ldots (B.36)$$

or the equivalent equation:

$$\underline{n} \cdot (\nabla \times \underline{b}) = 0 \ . \qquad \ldots (B.37)$$

Now making use of the gauge transformation :

$$\underline{A} = \underline{A}' + \nabla \psi \ ,$$

where we chose ψ to satisfy :

$$\underline{n} \times \nabla \psi = \underline{n} \times \underline{b} \ , \qquad \ldots (B.38)$$

we obtain :

$$\underline{n} \times (\underline{A}' - \underline{\xi} \times \underline{B}_V) = 0 \ . \qquad \ldots (B.39)$$

Note that $\nabla_s \cdot (\underline{n} \times \nabla \psi) = 0$ for all ψ .

It now remains to prove that equation (B.38) always has a solution. For this purpose it is convenient to introduce orthogonal curvilinear co-ordinates u, v, w, with the base unit vectors $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, and the scale factors h_1 , h_2 , h_3 . Let the surface S in these co-ordinates be given by $w = w_0 =$ constant, thus $\underline{a}_3 = \underline{n}$. Moreover we write:

$$\underline{b} = \underline{a}_i \, b_i \ , \qquad \ldots (B.40)$$

and equation (B.37) reads:

$$\frac{\partial}{\partial u} (h_2 b_2) - \frac{\partial}{\partial v} (h_1 b_1) = 0 \ . \qquad \ldots (B.41)$$

we also have :

$$\nabla \psi = \underline{a}_1 \frac{1}{h_1} \frac{\partial \psi}{\partial u} + \underline{a}_2 \frac{1}{h_2} \frac{\partial \psi}{\partial v} + \underline{n} \frac{1}{h_3} \frac{\partial \psi}{\partial w} \ , \qquad \ldots (B.42)$$

and equation (B.38) can be written as :

$$b_1 = \frac{1}{h_1} \frac{\partial \psi}{\partial u} \ , \quad b_2 = \frac{1}{h_2} \frac{\partial \psi}{\partial v} \ . \qquad \ldots (B.43)$$

Thus we obtain :

$$\psi = \int b_1 h_1 \, du \quad \text{or} \quad \psi = \int b_2 h_2 \, dv \qquad \ldots (B.44)$$

and equation (B.41) ensures that either of these expressions for ψ gives the same result. Thus for a given $\underline{b}$ we can always prescribe ψ which will give the proper gauge transformation. Equation (B.34) can therefore in general be taken to be the equation which relates $\underline{\xi}$ to $\underline{A}$ at the discontinuity surface.

Remark: We can look at this problem in a different way. Let the surface of discontinuity be replaced by a transition region of finite thickness over which all changes occur continuously. The quantity in the plasma region corresponding to $\nabla \times \underline{A}$ in the vacuum region clearly is $\underline{Q} = \nabla \times (\underline{\xi} \times \underline{B})$. By moving through this transition region from the plasma towards the vacuum $\underline{Q}$ changes continuously into $\nabla \times \underline{A}$. Since it is the normal component of the magnetic field which has to be continuous this requires that the parallel component of the vector potential has to be continuous, or

$$n \times (\underline{\xi} \times \underline{B}) \rightarrow \underline{n} \times \underline{A} .$$

This is the physical interpretation of equation (2.23).

APPENDIX C

THE EQUATION OF MOTION IN THE PERTURBED STATE

There are at least two obvious ways of obtaining this equation. First, one can calculate all quantities in the perturbed state at the point $\underline{r}' = \underline{r}^o + \underline{\xi}$, and substitute for these quantities in equation (2.1) with ∇ replaced by $\nabla' = \partial/\partial\underline{r}'$. The second way is to calculate all quantities in the perturbed state at $\underline{r}^o$, in which case $\nabla = \partial/\partial\underline{r}^o$ has to be used when substituting into equation (2.1). These two ways resemble some of the difference between an Eulerian and Lagrangian description of the fluid motion. We shall first prove that both these ways yield the same final result.

We start from the equation of motion which can be written as

$$\rho \frac{dv}{dt} = - \nabla \cdot \underset{\sim}{T} \qquad \text{... (C.1)}$$

where

$$\underset{\sim}{T} = \underset{\sim}{I}(p + \frac{B^2}{2}) - \underline{B}\underline{B} \;, \qquad \text{... (C.2)}$$

and $\underset{\sim}{I}$ is the unit dyadic (tensor). For simplicity we have omitted the term $-\rho\nabla\psi$ in equation (C.1). We now write:

$$\begin{aligned} \rho(\underline{r}^o + \underline{\xi}) &= \rho_o + \delta\rho = \rho_o + \Delta\rho + \underline{\xi}\cdot\nabla\rho_o \;, \\ \underline{v}(\underline{r}^o + \underline{\xi}) &= v_o + \delta\underline{v} = \underline{v}_o + \Delta\underline{v} + \underline{\xi}\cdot\nabla\underline{v}_o \;, \\ \underset{\sim}{T}(r^o + \underline{\xi}) &= \underset{\sim}{T}_o + \delta\underset{\sim}{T} = \underset{\sim}{T}_o + \Delta\underset{\sim}{T} + \underline{\xi}\cdot\nabla\underset{\sim}{T}_o \;, \end{aligned} \qquad \text{... (C.3)}$$

Here ρ_o, $\underline{v}_o$ and $\underset{\sim}{T}_o$ are the respective quantities in the unperturbed state calculated at $\underline{r}^o$. Furthermore, $\varphi(\underline{r}^o + \underline{\xi})$ etc. are the same quantities in the perturbed state calculated at $\underline{r}^o + \underline{\xi}$. Finally, $\rho_o + \Delta\rho$ etc. represents the zero and first order quantities with respect to $\underline{\xi}$ for the perturbed state calculated at the 'original' position $\underline{r}^o$. Note that the explicit time dependence is in both cases functionally the same and is calculated at constant $\underline{r}^o$. (See the discussion following equation (A.5)).

(i) By substituting for the perturbed quantities evaluated at $\underline{r}^o$ from equations (C.3), the linearised equation resulting from equation (2.1) is :

$$\rho_o \frac{d\underline{v}_o}{dt} + \Delta\rho \frac{d}{dt}\underline{v}_o + \rho_o \frac{d}{dt}\Delta\underline{v} + \rho_o(\Delta\underline{v})\cdot\nabla\underline{v}_o = -\nabla\cdot(\underset{\sim}{T}_o + \Delta\underset{\sim}{T}) \;. \qquad \text{... (C.4)}$$

Moreover, we have :

$$\rho_o \frac{d\underline{v}_o}{dt} = -\nabla\cdot\underset{\sim}{T}_o \qquad \text{... (C.5)}$$

Thus equation (C.4) reduces to :

$$\Delta\rho \frac{d}{dt}\underline{v}_o + \rho_o \frac{d}{dt}(\Delta\underline{v}) + \rho_o(\Delta\underline{v})\cdot\nabla\underline{v}_o = -\nabla\cdot(\Delta\underset{\sim}{T}) \;. \qquad \text{... (C.6)}$$

Now by using eqs.(C.3) again, and introducing the perturbed quantities evaluated at $\underline{r}^o + \underline{\xi}$, we find :

$$\delta\rho \frac{d}{dt} \underline{v}_o + \rho_o \frac{d}{dt} \delta \underline{v} = - \nabla . (\Delta \underset{\sim}{T})$$

$$+ \underline{\xi} \cdot \nabla\rho_o \frac{d\underline{v}_o}{dt} + \rho_o \frac{d}{dt} (\underline{\xi} \cdot \nabla \underline{v}_o) + \rho_o \underline{\xi} \cdot \nabla \underline{v}_o \cdot \nabla \underline{v}_o - \rho_o \delta \underline{v} \cdot \nabla \underline{v}_o \; . \quad \text{... (C.7)}$$

Furthermore, we have:

$$\underline{\xi} \cdot \nabla \rho_o \frac{d\underline{v}_o}{dt} + \rho_o \frac{d}{dt} (\underline{\xi} \cdot \nabla \underline{v}_o) + \rho_o \underline{\xi} \cdot \nabla \underline{v}_o \cdot \nabla \underline{v}_o - \rho_o \delta\underline{v} \cdot \nabla \underline{v}_o$$

$$= \underline{\xi} \cdot \nabla\left(\rho_o \frac{d\underline{v}_o}{dt}\right) = -\underline{\xi} \cdot \nabla (\nabla \cdot \underset{\sim}{T}_o) = -\nabla \cdot (\underline{\xi} \cdot \nabla \underset{\sim}{T}_o) + \nabla \underline{\xi} : \nabla \underset{\sim}{T}_o \; . \quad \text{... (C.8)}$$

Here we have used the facts that

$$\delta\underline{v} = \frac{d\underline{\xi}}{dt} \quad \text{and} \quad \frac{d}{dt}\nabla = \nabla \frac{d}{dt} - \nabla \underline{v}_o \cdot \nabla \; ;$$

equation (C.5) has also been used. Thus from equations (C.7) and (C.8) we obtain:

$$\delta\rho \frac{d}{dt} \underline{v}_o + \rho_o \frac{d}{dt} \delta\underline{v} = - \nabla \cdot (\delta T) + \nabla\underline{\xi} : \nabla \underset{\sim}{T}_o = -\nabla' \cdot (\underset{\sim}{T}_o + \delta \underset{\sim}{T}) + \nabla \cdot \underset{\sim}{T}_o \; ,$$

or :

$$\rho_o \frac{d\underline{v}_o}{dt} + \delta\rho \frac{d}{dt} \underline{v}_o + \rho_o \frac{d}{dt} \delta\underline{v} = -\nabla' \cdot (\underset{\sim}{T}_o + \delta\underset{\sim}{T}) \; . \quad \text{... (C.9)}$$

Equation (C.9) is exactly the equation one obtains by substituting into eq.(C.1) for the perturbed quantities evaluated at $\underline{r}^o + \underline{\xi}$ and by replacing ∇ by $\nabla' = \frac{\partial}{\partial \underline{r}'}$. Thus the proof is completed.

(ii) Now we proceed to obtain the actual equation of motion explicit in terms of $\underline{\xi}$ by starting from equation (C.9) and substituting for all the perturbed quantities by equations (2.14) - (2.17). Notice that $\frac{d}{dt}\delta\underline{v}$ is given by equation (A.5).

$$\delta\rho \frac{d}{dt}\underline{v} + \rho \frac{d}{dt} \delta\underline{v} = -\rho \nabla \cdot \underline{\xi} \frac{d}{dt}\underline{v} + \rho \left\{ 2\,\underline{v} \cdot \nabla \frac{\partial\underline{\xi}}{\partial t} + \underline{v} \cdot \nabla(\underline{v} \cdot \nabla\underline{\xi}) + \frac{\partial \underline{v}}{\partial t} \cdot \nabla\underline{\xi} + \frac{\partial^2\underline{\xi}}{\partial t^2} \right\} . \quad \text{... (C.10)}$$

The subscripts zero, indicating unperturbed quantities, have been omitted in equation (C.10). This will also be done in the following calculations. Now we add :

$$\left[\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho\underline{v})\right] \underline{v} \cdot \nabla\underline{\xi} = 0 \; , \quad \text{... (C.11)}$$

to the right hand side of equation (C.10). Thus we obtain, after rearranging the terms ,

$$\delta\rho \frac{d}{dt} \underline{v}_o + \rho_o \frac{d}{dt} \delta\underline{v} =$$

$$\rho \frac{\partial^2\underline{\xi}}{\partial t^2} + 2\rho\underline{v} \cdot \nabla \frac{\partial\underline{\xi}}{\partial t} + \frac{\partial}{\partial t}(\rho\underline{v}) \cdot \nabla\underline{\xi} - \nabla \cdot \left\{\underline{\xi}\rho \frac{d\underline{v}}{dt} - \rho\underline{v}\,\underline{v} \cdot \nabla\xi\right\} + \underline{\xi} \cdot \nabla \left(\rho \frac{dv}{dt}\right) . \quad \text{... (C.12)}$$

Furthermore, we have :

$$\delta\underset{\sim}{T} = \underset{\sim}{I}(\delta p + \underline{B} \cdot \delta\underline{B}) - \delta\underline{B}\,\underline{B} - \underline{B}\,\delta\underline{B} \quad \text{... (C.13)}$$

and

$$\nabla \cdot \delta\underset{\sim}{T} = \nabla(\delta p + \underline{B} \cdot \delta\underline{B}) - \nabla \cdot \delta\underline{B}\,\underline{B} - \delta\underline{B} \cdot \nabla\underline{B} - \underline{B} \cdot \nabla\delta\underline{B} \; , \quad \text{... (C.14)}$$

$$\delta p = -\gamma p \nabla \cdot \underline{\xi} \;, \qquad \text{... (C.15)}$$

$$\delta\underline{B} = \nabla \times (\underline{\xi} \times \underline{B}) + \underline{\xi} \cdot \nabla \underline{B} = \underline{Q} + \underline{\xi} \cdot \nabla \underline{B} \;. \qquad \text{... (C.16)}$$

Thus from equations (C.14) – (C.16) it follows that :

$$- \nabla \cdot \delta \underset{\sim}{T} = \nabla(\gamma p \nabla \cdot \underline{\xi} - \underline{B} \cdot \underline{Q}) + \underline{Q} \cdot \nabla \underline{B} + \underline{B} \cdot \nabla \underline{Q} - \nabla(\underline{B}\underline{\xi} : \nabla \underline{B}) + \nabla \cdot (\underline{\xi} \cdot \nabla \underline{B})\underline{B} + \underline{\xi} \cdot \nabla \underline{B} \cdot \nabla \underline{B}$$
$$+ \underline{B} \cdot \nabla(\xi \cdot \nabla \underline{B}) = \nabla(\gamma p \nabla \cdot \underline{\xi} - \underline{B} \cdot \underline{Q}) + \underline{Q} \cdot \nabla \underline{B} + \underline{B} \cdot \nabla \underline{Q} + \nabla \underline{\xi} \cdot \nabla\left(\frac{B^2}{2}\right) + \underline{\xi} \cdot \nabla \nabla \left(\frac{B^2}{2}\right) + \nabla \cdot \left\{\underline{\xi} \cdot \nabla(\underline{B}\,\underline{B})\right\}$$
$$\text{... (C.17)}$$

Notice that we have

$$\nabla(\underline{B}\,\underline{\xi} : \nabla \underline{B}) = \nabla\left\{\underline{\xi} \cdot \nabla\left(\frac{B^2}{2}\right)\right\} = \nabla \underline{\xi} \cdot \nabla\left(\frac{B^2}{2}\right) + \underline{\xi} \cdot \nabla \nabla\left(\frac{B^2}{2}\right)$$

and

$$\nabla \cdot \left\{\underline{\xi} \cdot \nabla(\underline{B}\,\underline{B})\right\} = \nabla \cdot \left\{\underline{\xi} \cdot \nabla \underline{B}\,\underline{B}\right\} + \underline{B} \cdot \nabla(\underline{\xi} \cdot \nabla \underline{B}) = \nabla \cdot (\underline{\xi} \cdot \nabla \underline{B})\underline{B} + \underline{\xi} \cdot \nabla \underline{B} \cdot \nabla \underline{B} + \underline{B} \cdot \nabla(\underline{\xi} \cdot \nabla \underline{B}) \;.$$

Furthermore, we have:

$$\nabla \underline{\xi} : \nabla \underset{\sim}{T}_o = \nabla \underline{\xi} \,.\, \nabla p + \nabla \underline{\xi} \cdot \nabla\left(\frac{B^2}{2}\right) - \nabla \underline{\xi} : \nabla(\underline{B}\underline{B}) = \nabla(\underline{\xi} \cdot \nabla p) - \underline{\xi} \cdot \nabla\nabla p + \nabla \underline{\xi} \cdot \nabla\left(\frac{B^2}{2}\right) - \nabla \cdot \left\{\underline{\xi} \cdot \nabla(\underline{B}\,\underline{B})\right\}$$
$$+ \underline{\xi} \cdot \nabla\left\{\nabla \cdot (B\,B)\right\} \qquad \text{... (C.18)}$$

By adding equation (C.17) and (C.18) we get:

$$- \nabla \cdot \delta T + \nabla \underline{\xi} : \nabla \underset{\sim}{T}_o = \nabla(\gamma p \nabla \cdot \underline{\xi} + \underline{\xi} \cdot \nabla p - \underline{B} \cdot \underline{Q}) + \underline{Q} \cdot \nabla B + \underline{B} \cdot \nabla Q - \underline{\xi} \cdot \nabla\left\{\nabla\left(p + \frac{B^2}{2}\right)\right.$$
$$\left. - \nabla \cdot (\underline{B}\,\underline{B})\right\} = \nabla(\gamma p \nabla \cdot \underline{\xi} + \underline{\xi} \cdot \nabla p - \underline{B} \cdot \underline{Q}) + \underline{Q} \cdot \nabla \underline{B} + \underline{B} \cdot \nabla \underline{Q} + \underline{\xi} \cdot \nabla\left(\rho \frac{dv}{dt}\right) \;. \qquad \text{... (C.19)}$$

Thus combining the results of equations (C.12) and (C.19) we obtain :

$$\rho \frac{\partial^2 \xi}{\partial t^2} + 2 \rho \underline{v} \cdot \nabla \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial t}(\rho \underline{v}) \cdot \nabla \underline{\xi} = \underline{F}(\underline{\xi}) \qquad \text{... (C.20)}$$

where:

$$\underline{F}(\underline{\xi}) = \nabla(\gamma p \nabla \cdot \underline{\xi} + \underline{\xi} \cdot \nabla p - \underline{B} \cdot \underline{Q}) + \underline{Q} \cdot \nabla \underline{B} + \underline{B} \cdot \nabla \underline{Q} + \nabla \cdot \left\{\underline{\xi}\rho \frac{dv}{dt} - \rho \underline{v}\,\underline{v} \cdot \nabla \underline{\xi}\right\} . \qquad \text{... (C.21)}$$

Notice also that

$$- \nabla(\underline{B} \cdot \underline{Q}) + \underline{Q} \cdot \nabla \underline{B} + \underline{B} \cdot \nabla \underline{Q} = - \underline{Q} \times \nabla \times \underline{B} - \underline{B} \times \nabla \times \underline{Q} \;,$$

and that equation (C.21) can be re-written by using this vector identity.

Remark: A conservative external force field can easily be included in the equation of motion. This means that we must add a term $-\rho \nabla \psi$ to the right hand side of equation (C.1). By evaluating this term at

$$\underline{r}' = \underline{r}^o + \underline{\xi}$$

we obtain:

$$- \rho(\underline{r}^o + \underline{\xi})\, \nabla' \psi(\underline{r}^o + \underline{\xi}) = - \rho \nabla \psi - \rho \nabla(\underline{\xi} \cdot \nabla \psi) - \delta \rho \nabla \psi + \rho \nabla \underline{\xi} \cdot \nabla \psi$$

$$= - \rho \nabla \psi - \Delta \rho \nabla \psi - \underline{\xi} \cdot \nabla(\rho \nabla \psi) = - \rho \nabla \psi + \nabla \cdot (\rho \underline{\xi}) \nabla \psi - \underline{\xi} \cdot \nabla(\rho \nabla \psi) . \qquad \text{... (C.22)}$$

Thus the effect of a conservative external force field is to add a term $\nabla \cdot (\rho \underline{\xi}) \nabla \psi$ to the expression for $F(\underline{\xi})$ equation (C.21).

APPENDIX D

PROOF OF THE SELF-ADJOINTNESS OF THE OPERATOR $\underline{F}$

We shall here give a direct proof for the self-adjointness of the operator $\underline{F}$ appearing in equation (2.26). For convenience we put:

$$\underline{F}(\underline{\xi}) = \underline{F}_1(\underline{\xi}) + \underline{F}_2(\underline{\xi}), \qquad \text{... (D.1)}$$

where:

$$\underline{F}_1(\underline{\xi}) = (\nabla\times\underline{Q})\times\underline{B} + \nabla(\gamma p \nabla\cdot\underline{\xi}) - \nabla\cdot(\rho\underline{v}\,\underline{v}\cdot\nabla\underline{\xi}) - \rho\underline{\xi}\cdot\nabla\nabla\psi, \qquad \text{... (D.2)}$$

and:

$$\underline{F}_2(\underline{\xi}) = \nabla(\underline{\xi}\cdot\nabla p) - \underline{Q}\times(\nabla\times\underline{B}) + \nabla\cdot\left(\rho\,\underline{\xi}\frac{d\underline{v}}{dt}\right) + \nabla\cdot(\rho\underline{\xi}\,\nabla\psi), \qquad \text{... (D.3)}$$

where we have used the fact that

$$\nabla\cdot(\rho\underline{\xi})\nabla\psi = \nabla\cdot(\rho\underline{\xi}\nabla\psi) - \rho\underline{\xi}\cdot\nabla\nabla\psi .$$

It is easily seen that we have :

$$\begin{aligned}\underline{\eta}\cdot\underline{F}_1(\underline{\xi}) &= \nabla\cdot\left[\left\{\underline{\eta}\times\underline{B}\right\}\times\underline{Q} + \underline{\eta}\gamma p\,\nabla\cdot\underline{\xi} - \rho\underline{v}\underline{\eta}\underline{v}:\nabla\underline{\xi}\right] \\ &- \left\{\nabla\times(\underline{\eta}\times\underline{B})\right\}\cdot\left\{\nabla\times(\underline{\xi}\times\underline{B})\right\} - \gamma p\,\nabla\cdot\underline{\eta}\;\;\nabla\cdot\underline{\xi} \\ &+ \rho(\underline{v}\cdot\nabla\underline{\xi})\cdot(\underline{v}\cdot\nabla\underline{\eta}) - \rho\underline{\eta}\underline{\xi}:\nabla\nabla\psi .\end{aligned} \qquad \text{... (D.4)}$$

For $\underline{F}_2$ we have :

$$\begin{aligned}\underline{\eta}\cdot\underline{F}_2(\underline{\xi}) - \underline{\xi}\cdot\underline{F}_2(\underline{\eta}) &= \underline{\eta}\cdot\underline{J}\times\left\{\nabla\times(\underline{\xi}\times\underline{B})\right\} + \underline{\eta}\cdot\nabla(\underline{\xi}\cdot\nabla p) + \underline{\eta}\nabla:\left(\underline{\xi}\rho\frac{d\underline{v}}{dt}\right) + \underline{\eta}\nabla:(\underline{\xi}\rho\nabla\psi) \\ &- \underline{\xi}\cdot\underline{J}\times\left\{\nabla\times(\underline{\eta}\times\underline{B})\right\} - \underline{\xi}\cdot\nabla(\underline{\eta}\cdot\nabla p) - \underline{\xi}\nabla:\left(\underline{\eta}\rho\frac{d\underline{v}}{dt}\right) - \underline{\xi}\nabla:(\underline{\eta}\rho\nabla\psi),\end{aligned} \qquad \text{... (D.5)}$$

where as usual :

$$\underline{J} = \nabla\times\underline{B} .$$

By expanding the curl we have :

$$\underline{\eta}\cdot\underline{J}\times\left\{\nabla\times(\underline{\xi}\times\underline{B})\right\} = \underline{\eta}\cdot\underline{J}\times\left\{\underline{B}\cdot\nabla\underline{\xi} - \underline{\xi}\cdot\nabla\underline{B} - \underline{B}\nabla\cdot\xi\right\}, \qquad \text{... (D.6)}$$

$$\underline{\xi}\cdot\underline{J}\times\left\{\nabla\times(\underline{\eta}\times\underline{B})\right\} = \underline{\xi}\cdot\underline{J}\times\left\{\underline{B}\cdot\nabla\underline{\eta} - \underline{\eta}\cdot\nabla\underline{B} - \underline{B}\,\nabla\cdot\underline{\eta}\right\} . \qquad \text{... (D.7)}$$

By making use of the equation of motion for the unperturbed state we obtain :

$$-\underline{\eta}\cdot\underline{J}\times\underline{B}\;\nabla\cdot\underline{\xi} - \underline{\xi}\cdot\nabla(\underline{\eta}\cdot\nabla p) = -\;\nabla\cdot(\underline{\xi}\underline{\eta}\cdot\nabla p) - \nabla\cdot\underline{\xi}\underline{\eta}\cdot\left(\rho\frac{d\underline{v}}{dt} + \rho\nabla\psi\right) \qquad \text{... (D.8)}$$

and

$$\underline{\xi}\cdot\underline{J}\times\underline{B}\nabla\cdot\underline{\eta} + \underline{\eta}\cdot\nabla(\underline{\xi}\cdot\nabla p) = \nabla\cdot(\underline{\eta}\underline{\xi}\cdot\nabla p) + \nabla\cdot\underline{\eta}\,\underline{\xi}\cdot\left(\rho\frac{d\underline{v}}{dt} + \rho\nabla\psi\right). \qquad \text{... (D.9)}$$

Further we obtain :

$$\underline{\eta}\cdot\underline{J}\times(\underline{B}\cdot\nabla)\underline{\xi} = \nabla\cdot\{\underline{B}\underline{\eta}\cdot\underline{J}\times\underline{\xi}\} - \underline{\xi}\cdot\underline{\eta}\times(\underline{B}\cdot\nabla)\underline{J} - \underline{J}\times\underline{\xi}\cdot(\underline{B}\cdot\nabla)\underline{\eta} \qquad \text{... (D.10)}$$

By taking the curl of the equation of motion, we obtain :

$$\underline{B}\cdot\nabla\underline{J} = \underline{J}\cdot\nabla\underline{B} + \nabla\times\left\{\rho\frac{d\underline{v}}{dt} + \rho\nabla\psi\right\}. \qquad \text{... (D.11)}$$

From equations (D.10) and (D.11) we obtain:

$$\underline{\eta}\cdot\underline{J}\times(B\cdot\nabla)\underline{\xi}-\underline{\xi}\cdot\underline{J}\times(\underline{B}\cdot\nabla)\underline{\eta} = \nabla\cdot\{\underline{B}\underline{\eta}\cdot\underline{J}\times\underline{\xi}\}-\underline{\xi}\cdot\underline{\eta}\times(\underline{J}\cdot\nabla)\underline{B}-\underline{\xi}\cdot\underline{\eta}\times\left[\nabla\times\left\{\rho\frac{d\underline{v}}{dt}+\rho\nabla\psi\right\}\right] .$$

... (D.12)

By making use of the results given in equations (D.6) – (D.9) and equation (D.12) we obtain :

$$\underline{\eta}\cdot\underline{F}_2(\underline{\xi})-\underline{\xi}\cdot\underline{F}_2(\underline{\eta})=\nabla\cdot\left\{(\underline{\eta}\underline{\xi}-\underline{\xi}\underline{\eta})\cdot\nabla p+\underline{B}\underline{\eta}\cdot\underline{J}\times\underline{\xi}\right\}+\underline{\xi}\cdot\underline{J}\times(\underline{\eta}\cdot\nabla)\underline{B}-\underline{\eta}\cdot\underline{J}\times(\underline{\xi}\cdot\nabla)\underline{B}$$

$$-\underline{\xi}\cdot\underline{\eta}\times(\underline{J}\cdot\nabla)\underline{B}+\nabla\cdot\underline{\eta}\underline{\xi}\cdot\underline{W}-\nabla\cdot\underline{\xi}\underline{\eta}\cdot\underline{W}+\underline{\eta}\nabla:(\underline{\xi}\underline{W})-\underline{\xi}\nabla:(\underline{\eta}\underline{W})-\underline{\xi}\cdot\underline{\eta}\times(\nabla\times\underline{W}) ,$$

... (D.13)

where we have put:

$$\rho\frac{d\underline{v}}{dt}+\rho\nabla\psi=\underline{W} .$$

Now :

$$\underline{\eta}\nabla:(\underline{\xi}\,\underline{W})=\underline{\eta}\cdot\underline{W}\;\nabla\cdot\underline{\xi}+\underline{\eta}\,\underline{\xi}:\nabla\underline{W} \qquad \text{... (D.14)}$$

$$\underline{\xi}\nabla:(\underline{\eta}\underline{W})=\underline{\xi}\cdot\underline{W}\;\nabla\cdot\underline{\eta}+\underline{\xi}\,\underline{\eta}:\nabla\underline{W} . \qquad \text{... (D.15)}$$

Furthermore for any vector $\underline{a}$ we have :

$$\underline{\eta}\underline{\xi}:\nabla\underline{a}-\underline{\xi}\underline{\eta}:\nabla\underline{a}-\underline{\xi}\cdot\underline{\eta}\times(\nabla\times\underline{a}) \equiv 0 , \qquad \text{... (D.16)}$$

a vector identity which is easily proved by standard methods. Thus by making use of equations (D.14) – (D.16) we see that all terms in equation (D.13) containing $\underline{W}$ cancel. By rearranging the remaining terms we obtain :

$$\underline{\eta}\cdot\underline{F}_2(\underline{\xi})-\underline{\xi}\cdot\underline{F}_2(\underline{\eta}) = \nabla\cdot\left\{(\underline{\eta}\underline{\xi}-\underline{\xi}\underline{\eta})\cdot\nabla p+\underline{B}\underline{\eta}\cdot\underline{J}\times\underline{\xi}\right\}$$

$$\underline{\eta}\cdot\left\{\nabla\underline{B}\cdot(\underline{\xi}\times\underline{J})-\underline{J}\times(\underline{\xi}\cdot\nabla)\underline{B}+\underline{\xi}\times(\underline{J}\cdot\nabla)\underline{B}\right\} , \qquad \text{... (D.17)}$$

The expression :

$$\underline{G}=\left\{(\underline{J}\times\underline{\xi})\times\nabla\right\}\times\underline{B} ,$$

can be expanded in two different ways. We first obtain :

$$\underline{G}=\left\{\underline{\xi}(\underline{J}\cdot\nabla)-\underline{J}(\underline{\xi}\cdot\nabla)\right\}\times\underline{B}=\underline{\xi}\cdot(\underline{J}\cdot\nabla)\underline{B}-\underline{J}\times(\underline{\xi}\cdot\nabla)\underline{B} ,$$

and secondly :

$$\underline{G}=\nabla\underline{B}\cdot\underline{J}\times\underline{\xi} ,$$

where the last identity is obtained readily by expanding the expression for $\underline{G}$ with $\underline{J}\times\underline{\xi}$ as one quantity, and using $\nabla\cdot\underline{B}=0$. By identifying the two expressions for $\underline{G}$ we obtain :

$$\nabla\underline{B}\cdot\underline{J}\times\underline{\xi}-\underline{\xi}\times(\underline{J}\cdot\nabla)\underline{B}+\underline{J}\times(\underline{\xi}\cdot\nabla)\underline{B} \equiv 0 , \qquad \text{... (D.18)}$$

(see also R. Kulsrud, reference 17)

Now by combining equations (C.17) and (C.18), we obtain :

$$\underline{\eta}\cdot\underline{F}_2(\underline{\xi})-\underline{\xi}\cdot\underline{F}_2(\underline{\eta})=\nabla\cdot\left\{(\underline{\eta}\underline{\xi}-\underline{\xi}\underline{\eta})\cdot\nabla p+\underline{B}\underline{\eta}\cdot\underline{J}\times\underline{\xi}\right\} . \qquad \text{... (D.19)}$$

Furthermore, from equations (D.19) and (D.4) we obtain :

$$\underline{\eta}\cdot\underline{F}(\underline{\xi})-\underline{\xi}\cdot\underline{F}(\underline{\eta})=\nabla\cdot\Big[(\underline{\eta}\times\underline{B})\times\underline{Q}(\underline{\xi})+\underline{\eta}(\gamma p\nabla\cdot\underline{\xi}+\underline{\xi}\cdot\nabla p)-(\underline{\xi}\times\underline{B})\times\underline{Q}(\underline{\eta})$$
$$-\underline{\xi}(\gamma p\nabla\cdot\underline{\eta}+\underline{\eta}\cdot\nabla p)+\underline{B}\,\underline{\eta}\cdot\underline{J}\times\underline{\xi}+\rho\,\underline{v}(\underline{\xi}\underline{v}:\nabla\underline{\eta}-\underline{\eta}\underline{v}:\nabla\underline{\xi})\Big] \quad \ldots \text{(D.20)}$$

where :

$$\underline{Q}(\underline{\xi})=\nabla\times(\underline{\xi}\times\underline{B}) \quad \text{and} \quad \underline{Q}(\underline{\eta})=\nabla\times(\underline{\eta}\times\underline{B}) \ .$$

By integrating equation (D.20) over the plasma region V_p and using Gauss' theorem we obtain :

$$\int_{V_p}\underline{\eta}\cdot\underline{F}(\underline{\xi})dV_p-\int_{V_p}\underline{\xi}\cdot\underline{F}(\underline{\eta})\,dV_p=\int_S\Big\{\underline{n}\cdot\underline{\xi}\Big(\underline{B}\cdot\underline{Q}(\underline{\eta})-\gamma p\nabla\cdot\underline{\eta}-\underline{\eta}\cdot\nabla p\Big)$$
$$-\underline{n}\cdot\underline{\eta}\Big(\underline{B}\cdot\underline{Q}(\underline{\xi})-\gamma p\nabla\cdot\underline{\xi}-\underline{\xi}\cdot\nabla p\Big)+\rho\underline{n}\cdot\underline{v}(\underline{\xi}\underline{v}:\nabla\underline{\eta}-\underline{\eta}\underline{v}:\nabla\underline{\xi})\,dS, \quad \ldots \text{(D.21)}$$

where S is the surface bounding the plasma. The double vector products containing $\underline{Q}$ have been expanded and the boundary condition $\underline{n}\cdot\underline{B}=0$ has been used. Notice that $\underline{n}$ is the surface unit normal pointing outwards from the plasma region.

In the following, two different situations are considered :

(1) First we assume the plasma to be contained within a rigid wall having infinite conductivity. If there are buried conductors in the plasma region these are also assumed to have infinite conductivity and to be rigid. The proper boundary conditions to be used at the wall and the conductor surfaces are $\underline{n}\cdot\underline{B}=0$ (already used), $\underline{n}\cdot\underline{\xi}=\underline{n}\cdot\underline{\eta}=0$ and $\underline{n}\cdot\underline{v}=0$. It is then readily seen from equation (D.21) that :

$$\int_{V_p}\underline{\eta}\cdot\underline{F}(\underline{\xi})\,dV_p=\int_{V_p}\underline{\xi}\cdot\underline{F}(\underline{\eta})\,dV_p \ . \quad \ldots \text{(D.22)}$$

Thus we have proved that $\underline{F}$ is self adjoint in this case.

(2) In the second case we consider the situation where the plasma region is separated from the vacuum region by a surface of discontinuity carrying a sheet current. Although mathematically the thickness of the sheet is zero this is of course not the case in the actual physical system, where the sheet corresponds to a transition region of finite thickness. For physical reasons all quantities appearing in the equations must be continuous across such a transitio region. Furthermore we have by definition that the mass density ρ is zero on the vacuum side of the transition region. We now extend the volume integration in equation (D.21) to include the transition region. Thus we have $\rho = 0$ over the surface S which is bounding the volume V_p. Therefore in the integration over S the terms having ρ as a common factor vanish. For the other terms we have already obtained the value on the vacuum side of the region in terms of vacuum quantities given

by eq.(2.22). By making use of this equation we obtain :

$$\int_{V_p} \underline{\eta}\cdot\underline{F}(\underline{\xi})dV_p - \int_{V_p} \underline{\xi}\cdot\underline{F}(\underline{\eta})dV_p = \int_{S_p} \left\{ \underline{n}\cdot\underline{\xi}[\underline{B}_V\cdot\nabla\times\underline{A}(\underline{\eta})] - \underline{n}\cdot\underline{\eta}[\underline{B}_V\cdot\nabla\times\underline{A}(\underline{\xi})] \right\} dS \ . \qquad \ldots (D.23)$$

By making use of the boundary condition equation (2.23), which relates $\underline{A}(\underline{\xi})$ to $\underline{\xi}$ at the boundary, we obtain :

$$\int_{V_p} \underline{\eta}\underline{F}(\underline{\xi})dV_p - \int_{V_p} \underline{\xi}\cdot\underline{F}(\underline{\eta})dV_p = \int_{S_p} \underline{n}\cdot \left\{ \underline{A}(\underline{\eta})\times[\nabla\times\underline{A}(\underline{\xi})] - \underline{A}(\underline{\xi})\times[\nabla\times\underline{A}(\underline{\eta})] \right\} dS \ . \qquad \ldots (D.24)$$

All the quantities appearing in the last surface integral are continuously differentiable in the vacuum region. Furthermore, we assume the vacuum region to be bounded by a rigid external boundary S_E of either infinite conductivity or infinite resistivity. In between the plasma surface and this external boundary we can have a rigid conductor system of infinite conductivity. To be precise we shall define the vacuum region V_V to be the region inbetween S_p and S_E minus the volume occupied by the conductor system. We shall call the surfaces of the conductors S_C. Thus by application of Gauss' theorem we obtain the following result valid for any differentiable vector field $\underline{G}$:

$$\int_{S_p} \hat{\underline{n}}\cdot\underline{G}\, dS_p = \int_{V_V} \nabla\cdot\underline{G}\, dV_V - \int_{S_E} \hat{\underline{n}}\cdot\underline{G}\, dS_E - \int_{S_C} \hat{\underline{n}}\cdot\underline{G}\, dS_C \ . \qquad \ldots (D.25)$$

Here $\hat{\underline{n}}$ is the surface unit normal pointing outward from the vacuum region. We can now calculate the left hand side of equation (D.24). Let $\underline{G}$ be given by

$$\underline{G} = \underline{A}(\underline{\eta}) \times \left[\nabla\times\underline{A}(\underline{\xi})\right] - \underline{A}(\underline{\xi}) \times \left[\nabla\times\underline{A}(\underline{\eta})\right] . \qquad \ldots (D.26)$$

It is then readily obtained that :

$$\nabla\cdot\underline{G} = 0 \ . \qquad \ldots (D.27)$$

Notice that $\nabla\times(\nabla\times\underline{A}) = 0$, since no currents can flow in vacuum. Furthermore $\hat{\underline{n}}\times\underline{A} = 0$ is the proper boundary condition at the surface S_C and also over S_E in the case where S_E is of infinite conductivity. Hence, it is readily obtained that :

$$\int_{S_p} \hat{\underline{n}}\cdot\underline{G}\, dS_p = \int_{S_E} \hat{\underline{n}}\cdot\underline{G}\, dS_E = 0 \qquad \ldots (D.28)$$

Notice that $\hat{n} = -\underline{n}$ in equation (D.24). From equations (D.25) - (D.28) it follows that the right hand side of equation (D.24) is zero. Thus F is proved to be self adjoint in this case too.

Furthermore, in the case of a boundary of infinite resistivity surrounding the vacuum vessel the volume integration can be extended outside the vacuum region. And the surface of integration can be taken to be sufficiently far away from the plasma region to ensure that $\nabla \times \underline{A}$ is effectively zero over the surface of integration. The essential assumption is therefore that the perturbed vacuum magnetic field is not in contact with material of finite conductivity anywhere. Physically the presence of such material would dissipate energy because the magnetic field would penetrate into it and generate currents which would be dissipated in the resistive medium. Mathematically this is expressed in a non-zero contribution to the left hand side of equation (D.24) and consequently non-self adjointness of the operator $\underline{F}$.

APPENDIX E

THE LANGRANGIAN AND HAMILTONIAN FORMULATION

We start by listing the general properties of a system described by the field variables $\eta_i(\underline{x}, t)$, $i = 1, 2, 3$, where $\underline{x} = \underline{\ell}_1 x_1 + \underline{\ell}_2 x_2 + \underline{\ell}_3 x_3$, is the position vector and $\underline{\ell}_1, \underline{\ell}_2, \underline{\ell}_3$ is assumed to be an orthomal set of base vectors. We assume that there exists a Langrangian density of the functional form :

$$\mathcal{L} = \mathcal{L}\left(\eta_i, \frac{\partial \eta_i}{\partial x_k}, \frac{\partial \eta_i}{\partial t}, x_k, t\right). \qquad \text{... (E.1)}$$

The Lagrangian L of the system is then given by:

$$L = \int_V \mathcal{L}\, dV, \qquad \text{... (E.2)}$$

where V is the volume over which the Lagrangian density is defined. The correct equations of motion determining the field-variables are the Euler-Lagrange equations for the variational problem :

$$\delta \int_{t_1}^{t_2} L\, dt = 0. \qquad \text{... (E.3)}$$

The quantities to be varied are the field-variables and the variation is subjected to the conditions :

$$\delta\eta_i(\underline{x}, t_1) = \delta\eta_i(\underline{x}, t_2) = 0. \qquad \text{... (E.4)}$$

Normally the additional condition

$$\delta\eta_i(\underline{x}_s, t) = 0, \qquad \text{... (E.5)}$$

is also imposed. Here $\underline{x} = \underline{x}_s$ is the equation for the surface S bounding the volume V, over which the integration is performed. However, if we have

$$\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta_i}{\partial x_k}\right)}\right]_{\underline{x} = \underline{x}_s} = 0, \qquad \text{... (E.6)}$$

Then we can also deal with the free boundary case and remove condition (E.5).

The Euler-Lagrange equations can now be written in a concise form as one vector equation, namely

$$\frac{\delta L}{\delta \underline{\eta}} = \frac{d}{dt}\frac{\delta L}{\delta \dot{\underline{\eta}}}. \qquad \text{... (E.7)}$$

Here we have

$$\frac{\delta}{\delta \underline{\eta}} = \underline{\ell}_i \frac{\delta}{\delta \eta_i}, \quad \frac{\delta}{\delta \dot{\underline{\eta}}} = \underline{\ell}_i \frac{\delta}{\delta \dot{\eta}_i}, \quad \dot{\eta}_i = \frac{\partial \eta_i}{\partial t} \text{ and } \underline{\eta} = \ell_i \eta_i .$$

Furthermore the concept of the functional derivative is used and is defined as follows:

$$\frac{\delta L}{\delta \eta_i} = \frac{\partial \mathcal{L}}{\partial \eta_i} - \frac{d}{dx_j} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \eta_i}{\partial x_j}\right)} . \qquad \text{... (E.8)}$$

In vector notation the functional derivative can be written as:

$$\frac{\delta L}{\delta \underline{\eta}} = \frac{\partial \mathcal{L}}{\partial \underline{\eta}} - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial (\nabla \underline{\eta})}\right) , \qquad \text{... (E.9)}$$

where the dyadic

$$\frac{\partial \mathcal{L}}{\partial (\nabla \underline{\eta})} = \underline{\ell}_k \, \underline{\ell}_j \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \eta_i}{\partial x_k}\right)} . \qquad \text{... (E.10)}$$

The Hamiltonian formalism is now obtained by defining the momemntum density $\underline{\pi}$, which is cononical conjugate to $\underline{\xi}$ as :

$$\underline{\pi} = \frac{\partial \mathcal{L}}{\partial \dot{\underline{\xi}}} . \qquad \text{... (E.11)}$$

The Hamiltonian density is then given by

$$\mathcal{H} = \underline{\pi} \cdot \dot{\underline{\xi}} - \mathcal{L} \qquad \text{... (E.12)}$$

Accordingly the Hamiltonian H is given by

$$H = \int_{V_p} \mathcal{H} \, dV_p . \qquad \text{... (E.13)}$$

The resulting Hamiltonian equations of motion in concise vector notations read :

$$\frac{\partial \underline{\xi}}{\partial t} = \frac{\delta H}{\delta \underline{\pi}} , \qquad \text{... (E.14)}$$

$$\frac{\partial \underline{\pi}}{\partial t} = - \frac{\delta H}{\delta \underline{\xi}} . \qquad \text{... (E.15)}$$

Let us consider the Lagrangian density :

$$\mathcal{L}_1 = \frac{1}{2} \rho \left(\frac{\partial \underline{\xi}}{\partial t}\right)^2 - \rho \underline{\xi} \, \underline{v} : \nabla \frac{\partial \underline{\xi}}{\partial t} + \frac{1}{2} \underline{\xi} \cdot \underline{F} (\underline{\xi}) - \frac{1}{2} \frac{\partial^2 \rho}{\partial t^2} \underline{\xi} \cdot \underline{\xi} . \qquad \text{... (E.16)}$$

It is readily shown that the Euler-Lagrange equations for the variational problem, equation (E.3), in this case gives the equation of motion, equation (2.26). Furthermore, it follows directly from equations (E.11) and (E.12) that

$$\underline{\pi}_1 = \rho \frac{\partial \underline{\xi}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{\xi} - \frac{\partial \rho}{\partial t} \underline{\xi} , \qquad \text{... (E.17)}$$

$$\mathcal{H}_1 = \frac{1}{2\rho} \left\{ \underline{\pi}_1 - \rho \underline{v} \cdot \nabla \underline{\xi} + \frac{\partial \rho}{\partial t} \underline{\xi} \right\}^2 - \frac{1}{2} \underline{\xi} \cdot \underline{F}(\underline{\xi}) + \frac{1}{2} \frac{\partial^2 \rho}{\partial t^2} \underline{\xi} \cdot \underline{\xi} . \qquad \text{... (E.18)}$$

From equation (E.18) we obtain Hamilton equations of motion :

$$\frac{\partial \underline{\xi}}{\partial t} = \frac{1}{\rho} \left(\underline{\pi}_1 - \rho \underline{v} \cdot \nabla \underline{\xi} + \frac{\partial \rho}{\partial t} \underline{\xi} \right) , \qquad \text{... (E.19)}$$

$$\frac{\partial \underline{\pi}_1}{\partial t} = - \rho \underline{v} \cdot \nabla \left[\frac{1}{\rho} \left(\underline{\pi}_1 - \rho \underline{v} \cdot \nabla \underline{\xi} - \frac{\partial \rho}{\partial t} \underline{\xi} \right) \right] + \underline{F}(\underline{\xi}) - \frac{\partial^2 \rho}{\partial t^2} \underline{\xi} . \qquad \text{... (E.20)}$$

Equation (E.19) is a re-statement of equation (E.17). It is easily seen that by eliminating $\underline{\pi}$ between equations (E.19) and (E.20) the original equation of motion equation (2.26) is obtained. The result here is closely related to the result given by Frieman and Rotenberg[4]. Their result is recovered by putting $\frac{\partial}{\partial t}$ on all unperturbed quantities equal to zero. However, the Lagrangian density is not uniquely defined, as one can always add terms which can be written as a divergence or the time derivative of another quantity (provided they integrate to zero when subjected to the conditions given in equations (E.4)–(E.6)). This enables us to spot a simple form of the Lagrangian density namely :

$$\mathcal{L}_2 = \frac{1}{2}\rho\left(\frac{\partial \underline{\xi}}{\partial t}\right)^2 + \rho\frac{\partial \underline{\xi}}{\partial t}\,\underline{v} : \nabla\,\underline{\xi} + \frac{1}{2}\,\underline{\xi}\cdot\underline{F}(\underline{\xi}) \;. \qquad \ldots \text{(E.21)}$$

From this Lagrangian density we obtain :

$$\underline{\pi}_2 = \rho\,\frac{\partial \underline{\xi}}{\partial t} + \rho\,\underline{v}\cdot\nabla\underline{\xi}\;, \qquad \ldots \text{(E.22)}$$

$$\mathcal{H}_2 = \frac{1}{2\rho}\left[\underline{\pi}_2 - \rho\,\underline{v}\cdot\nabla\underline{\xi}\right]^2 - \frac{1}{2}\,\underline{\xi}\cdot\underline{F}(\underline{\xi}) \;. \qquad \ldots \text{(E.23)}$$

The corresponding Hamilton's equations of motion are :

$$\frac{\partial \underline{\xi}}{\partial t} = \frac{1}{\rho}\left(\underline{\pi}_2 - \rho\,\underline{v}\cdot\nabla\,\underline{\xi}\right) \qquad \ldots \text{(E.24)}$$

$$\frac{\partial \underline{\pi}_2}{\partial t} = -\,\nabla\cdot\left[\rho\,\underline{v}\;\frac{\underline{\pi}_2 - \rho\,\underline{v}\cdot\nabla\,\underline{\xi}}{\rho}\right] + \underline{F}(\underline{\xi}) \qquad \ldots \text{(E.25)}$$

There exists of course a canonical transformation which transforms :

$$\underline{\xi}\,,\underline{\pi}_1\,,\mathcal{H}_1 \;\rightleftarrows\; \underline{\xi}\,,\underline{\pi}_2\,,\;\mathcal{H}_2 \;.$$

The following function is a generating function for this transformation ,

$$G = \underline{\xi}\cdot\underline{\pi}_2 - \frac{1}{2}\,\frac{\partial\rho}{\partial t}\,\underline{\xi}\cdot\underline{\xi}\;. \qquad \ldots \text{(E.26)}$$

In the last Hamiltonian representation we notice that both the Hamiltonian density and the Hamiltonian equations of motion have a very simple form. Apart from the fact that $\rho\underline{v}$ can be taken outside the divergence operation in equation (E.25) for the stationary case, there is no longer any significant formal difference between a system in an arbitrary state of motion and a stationary system.

Note that when the Euler-Lagrange equations are derived, one of the conditions listed in equation (E.5) and (E.6) has to be used in addition to equation (E.4). Equation (E.5) corresponds to the case where the plasma is surrounded by a rigid boundary. Equation (E.6) corresponds to the case where the plasma is surrounded by a vacuum magnetic field. In either case the same equation of motion is recovered. Note that $\underline{F}(\underline{\xi}) \to 0$ as one approaches the vacuum region.

Notice also that we have

$$\frac{d}{dt} H = \frac{\partial H}{\partial t} + \int_{\Sigma} \underline{n} \cdot \underline{S} \, d\sigma \qquad \text{... (E.27)}$$

where

$$\underline{S} = \dot{\underline{\xi}} \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \underline{\xi})} \qquad \text{... (E.28)}$$

and Σ is the surface bounding the volume under consideration. Therefore if the normal component of $\underline{S}$ is zero over the surface Σ then :

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} . \qquad \text{... (E.29)}$$

This is the case if $\mathcal{L}$ is given by equation (E.21) and the surface Σ is a surface bounding the plasma from a vacuum region since $\underline{F}(\underline{\xi})$ and ρ are both zero at such a surface.

APPENDIX F

THE MTS-EXPANSION - MATHEMATICAL DETAILS

(a) THE ORDERING PROCEDURE

We start from equation (4.2) and shall solve this equation by an expansion technique using the method of multiple time scales (MTS). It is, however, necessary to make some assumptions about the order of magnitude of the growth rate of any instability present in the static system. For this purpose we first assume the unperturbed quantities to be described by the following expansion scheme :

$$\rho = \rho_o + \varepsilon^2 \rho_2 + \varepsilon^4 \rho_4 + \dots \quad , \qquad \dots \text{(F.1)}$$

$$\underline{v}/\omega_s = 0 + \varepsilon^2 \underline{v}_2 + \varepsilon^4 \underline{v}_4 + \dots , \qquad \dots \text{(F.2)}$$

$$\underline{F}(\underline{\xi}) = \underline{F}_e(\underline{\xi}) + \varepsilon^2 \, \tilde{\underline{F}}_2(\underline{\xi}) + \varepsilon^4 \, \tilde{\underline{F}}_4(\underline{\xi}) + \dots . \qquad \dots \text{(F.3)}$$

It is also useful to divide the perturbations $\underline{\xi}$ into two subsets, the stable perturbations and the unstable perturbations, by the following definition.

A perturbation $\underline{\xi}$ belongs to the set of stable perturbations provided $\delta\bar{W}_o > 0$, otherwise it belongs to the unstable perturbations, where

$$\delta\bar{W}_o = - \lim_{\varepsilon \to o} \int_{V_p} \underline{\xi} \cdot \underline{F}(\underline{\xi}) \, dV_p \, , \qquad \dots \text{(F.4)}$$

and V_p is the volume of the plasma. Notice that the static equilibrium problem is described by the limit $\varepsilon \to 0$. The oscillatory terms (in time) represented by the expansion in ε^2 is assumed to be periodic with period $T_s = \frac{2\pi}{\omega_s}$. We shall refer to ω_s as the stabilising frequency.

Introducing the multiple timescales : $\tau_o, \tau_1, \tau_2 \dots$, we expand $\underline{\xi}$ and $\frac{\partial}{\partial \tau}$,

$$\underline{\xi} = \underline{\xi}_o + \varepsilon \, \underline{\xi}_1 + \varepsilon^2 \, \underline{\xi}_2 + \dots , \qquad \dots \text{(F.5)}$$

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau_o} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + \dots . \qquad \dots \text{(F.6)}$$

By assumption T_s is the fastest timescale in our problem, we therefore take τ_o to be the time variation on this timescale and $\tau_n = \varepsilon^n \tau_o$. Hence all unperturbed quantities are periodic functions of τ_o only, (in addition to the space dependence).

Now we have

$$\frac{\partial^2}{\partial \tau^2} = \frac{\partial^2}{\partial \tau_o^2} + \varepsilon \, 2 \frac{\partial^2}{\partial \tau_1 \partial \tau_o} + \varepsilon^2 \left(\frac{\partial^2}{\partial \tau_1^2} + 2 \frac{\partial^2}{\partial \tau_2 \partial \tau_o} \right) + \varepsilon^3 2 \left(\frac{\partial^2}{\partial \tau_2 \partial \tau_1} + \frac{\partial^2}{\partial \tau_3 \partial \tau_o} \right) + \dots . \qquad \dots \text{(F.7)}$$

In the subsequent analysis the discussion is limited to the effect of dynamic stabilisation on the unstable modes. We make the provisional assumption that the growth rates of these modes are small, at least of order ε^2 compared to ω_s. This means that we have :

$$\omega_s^{-2}\ \underline{F}_e(\underline{\xi}) = 0\ (\varepsilon^2 \rho_o \underline{\xi}_o)\ . \qquad \ldots (F.8)$$

By substituting for all quantities in equation (4.2) by equations (F.1) - (F.6) we obtain an infinite number of equations by equating the coefficients of different powers of ε. Starting from the zeroth order and seeking the solutions with no secular terms, we successively obtain :

$$\varepsilon^0 : \quad \frac{\partial^2 \underline{\xi}_o}{\partial \tau_o^2} = 0 \Longrightarrow \underline{\xi}_o = \underline{\xi}_o(\tau_1\ , \tau_2\ , \ \ldots\ )\ , \qquad \ldots (F.9)$$

$$\varepsilon^1 : \quad \frac{\partial^2 \underline{\xi}_1}{\partial \tau_o^2} = 0 \Longrightarrow \underline{\xi}_1 = \underline{\xi}_1(\tau_1\ , \tau_2\ , \ \ldots\ )\ , \qquad \ldots (F.10)$$

$$\varepsilon^2 : \quad \frac{\partial^2 \xi_o}{\partial \tau_1^2} = \frac{1}{\varepsilon^2 \omega_s^2}\ \underline{F}_e(\underline{\xi}_o)\ , \qquad \ldots (F.11)$$

and

$$\rho_o \frac{\partial^2 \underline{\xi}_2}{\partial \tau_o^2} = - \frac{\partial}{\partial \tau_o}(\rho_o \underline{v}_2) \cdot \nabla\ \underline{\xi}_o + \frac{1}{\omega_s^2}\ \underline{\tilde{F}}_2(\underline{\xi}_o) + \frac{1}{\omega_s^2}\ \underline{F}_e\,(\underline{\xi}_2)\ . \qquad \ldots (F.12)$$

In obtaining equations (F.11) and (F.12) we have assumed $\underline{\xi}_1$ to belong to the unstable modes which is reasonable since equation (F.10) shows that $\underline{\xi}_1$ is independent of τ_o. However, notice that $\underline{F}_e(\underline{\xi}_2)$ is not necessarily small even if $\underline{F}_e(\underline{\xi}_o)$ is. This is because $\underline{\xi}_2$ does not in general belong to the unstable modes and therefore $\underline{F}_e(\underline{\xi}_2)$ has to be treated as a zeroth order quantity. The important result so far is, however, obtained in equation (F.11), which shows that if the growth rate is large, of order $\varepsilon^2\omega_s$, then the oscillations imposed on the system do not have any significant effect on the instabilities. Equation (F.11) can be brought into a form which yields the usual energy principle[5], and any instability present with a growth rate of this order will develop independently of the imposed oscillations.

(b) <u>DETAILS OF THE GENERAL CASE</u>

Referring to the preceding discussion we now assume :

$$\omega_s^{-2}\ \underline{F}_e(\underline{\xi}_o) = 0(\varepsilon^4 \rho_o \underline{\xi}_o)\ , \qquad \ldots (F.13)$$

in the subsequent analysis. Notice that this is analogous to the ordering necessary for dynamic stabilisation of the inverted pendulum, see Section (c) of this appendix. We could continue the process we have already started and derive the desired results, but as they are more easily obtained by changing the expansion parameter from ε^2 to ε we shall do so. Thus instead of equations (F.1) - (F.3) and equations (F.5) and (F.6) we write :

$$\rho = \rho_o + \varepsilon\, \rho_1 + \varepsilon^2 \rho_2 + \dots , \qquad \dots \text{(F.14)}$$

$$\underline{v}/\omega_s = 0 + \varepsilon\, \underline{v}_1 + \varepsilon^2 \underline{v}_2 + \dots , \qquad \dots \text{(F.15)}$$

$$\underline{F}(\underline{\xi}) = \underline{F}_e(\underline{\xi}) + \varepsilon\, \underline{\tilde{F}}_1(\underline{\xi}) + \varepsilon^2\, \underline{\tilde{F}}_2(\underline{\xi}) + \dots , \qquad \dots \text{(F.16)}$$

$$\underline{\xi} = \underline{\xi}_o + \varepsilon\, \underline{\xi}_1 + \varepsilon^2 \underline{\xi}_2 + \dots , \qquad \dots \text{(F.17)}$$

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\tau_o} + \varepsilon \frac{\partial}{\partial\tau_1} + \varepsilon^2 \frac{\partial}{\partial\tau_2} + \dots . \qquad \dots \text{(F.18)}$$

We then get :

$$\rho \frac{\partial^2 \underline{\xi}}{\partial\tau^2} = \rho_o \frac{\partial^2 \underline{\xi}_o}{\partial\tau_o^2} + \varepsilon \left\{ \rho_1 \frac{\partial^2 \underline{\xi}_o}{\partial\tau_o^2} + \rho_o \frac{\partial^2 \underline{\xi}_1}{\partial\tau_o^2} + 2\rho_o \frac{\partial^2 \underline{\xi}_o}{\partial\tau_1 \partial\tau_o} \right\}$$

$$+ \varepsilon^2 \left\{ \rho_2 \frac{\partial^2 \underline{\xi}}{\partial\tau_o^2} + \rho_1 \left(\frac{\partial^2 \underline{\xi}_1}{\partial\tau_o^2} + 2 \frac{\partial^2 \underline{\xi}_o}{\partial\tau_1 \partial\tau_o} \right) + \rho_o \left(\frac{\partial^2 \underline{\xi}_o}{\partial\tau_1^2} + 2 \frac{\partial^2 \underline{\xi}_o}{\partial\tau_2 \partial\tau_o} + 2 \frac{\partial^2 \underline{\xi}_1}{\partial\tau_1 \partial\tau_o} \right) \right\}$$

$$+ \varepsilon^3 \Big(\dots \Big) + \dots , \qquad \dots \text{(F.19)}$$

$$2\, \rho \frac{\underline{v}}{\omega_s} \cdot \nabla \frac{\partial \underline{\xi}}{\partial\tau} = \varepsilon\, 2\, \rho_o \underline{v}_1 \cdot \nabla \frac{\partial \underline{\xi}_o}{\partial\tau_o}$$

$$+ \varepsilon^2 \left\{ 2\, \rho_1 \underline{v}_1 \cdot \nabla \frac{\partial \underline{\xi}_o}{\partial\tau_o} + 2\, \rho_o \underline{v}_1 \cdot \nabla \frac{\partial \underline{\xi}_1}{\partial\tau_o} + 2\, \rho_o \underline{v}_1 \cdot \nabla \frac{\partial \underline{\xi}_o}{\partial\tau_1} \right\} + \dots , \qquad \dots \text{(F.20)}$$

$$\frac{\partial}{\partial\tau} \left(\rho \frac{\underline{v}}{\omega_s} \right) \cdot \nabla \underline{\xi} = \varepsilon \frac{\partial}{\partial\tau_o} (\rho_o \underline{v}_1) \cdot \nabla \underline{\xi}_o$$

$$+ \varepsilon^2 \left\{ \frac{\partial}{\partial\tau_o} (\rho_1 \underline{v}_1) \cdot \nabla \underline{\xi}_o + \frac{\partial}{\partial\tau_o} (\rho_o \underline{v}_1) \cdot \nabla \underline{\xi}_1 \right\} + \dots , \qquad \dots \text{(F.21)}$$

$$\underline{F}(\underline{\xi}) = \underline{F}_e(\underline{\xi}_o) + \varepsilon\, \underline{F}_e(\underline{\xi}_1) + \varepsilon^2\, \underline{F}_e(\underline{\xi}_2) + \dots$$

$$+ \varepsilon\, \underline{\tilde{F}}_1(\underline{\xi}_o) + \varepsilon^2\, \underline{\tilde{F}}_1(\underline{\xi}_1) + \dots$$

$$+ \varepsilon^2\, \underline{\tilde{F}}_2(\underline{\xi}_o) + \dots$$

$$\vdots \qquad \dots \text{(F.22)}$$

In the following procedure we make use of the extra freedom, introduced into the system by the MTS scheme to eliminate secular terms in order to preserve the asymptotic character of the solution for large τ. By successively equating coefficients of different powers of ε we obtain ,

$$\rho_o \frac{\partial^2 \underline{\xi}_o}{\partial\tau_o^2} = 0 \implies \underline{\xi}_o = \underline{\xi}_o(\tau_1, \tau_2, \dots) , \qquad \dots \text{(F.23)}$$

$$\rho_o \frac{\partial^2 \underline{\xi}_1}{\partial\tau_o^2} = \underline{F}_1(\underline{\xi}_o) + \frac{1}{\omega_s^2} \underline{F}_e(\underline{\xi}_1) , \qquad \dots \text{(F.24)}$$

where

$$\underline{F}_1'(\underline{\xi}_o) = -\frac{\partial}{\partial \tau_o}(\rho_o \underline{v}_1) \cdot \nabla \underline{\xi}_o + \frac{1}{\omega_s^2} \underline{\tilde{F}}_1(\underline{\xi}_o) \ , \qquad \ldots \text{(F.25)}$$

$$\rho_1 \frac{\partial^2 \underline{\xi}_1}{\partial \tau_o^2} + \rho_o \frac{\partial^2 \underline{\xi}_o}{\partial \tau_1^2} + 2 \rho_o \frac{\partial^2 \underline{\xi}_1}{\partial \tau_1 \partial \tau_o} + \frac{\partial}{\partial \tau_o}(\rho_1 \underline{v}_1) \cdot \nabla \underline{\xi}_o$$

$$+ \frac{\partial}{\partial \tau_o}(\rho_o \underline{v}_1) \cdot \nabla \underline{\xi}_1 + 2 \rho_o \underline{v}_1 \ \nabla \frac{\partial \underline{\xi}_1}{\partial \tau_o} + 2 \rho_o \underline{v}_1 \cdot \nabla \frac{\partial \underline{\xi}_o}{\partial \tau_1}$$

$$= \frac{1}{\varepsilon^2 \omega_s^2}\left\{\underline{F}_e(\underline{\xi}_o) + \varepsilon^2 \underline{F}_e(\underline{\xi}_2)\right\} + \frac{1}{\omega_s^2}\left\{\underline{\tilde{F}}_2(\underline{\xi}_o) + \underline{\tilde{F}}_1(\underline{\xi}_1)\right\}. \qquad \ldots \text{(F.26)}$$

These are the equations to the first significant order which we are now going to discuss. It is convenient to allow for complex displacements $\underline{\xi}_o$, and as usual $\underline{\xi}_o^*$ is the complex conjugate of $\underline{\xi}_o$. We then proceed by multiplying equation (F.26) by $\underline{\xi}_o^*$ and integrating over the instantaneous volume V_p. In this process the following terms arise which need separate discussion. First we have

$$\int_{V_p} \underline{\xi}_o^* \cdot \rho_1 \frac{\partial^2 \underline{\xi}_1}{\partial \tau_o^2} d\underline{r} = \int_{V_p} \frac{\partial}{\partial \tau_o}\left(\underline{\xi}_o^* \rho_1 \frac{\partial \underline{\xi}_1}{\partial \tau_o}\right) + \int_{V_p} \nabla \cdot \left(\rho_o \underline{v}_1 \ \underline{\xi}_o^* \cdot \frac{\partial \underline{\xi}_1}{\partial \tau_o}\right) d\underline{r}$$

$$- \int_{V_p} \frac{\partial}{\partial \tau_o}\left(\underline{\xi}_1 \rho_o \underline{v}_1 : \nabla \underline{\xi}_o^*\right) d\underline{r} + \int_{V_p} \underline{\xi}_1 \frac{\partial}{\partial \tau_o}(\rho_o \underline{v}_1) : \nabla \underline{\xi}_o^* d\underline{r} - \int \underline{\xi}_o^* \rho_o \underline{v}_1 : \nabla \frac{\partial \underline{\xi}_1}{\partial \tau_o} d\underline{r} . \quad \ldots \text{(F.27)}$$

Notice that in deriving equation (F.27) use has been made of the fact that :

$$\frac{\partial \rho_1}{\partial \tau_o} = - \nabla \cdot (\rho_o \underline{v}_1) \ ,$$

which is an obvious result from the equation of continuity. Furthermore, we have

$$\int_{V_p} \underline{\xi}_o^* \frac{\partial}{\partial \tau_o}(\rho_o \underline{v}_1) : \nabla \underline{\xi}_1 d\underline{r} = \int_{V_p} \left\{\frac{\partial}{\partial \tau_o}\left(\underline{\xi}_o^* \rho_o \underline{v}_1 : \nabla \underline{\xi}_1\right) - \underline{\xi}_o^* \rho_o \underline{v}_1 : \nabla \frac{\partial \underline{\xi}_1}{\partial \tau_o}\right\} d\underline{r} . \quad \ldots \text{(F.28}$$

Now the following formulae is useful :

$$\frac{d}{dt}\int_{V(t)} f(\underline{r},t) dV = \int_{V(t)} \frac{\partial f}{\partial t} dV + \int_{S(t)} \underline{n} \cdot \underline{v} \, f ds \ . \qquad \ldots \text{(F.29)}$$

Here $S(t)$ is the surface enclosing the volume $V(t)$ and the volume integral as well as the surface integral have to be calculated at the instantaneous value of t. Moreover $\underline{n} \cdot \underline{v}$ is the normal material velocity at which the surface $S(t)$ is moving, see also C. Truesdell[15]. If $f(\underline{r},t)$ (which can be an arbitrary tensor point function of $\underline{r}$) and $V(t)$ are periodic in time with the period T, then we obtain the following important result :

$$\frac{1}{T}\int_o^T \left\{\int_{V(t)} \frac{\partial f}{\partial t} dV\right\} dt = -\frac{1}{T}\int_o^T \left\{\int \underline{n} \cdot \underline{v} \, f ds\right\} dt \ . \qquad \ldots \text{(F.30)}$$

In particular if $\underline{n}\cdot\underline{v}$ or f vanishes over the surface S for all t we obtain :

$$\frac{1}{T}\int_0^T \left\{ \int \frac{\partial f}{\partial t}\, dV \right\} dt = 0 \; . \qquad \text{... (F.31)}$$

Now by integrating equation (F.26) over one period in τ_o (after having multiplied by $\underline{\xi}_o^*$ and integrated over the volume) and making use of equations (F.27)-(F.31), we obtain

$$-\,\omega^2 \overline{\int_{V_p}} \rho_o |\underline{\xi}_o|^2\, dr = \frac{1}{\omega_s^2\varepsilon^2} \overline{\int_{V_p}} \underline{\xi}_o^* \cdot \underline{F}_e(\underline{\xi}_o)\, d\underline{r} + \frac{1}{\omega_s^2} \overline{\int_{V_p}} \underline{\xi}_o^* \cdot \underline{F}_e(\underline{\xi}_2)\, d\underline{r}$$

$$+ \frac{1}{\omega_s^2} \overline{\int_{V_p}} \underline{\xi}_o^* \cdot \underline{\tilde{F}}_2(\underline{\xi}_o)\, d\underline{r} + \overline{\int_{V_p}} \left\{ \frac{1}{\omega_s^2} \underline{\xi}_o^* \cdot \underline{\tilde{F}}_1(\underline{\xi}_1) - \underline{\xi}_1 \frac{\partial}{\partial\tau_o}(\rho_o \underline{v}_1) : \nabla \underline{\xi}_o^* \right\} d\underline{r} \; . \qquad \text{... (F.32)}$$

Furthermore, by taking the complex conjugate of equation (F.24) multiplying by $\underline{\xi}_1$, integrating over space and averaging over τ_o, we obtain :

$$\overline{\int_{V_p}} \rho_o\, \underline{\xi}_1 \cdot \frac{\partial^2 \underline{\xi}_1^*}{\partial\tau_o^2}\, d\underline{r} - \frac{1}{\omega_s^2} \overline{\int_{V}} \underline{\xi}_1 \cdot \underline{F}_e(\underline{\xi}_1^*) dr$$

$$= \overline{\int_{V_p}} \left\{ \frac{1}{\omega_s^2} \underline{\xi}_1 \cdot \underline{\tilde{F}}_1(\underline{\xi}_o^*) - \underline{\xi}_1 \frac{\partial}{\partial\tau_o}(\rho_o \underline{v}_1) : \nabla \underline{\xi}_o^* \right\} d\underline{r} . \qquad \text{... (F.33)}$$

By combining equations (F.32) and (F.33) we obtain :

$$\omega^2 \overline{\int_{V_p}} \rho_o |\underline{\xi}_o|^2\, d\underline{r} = -\frac{1}{\omega_s^2\varepsilon^2} \overline{\int_{V_p}} \underline{\xi}_o^* \cdot (\underline{F}_e + \varepsilon^2 \underline{\tilde{F}}_2)(\underline{\xi}_o)\, d\underline{r} - \overline{\int_{V_p}} \rho_o\, \underline{\xi}_1 \cdot \frac{\partial^2 \underline{\xi}_1^*}{\partial\tau_o^2}\, d\underline{r} + \frac{1}{\omega_s^2} \overline{\int_{V_p}} \underline{\xi}_1 \cdot \underline{F}_e(\underline{\xi}_1) d\underline{r}$$

$$- \frac{1}{\omega_s^2} \overline{\int_{V_p}} \underline{\xi}_o^* \cdot \underline{\tilde{F}}_1(\underline{\xi}_1)\, d\underline{r} + \frac{1}{\omega_s^2} \overline{\int_{V_p}} \underline{\xi}_1 \cdot \underline{\tilde{F}}_1(\underline{\xi}_o^*)\, d\underline{r} - \overline{\int_{V_p}} \underline{\xi}_o^* \cdot \underline{F}_e(\underline{\xi}_2)\, d\underline{r} \; . \qquad \text{... (F.34)}$$

We have assumed that the time variation on the τ_1 timescale is of the form $e^{i\omega\tau_1}$, and we are seeking normal mode solutions of this form. This is made possible because equations (F.24) and (F.26) are dependent on τ_1 only through $\underline{\xi}_o$ and $\underline{\xi}_1$. Moreover $\underline{\xi}_1$ can be regarded as determined in terms of $\underline{\xi}_o$ by equation (F.24). By using the self-adjointness property of $\underline{F}$ we see that the final term in equation (F.34) is small, at least of the order ε^2. Furthermore the fourth and the fifth terms cancel. Notice that the boundary conditions for the expanded operator $\underline{F}_e$ and $\underline{\tilde{F}}_n$ differ from the boundary conditions for the total $\underline{F}$. In the case of a sharp boundary we have $\underline{n}\cdot\underline{B} = 0$ in general at the interface. However, we must allow for $\underline{n}\cdot\underline{B}_n \neq 0$ in general and in particular $\underline{n}\cdot\underline{B}_o$ can be different from zero. This means that the $\underline{B}_o$ part of the magnetic field will cross the surface of integration. Therefore the proper boundary condition to replace equation (2.23) is:

$$\underline{n} \times \underline{A}_n = \underline{n} \times (\underline{\xi} \times \underline{B}_n) = \underline{\xi}\ \underline{n} \cdot \underline{B}_n - \underline{B}_n\ \underline{n} \cdot \underline{\xi}\ , \qquad \text{... (F.35)}$$

where :

$$\underline{B}_{vac} = \Sigma\ \varepsilon^n\ \underline{B}_n \quad , \quad (\underline{\xi} \times \underline{B}_n \rightarrow \underline{A}_n)\ , \qquad \text{... (F.36)}$$

see also the final part of Appendix B.

We shall now make the assumption that all time dependences in the unperturbed quantities are sinusoidal in leading order. Therefore we can write equation (F.24) as :

$$\frac{\partial^2 \underline{\xi}_1}{\partial \tau_o^2} - \frac{1}{\rho_o \omega_s^2} \underline{F}_e(\underline{\xi}_1) = \frac{1}{\rho_o} \underline{F}_1'(\underline{\xi}_o) = A \sin \tau_o + B \cos \tau_o\ . \qquad \text{... F.37)}$$

This is an inhomogeneous equation having a particular solution which is also sinusoidal in τ_o. In implicit form it is given by :

$$\underline{\xi}_1 = - \frac{1}{\rho_o} \underline{F}_1'(\underline{\xi}_o) - \frac{1}{\rho_o \omega_s^2} \underline{F}_e(\underline{\xi}_1)\ . \qquad \text{... (F.38)}$$

The general solution of equation (F.38) is obtained by adding to this solution any solution of the homogeneous equation :

$$\frac{\partial^2 \underline{\xi}_1}{\partial \tau_o^2} = \frac{1}{\rho_o \omega_s^2} \underline{F}_e(\underline{\xi}_1)\ . \qquad \text{... (F.39)}$$

This equation is identical in form to equation (F.11). Having basically assumed that there is no unstable solution on the τ_o timescale, the only possible non-trivial solution of equation (F.39) is a stable solution with a frequency of the same order as ω_s. Such a solution is, as far as stability is concerned, only interesting from a resonance point of view. This problem will be the subject of a separate investigation. In the following analysis we shall therefore regard $\underline{\xi}_1$ as being determined in terms of $\underline{\xi}_o$ by equation (F.38). We then have:

$$\frac{\partial^2 \underline{\xi}_1}{\partial \tau_o^2} = - \underline{\xi}_1 \qquad \text{... (F.40)}$$

and equation (F.34) can be re-written as :

$$\omega^2 \overline{\int_{V_p}} \rho_o\ |\underline{\xi}_o|^2\ d\underline{r} = \overline{\int_{V_p}} \frac{1}{\rho_o} \left| \underline{F}'(\underline{\xi}_o) + \frac{1}{\omega_s^2} \underline{F}_e(\underline{\xi}_1) \right|^2 d\underline{r}$$

$$- \frac{1}{\omega_s^2 \varepsilon^2} \overline{\int_{V_p}} \underline{\xi}_o^* \cdot \left(\underline{F}_e + \varepsilon^2 \underline{\tilde{F}}_2\right)(\underline{\xi}_o)\ d\underline{r} + \frac{1}{\omega_s^2} \overline{\int_{V_p}} \underline{\xi}_1 \cdot \underline{F}_e(\underline{\xi}_1^*)\ d\underline{r}\ . \qquad \text{... (F.41)}$$

By making use of equation (F.38), equation (F.41) can also be re-written as :

$$\omega^2 \overline{\int_V} \rho_o\ |\underline{\xi}_o|^2\ d\underline{r} = \delta\ \bar{W}_D + \delta\ \bar{W}_O + \delta\ \bar{W}_E\ , \qquad \text{... (F.42)}$$

where :

$$\delta \bar{W}_D = \int_{V_p}^{-} \frac{1}{\rho_o} \left| \underline{F}'(\underline{\xi}_o) \right|^2 d\underline{r} , \qquad \text{... (F.43)}$$

$$\delta \bar{W}_o = - \frac{1}{\omega_s^2 \varepsilon^2} \int_{V_p}^{-} \underline{\xi}_o^* \cdot \left(\underline{F}_e + \varepsilon^2 \tilde{\underline{F}}_2 \right) (\underline{\xi}_o) \, d\underline{r} , \qquad \text{... (F.44)}$$

$$\delta \bar{W}_E = \int_{V_p}^{-} \frac{1}{\rho_o \omega_s^2} \underline{F}'(\underline{\xi}_o^*) \, \underline{F}_e(\underline{\xi}_1) \, d\underline{r} . \qquad \text{... (F.45)}$$

By using equation (F.38) once more we can also re-write the expression for $\delta \bar{W}_E$

$$\delta \bar{W}_E = - \int_{V_p}^{-} \frac{1}{\omega_s^2} \underline{\xi}_1^* \cdot \underline{F}_e(\underline{\xi}_1) \, d\underline{r} - \int_{V_p}^{-} \frac{1}{\rho_o \omega_s^4} \left| \underline{F}_e(\underline{\xi}_1) \right|^2 d\underline{r} . \qquad \text{... (F.46)}$$

(c) THE INVERTED PENDULUM

The motion of a pendulum where the support oscillates sinusoidally in the vertical direction, with an amplitude a and a frequency ω_s is described by the following equation:

$$\frac{d^2\theta}{dt^2} + \lambda \frac{d\theta}{dt} + \left\{ \frac{g}{\ell} - \frac{a}{\ell} \omega_s^2 \sin \omega_s t \right\} \sin\theta = 0 . \qquad \text{... (F.47)}$$

Here ℓ is the length of the pendulum, g is the acceleration due to gravity, and θ is an angle measuring the deviation from a vertical line through the support. The motion is assumed to be plane, and $\theta = \pi$ is the unstable equilibrium. The damping force proportional to λ is neglected in the subsequent analysis.

Introducing the non-dimensional time $\tau = \omega_s t$ we obtain :

$$\frac{d^2}{d\tau^2} \theta + (k^2\varepsilon^2 - \varepsilon \sin\tau) \sin\theta = 0 , \qquad \text{... (F.48)}$$

where

$$\varepsilon = \frac{a}{\ell} , \quad \varepsilon^2 k^2 = \frac{g}{\ell \omega_s^2} = \frac{\omega_o^2}{\omega_s^2} .$$

Notice that $\omega_o = \sqrt{\frac{g}{\ell}}$ is the natural frequency of the pendulum for small oscillations, and in the unstable position it represents the initial growth rate for a small displacement from the equilibrium.

We shall now investigate the stability of the equilibrium $\theta = \pi$ in the case where $\varepsilon \ll 1$ and k is of order unity. Thus, writing :

$$\theta = \pi + \alpha, \qquad \text{... (F.49)}$$

where α is a small quantity, we obtain the following linearised equation, starting from equation (F.48),

$$\frac{d^2}{d\tau^2} \alpha - (\varepsilon^2 k^2 - \varepsilon \sin\tau) \alpha = 0 . \qquad \text{... (F.50)}$$

Now, carrying out the MTS expansion similarly to equations (F.9) - (F.12) we obtain :

$$(0) \quad \frac{\partial^2 \alpha_0}{\partial \tau_0^2} = 0 \implies \alpha_0 = a\tau_0 + b \implies a = 0 , \qquad \ldots \text{(F.51)}$$

$$(1) \quad 2\frac{\partial^2 \alpha_0}{\partial \tau_1 \partial \tau_0} + \frac{\partial^2 \alpha_1}{\partial \tau_0^2} = \alpha_0 \sin \tau_0 \implies \alpha_1 = \alpha_0 \sin \tau_0 , \qquad \ldots \text{(F.52)}$$

$$(2) \quad \left(\frac{\partial^2 \alpha_2}{\partial \tau_0^2} + 2\frac{\partial^2 \alpha_1}{\partial \tau_0 \partial \tau_1} + \frac{\partial^2 \alpha_0}{\partial \tau_1^2} \right) = -k^2 \alpha_0 + \alpha_1 \sin \tau_0 . \qquad \ldots \text{(F.53)}$$

By multiplying equation (F.53) by $d\tau_0$ and integrating from 0 to 2π we obtain :

$$\frac{\partial^2}{\partial \tau_1^2} \alpha_0 + \left(\frac{1}{2} - k^2 \right) \alpha_0 = 0 . \qquad \ldots \text{(F.54)}$$

Thus the condition for stable solutions is

$$\frac{1}{2} > k^2 . \qquad \ldots \text{(F.55)}$$

We now write equation (F.50) in a more general form ,

$$\frac{d^2}{d\tau^2} \alpha - \left(\varepsilon^2 F_e + \varepsilon \tilde{F}(\tau_0) \right) \alpha = 0 , \qquad \ldots \text{(F.56)}$$

where $\varepsilon^2 F_e$ represents the driving force of the instability and $\varepsilon \tilde{F}(\tau_0)$ represents the oscillatory force imposed on the system. By taking $\tilde{F}(\tau_0)$ to be sinusoidal the stability condition, equation (F.55), now reads:

$$\overline{\left\{ \tilde{F}(\tau_0) \right\}^2} > F_e . \qquad \ldots \text{(F.57)}$$

where $\bar{x}$ means the average value of x over one period in τ_0 .

APPENDIX G

THE ORTHOGONALITY OF THE EIGENFUNCTIONS $\underline{\xi}_{on}$

It is easily seen from equation (F.41) that ω^2 must be real. Moreover $\underline{\xi}_1$ is determined in terms of $\underline{\xi}_o$ by equation (F.38) and consequently $\underline{\xi}_o$ is determined by equation (F.26). By seeking the normal mode solutions of the form $\underline{\xi}_o(r)\, e^{i\omega\tau_1}$ we reduce the problem of solving equation (F.26) to that of an eigen value problem where ω^2 is the eigenvalue. To each eigenvalue ω_n^2 there corresponds at least one eigenfunction $\underline{\xi}_{on}$. We shall prove that the eigenfunctions belonging to different eigenvalues are orthogonal, that is if $\omega_n^2 \neq \omega_k^2$ then:

$$\int_{V_p} \rho_o\, \underline{\xi}^*_{on} \cdot \underline{\xi}_{ok}\, d\underline{r} = 0 \; . \qquad \text{... (G.1)}$$

We start by proving that the following relation exists :

$$\int_{V_p} \underline{\xi}^*_{1n} \cdot \underline{F}'_1(\underline{\xi}_{ok}) d\underline{r} = \int_{V_p} \underline{\xi}_{1k} \cdot \underline{F}'_1(\underline{\xi}^*_{on})\, d\underline{r} \; , \qquad \text{... (G.2)}$$

where

$$\underline{\xi}_{1k} = \underline{\xi}_1(\underline{\xi}_{ok}) \quad \text{and} \quad \underline{\xi}^*_{1n} = \underline{\xi}_1(\underline{\xi}^*_{on}) \; .$$

From equation (F.24) we obtain by taking $\underline{\xi}_1 = \underline{\xi}_{1k}$, multiplying through by $\underline{\xi}^*_{1n}$ and integrating over the volume and τ_o,

$$\int_{V_p} \underline{\xi}^*_{1n} \cdot \underline{F}'_1(\underline{\xi}_{ok})\, d\underline{r} = \int_{V_p} \rho_o\, \underline{\xi}^*_{1n} \cdot \frac{\partial^2}{\partial\tau_o^2} \underline{\xi}_{1k}\, d\underline{r} - \frac{1}{\omega^2} \int_{V_p} \underline{\xi}^*_{1n} \cdot \underline{F}_e(\underline{\xi}_{1k})\, d\underline{r} \; . \qquad \text{... (G.3)}$$

By interchanging $\underline{\xi}_{1k}$ and $\underline{\xi}^*_{1n}$ we also obtain :

$$\int_{V_p} \underline{\xi}_{ok} \cdot \underline{F}'_1(\underline{\xi}^*_{1n})\, d\underline{r} = \int_{V_p} \rho_o\, \underline{\xi}_{1k} \cdot \frac{\partial^2}{\partial\tau_o^2} \underline{\xi}^*_{1n}\, d\underline{r} - \frac{1}{\omega_s^2} \int_{V_p} \underline{\xi}_{1k} \cdot \underline{F}_e(\underline{\xi}^*_{1n})\, d\underline{r} \; . \qquad \text{... (G.4)}$$

Furthermore we have

$$\rho_o\, \underline{\xi}^*_{1n} \cdot \frac{\partial^2}{\partial\tau_o^2}\, \underline{\xi}_{1k} = \frac{\partial}{\partial\tau_o}\left(\rho_o\, \underline{\xi}^*_{1n} \cdot \frac{\partial}{\partial\tau_o}\, \underline{\xi}_{1k}\right) - \rho_o\, \frac{\partial \underline{\xi}^*_{1n}}{\partial\tau_o} \cdot \frac{\partial \underline{\xi}_{1k}}{\partial\tau_o} \; . \qquad \text{... G.5)}$$

with a similar expression for $\rho_o\, \xi_{1k} \cdot \frac{\partial^2}{\partial\tau_o^2}\, \xi^*_{1n}$. Thus applying the formulae given by equation (F.31) and using the self-adjointness property of $\underline{F}_e$, equation (G.2) is readily obtained.

In equation (F.26) we now put $\underline{\xi}_o = \underline{\xi}_{ok}$ with the corresponding ω_k^2, we multiply by $\underline{\xi}^*_{on}$ and integrate over the volume and τ_o. This gives :

$$- \omega_k^2 \int_{V_p} \rho_o \, \xi_{on}^* \cdot \xi_{ok} \, dr = \frac{1}{\omega_s^2 \varepsilon^2} \int_{V_p} \underline{\xi}_{on}^* \cdot \left(\underline{F}_e + \varepsilon^2 \tilde{\underline{F}}_2 \right) (\underline{\xi}_{ok}) d\underline{r} + \int_{V_p} \underline{\xi}_{1k} \cdot \underline{F}_1' (\underline{\xi}_{on}^*) d\underline{r} + \delta \bar{W}_D \quad \ldots \text{(G.6)}$$

By interchanging $\underline{\xi}_{ok}$ and $\underline{\xi}_{on}^*$ we also obtain equation (G.5) with $\underline{\xi}_{ok}$ and $\underline{\xi}_{on}^*$ interchanged and ω_k^2 replaced by ω_n^2. Notice that ω_k^2 and ω_n^2 are real. By using the self-adjointness property of $\underline{F}$ and equation (G.2), we get

$$(\omega_n^2 - \omega_k^2) \int_V \rho_o \, \underline{\xi}_{on}^* \cdot \underline{\xi}_{ok} \, dr = 0 \, . \quad \ldots \text{(G.7)}$$

This completes the proof, and we have all eigenfunctions belonging to different eigenvalues mutually orthogonal. If there are some ω_n^2 belonging to more than one, say $n > 1$, eigenfunctions then it is well-known that they can be chosen so as to be mutually orthogonal. Thus there is no loss in generality by assuming all the eigenfunctions to belong to an orthonormal set, in which case :

$$\int_V \rho_o \, \underline{\xi}_{on}^* \cdot \underline{\xi}_{ok} \, d\underline{r} = \delta_{nk} \quad . \quad \ldots \text{(G.8)}$$

APPENDIX H

AN "ENERGY PRINCIPLE"

(a) A NECESSARY AND SUFFICIENT CONDITIONS FOR STABILITY

We consider the quantity

$$\omega^2 = \frac{\delta \bar{W}(\underline{\xi}_o^*, \underline{\xi}_o)}{\bar{K}(\underline{\xi}_o^*, \underline{\xi}_o)} \quad , \qquad \text{... (H.1)}$$

see equations (4.12) – (4.19). $\bar{K}(\underline{\xi}_o^*, \underline{\xi}_o)$ is positive definite and therefore if the system is unstable then there must exist some $\underline{\xi}_o$ for which $\delta\bar{W} < 0$. That is, $\delta\bar{W} < 0$ for some admissible $\underline{\xi}_o$ is a necessary condition for exponential instability, or $\delta\bar{W} \geq 0$ for all admissible $\underline{\xi}_o$ is a sufficient condition for exponential stability.

We now assume that the set of eigenfunctions $\underline{\xi}_{on}$ is complete in the sense that an arbitrary admissible $\underline{\xi}_o$ can be represented by an eigenfunction expansion :

$$\underline{\xi}_o = \Sigma\, a_n\, \underline{\xi}_{on} \; . \qquad \text{... (H.2)}$$

From equations (G.6) and (G.8) it follows that we can now write

$$\delta\bar{W}(\underline{\xi}_o, \underline{\xi}_o^*) = \Sigma\, |a_n|^2\, \omega_n^2 \; . \qquad \text{... (H.3)}$$

We can then make the following conclusion. If there exists an admissible $\underline{\xi}_o$ for which $\delta\bar{W}(\underline{\xi}_o, \underline{\xi}_o^*) < 0$, then there also exists at least one $\omega_n^2 < 0$, and the system is exponentially unstable. Thus under the assumed conditions :

<u>$\delta\bar{W} \geq 0$ for all admissible $\underline{\xi}_o$ is a necessary and sufficient condition for exponential stability on the τ_1 timescale.</u>

This proves Theorems 1 and 2 in Section 5.

(b) ON THE POSITIVE DEFINITENESS OF $\delta\bar{W}_E$

From equation (F.45) we have :

$$\delta\bar{W}_E = \int_V \frac{1}{\omega_s^2 \rho_o}\, \underline{F}'(\underline{\xi}_o) \cdot \underline{F}_e(\underline{\xi}_1^*)\, \underline{dr} \;, \qquad \text{... (H.4)}$$

and from equation (4.18) :

$$\underline{\xi}_1^* = -\, \underline{\eta}_o^* - \frac{1}{\omega_s^2 \rho_o}\, \underline{F}_e(\underline{\xi}_1^*) \qquad \text{... (H.5)}$$

where we have put :

$$\underline{\eta}_o \overset{\text{def}}{=} \frac{1}{\rho_o}\, \underline{F}_1'(\underline{\xi}_o) \; . \qquad \text{... (H.6)}$$

Moreover we shall define $\underline{\eta}_n$, n > 0, by the recursion formulae :

$$\underline{\eta}_n \overset{\text{def}}{=} \frac{1}{\omega_s^2 \rho_o}\, \underline{F}_e(\underline{\eta}_{n-1}) = \left(\frac{1}{\omega_s^2 \rho_o}\, \underline{F}_e \right)^n (\underline{\eta}_o) \;, \qquad \text{... (H.7)}$$

where (n times)

$$\left(\frac{1}{\omega_s^2\rho_o}\underline{F}_e\right)^n(\underline{\xi}) = \frac{1}{\omega_s^2\rho_o}\underline{F}_e\left\{\frac{1}{\omega_s^2\rho_o}\underline{F}_e\left[\ldots\ \frac{1}{\omega_s^2\rho_o}\underline{F}_e(\underline{\xi})\ldots\right]\right\} . \qquad \ldots \text{(H.8)}$$

The following recursion formulae is readily obtained by using equation (H.5) and the self-adjointness property of $\underline{F}_e$,

$$\int_{V_p}^{-} \underline{\eta}_n \cdot \underline{F}_e\left\{\left(\frac{1}{\omega_s^2\rho_o}\underline{F}_e\right)^n(\underline{\xi}_1^*)\right\} d\underline{r} = -\int_{V_p}^{-}\underline{\eta}_n^* \cdot \underline{F}_e(\underline{\eta}_n)\, d\underline{r}$$

$$+\int_{V_p}^{-}\frac{1}{\omega_s^2\rho_o}\left|\underline{F}_e(\underline{\eta}_n)\right|^2 d\underline{r} + \int_{V_p}^{-}\underline{\eta}_{n+1}\cdot\underline{F}_e\left\{\left(\frac{1}{\omega_s^2\rho_o}\underline{F}_e\right)^{n+1}(\underline{\xi}_1^*)\right\}d\underline{r} . \qquad \ldots \text{(H.9)}$$

We can now obtain the following expression for $\delta\overline{W}_E$ by applying the recursion formula given by equation (H.9) n times,

$$\omega_s^2\,\delta\overline{W}_E = \sum_{k=o}^{n}\left\{-\int_{V_p}^{-}\underline{\eta}_k^*\cdot\underline{F}_e(\underline{\eta}_k)\,d\underline{r} + \int_{V_p}^{-}\frac{1}{\omega_s^2\rho_o}\left|\underline{F}_e(\underline{\eta}_k)\right|^2 d\underline{r}\right\} + R_{n+1} , \qquad \ldots \text{(H.10)}$$

where

$$R_{n+1} = \int_{V_p}^{-}\underline{\eta}_{n+1}\cdot\underline{F}_e\left\{\left(\frac{1}{\omega_s^2\rho_o}\underline{F}_e\right)^{n+1}(\underline{\xi}_1^*)\right\} . \qquad \ldots \text{H.11}$$

Now for each k , $\underline{\eta}_k$ can belong either to the class of stable or unstable perturbations. Suppose that $\underline{\eta}_k$ for $0 \leqslant k \leqslant n$ belongs to the class of stable perturbations and that $\underline{\eta}_{n+1}$ belongs to the class of unstable perturbations. Then by definition :

$$-\int_{V_p}\underline{\eta}_k^*\,\underline{F}_e(\underline{\eta}_k)d\underline{r} > 0 \quad \text{for} \quad 0 \leqslant k \leqslant n .$$

Since the remainder, equation (H.11) in this case is small, of order ε^2, it follows that $\delta\overline{W}_E > 0$. However, the remainder, equation (H.11), can be small, of order ε, even if $\underline{\eta}_{n+1}$ belongs to the class of stable perturbations; in which case we can make the same conclusion. For $n \to \infty$ we can have two cases :

(i) The series generated for $\delta\overline{W}_E$ converges and

$$\lim_{n\to\infty} R_{n+1} = 0 .$$

In this case we again have the result $\delta\overline{W}_E > 0$.

(ii) If the series generated for $\delta\overline{W}_E$ does not converge then we cannot make any conclusions about the sign of $\delta\overline{W}_E$.

(c) THE POTENTIAL ENERGY

We shall refer to the integral :

$$\delta W = -\frac{1}{2}\int_{V_p} \underline{\xi}^* \cdot \underline{F}(\underline{\xi})\, dV_p \,, \qquad \ldots \text{(H.12)}$$

as the potential energy. By making use of equations (D.3) and (D.4) we obtain L

$$\underline{\xi}^* \cdot \underline{F}(\underline{\xi}) = \nabla \cdot \left\{ -\underline{\xi}^* \underline{B} \cdot \underline{Q} + \underline{\xi}^* \gamma p \nabla \cdot \underline{\xi} + \underline{\xi}^* \underline{\xi} \cdot \nabla p + \underline{B}\, \underline{\xi}^* \cdot \underline{Q} - \rho \underline{v}\, \underline{\xi}^* \underline{v} : \nabla \underline{\xi} \right\}$$

$$- |\underline{Q}|^2 - \gamma p |\nabla \cdot \underline{\xi}|^2 + \rho |\underline{v} \cdot \nabla \underline{\xi}|^2 - \rho \underline{\xi}^* \underline{\xi} : \nabla\nabla\Psi - \nabla \cdot \underline{\xi}^* \underline{\xi} \cdot \nabla p + \underline{Q} \cdot \underline{\xi}^* \times (\nabla \times \underline{B})$$

$$+ \underline{\xi}^* \nabla : \left\{ \underline{\xi} \left[-\nabla p + (\nabla \times \underline{B}) \times \underline{B} \right] \right\} . \qquad \ldots \text{(H.13)}$$

Furthermore, we write,

$$G = -\nabla \cdot \underline{\xi}^* \underline{\xi} \cdot \nabla p + \underline{Q} \cdot \underline{\xi}^* \times (\nabla \times \underline{B}) + \underline{\xi}^* \nabla : \left\{ \underline{\xi} \left[-\nabla p + (\nabla \times \underline{B}) \times \underline{B} \right] \right\} \ldots \text{(H.14)}$$

We note that one can write :

$$Q = \nabla \times (\underline{\xi} \times \underline{B}) = \underline{B} \cdot \nabla \underline{\xi} - \underline{\xi} \cdot \nabla \underline{B} - \underline{B} \nabla \cdot \underline{\xi} \,, \qquad \ldots \text{(H.15)}$$

$$(\nabla \times \underline{B}) \times \underline{B} = \underline{B} \cdot \nabla \underline{B} - \nabla \frac{B^2}{2} \,, \qquad \ldots \text{(H.16)}$$

$$\underline{\xi}^* \times (\nabla \times \underline{B}) = \nabla \underline{B} \cdot \underline{\xi}^* - \underline{\xi}^* \cdot \nabla \underline{B} \,, \qquad \ldots \text{(H.17)}$$

$$-\nabla p + (\nabla \times \underline{B}) \times \underline{B} = -\nabla\left(p + \frac{B^2}{2}\right) + \underline{B} \cdot \nabla \underline{B} = \underline{a} \,, \qquad \ldots \text{(H.18)}$$

$$\underline{\xi}^* \nabla : (\underline{\xi}\, a) = \underline{\xi}^* \cdot \underline{a}\ \nabla \cdot \underline{\xi} + \underline{\xi}^* \underline{\xi} : \nabla \underline{a} \,. \qquad \ldots \text{(H.19)}$$

Hence by using equations (H.15) - (H.19) we can re-write equation (H.14) as

$$G = -\nabla \cdot \underline{\xi}^* \underline{\xi} \cdot \nabla p + \underline{\xi}^* \underline{B} : \nabla \underline{B}\, \nabla \cdot \underline{\xi} - \underline{\xi}^* \cdot \nabla \left(p + \frac{B^2}{2}\right) \nabla \cdot \underline{\xi} + \underline{\xi}^* \underline{\xi} : \nabla(\underline{B} \,.\, \nabla \underline{B}) - \underline{\xi}^* \underline{\xi} : \nabla\nabla \left(p + \frac{B^2}{2}\right)$$

$$+ \left(\underline{B} \cdot \nabla \underline{\xi} - \underline{\xi} \cdot \nabla \underline{B} - \underline{B}\, \nabla \cdot \underline{\xi}\right) \cdot \left(\nabla \underline{B} \cdot \underline{\xi}^* - \underline{\xi}^* \cdot \nabla \underline{B}\right) = -\underline{\xi}^* \underline{\xi} : \nabla\nabla \left(p + \frac{B^2}{2}\right) + |\underline{\xi} \cdot \nabla \underline{B}|^2 - \nabla \cdot \underline{\xi}^* \underline{\xi} \cdot \nabla p$$

$$- \nabla \cdot \underline{\xi}\, \underline{\xi}^* \cdot \nabla p - \underline{B} \cdot \nabla \underline{\xi} \cdot (\xi^* \cdot \nabla) \underline{B} - \underline{B} \cdot \nabla \underline{\xi}^* \cdot (\underline{\xi} \cdot \nabla) \underline{B} + \nabla \cdot \left\{ \underline{B}\, \underline{\xi} \cdot \nabla \underline{B} \cdot \underline{\xi}^* \right\} \qquad \ldots \text{(H.20)}$$

By making use of equation (H.20) we may write equation (H.13) as :

$$\underline{\xi}^* \cdot \underline{F}(\xi) = \nabla \cdot \left\{ -\underline{\xi}^* \underline{B} \,.\, \underline{Q} + \underline{\xi}^* \gamma p \nabla \cdot \underline{\xi} + \underline{\xi}^* \underline{\xi} \cdot \nabla p + \underline{B}\, \underline{\xi}^* \cdot \underline{Q} - \rho \underline{v}\, \underline{\xi}^* \underline{v} : \nabla \underline{\xi} + \underline{B}\, \underline{\xi} \cdot \nabla \underline{B} \cdot \underline{\xi}^* \right\}$$

$$- |\underline{Q}|^2 - \gamma p |\nabla \cdot \underline{\xi}|^2 + \rho |\underline{v} \cdot \nabla \underline{\xi}|^2 - \rho \underline{\xi}^* \underline{\xi} : \nabla\nabla\Psi - \underline{\xi}^* \underline{\xi} : \nabla\nabla \left(p + \frac{B^2}{2}\right) + |\underline{\xi} \cdot \nabla \underline{B}|^2$$

$$- \left\{ \underline{\xi} \cdot \nabla p\, \nabla \cdot \underline{\xi}^* + \underline{B} \cdot \nabla \underline{\xi}^* \cdot (\underline{\xi} \cdot \nabla) \underline{B} \right\} - \left\{ \underline{\xi} \cdot \nabla p\, \nabla \cdot \underline{\xi}^* + \underline{B} \cdot \nabla \underline{\xi}^* \cdot (\underline{\xi} \cdot \nabla) \underline{B} \right\}^* . \qquad \ldots \text{(H.21)}$$

Furthermore by expanding the expression for $\underline{Q}$ we obtain :

$$|\underline{Q}|^2 = |\underline{B}\cdot\nabla\underline{\xi}|^2 + |\underline{\xi}\cdot\nabla\underline{B}|^2 + \underline{B}^2|\nabla\cdot\underline{\xi}|^2 + \left\{\underline{B}\underline{\xi}:\nabla\underline{B}\,\nabla\cdot\underline{\xi}^* - \underline{B}\underline{B}\cdot\nabla\underline{\xi}\,\nabla\cdot\underline{\xi}^* - \underline{B}\cdot\nabla\underline{\xi}\cdot(\underline{\xi}^*\cdot\nabla)\underline{B}\right\} + \left\{\ \cdots\ \right\}^* ; \qquad \ldots\ (H.22)$$

Note that $\underline{B}\underline{\xi}:\nabla\underline{B} = \underline{\xi}\cdot\nabla\,\frac{B^2}{2}$.

From equations (H.21) and (H.22) we obtain

$$\underline{\xi}^*\cdot\underline{F}(\underline{\xi}) = \nabla\cdot\left\{-\underline{\xi}^*\underline{B}\cdot\underline{Q} + \underline{\xi}^*\gamma p\nabla\cdot\underline{\xi} + \underline{\xi}^*\underline{\xi}\cdot\nabla p + \underline{B}\left(\underline{\xi}^*\cdot\underline{Q} + \underline{\xi}\cdot\nabla\underline{B}\cdot\underline{\xi}^*\right) - \rho\underline{v}\underline{v}\underline{\xi}^*:\nabla\underline{\xi}\right\}$$
$$- |\underline{B}\cdot\nabla\underline{\xi}|^2 - (B^2+\gamma p)|\nabla\cdot\underline{\xi}|^2 + \rho|\underline{v}\cdot\nabla\underline{\xi}|^2 - \underline{\xi}^*\underline{\xi}:\left\{\nabla\nabla\left(p+\frac{B^2}{2}\right) + \rho\nabla\nabla\Psi\right\}$$
$$- \left\{\underline{\xi}\cdot\nabla\left(p+\frac{B^2}{2}\right) - \underline{B}\underline{B}:\nabla\underline{\xi}\right\}\nabla\cdot\underline{\xi}^* - \left\{\underline{\xi}^*\cdot\nabla\left(p+\frac{B^2}{2}\right) - \underline{B}\underline{B}:\nabla\underline{\xi}^*\right\}\nabla\cdot\underline{\xi}. \qquad \ldots\ (H.23)$$

By integrating equation (H.23) over the plasma volume including the 'surface layer' we obtain by making use of Gauss' theorem and the boundary conditions $\underline{n}\cdot\underline{B} = 0$ at the surface S ,

$$-\int_{V_p}\underline{\xi}^*\cdot\underline{F}(\underline{\xi})\,dV_p = -\int_S \underline{n}\cdot\underline{\xi}^*\left\{-\underline{B}\cdot\underline{Q} + \gamma p\nabla\cdot\underline{\xi} + \underline{\xi}\cdot\nabla p\right\}dS$$
$$+\int\left\{|\underline{B}\cdot\nabla\underline{\xi}|^2 + (B^2+\gamma p)|\nabla\cdot\underline{\xi}|^2 - \rho|\underline{v}\cdot\nabla\underline{\xi}|^2 + \underline{\xi}^*\underline{\xi}:\left\{\nabla\nabla\left(p+\frac{B^2}{2}\right) + \rho\nabla\nabla\Psi\right\}\right.$$
$$\left. + 2\,\mathrm{Re}\left\{\left[\underline{\xi}\cdot\nabla\left(p+\frac{B^2}{2}\right) - \underline{B}\underline{B}:\nabla\underline{\xi}\right]\nabla\cdot\underline{\xi}^*\right\}\right\}d\underline{r}\ . \qquad \ldots\ (H.24)$$

As a matter of definition, $\rho = 0$ on the vacuum side of the 'surface layer'. This has been used in obtaining equation (H.24). The surface integral in equation (H.24) can be re-written by using the boundary conditions equations (2.22) and (2.23).

$$-\int_S \underline{n}\cdot\underline{\xi}^*\left\{-\underline{B}\cdot\underline{Q} + \gamma p\nabla\cdot\underline{\xi} + \underline{\xi}\cdot\nabla p\right\}dS = \int_S\left\{\underline{n}\cdot\underline{\xi}^*\underline{B}_v\cdot\nabla\times\underline{A} + |\underline{n}\cdot\underline{\xi}|^2\,\underline{n}\cdot\left\langle\nabla\left(p+\frac{B^2}{2}\right)\right\rangle\right\}dS$$
$$= \int_S -\underline{n}\times\underline{A}\cdot(\nabla\times\underline{A}) + |\underline{n}\cdot\underline{\xi}|^2\underline{n}\cdot\left\langle\nabla\left(p+\frac{B^2}{2}\right)\right\rangle ds = \int_S |\underline{n}\cdot\underline{\xi}|^2\,\underline{n}\cdot\left\langle\nabla\left(p+\frac{B^2}{2}\right)\right\rangle ds$$
$$+ \int_{V_v}\nabla\cdot\left\{\underline{A}^*\times(\nabla\times\underline{A})\right\}d\underline{r} = \int_S |\underline{n}\cdot\underline{\xi}|^2\,\underline{n}\cdot\left\langle\nabla\left(p+\frac{B^2}{2}\right)\right\rangle ds + \int_{V_v}|\nabla\times\underline{A}|^2\,d\underline{r}\ . \qquad \ldots\ (H.25)$$

The same assumptions about the material present in the vacuum region as have been made in connection with the proof of the self-adjointness of $\underline{F}$ are used here as well. Finally we obtain :

$$- \int_{V_p} \underline{\xi}^* \cdot \underline{F}(\underline{\xi})\, dV_p = \int_{V_v} |\nabla \times \underline{A}|^2\, dV_v + \int_S |\underline{n} \cdot \underline{\xi}|^2\, \underline{n} \cdot \left\langle \nabla \left(p + \frac{B^2}{2}\right) \right\rangle ds$$

$$+ \int_{V_p} \left[|\underline{B} \cdot \nabla \underline{\xi}|^2 + (B^2 + \gamma p)|\nabla \cdot \underline{\xi}|^2 - \rho |\underline{v} \cdot \nabla \underline{\xi}|^2 + \underline{\xi}^* \underline{\xi} : \left\{ \nabla\nabla\left(p + \frac{B^2}{2}\right) + \rho \nabla\nabla\Psi \right\} \right.$$

$$\left. + 2\, \mathrm{Re} \left\{ \left[\underline{\xi} \cdot \nabla \left(p + \frac{B^2}{2}\right) - \underline{B}\underline{B} : \nabla \underline{\xi} \right] \nabla \cdot \underline{\xi}^* \right\} \right] d\underline{r} \,. \qquad \text{... (H.26)}$$

It should also be noted that it is now easy to construct another direct proof for the self-adjointness of $\underline{F}$ based on equation (H.26).

Instead of treating the sharp boundary problem directly and applying the boundary condition equation (2.22), one can proceed as follows: Divide the region over which the integration is performed into two sub-regions V_i and V_s the volume of a shell contained between the two surfaces S_v and S_p on the vacuum and plasma side respectively. The surfaces are taken to be everywhere parallel to the magnetic field lines. We now treat all quantities to be continuous over the region $V_i + V_s$. The sharp boundary case is now defined to be the formal limit $\delta \to 0$, where δ is a proper measure of the thickness of the shell. Thus in this limit S_v and S_i coincide in one surface which we denote by S.

The 'divergence' term appearing in equation (H.22) can be transformed into surface integrals. The contributions from integration over the surface S_i cancel because of equal contributions with opposite sign originating from the two regions V_i and V_s. Thus we obtain :

$$- \int_{V_s + V_p} \underline{\xi}^* \cdot \underline{F}(\underline{\xi}) d\underline{r} = \int_{S_v} \underline{n} \cdot \underline{R}(\underline{\xi}_1^*, \underline{\xi})\, dS_v + \int_{V_s} w(\underline{\xi}^*, \underline{\xi})\, d\underline{r}$$

$$+ \int_{V_p} w(\underline{\xi}^*, \underline{\xi})\, d\underline{r} \qquad \text{... (H.27)}$$

where

$$\underline{R}(\underline{\xi}^*, \underline{\xi}) = \underline{\xi}^*(\underline{B} \cdot \underline{Q} \quad \gamma p \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla p) - \underline{B}(\underline{Q} + \underline{\xi} \cdot \nabla \underline{B}) \cdot \underline{\xi}^* + \rho\, \underline{v}\,\underline{v}\,\underline{\xi}^* : \nabla \underline{\xi} \,, \qquad \text{... (H.28)}$$

and

$$w(\underline{\xi}^*, \underline{\xi}) = |\underline{B} \cdot \nabla \underline{\xi}|^2 + (B^2 + \gamma p)|\nabla \cdot \underline{\xi}|^2 - \rho |\underline{v} \cdot \nabla \underline{\xi}|^2 + \underline{\xi}^* \underline{\xi} : \left\{ \nabla\nabla\left(p + \frac{B^2}{2}\right) + \rho \nabla\nabla\varphi \right\}$$

$$+ \left\{ \underline{\xi} \cdot \nabla \left(p + \frac{B^2}{2}\right) - \underline{B}\,\underline{B} : \nabla \underline{\xi} \right\} \nabla \cdot \underline{\xi}^* + \left\{ \underline{\xi}^* \cdot \nabla\left(p + \frac{B^2}{2}\right) - \underline{B}\,\underline{B} : \nabla \underline{\xi}^* \right\} \nabla \underline{\xi} \,. \qquad \text{... (H.29)}$$

In the surface integral we note that the two terms :

$$- \underline{B}(\underline{Q} + \underline{\xi} \cdot \nabla \underline{B}) \cdot \underline{\xi}^* + \rho\, \underline{v}\,\underline{v}\,\underline{\xi}^* : \nabla \underline{\xi}$$

integrate to zero, the first of them because $\underline{n} \cdot \underline{B} = 0$ and the second because $\rho = 0$ over S_v; likewise p and ∇p is also zero, thus we obtain :

$$\int_{S_V} \underline{n} \cdot \underline{R}(\underline{\xi}^*, \underline{\xi})\, d\, S_V = \int_{S_V} \underline{n} \cdot \underline{\xi}^*\, \underline{B} \cdot \underline{Q}\, d\, S_V \; . \qquad \text{... (H.30)}$$

Note that $\underline{Q} = \nabla \times (\underline{\xi} \times \underline{B})$ is the perturbed magnetic field in the plasma region which corresponds to $\nabla \times \underline{A}$ in the vacuum region. Thus regarding the plasma region to be a pressureless fluid with no currents it is quite natural to take $\underline{A}$ to be the analytic continuation of $\underline{\xi} \times \underline{B}$ into this region. Hence over S_V we have,

$$\underline{A} = \underline{\xi} \times \underline{B} \; , \qquad \text{... (H.31)}$$

from which one easily obtains :

$$\underline{n} \times \underline{A} = - \underline{B}\, \underline{\xi} \cdot \underline{n} \; . \qquad \text{... (H.32)}$$

This should be compared with equation (2.23) and the remark at the end of Appendix B. It then readily follows that :

$$\int_{S_V} \underline{n} \cdot \underline{R}(\underline{\xi}^*, \underline{\xi})\, d\, S_V = \int \underline{\hat{n}} \cdot \underline{A}^* \times (\nabla \times \underline{A})\, d\, S_V = \int_{V_V} |\nabla \times \underline{A}|^2\, d\underline{r} \qquad \text{... (H.33)}$$

where $\underline{n} = - \underline{\hat{n}}$. We have also used the vector identity :

$$\nabla \cdot \left\{ \underline{A}^* \times (\nabla \times \underline{A}) \right\} = |\nabla \times \underline{A}|^2 - \underline{A}^* \cdot \nabla \times (\nabla \times \underline{A}) \; , \qquad \text{... (H.34)}$$

and the fact that $\nabla \times (\nabla \times \underline{A}) = 0$ in vacuum. Moreover, the boundary conditions used in connection with the self-adjointness proof of $\underline{F}$, (Appendix S), still apply.

The next step is to calculate the volume integral over V_S in the limit $\delta \to 0$. We note that derivatives along the surface remain finite, but we must allow for derivatives normal to the surface to be of the order δ^{-1}. The volume V_S is obviously proportional to δ.

Let us first consider the following integral :

$$\lim_{\delta \to 0} \int_{V_S} (B^2 + \gamma p)\, |\nabla \cdot \underline{\xi}|^2\, d\underline{r} \; .$$

The leading order contribution , I , from this integral, is given by :

$$I_1 = \lim_{\delta \to 0} \int_{V_S} (B^2 + \gamma p) \left| \frac{d\xi_n}{dn} \right|^2 d\underline{r} \; , \qquad \text{... (H.35)}$$

where

$$\frac{d}{dn} \overset{\text{def}}{=} \underline{n} \cdot \nabla \quad \text{and} \quad \xi_n = \underline{n} \cdot \underline{\xi} \; .$$

Now we have :

$$\frac{d\xi_n}{dn} = 0 \left(\frac{\langle \xi_n \rangle}{\delta} \right) ,$$

thus if ξ_n is not continuous across the surface this could give an arbitrarily large positive contribution to this integral. It therefore appears reasonable to assume that $\langle \xi_n \rangle = 0$ from which it follows that $I_1 = 0$.

We then consider

$$\lim_{\delta \to o} \int_{V_s} \rho |\underline{v} \cdot \nabla \underline{\xi}|^2 \, d\underline{r} \, ,$$

which has the leading order contribution :

$$I_2 = - \lim_{\delta \to o} \int \rho (\underline{v} \cdot \underline{n})^2 \left| \frac{d\xi_{\parallel}}{dn} \right|^2 d\underline{r} \, , \qquad \text{... (H.36)}$$

where $\xi_{\parallel}^2 = \{\underline{\xi} - \underline{n}\,\underline{n} \cdot \underline{\xi}\}^2$. If $\langle \xi_{\parallel} \rangle \neq 0$ this could give an arbitrarily large negative contribution. It is difficult to rule out the possibility that this is not associated with an instability. We shall, however, not be able to undertake any further discussion on this problem here, but we want to point out that in the particular case of periodic systems this term is on the average exactly cancelled by a stabilizing term coming from the left hand side of equation (2.26). The subsequent analysis is, however, based on the assumption that $\langle \xi_{\parallel} \rangle = 0$ as well. In addition we have $\langle p + \frac{B^2}{2} \rangle = 0$. It is then easily seen that all the other terms except

$$\underline{\xi}^* \underline{\xi} : \nabla \nabla \left(p + \frac{B^2}{2} \right)$$

give no contribution in the limit $\delta \to 0$. Thus we are left with :

$$\lim_{\delta \to o} \int_{V_s} \underline{\xi}\, \underline{\xi}^* : \nabla\nabla \left(p + \frac{B^2}{2} \right) dr \, ,$$

which leading order contribution is given by :

$$I_3 = \lim_{\delta \to o} \int |\underline{\xi}_n|^2 \frac{d}{dn} \frac{d}{dn} \left(p + \frac{B^2}{2} \right) d\underline{r} \, .$$

By integrating across the shell one easily obtains :

$$I_3 = \int_S |\underline{\xi}_n|^2 \, \underline{n} \cdot \langle \nabla \left(p = \frac{B^2}{2} \right) \rangle \, dS \, ; \qquad \text{... (H.37)}$$

notice that $\langle \underline{\xi}_n \rangle = 0$ and that $\frac{d\underline{\xi}_n}{dn}$ remains finite in the limit $\delta \to 0$. These results therefore give the same expression for

$$\int_{V_p} \underline{\xi}^* \cdot \underline{F}(\underline{\xi}) \, d\underline{r}$$

as previously obtained in equation (H.26).

Available from

HER MAJESTY'S STATIONERY OFFICE

49 High Holborn, London, W.C.1
13a Castle Street, Edinburgh 2
109 St. Mary Street, Cardiff CF1 1JW
Brazennose Street, Manchester M60 8AS
50 Fairfax Street, Bristol BS1 3DE
258 Broad Street, Birmingham 1
7-11 Linenhall Street, Belfast BT2 8AY

or through any bookseller.