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TRANSFER OF RADIATION IN A SPECTRAL LINE

by

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INTRODUCTION

The investigation of the transfer of radiation in a spectral line constitutes the theoretical foundation for the interpretation of the line spectra of stars (see refs. (1, 2)). This problem is important also for solar physics[3] and for the physics of interstellar matter[4]. The investigation of multiple scattering of light in a spectral line constitutes as well one of the important problems in gas discharge optics[5, 6, 7] and in other fields. This problem will be encountered also in the investigation of high-temperature plasmas[8,9]. It is quite natural that the theory of multiple scattering of radiation in a spectral line has long attracted attention. In papers published during the twenties and in most papers published during the thirties the assumption was made of a strict monochromatic scattering, i.e. it was assumed that the frequency of a quantum scattered by an atom is exactly equal to the frequency of the absorbed quantum. On the basis of this assumption numerous problems were solved and the theory of the formation of absorption lines in stellar spectra was developed in detail (see, e.g. refs. (10, 2)). However as long ago as 1929 Eddington[11] observed that if mechanisms are discovered which cause the quantum to change its frequency during scattering, the theory will have to be completely revised. Shortly thereafter, such mechanisms were in fact discovered[12-15]. However the rebuilding of radiative transfer theory including the frequency changes during scattering turned out to be by no means easy and is not yet completed.

Since the problem is a very complex one, an important step forward was the introduction of an approximation which considerably simplified the problem and at the same time took into account fairly adequately the above feature of scattering in a spectral line. This is the so-called approximation of complete frequency redistribution, according to which the probability of the re-radiation of a quantum at a given frequency is not dependent on the frequency of the absorbed quantum but is proportional to the absorption coefficient. This approximation was introduced towards the end of the thirties and the beginning of the forties (see in particular ref. 16). In 1944 L. Spitzer[17] gave a discussion of the causes of the non-monochromatic character of scattering and wrote down the transfer equation for scattering with complete frequency redistribution. He then arrived at the conclusion that it is necessary to construct the theory of the formation of absorption lines in stellar spectra on the basis of this approximation. Soon afterwards the transfer equation written in integral form in the approximation of complete frequency redistribution also appeared in the physical literature[18, 19] and gradually entered into general use[20-22].

It was found that the variation of the frequency of a quantum during scattering renders impossible the approximate reduction of the problem to a solution of the diffusion equation. It was therefore necessary to develope effective approximate methods for solving the transfer equation for resonance radiation. This is another problem which has not yet been completely solved. For the solution of individual problems variational methods were used[19, 23] as well as a modified form of Chandrasekar's method[3, 24] and the direct numerical solution of the principal integral equation[18, 25-28]. An approximate method proposed specially for the investigation of the transfer of radiation in a spectral line has been fairly widely adopted[29, 39].

In parallel with the approximate methods, rigorous methods of solving the transfer equation have evolved[31-34]. The application of these methods to the transfer of radiation with complete frequency distribution, which was started during the forties[35], has subsequently been developed to a considerable extent[36, 32, 37, 38].

Recently it has been pointed out[39, 40] that the specific features of scattering in a line due to possible variations in frequency are, from a mathematical point of view, related to the characteristic feature of the analytical properties of the Laplace transform of the resolvent of the principal integral equation. This feature is now discussed.

In the classical problems concerning the diffusion of particles and the scattering of light without change of frequency, the singularity of the Laplace transform of the resolvent farthest to the right is a simple pole situated in the left hand-plane on the real axis (see, e.g. refs. 41, 42, 34). This leads to an exponential decrease of the density of the diffuse radiation at some distance away from a source. However, with complete frequency redistribution during scattering and no absorption in the continuous spectrum, there is no pole and the negative part of the real axis $(-\infty, 0)$ is a branch line. This feature in particular determines the now fairly well-known fact that the density of the diffuse radiation at large distances from a source decreases relatively slowly (non-exponentially). This explanation of the mathematical nature of the phenomenon made it possible to investigate the asymptotic behaviour of the solution of the principal integral equation in deep layers (remote from a boundary of the medium). For the development of effective approximate methods of solution, a knowledge of the asymptotics is highly important. Finally, explicit expressions for the resolvent of the principal equation (for an infinite medium and half space) were determined quite recently[43,44]. Moreover, under fairly general assumptions about the distribution of the sources, the radiation intensity coming from a semi-infinite medium can be expressed in terms of the corresponding H-function, which has been determined in explicit form and tabulated[45].

Exact solutions for a fairly wide group of problems have thus been obtained. One might think that they will in the future be applied directly as well as for the verification of the accuracy of various approximate methods.

However, until quite recently, applications of the theory (these are of a considerable variety) were severely impeded by the absence of tables of the special functions through which the solutions are expressed. It would seem that this circumstance, in particular, is the cause of the present odd situation in which the experimenters (especially astrophysicist-observers) ignore the achievements of the theory and continue to use rough and, often, even incorrect solutions, even in those cases in which an exact solution is known.

Recently, the tabulation has been undertaken of the principal special functions encountered in the theory of resonance radiation in spectral lines. At present only the first results in this field have been published[46,47]. One may expect that the complete realization of the projected program will eliminate the above-mentioned difficulty in the application of the theory which has been developed.

Just as previously, we shall be interested in determining those properties of the solutions of the transfer equation which might facilitate its numerical solution and improve the accuracy of the approximate methods. The present paper is concerned with this range of problems. Simplifications in the structure of the radiation field are investigated which come about when true absorption plays only a small part, since then the act of photoexcitation of the atom is almost invariably followed by a spontaneous transition, i.e. reradiation of the quantum takes place. Some results obtained earlier[40,45] are special cases of the more general relations obtained here.

I

Very frequently the characteristic dimension of the region occupied by the gas scattering the light is considerably larger than the mean free path of the quantum having the frequency of the line centre. The first step in the investigation of the radiation field in such systems is the investigation of scattering in an idealised medium occupying the half space bounded by a plane.

The problem of the scattering of light in such a semi-infinite medium belongs amongst the infrequent problems in scattering theory whose exact solution can be obtained in closed form. Apart from the interest which this problem presents in itself, it assumes considerable importance from the point of view of determining the accuracy and the regions of validity of various approximate methods in transfer theory. Of great interest is the investigation of asymptotic properties of the solution of the problem, since these determine the simplifications characteristic, not only of such a semi-infinite medium, but also of media of more complex geometry.

The equation for the transfer of radiation in a spectral line for a plane layer, as is known, has the following form (see for instance ref. 32, chapter 8)

$$\eta \frac{dI(\tau, \eta, x)}{d\tau} = \alpha(x) I(\tau, \eta, x) - \frac{\lambda A}{2} \alpha(x) \times \times \int_{-\infty}^{\infty} \alpha(x')\, dx' \int_{-1}^{1} I(\tau, \eta', x')\, d\eta' - A\alpha(x) g(\tau), \quad \ldots (1)$$

where η is the cosine of the angle between the direction of the quantum and the external normal to the layer; x is the dimensionless frequency representing the distance from the line centre expressed in some suitable units such as Doppler widths or damping widths; τ, the optical depth in the frequency $x = 0$, i.e. the distance from the boundary of the medium along the normal, measured in quantum mean-free paths, the quantum having the frequency of the line centre; $\alpha(x)$, the ratio of the coefficient of absorption in the frequency x to the coefficient of absorption at the line centre; A, the normalisation constant

$$A \int_{-\infty}^{\infty} \alpha(x)\, dx = 1; \quad \ldots (2)$$

$\lambda \leqslant 1$, the survival probability of the quantum during the elementary scattering act; $I(\tau, \eta, x)$ the radiation intensity; $4\pi g(\tau)$, the power of the radiation sources, i.e. the total energy supplied by the sources in the whole line per unit time in a cylinder of unit optical length and of cross section 1 cm^2. It is assumed that the frequency distribution of the energy emitted by the sources is proportional to the absorption coefficient.

Equation (1) describes the scattering in a line in the approximation of complete frequency redistribution. The accuracy provided by these approximations is in practice sufficient for all applications (ref. 34, chapt. 8; 48, 49). In writing down equations (1) it has also been assumed that the absorption in the continuous spectrum is negligibly small and that the indicatrix of scattering is spherical.

As stated earlier, equation (1) will be investigated for the simplest case in which the optical thickness of the layer is infinitely large, i.e. where the gas scattering the quanta occupies a half space. Without loss of generality it can be assumed that no radiation from the outside falls on the boundary of the medium, so that

$$I(0, \eta, x)=0 \quad \text{при } \eta<0. \qquad \dots (3)$$

Let us introduce the source function $S(\tau)$:

$$S(\tau)=\frac{\lambda A}{2}\int_{-\infty}^{\infty}\alpha(x')dx'\int_{-1}^{1}I(\tau, \eta', x')d\eta'+Ag(\tau). \qquad \dots (4)$$

We note that $S(\tau)$ differs from the previously used function $B(\tau)$ (39, 40, 45, 50) by the factor A

$$S(\tau) = AB(\tau)$$

The normalisation of the source function assumed here is more convenient in several respects, in particular since $S(\tau)$ goes over immediately into Plank's function in the case of thermodynamic equilibrium.

Equation (1) with the boundary condition (3) is equivalent to the following integral equation for $S(\tau)$

$$S(\tau)=\frac{\lambda}{2}\int_{0}^{\infty}K(|\tau-\tau'|)S(\tau')d\tau'+Ag(\tau), \qquad \dots (5)$$

where

$$K(\tau)=A\int_{-\infty}^{\infty}\alpha^{2}(x)E_{1}(\tau\alpha(x))dx, \qquad \dots (6)$$

where $E_1(t)$ is the exponential integral function of the first order

$$E_{1}(t)=\int_{0}^{1}e^{-\frac{t}{\zeta}}\frac{d\zeta}{\zeta}. \qquad \dots (7)$$

Equation (5) will be investigated further below.

Re-writing (1) in the form

$$\frac{\eta}{\alpha(x)} \cdot \frac{dI}{d\tau} = I - S(\tau), \qquad \ldots (8)$$

it is seen that in reality the radiation intensity is not dependent on the individual values of η and x but only on the ratio

$$z = \frac{\eta}{\alpha(x)}. \qquad \ldots (9)$$

The intensity as a function of τ and z will be designated as previously by I. This should not lead to misunderstandings. For the intensity of the emitted radiation $I(0, z)$ we have

$$I(0, z) = \int_0^\infty e^{-\frac{\tau}{z}} S(\tau) \frac{d\tau}{z}. \qquad \ldots (10)$$

The kernel (6) of the principal integral equation can be brought readily into the form

$$K(\tau) = \int_0^\infty e^{-\frac{\tau}{z'}} G(z') \frac{dz'}{z'}, \qquad \ldots (11)$$

where

$$G(z) = 2A \int_{x(z)}^\infty \alpha^2(t)\, dt, \qquad \ldots (12)$$

where $x(z) = 0$ for $z \leq 1$ and $\alpha[x(z)] = \frac{1}{z}$ for $z > 1$.

In the work of the author[39, 40, 45] and of Nagirner[43, 44] the method used for investigating equation (5) is that suggested by V. V. Sobolev[51, 52] and K. Case[53, 54] and which is a further development of the familiar method of Ambartsumyan[31]. We quote those relations which will be necessary in what follows.

Let us introduce the function $P(\tau, z)$ which satisfies the equation

$$P(\tau, z) = \frac{\lambda}{2} \int_0^\infty K(|\tau - \tau'|) P(\tau', z)\, d\tau' + \frac{\lambda}{4\pi} e^{-\frac{\tau}{z}}. \qquad \ldots (13)$$

It has a simple physical meaning, namely, the quantity

$$p(\tau, \eta, x)\, d\omega dx = A\alpha(x) P\left(\tau, \frac{\eta}{\alpha(x)}\right) d\omega dx$$

represents the probability that a quantum absorbed at the depth τ will leave the medium through an angle across η to the external normal within the limits of the solid angle $d\omega$, having a frequency from x to $x + dx$. In addition to equation (13), the function $P(\tau, z)$ also satisfies the equation

$$\frac{\partial P(\tau, z)}{\partial \tau} = -\frac{1}{z} P(\tau, z) + \frac{\lambda}{4\pi} H(z) \Phi(\tau), \qquad \ldots (14)$$

where

$$\Phi(\tau) = 2\pi \int_0^\infty P(\tau, z) G(z) \frac{dz}{z}, \qquad \ldots (15)$$

and $H(z)$ is the solution of the non-linear integral equation

$$H(z)=1+\frac{\lambda}{2}zH(z)\int_0^\infty \frac{H(z')}{z+z'}G(z')dz', \qquad \dots (16)$$

where

$$H(z)=\frac{4\pi}{\lambda}P(0,\ z). \qquad \dots (17)$$

Equation (16) is a generalisation of the familiar equation of Ambartsumyan[31] (see also ref. 32) to the case of scattering with complete frequency redistribution.

It follows from (14) and (17) that

$$P(\tau,\ z)=\frac{\lambda}{4\pi}H(z)\left(e^{-\frac{\tau}{z}}+\int_0^\tau e^{-\frac{\tau-\tau'}{z}}\Phi(\tau')d\tau'\right). \qquad \dots (18)$$

On the basis of the probabilistic intepretation of $P(\tau, z)$, one can also write for the intensity of the emergent radiation, in addition to (10),

$$I(0,\ z)=\frac{4\pi}{\lambda}A\int_0^\infty P(\tau',\ z)g(\tau')\frac{d\tau'}{z}. \qquad \dots (19)$$

In a number of important special cases the integral on the right side of (19) can be expressed directly in terms of $H(z)$ with the aid of (14) and (17). In particular, this situation applies when $g(\tau)$ is represented by the product of a polynomial in τ with an exponential function (see, for instance, ref. 32, chapter 6).

For $g(\tau)$ of a given arbitrary form, the solution of the principal integral equation, that is, the determination of the intensity of the emitted radiation, reduces to the identification of the resolvent of this equation. It can be expressed by a function of one argument $\Phi(\tau)$. This function therefore plays a fundamental part in the scattering theory for a semi-infinite medium. Its explicit expression has the form

$$\Phi(\tau)=\frac{\lambda}{2}\int_0^\infty F_*(x)e^{-\tau x}\frac{dx}{H\left(\frac{1}{x}\right)}, \qquad \dots (20)$$

where

$$F_*(x)=\frac{1}{x}\cdot\frac{G\left(\frac{1}{x}\right)}{\left[1-\lambda\int_0^\infty \frac{G\left(\frac{1}{y}\right)dy}{y^2-x^2}\right]^2+\frac{\lambda^2\pi^2}{4x^2}G^2\left(\frac{1}{x}\right)}. \qquad \dots (21)$$

In the papers quoted above by D.I. Nagirner the asymptotic behaviour was also investigated of $\Phi(\tau)$ and $P(\tau, z)$ for $\tau \gg 1$. The principle results are quoted here.

When the importance of true absorption is large compared with that of the radiation escaping through the boundary, with an accuracy of the leading term for large τ

$$\Phi(\tau)=\frac{\frac{\lambda}{2}K(\tau)}{(1-\lambda)^{3/2}}. \qquad \dots (22)$$

On the other hand where the escape of radiation predominates over absorption,

$$\Phi_L(\tau)=\left(\frac{9}{2}\right)^{1/4}\frac{1}{\Gamma\left(\frac{1}{4}\right)}\tau^{-\frac{3}{4}}, \qquad \text{... (23)}$$

$$\Phi_D(\tau)=2\pi^{-3/4}\tau^{-\frac{1}{2}}(\ln\tau)^{1/4}, \qquad \text{... (24)}$$

where the first relation pertains to a Lorentz absorption coefficient

$$\alpha_L(x)=\frac{1}{1+x^2}, \qquad \text{... (25)}$$

and the second, to a pure Doppler absorption coefficient.

$$\alpha_D(x)=e^{-x^2}. \qquad \text{... (26)}$$

In (23) $\Gamma(s)$ is the Gamma-function.

The relative part played by the escape of radiation and true absorption is determined by the value of the ratio

$$\frac{\frac{\lambda}{2}L(\tau)}{1-\lambda}, \qquad \text{... (27)}$$

where

$$L(\tau)=\int_{\tau}^{\infty}K(\tau')\,d\tau'. \qquad \text{... (28)}$$

Equation (22) is valid when this ratio is small. On the other hand, (23) and (24) are appropriate in the opposite case when the value of (27) is much larger than unity.

For the probability of the escape of a quantum $P(\tau, z)$ from a large depth τ it was found that

$$P(\tau, z)=\frac{\lambda}{4\pi}zH(z)\Phi(\tau) \qquad \text{... (29)}$$

for $\tau/z \gg 1$ and

$$P(\tau, z)=\frac{\lambda}{4\pi}H(z)\left(1+\int_0^{\tau}\Phi(\tau')\,d\tau'\right), \qquad \text{... (30)}$$

when $\tau/z \ll 1$.

Below, it will be shown that in those cases in which the absorption is not large or is practically absent, one may eliminate some of the conditions imposed hitherto to obtain more general relations.

II

In this section we investigate in somewhat greater detail the probability of the escape of a quantum from great depths for cases in which true absorption is essentially absent, i.e. when it can be assumed that $\lambda = 1$.

We shall seek a particular solution of equation (14) for $\tau \gg 1$ in the form

$$P^{(0)}(\tau, z) = \frac{1}{4\pi} z H(z) \Phi(\tau) f\left(\frac{\tau}{z}\right). \qquad \text{... (31)}$$

Substituting (31) in (14) we obtain the equation determining f:

$$z \frac{\Phi'(\tau)}{\Phi(\tau)} f\left(\frac{\tau}{z}\right) + f'\left(\frac{\tau}{z}\right) = -f\left(\frac{\tau}{z}\right) + 1. \qquad \text{... (32)}$$

Let us consider first the most interesting case in practice, the Doppler absorption coefficient. In order to indicate that any given relation is correct only with this coefficient of absorption we shall use the index D. Similarly the index L will be used to indicate the various functions with a Lorentz absorption coefficient. The absence of an index will imply that the given relation is valid for both absorption coefficients.

It follows from (24) that for large τ

$$\frac{\Phi_D'(\tau)}{\Phi_D(\tau)} = -\frac{1}{2\tau}, \qquad \text{... (33)}$$

and (32) therefore takes the form

$$-\frac{1}{2} \frac{z}{\tau} f_D\left(\frac{\tau}{z}\right) + f_D'\left(\frac{\tau}{z}\right) = -f_D\left(\frac{\tau}{z}\right) + 1, \qquad \text{... (34)}$$

or

$$f_D'(t) + \left(1 - \frac{1}{2t}\right) f_D(t) - 1 = 0, \qquad \text{... (35)}$$

where we introduce the notation

$$t = \frac{\tau}{z}. \qquad \text{... (36)}$$

The solution of this equation satisfying the condition $f(\infty) = 1$ has the form

$$f_D(t) = 2\sqrt{t}\, e^{-t} \int_0^{\sqrt{t}} e^{+u^2} du, \qquad \text{... (37)}$$

so that

$$P_D^{(0)}(\tau, z) = \frac{1}{4\pi} z H_D(z) \Phi_D(\tau)\, 2\sqrt{\frac{\tau}{z}}\, e^{-\frac{\tau}{z}} \int_0^{\sqrt{\frac{\tau}{z}}} e^{u^2} du. \qquad \text{... (38)}$$

By exactly the same method it is shown that in the case of a Lorentz coefficient of absorption for $\tau \gg 1$ and $\lambda = 1$ equation (14) has a particular solution of the form

$$P_L^{(0)}(\tau, z) = \frac{1}{4\pi} z H_L(z) \Phi_L(\tau) \left(\frac{\tau}{z}\right)^{3/4} e^{-\frac{\tau}{z}} \int_0^{\tau/z} e^u \frac{du}{u^{3/4}}. \qquad \text{... (39)}$$

The function $P^{(o)}(\tau, z)$ represents a particular solution of equation (14). In order to obtain a general solution of this equation we must add to $P^{(o)}(\tau, z)$ a general solution of the homogeneous equation corresponding to (14). The general solution of this homogeneous equation has the form

$$P^{(1)}(\tau, z) = Q(z)\, e^{-\frac{\tau}{z}}, \qquad \text{... (40)}$$

where Q(z) is an arbitrary function of z, and the general solution of equation (14) itself is therefore

$$P(\tau, z)=P^{(0)}(\tau, z)+P^{(1)}(\tau, z). \qquad \ldots (41)$$

It can be shown that in order to obtain a solution of (14) having the correct physical interpretation, the function Q(z) must be taken equal to

$$Q(z)=\frac{1}{4\pi}\cdot\frac{1}{1-z^2\int\limits_0^\infty \frac{G(z')\,dz'}{z^2-z'^2}}. \qquad \ldots (42)$$

Thus, for $\lambda = 1$ we have the following asymptotic expression for the probability of escape of a quantum from great depths, valid for all $z \geqslant 0$:

$$P(\tau, z)=\frac{1}{4\pi}zH(z)\Phi(\tau)f\left(\frac{\tau}{z}\right)+\frac{1}{4\pi}\cdot\frac{e^{-\frac{\tau}{z}}}{1-z^2\int\limits_0^\infty \frac{G(z')\,dz'}{z^2-z'^2}}. \qquad \ldots (43)$$

Suppose the sources of radiation are concentrated within the limits of an infinitely thin layer situated at a depth τ, i.e. suppose

$$g(\tau')=\delta(\tau-\tau'), \qquad \ldots (44)$$

where δ is a delta-function. It then follows from (19) that the intensity of the escaping radiation as a function of the frequency x and the angular variable η is given by the expression ($\lambda = 1$)

$$I(0, \eta, x)=4\pi AP\left(\tau, \frac{\eta}{\alpha(x)}\right)\frac{\alpha(x)}{\eta}. \qquad \ldots (45)$$

Fig. 1 shows the contour of the emission line so formed, constructed on the basis of (45) and (43) for a Doppler coefficient of absorption and $\tau = 10^3$. It is assumed that the radiation emerges along the normal so that $\eta = 1$. The line has the characteristic double-peaked form. The position of the intensity maxima is determined by the condition $\tau/z \sim 1$. The conclusion that with a distribution of sources at great depths a line must have such a shape, has been drawn earlier[39, 40]. We have now succeeded in determining its contour.

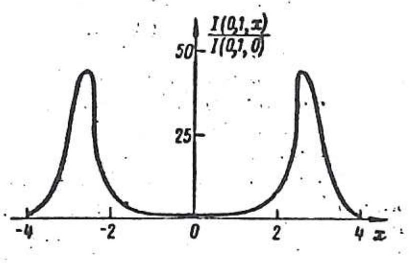

Fig. 1

Now let us investigate a characteristic simplification which arises in the structure of the radiation field when the probability of survival of a quantum, λ, differs only slightly from unity. It is important to note that in astrophysical problems one is usually faced, in particular, with this situation. Thus, in the scattering of L_α - quanta in the chromosphere λ differs from unity by a value of the order of 10^{-5} - 10^{-6}, and values of $1-\lambda$ of the order of 10^{-3} - 10^{-4} are typical for most problems.

Suppose a quantum is absorbed at some depth τ. With a probability λ it will be re-radiated and after undergoing, generally speaking, a number of scatterings it will leave the medium through its boundary. It is also not impossible that it gets "stuck" in the medium, suffering at some stage actual absorption. The function $p(\tau, \eta, x)$ introduced above represents the probability of the escape of the quantum from the depth τ through an angle across η in the frequency x. It is obvious that the total escape probability is equal to the integral of this function over all frequencies and angles. Let us denote this total probability by $P(\tau)$. We have

$$P(\tau)=2\pi\int_{-\infty}^{\infty}dx\int_{0}^{1}p(\tau,\eta,x)\,d\eta= \\ =2\pi A\int_{-\infty}^{\infty}\alpha(x)\,dx\int_{0}^{1}P\left(\tau,\frac{\eta}{\alpha(x)}\right)d\eta. \qquad \ldots (46)$$

The substitution $\eta/\alpha(x) = z$ gives

$$P(\tau)=2\pi A\int_{-\infty}^{\infty}\alpha^2(x)\,dx\int_{0}^{\frac{1}{\alpha(x)}}P(\tau,z)\,dz. \qquad \ldots (47)$$

Changing the order of integration we obtain

$$P(\tau)=2\pi\int_{0}^{\infty}P(\tau,z)\,G(z)\,dz. \qquad \ldots (48)$$

The quantity $P(\tau)$ can be readily expressed in terms of the function $\Phi(\tau)$. In fact, multiplying (19) by $2\pi G(z)$ and integrating over z we obtain

$$\frac{dP(\tau)}{d\tau}=-2\pi\int_{0}^{\infty}P(\tau,z)\,G(z)\frac{dz}{z}+\frac{\lambda}{2}H_0\Phi(\tau), \qquad \ldots (49)$$

where

$$H_0=\int_{0}^{\infty}H(z)\,G(z)\,dz. \qquad \ldots (50)$$

The first term on the right side of (49), according to (15), is in fact the same as $\Phi(\tau)$. Moreover, as can be readily seen (see ref. 45)

$$H_0=\frac{2}{\lambda}(1-\sqrt{1-\lambda}). \qquad \ldots (51)$$

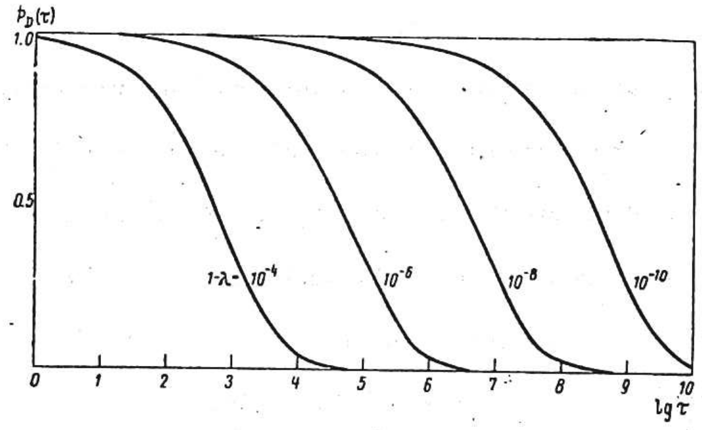

Fig. 2

Therefore, instead of (49) we have

$$\frac{dP(\tau)}{d\tau} = -\sqrt{1-\lambda}\,\Phi(\tau) \qquad \ldots (52)$$

and finally

$$P(\tau) = \sqrt{1-\lambda}\int_{\tau}^{\infty} \Phi(\tau')\,d\tau'. \qquad \ldots (53)$$

Since

$$P(0) = 2\pi\int_{0}^{\infty} P(0, z)\,G(z)\,dz = \frac{\lambda}{2}\int_{0}^{\infty} H(z)\,G(z)\,dz = \frac{\lambda}{2}\,H_0 = 1 - \sqrt{1-\lambda}, \qquad \ldots (54)$$

the function $P(\tau)$ can also be put into the form

$$P(\tau) = 1 - \sqrt{1-\lambda}\left(1 + \int_{0}^{\tau} \Phi(\tau')\,d\tau'\right). \qquad \ldots (55)$$

The curves $P_D(\tau)$ as a function of $\log \tau$ for a number of λ-values are given in Fig. 2. In constructing this figure we have used the results of the numerical solution of the transfer equation obtained by Avrett and Hummer[28]. The diagram shows that for λ close to unity there are τ values at which $P(\tau)$ differs little from unity. As τ increases a region is ultimately reached where $P(\tau)$ begins to decrease rapidly. It is understandable that the reason for this rapid decrease of $P(\tau)$ for large τ depends on the fact that the absorption processes begin to predominate over the escape of radiation. Therefore the substantial decrease of $P(\tau)$ begins when the probability of the escape of a quantum from a given depth without subsequent scatterings, equal to

$$\frac{\lambda A}{2}\int_{-\infty}^{\infty} \alpha(x')\,dx' \int_{0}^{1} e^{-\frac{\alpha(x')}{\eta}\tau}\,d\eta = \frac{\lambda}{2}\int_{0}^{\infty} e^{-\frac{\tau}{z}}\,G(z)\,dz = \frac{\lambda}{2}\,L(\tau), \qquad \ldots (56)$$

becomes comparable with the probability equal to $1-\lambda$, that the quantum is destroyed by true absorption in a single act of scattering.

If λ is close to unity, then for quanta absorbed at such depths that $\lambda/2\,L(\tau) \gg 1-\lambda$ the part played by absorption is small compared to that played by the escape of radiation. In particular, for $\tau \gg 1$ situated in this region it follows from (55) and (33) that

$$P_L(\tau) = 1 - \sqrt{1-\lambda}\,\frac{4\left(\frac{9}{2}\right)^{1/4}}{\Gamma\left(\frac{1}{4}\right)}\,\tau^{1/4}, \qquad \dots (57)$$

and (55) and (24) give

$$P_D(\tau) = 1 - \sqrt{1-\lambda}\,4\pi^{-\frac{3}{4}}\tau^{1/2}(\ln\tau)^{1/4}. \qquad \dots (58)$$

Using the circumstance that for $\tau \gg 1$ (see ref. 39)

$$L_L(\tau) = \frac{2}{3\sqrt{\pi}}\tau^{-\frac{1}{2}}, \qquad \dots (59)$$

$$L_D(\tau) = \frac{1}{2\sqrt{\pi}\,\tau\sqrt{\ln\tau}}, \qquad \dots (60)$$

(57) and (58) can also be rewritten thus

$$P(\tau) = 1 - C\sqrt{\frac{1-\lambda}{L(\tau)}}, \qquad \dots (61)$$

where the value of the constant C depends on the shape of the contour:

$$C_L = \frac{2^{9/4}}{\pi^{1/4}\Gamma\left(\frac{1}{4}\right)} = 0.984, \qquad \dots (62)$$

$$C_D = \frac{2^{3/2}}{\pi} = 0.900. \qquad \dots (63)$$

For τ such that $\lambda/2\,L(\tau) \ll 1-\lambda$, it is possible to use the asymptotics for $\Phi(\tau)$ relating to the case $\lambda < 1$. We then find from (53), taking into account (22), that

$$P(\tau) = \frac{\frac{\lambda}{2}L(\tau)}{1-\lambda}. \qquad \dots (64)$$

Thus, whatever the coefficient of absorption, the probability of escape of a quantum from a sufficiently large depth is proportional to the probability of its escape from this depth without scattering. The proportionality coefficient $(1-\lambda)^{-1}$ is equal to the mean number of scatterings of the quantum in an infinite medium.

This result admits of a simple physical interpretation. Rewriting (64) in the form

$$P(\tau) = \frac{\lambda}{2}L(\tau) + \lambda\frac{\lambda}{2}L(\tau) + \lambda^2\frac{\lambda}{2}L(\tau) + \dots, \qquad \dots (65)$$

we arrive at the conclusion that the total probability of escape of a quantum from a given depth τ is made up of the probability of direct escape from this depth $\lambda/2\,L(\tau)$, plus the probability of an escape after one $(\lambda\frac{\lambda}{2}L(\tau))$, two $(\lambda^2\frac{\lambda}{2}L(\tau))$ and so forth scatterings from this depth. This feature of scattering with complete frequency redistribution is the basis of one of the methods used for the approximate solution of the transfer equation for this type of scattering(29, 30).

Expressions (61) and (64) lead one to consider $P(\tau)$, not as a function of τ, but as a function of the argument

$$u = \frac{\frac{\lambda}{2} L(\tau)}{1-\lambda}. \qquad \ldots (66)$$

Let us denote the total probability of escape of a quantum from a given depth τ, for the given λ-value, considered as a function of the variable u and the parameter λ, by $F(u, \lambda)$. Thus

$$P(\tau, \lambda) = F(u, \lambda), \qquad \ldots (67)$$

where u and τ are related by relation (66). Schematic curves of $F(u, \lambda)$ for a number of λ values are shown in Fig. 3. The introduction of the function $F(u, \lambda)$, as we now see, makes it possible to bring out one important feature of the scattering with complete frequency redistribution.

Let us list the properties of $F(u, \lambda)$ following directly from its definition. It follows from (66) that for $\tau \to \infty$ the value of u tends toward zero. Therefore

$$F(0, \lambda) = 0. \qquad \ldots (68)$$

Formula (64) permits one to affirm that

$$F'(0, \lambda) = 1. \qquad \ldots (69)$$

Since $L(0) = 1$, the largest value of u for a given λ is, according to (66), equal to

$$u_{max} = \frac{\lambda}{2(1-\lambda)}. \qquad \ldots (70)$$

Yet

$$P(0, \lambda) = \frac{\lambda}{2} H_0 = 1 - \sqrt{1-\lambda}. \qquad \ldots (71)$$

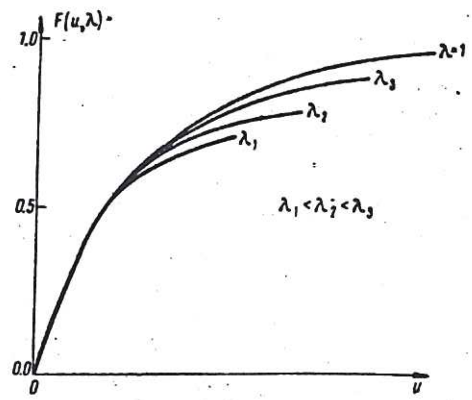

Fig. 3

Therefore assuming $\tau = 0$ in (67) we obtain

$$F\left(\frac{\lambda}{2(1-\lambda)},\ \lambda\right)=1-\sqrt{1-\lambda}. \qquad \text{... (72)}$$

Thus, as u increases from zero to u_{max} (i.e. as τ decreases from ∞ to 0), $F(u,\lambda)$ increases monotonically from zero to $1 - \sqrt{1-\lambda}$. The greatest interest is presented by the fact that $F(u,\lambda)$ for $\lambda \rightarrow 1$ tends toward some limiting function which we denote by $F(u)$:

$$F(u)=\lim_{\lambda\to 1} F(u,\lambda). \qquad \text{... (73)}$$

It follows from (68), (69) and (72) that

$$F(0)=0;\ F'(0)=1;\ F(\infty)=1,$$

and (61) shows that for $u \gg 1$

$$F(u)=1-\frac{C}{\sqrt{2u}}+\dots \qquad \text{... (74)}$$

The existence of the limit (73) indicates that for nearly-conservative scattering with complete frequency redistribution the following similarity principle applies: the total probability of escape of a quantum from the depth $\tau = \tau_1$ from a medium with $\lambda = \lambda_1$ is equal to the total probability of escape of a quantum from a medium with $\lambda = \lambda_2$ from a depth $\tau = \tau_2$, where τ_1 and τ_2 are related by

$$\frac{L(\tau_1)}{1-\lambda_1}=\frac{L(\tau_2)}{1-\lambda_2}. \qquad \text{... (75)}$$

This principle is exceedingly useful for the numerical solution of the principal integral equation. It makes it possible to use the solution obtained for one value of λ as a very useful approximation to the solution for a second λ.

The determination in explicit form of the function $F(u)$, at least for the two simplest absorption coefficients, the Lorentz and the Doppler coefficients, constitutes an exceedingly important problem which nevertheless remains unsolved.

To obtain the graph $F_D(u)$ we used the numerical results of Avrett and Hummer[28]. These results constitute simultaneously an illustration of the usefulness of the similarity principle.

Avrett and Hummer, using numerical methods, obtained the solution of the principle integral equation with uniform distribution of the sources and a λ differing only slightly from unity. They considered the equation

$$S(\tau)=\frac{\lambda}{2}\int_0^\infty K(|\tau-\tau'|)S(\tau')\,d\tau'+1-\lambda. \qquad \text{... (76)}$$

It is easily shown that the solution of this equation is

$$S(\tau)=\sqrt{1-\lambda}\left(1+\int_0^{\tau}\Phi(\tau')\,d\tau'\right). \qquad \ldots (77)$$

In fact differentiation of (76) gives

$$S'(\tau)=\frac{\lambda}{2}\int_0^{\infty}K(|\tau-\tau'|)\,S'(\tau')\,d\tau'+S(0)\,\frac{\lambda}{2}K(\tau). \qquad \ldots (78)$$

But

$$\Phi(\tau)=\frac{\lambda}{2}\int_0^{\infty}K(|\tau-\tau'|)\,\Phi(\tau')d\tau'+\frac{\lambda}{2}K(\tau). \qquad \ldots (79)$$

This follows for instance from (13) and (15). Comparison of (78) and (79) gives therefore

$$S'(\tau)=S(0)\Phi(\tau), \qquad \ldots (80)$$

and consequently

$$S(\tau)=S(0)\left(1+\int_0^{\tau}\Phi(\tau')\,d\tau'\right). \qquad \ldots (81)$$

Considering the relation

$$\int_0^{\infty}\Phi(\tau')\,d\tau'=\frac{1}{\sqrt{1-\lambda}}-1, \qquad \ldots (82)$$

which follows from (55) for $\tau = \infty$, we arrive at (77).

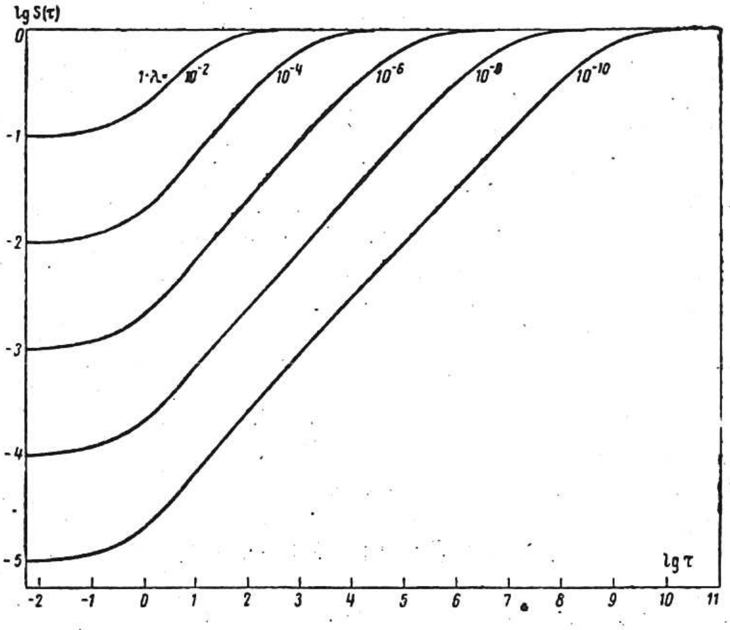

Fig. 4

We note that in this author's paper[45] it is asserted that the function (81) is a solution of the homogeneous equation corresponding to (76), i.e. a solution of Milne's problem. This assertion is erroneous. In fact, for $\lambda < 1$ the homogeneous equation has only the trivial solution, and the solution of Milne's problem determined in ref. 45 is correct only in the conservative case $\lambda = 1$.

Avrett and Hummer calculated $S(\tau)$ for the Doppler absorption coefficient for values of $1 - \lambda$ equal to 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8} and 10^{-10}. Their results are given in Fig. 4.

From (77) and (55) it follows that

$$S(\tau) = 1 - P(\tau), \qquad \text{... (83)}$$

and, since in Avrett and Hummer's calculations $1 - \lambda \ll 1$, one can put approximately

$$S(\tau) = 1 - F(u), \qquad \text{... (84)}$$

where u is given by (66).

In a paper by the author and Shcherbakov[47], a table is given of the values of $L_D(\tau)$ for $\tau \lesssim 100$, and the following asymptotic expansion, valid for $\tau \gg 1$, is obtained

$$L_D(\tau) = \frac{1}{\sqrt{\pi}\,\tau\sqrt{\ln\tau}}\left(0.500 - \frac{0.269}{\ln\tau} + \frac{0.573}{\ln^2\tau} - \frac{1.566}{\ln^3\tau} + \ldots\right). \qquad \text{... (85)}$$

Using this data it is possible from (66), for given λ and any τ, to calculate the corresponding u value with an accuracy sufficient for our purposes.

Let us consider, for example, $\lambda = 1 - 10^{-10}$. For $\tau < 10^8$, we have $u > 5$ and $F(u)$ may be found from equation (74). It then follows from (84) that

$$S(\tau) = \frac{C}{\sqrt{2u}} = \frac{C\sqrt{1-\lambda}}{\sqrt{L(\tau)}}. \qquad \text{... (86)}$$

Substituting the values of the constants we find that $S(\tau) = \dfrac{0.900 \cdot 10^{-5}}{\sqrt{L_D(\tau)}}$. The values of $S(\tau)$ calculated from this simple expression agree splendidly with the results of the laborious calculations by Avrett and Hummer.

Equation (86) should give the largest error when τ is small. For $\tau = 0$ equation (77) gives $S(0) = \sqrt{1 - \lambda}$, whereas from (86) we have $S(0) = C\sqrt{1 - \lambda}$. Thus with a Doppler absorption coefficient the error in the value of $S(\tau)$ which we commit by using expression (86) is, even in this unfavourable case, not more than 10%. It is well to recall here that the transfer equation which we solve is an approximate one and therefore higher accuracy is hardly necessary (on this subject see ref. 48). We note that for $\tau > 1$, the accuracy of (86) increases rapidly as τ, increases, so that over a major part of the interval $\log \tau \leqslant 8$, the curve of $\log S(\tau)$ representing (86) for $\lambda = 1 - 10^{-10}$ is indistinguishable, on the scale of Fig. 4, from the curve calculated by Avrett and Hummer.

For $\log \tau > 8$ the value of u becomes small and equation (86) can no longer be used. However for $u \ll 1$ we have $F(u) = u$. This follows from the circumstance that $F'(0) = 1$ and $F(0) = 0$. Making use of this fact it is easy to sketch the curve $\log S(\tau) \gtrsim 10$. Since the curve is continuous, on the interval $8 < \log \tau < 10$ it can be plotted (without great accuracy it is true) by simple graphical interpolation. However one can also proceed in the opposite way by picking off from the calculated curve the values of $S(\tau)$ and using them to determine $F(u)$. By this method the curve $F_D(u)$ shown in Fig. 5 was obtained.

Hitherto we have considered only the curve corresponding to $1 - \lambda = 10^{-10}$. However, having the graph $F_D(u)$ obtained with its help, and using for large values of u equation (74), curves can be easily constructed corresponding to values of $1 - \lambda$ equal to 10^{-8}, 10^{-6}, 10^{-4} and 10^{-2} without resorting to a numerical solution of the principal integral equation. It is sufficient to use the similarity principle. The curves obtained by this method coincide with those shown in Fig. 4.

We note that in the paper by Avrett and Hummer the principal integral equation is solved, not only for the half space, but also for plane layers of large thickness, and taking into account not only the Doppler, but also other, contours of the absorption coefficient, such as the Lorentz and the Voigt contours. It is true that most of the results given in that paper cannot be obtained by methods other than by applying numerical techniques. It may be hoped that careful analysis of the results of that paper will make it possible to detect the existence of some simplification in the structure of the radiation field with nearly-conservative scattering, and also in more complex cases than the one investigated above.

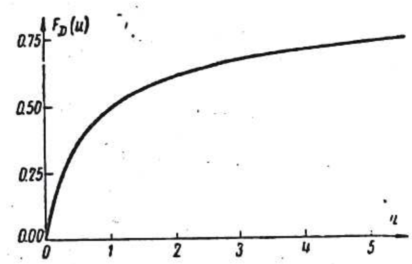

Fig. 5.

IV.

Let us now use the results of the preceding section to show that in the case of pure scattering the following asymptotic expression, valid for $\tau \gg 1$, independently of the contours of the absorption coefficient, holds good

$$\Phi(\tau)=\frac{C}{2}K(\tau)\left[L(\tau)\right]^{-\frac{3}{2}}, \qquad \cdots (87)$$

where C is some constant close to unity, whose value is different for different $\alpha(x)$. In the case of Lorentz and Doppler coefficients the values of C are given by equation (62) and (63).

Suppose $\Phi(\tau,\lambda)$ is the Φ function for some λ value, and $\Phi(\tau,1)$ the same function for $\lambda = 1$. We represent $\Phi(\tau,\lambda)$ in the form

$$\Phi(\tau,\lambda)=\frac{\lambda}{2}\cdot\frac{K(\tau)}{(1-\lambda)^{3/2}}F'\left(\frac{\frac{\lambda}{2}L(\tau)}{1-\lambda}\right)-F_1(\tau,\lambda). \qquad \cdots (88)$$

where $F(u)$ is the limiting function introduced in the preceding section. Such a representation of $\Phi(\tau,\lambda)$ is, it is true, always possible since the form of the function $F_1(\tau,\lambda)$ is in no way fixed. Yet the reason why $\Phi(\tau,\lambda)$ is conveniently represented in this form must be clear from the above discussion.

In (88), λ tends toward unity. It follows from (74) that for $\lambda \to 1$

$$F'\left(\frac{\frac{\lambda}{2}L(\tau)}{1-\lambda}\right)=C\left[\frac{1-\lambda}{\lambda L(\tau)}\right]^{3/2}+\ldots \qquad \cdots (89)$$

Substituting this expression in (88) we obtain, in the limit

$$\Phi(\tau,1)=\frac{C}{2}K(\tau)\left[L(\tau)\right]^{-\frac{3}{2}}-F_1(\tau,1). \qquad \cdots (90)$$

With the aid of relation (82) it can be easily shown that for $\tau \gg 1$ the second term on the right side can be neglected in comparison with the first. In fact, we obtain from (88)

$$\begin{aligned}\int_0^\infty \Phi(\tau,\lambda)\,d\tau &= \frac{1}{(1-\lambda)^{3/2}}\int_0^\infty \frac{\lambda}{2}K(\tau)F'\left(\frac{\frac{\lambda}{2}L(\tau)}{1-\lambda}\right)d\tau-\int_0^\infty F_1(\tau,\lambda)\,d\tau=\\ &=\frac{1}{\sqrt{1-\lambda}}F\left(\frac{\frac{\lambda}{2}L(0)}{1-\lambda}\right)-\frac{F(0)}{\sqrt{1-\lambda}}-\int_0^\infty F_1(\tau,\lambda)\,d\tau=\\ &=\frac{1}{\sqrt{1-\lambda}}F\left(\frac{\lambda}{2(1-\lambda)}\right)-\int_0^\infty F_1(\tau,\lambda)\,d\tau.\end{aligned} \qquad \cdots (91)$$

The relation (82) therefore takes the form

$$\int_0^\infty F_1(\tau,\lambda)\,d\tau=1-\frac{1}{\sqrt{1-\lambda}}+\frac{1}{\sqrt{1-\lambda}}F\left(\frac{\lambda}{2(1-\lambda)}\right). \qquad \cdots (92)$$

Passing here to the limit $\lambda \to 1$ and using (74) we finally obtain that

$$\int_0^\infty F_1(\tau, 1)\, d\tau = 1 - C. \quad \ldots (93)$$

Thus, for $\lambda \to \infty$ the function $F_1(\tau, 1)$ decreases sufficiently rapidly to ensure convergence of this integral. Yet the first term on the right hand side of (90) decreases more slowly than $F_1(\tau, 1)$ since the integral

$$\frac{1}{2}\int_0^\tau K(\tau')\,[L(\tau')]^{-\frac{3}{2}}\, d\tau' = \frac{1}{\sqrt{L(\tau)}} - 1 \quad \ldots (94)$$

diverges for $\tau \to \infty$, so that $L(\tau)$ tends toward zero as $\tau \to \infty$. Therefore for $\tau >> 1$, the second term in (90) can in fact be neglected in comparison with the first and we arrive at expression (87).

We note that the functions $K_D(\tau)$ and $L_D(\tau)$ can be assumed to be known, since they were investigated and tabulated previously[47].

In conclusion, it may be shown that the function $\Phi(\tau)$ in the case of pure scattering can be put into the form

$$\Phi(\tau) = \frac{1}{2} K(\tau)\,[L(\tau)]^{-\frac{3}{2}}\, q(\tau), \quad \ldots (95)$$

where

$$q(0) = 1,$$
$$q(\infty) = C.$$

It is important that the value of C differs little from unity. It may be inferred that C_L is not by accident closer to unity than C_D and that, generally speaking, C will differ less from unity the more slowly $\alpha(x)$ tends toward zero as $|x| \to \infty$. The theory of the monotonic nature of $q(\tau)$ would seem exceedingly attractive, although any proof is as yet lacking.

The author is indebted to D. Hummer for communicating the results of the numerical solution of the transfer equation privately before they were published.

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