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# STABILITY OF A PLASMA WITH FINITE β

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# STABILITY OF A PLASMA WITH FINITE $\beta$.

by

J. ANDREOLETTI

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ABSTRACT

A necessary stability condition is obtained for an arbitrary system without magnetic surface, and a sufficient condition for a system with isotropic pressure and without parallel current. These two conditions tend toward the necessary and sufficient condition for very small and very large $\beta$ respectively. From this, the critical $\beta$ is evaluated in a magnetic well, showing the importance of the transverse depth.

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According to established usage, $\beta$ is the ratio of the plasma energy to the magnetic energy; in general, $\beta$ will be defined as $\beta \equiv \mathrm{Max}\ \beta(\bar{r})$, where $\beta(\bar{r})$ is the ratio of the energy densities at the point $\bar{r}$ .

Necessary Stability Condition for an Arbitrary System without Magnetic Surfaces.

In a stability theory to the first order in $\beta$ only strict interchange perturbations $\left[\delta^1_E\bar{B} = \nabla\wedge(\bar{\xi}\wedge\bar{B}) = 0\right]$ are of importance. In the general case the consideration of these special perturbations, which are permitted in a topology without magnetic surfaces, will lead to a necessary stability condition. The expression for the Kruskal-Oberman functional(1) then is: $\left[\text{we use } \langle\chi\rangle \equiv \int (d\ell/u_{\parallel})\ \chi \Big/ \int (d\ell/u_{\parallel})\right]$

$$\delta^2 W\left[\bar{\xi}_\perp, \bar{\xi}_\perp\right] = \frac{1}{2}\int d\tau \Big\{ -(\bar{\xi}_\perp\cdot\nabla p_\perp)\frac{\delta B}{B} + (\bar{\xi}_\perp\cdot\nabla p_\parallel)\frac{\delta d\ell}{d\ell}$$

$$+ 2p_\perp\left(\frac{\delta B}{B}\right)^2 + (p_\perp - p_\parallel)\frac{\delta B}{B}\ \frac{\delta d\ell}{d\ell}$$

$$-\int \Sigma\, m\,\frac{B}{u_\parallel}\, d\varepsilon\, d\nu\, \Theta\, F_\varepsilon\left[\left\langle -u_\parallel^2\frac{\delta d\ell}{d\ell} + \nu B\frac{\delta B}{B}\right\rangle^2 - \nu^2 B^2\left\langle\frac{\delta B}{B}\right\rangle^2\right]\Big\}$$

with

$$\frac{\delta B}{B} = \bar{\xi}_\perp\cdot\frac{\nabla B}{B} \quad\text{and}\quad \frac{\delta d\ell}{d\ell} = -\bar{\xi}_\perp\cdot\frac{\bar{R}}{R^2} = -\bar{\xi}_\perp\cdot\frac{\nabla\left(\frac{B^2}{2\mu_o} + p_\perp\right)}{\frac{B^2}{\mu_o} + p_\perp - p_\parallel}$$

In magnetic coordinates $(\varphi^1, \varphi^2)$ such that $\bar{B} = \nabla\varphi^1\wedge\nabla\varphi^2$, one has:

$$\delta^2 W = \frac{1}{2}\int d\Phi\, \alpha_{ij}\ (\Phi)\ \delta\varphi^i\,\delta\varphi^j$$

with

$$\alpha_{ij} = \int\frac{d\ell}{B}\Big\{ -\partial_i p_\perp\frac{\partial_j B}{B} - \partial_i p_\parallel\frac{\partial_j\left(\frac{B^2}{2\mu_o} + p_\perp\right)}{\frac{B^2}{\mu_o^2} + p_\perp - p_\parallel} + 2p_\perp\frac{\partial_i B}{B}\ \frac{\partial_j B}{B}$$

$$+ (p_\parallel - p_\perp)\frac{\partial_i B}{B}\ \frac{\partial_j\left(\frac{B^2}{2\mu_o} + p_\perp\right)}{\frac{B^2}{\mu_o} + p_\perp - p_\parallel}$$

$$+\int \Sigma m\, d\varepsilon\, d\nu\,\frac{B}{u_\parallel}\,\Theta\, F_\varepsilon\Big[\nu^2 B^2\frac{\partial_i B}{B}\frac{\partial_j B}{B} - \left\langle \nu B\frac{\partial_i B}{B} + u_\parallel^2\frac{\partial_i\left(\frac{B^2}{2\mu_o} + p_\perp\right)}{\frac{B^2}{\mu_o^2} + p_\perp - p_\parallel}\right\rangle$$

$$\times\left\langle \nu B\frac{\partial_j B}{B} + u_\parallel^2\frac{\partial_j\left(\frac{B^2}{2\mu_o} + p_\perp\right)}{\frac{B^2}{\mu_o^2} + p_\perp - p_\parallel}\right\rangle\Big]\Big\}$$

The necessary stability condition is that, for each magnetic line of force $\bar{\Phi}$, the tensor $\alpha_{ij}(\bar{\Phi})$ should be positive definite, namely:

$$\alpha_{11} \text{ to } \alpha_{22} > 0 \quad \text{and} \quad \alpha_{11}\,\alpha_{22} - \alpha_{12}^2 \geqslant 0.$$

If we are not interested in having $p_\perp$ and $p_\parallel$ in the above expression, the expression for the necessary condition can be transformed (integration by parts in $\varepsilon$ : $F \to F_\varepsilon$) ; one thus obtains a very simple expression which depends on the distribution function itself:

$$\alpha_{ij}\,\delta\varphi^i\,\delta\varphi^j = \boxed{\int \Sigma m\, J_\varepsilon\, d\varepsilon\, d\nu\, (-\,F_{\varepsilon/\nu,J}) \left(\frac{\delta_\Phi J}{J_\varepsilon}\right)^2 \geqslant 0}$$

Further, the expressions for the gradients of the particle number density $N(\bar{\Phi})$ on a flux tube, and of the invariant $J(\bar{\Phi}, \varepsilon, \nu)$ are:

$$\partial_\Phi N = \int \Sigma J_\varepsilon\, d\varepsilon\, d\nu\, (-\,F_{\varepsilon/\nu,J})\, \frac{\partial_\Phi J}{J_\varepsilon}$$

$$\partial_\Phi J = \int \frac{d\ell}{u_\parallel} \left\{ -\left( \nu B + \frac{u_\parallel^2}{1 + \dfrac{p_\perp - p_\parallel}{\dfrac{B^2}{\mu_0}}} \right) \frac{\partial_\Phi \left( \dfrac{B^2}{2\mu_0} + p_\perp \right)}{\dfrac{B^2}{\mu_0}} + \nu B\, \frac{\partial_\Phi p_\perp}{\dfrac{B^2}{\mu_0}} \right\}$$

This result, obtained for an arbitrary $\beta$, is formally identical to that obtained in the small $\beta$ theory[(2)]. Here, however, the relationship between $(\partial_\Phi d\ell/d\ell)$ and $(\partial_\Phi B/B)$ (which comes into $\partial_\Phi J$), that is, the relationship between the curvature of the magnetic field lines and the gradient of $B$ is influenced by the presence of the plasma.

In the limit of small $\beta$ it is shown[(3)] that the strict interchange perturbations $(\delta_E^1\bar{B} = 0)$ lead to a necessary and sufficient stability condition. This theory is valid for $\beta \ll 1$. As $\beta$ increases, the perturbations which minimize the functional $\delta^2 W[\bar{\xi}, \bar{\xi}]$ may depart progressively from strict interchanges. Thus, the expression obtained by considering only these displacements yields a necessary stability condition which departs progressively from the necessary and sufficient condition as $\beta$ increases.

When the magnetic configuration is such that $J(\varepsilon, \nu, \bar{\Phi})$ decreases towards the exterior of the system, distributions with $F_{\varepsilon/\nu,J} < 0$ make it possible simultaneously to satisfy the necessary stability condition and the confinement condition [$N(\bar{\Phi})$ decreasing outwards]. One observes further that as $[(B^2/2\mu_0) + p_\perp]$ increases toward the exterior of the system, $J(\varepsilon, \nu, \bar{\Phi})$ decreases in this direction, whatever the ratio $\nu/\varepsilon$ (by considering only the case where $p_\parallel - p_\perp < B^2/\mu_0$ i.e. excluding systems which are unstable to Alfvén waves).

Sufficient Stability Condition for a System with
Isotropic Pressure and without Parallel Current

By decomposing the displacement vector $\bar{\xi}$ into a component $\bar{\xi}_n$ normal to the plane $(\bar{J},\bar{B})$ and a component $\bar{\xi}_t$ lying in this plane, and after several integrations by parts, the Kruskal-Oberman functional can be written in the following form: (putting $\bar{u} \equiv \nabla\cdot\bar{\bar{p}}/|\nabla\cdot\bar{\bar{p}}|$)

$$\delta^2 W = \frac{1}{2}\int d\tau \left\{ \frac{1}{\mu_o}|\bar{Q} - \bar{\xi}_n \wedge \mu_o \bar{J}|^2 - 2|\bar{\xi}_n|^2 \frac{\bar{J}^2 \cdot \bar{B}^2}{|\nabla\cdot\bar{\bar{p}}|^2}\frac{\bar{R}}{R^2}\cdot(\nabla\cdot\bar{\bar{p}}) \right.$$

$$+ 2|\bar{\xi}_n|^2 \frac{\bar{J}\cdot\bar{B}}{|\nabla\cdot\bar{\bar{p}}|^2}((\nabla\cdot\bar{\bar{p}})\cdot\nabla\bar{B})\cdot\bar{J} + |\bar{\xi}_n|^2(\bar{J}\cdot\bar{B})\,\bar{u}\cdot\nabla\wedge\bar{u}$$

$$+ \nabla\cdot(p - \bar{n}\bar{n})\cdot[\nabla\wedge(\bar{\xi}_n\wedge\bar{\xi}) + \bar{\xi}\cdot\nabla\bar{\xi} - \bar{\xi}\nabla\cdot\bar{\xi}]$$

$$- p - [(\bar{n}\cdot\nabla\bar{\xi})^2 - (\bar{n}\bar{n} : \nabla\bar{\xi})^2] + 2p + (\bar{n}\bar{n} : \nabla\bar{\xi} - \nabla\cdot\bar{\xi})^2$$

$$+ \int \Sigma m \, d\varepsilon \, d\nu \frac{B}{u_{\parallel}} \Theta F_{\varepsilon} \nu^2 B^2 (\bar{n}\bar{n} : \nabla\bar{\xi} - \nabla\cdot\bar{\xi})^2$$

$$\left. - \int \Sigma m \, d\varepsilon \, d\nu \frac{B}{u_{\parallel}} \Theta F_{\varepsilon} \langle - u_{\parallel}^2(\bar{n}\bar{n} : \nabla\bar{\xi}) + \nu B(\bar{n}\bar{n} : \nabla\bar{\xi} - \nabla\cdot\bar{\xi})\rangle^2 \right\}$$

For a system without parallel current $(J_{\parallel} - 0)$ and with an isotropic distribution $(F_{\nu} = 0)$ one has:

$$\delta^2 W = \frac{1}{2}\int d\tau \left\{ \frac{1}{\mu_o}|\bar{Q} - \bar{\xi}_n\wedge\mu_o\bar{J}|^2 - 2\mu_o|\bar{\xi}_n|^2 \frac{\bar{J}^2}{|\nabla p|^2}\nabla p\cdot\nabla\left(p + \frac{B^2}{2\mu_o}\right) \right.$$

$$\left. - \int \Sigma m \, d\varepsilon \, d\nu \frac{B}{u_{\parallel}} \Theta F_{\varepsilon} \langle - u_{\parallel}^2(\bar{n}\bar{n} : \nabla\bar{\xi}) + \nu B(\bar{n}\bar{n} : \nabla\bar{\xi} - \nabla\cdot\bar{\xi})\rangle^2 \right\}$$

Moreover, if the distribution is a decreasing function of $\varepsilon (F_{\varepsilon} \leqslant 0)$ a sufficient stability condition is that at each point:

$$\boxed{\nabla p \cdot \nabla\left(p + \frac{B^2}{2\mu_o}\right) \leqslant 0}$$

This result is analogous to that obtained by Hain, Lüst and Schlüter[(4)]. The functional used is different and, in particular, we have not assumed that the most unstable perturbations are incompressible.

Evaluation of the Critical $\beta$ in Magnetic Wells.

The necessary stability condition yielded by considering interchanges is valid for any system, and tends toward the necessary and sufficient condition as $\beta$ becomes small. It can be satisfied for a large group of distributions when $J(\varepsilon, \nu, \Phi)$ decreases outwards.

The sufficient stability condition based on the direction of curvature of the field lines is valid for a system with an initially isotropic pressure, and tends toward the necessary and sufficient condition as $\beta$ becomes very large. Indeed, whilst it is known, on the one hand, that for $\beta = \infty$ (complete separation between gas and magnetic field by an interface) the necessary and sufficient condition reduces to the sign of the curvature[5]; on the other hand, when tending toward this limiting case, the restrictions: $J_{\parallel}$ zero and $\bar{\bar{p}}$ isotropic, tend to be always satisfied.

The above conditions show that the sign of the curvature plays a dominant part in the necessary and sufficient condition over the entire range of $\beta$. By considering only systems with moderate anisotropy, the critical $\beta$ can be evaluated for a magnetic well, by requiring that the radius of curvature of the magnetic field lines deformed by the presence of the plasma, be directed towards the outside, that is, that the "total perpendicular pressure" $[(B^2/2\mu_0) + p_{\perp}]$ should have no maximum within the plasma. One thus arrives at the following estimate:

$$\beta_c \sim \frac{\delta_{\perp}\left(\frac{B^2}{2\mu_0}\right)}{\frac{B^2}{2\mu_0}}$$

$\delta_{\perp}$ indicates a transverse variation between the edge and the centre of the plasma. A lower bound for $\beta_c$ will be obtained by evaluating it for the initial vacuum field.

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