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STABILITY OF MAGNETIC WELLS FOR A PLASMA AT FINITE PRESSURE

by

J. ANDRELOETTI

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ABSTRACT

The influence of the curvature of the magnetic field lines and consequently of the transverse depth of a magnetic well on stability has been demonstrated in the limit of weak anisotropy(1). In this paper we demonstrate in a similar form the influence of the curvature and of the anisotropy which are related to the transverse depth of the well and to its longitudinal depth respectively.

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The stability of the magnetohydrodynamic motion of a plasma (zero order in $\frac{m}{q}$) depends on the sign of the Kruskal Oberman functional. Since it is not possible to solve the corresponding Euler-Lagrange problem we use a method of grouping the terms of the functional; we seek by simple algebraic rearrangement or by integrations by parts to form terms whose bi-linear part in $\bar{\xi}$ is a square. If the entire functional can be reduced to terms of this type, the coefficients of these terms yield sufficient stability conditions.

An interesting grouping is the following (the notations are the same as those used in[(1)]); moreover,

$$c \equiv \int \sum m \, d\varepsilon \, d\nu \, \frac{B}{u_{\parallel}} \, \Theta \, F_{\varepsilon} \nu^2 B^2 ;$$

$$\begin{aligned}
\delta^2 W = \tfrac{1}{2} \int d\tau \Bigg\{ & \left(\frac{B^2}{\mu_o} - p_- \right) \left(\frac{\bar{Q}_{\perp}}{B} \right)^2 + \left(\frac{B^2}{\mu_o} + 2p_+ + c \right) \left(2\, \bar{\xi} \cdot \frac{\bar{R}}{R^2} + \nabla \cdot \bar{\xi} \right)^2 \\
& + 2 \left[- \frac{\bar{J}^2 \bar{B}^2}{|\nabla \cdot \bar{\bar{p}}|^2} \frac{\bar{R}}{R^2} \cdot (\nabla \cdot \bar{\bar{p}}) + \bar{J} \cdot \bar{B} \left(\frac{(\nabla \cdot \bar{\bar{p}}) \cdot (\nabla \bar{B}) \cdot \bar{J}}{|\nabla \cdot \bar{\bar{p}}|^2} + \tfrac{1}{2} \bar{u} \cdot \nabla \wedge \bar{u} \right) \right] |\bar{\xi}_n|^2 \\
& + \frac{1}{\mu_o} \left[(\bar{\xi}_n \wedge \mu_o \bar{J}_{\parallel})^2 - 2\bar{Q}_{\perp} \cdot \bar{\xi}_n \wedge \mu_o \bar{J}_{\parallel} \right] - \frac{p_-}{\frac{B^2}{\mu_o}} \bar{J}_{\parallel} \cdot \bar{\xi} \wedge \bar{Q}_{\perp} \\
& + \nabla \cdot (p_- \, \bar{n}\bar{n}) \cdot \nabla \wedge (\bar{\xi}_n \wedge \bar{\xi}) - \left(\bar{\xi} \cdot \nabla p_- + 2p_- \nabla \cdot \bar{\xi} + p_- \bar{\xi} \cdot \frac{\bar{R}}{R^2} \right) \left(\bar{\xi} \cdot \frac{\bar{R}}{R^2} \right) \\
& - (2p_+ + c) \left(\bar{\xi} \cdot \frac{\bar{R}}{R^2} \right) \left(3\bar{\xi} \cdot \frac{\bar{R}}{R^2} + 2\nabla \cdot \bar{\xi} \right) \\
& - \int \sum m \, d\varepsilon \, d\nu \, \frac{B}{u_{\parallel}} \, \Theta \, F_{\varepsilon} \left\langle u_{\parallel}^2 \left(\bar{\xi} \cdot \frac{\bar{R}}{R^2} \right) - \nu B \left(\bar{\xi} \cdot \frac{\bar{R}}{R^2} + \nabla \cdot \bar{\xi} \right) \right\rangle^2 \Bigg\} .
\end{aligned}$$

For states with isotropic distributions the following will be the sufficient stability condition

$$- \frac{\bar{R}}{R^2} \cdot \nabla p + \frac{(\bar{J} \cdot \bar{B}) \bar{J} \cdot (\nabla p \cdot \nabla \bar{B})}{\bar{J}^2 \bar{B}^2} > 0 .$$

For the states without parallel current the functional is reduced to the following expression ($-\int F_{\varepsilon}$ is a condensed notation for the integral term in ε and ν)

$$\begin{aligned}
\delta^2 W = \tfrac{1}{2} \int d\tau \Bigg\{ & \left(\frac{B^2}{\mu_o} - p_- \right) \left(\frac{\bar{Q}_{\perp}}{B} \right)^2 + \left(\frac{B^2}{\mu_o} + 2p_+ + c \right) \left(2\bar{\xi} \cdot \frac{\bar{R}}{R^2} + \nabla \cdot \bar{\xi} \right)^2 \\
& - 2 \frac{\bar{R}}{R^2} \cdot (\nabla \cdot \bar{\bar{p}}) \, |\bar{\xi}_n|^2 + p_- \frac{\bar{R}}{R^2} \cdot \nabla \wedge (\bar{\xi}_n \wedge \bar{\xi}) \\
& - \left(\bar{\xi} \cdot \nabla p_- + 2p_- \nabla \cdot \bar{\xi} + p_- \bar{\xi} \cdot \frac{\bar{R}}{R^2} \right) \left(\bar{\xi} \cdot \frac{\bar{R}}{R^2} \right) \\
& - (2p_+ + c) \left(\bar{\xi} \cdot \frac{\bar{R}}{R^2} \right) \left(3\bar{\xi} \cdot \frac{\bar{R}}{R^2} + 2\, \nabla \cdot \bar{\xi} \right) - \int F_{\varepsilon} \Bigg\} .
\end{aligned}$$

The first two terms correspond to anisotropy instabilities, that is, the instability of the Alfvén wave and that of the "mirror" wave respectively; in effect, in the limit of a homogeneous plasma one has

$$\delta^2 W = \tfrac{1}{2} \int d\tau \left\{ \left(\frac{B^2}{\mu_0} - p_- \right) \left(\frac{\delta \bar{B}_\perp}{B} \right)^2 + \left(\frac{B^2}{\mu_0} + 2p_+ + c \right) \left(\frac{\delta \bar{B}_\parallel}{B} \right)^2 - \int F_\varepsilon \right\} .$$

The third term corresponds to the interchange instability; one has in the limit of a plasma with isotropic distribution

$$\delta^2 W = \tfrac{1}{2} \int d\tau \left\{ \frac{1}{\mu_0} |\bar{Q} - \bar{\xi}_n \wedge \mu_0 \bar{J}|^2 - 2 \frac{\bar{R}}{R^2} \cdot \nabla p |\bar{\xi}_n|^2 - \int F_\varepsilon \right\} .$$

The terms which do not reduce to the form of a square are zero when the product of the anisotropy with the curvature $[\sim p_- \, (I/R)]$ is zero.

In the limit of a state with high β $(\beta \gg 1)$ the anisotropy is weak, and the condition involving the direction of the curvature tends toward the necessary and sufficient stability condition. In the limit of a state with small β $(\beta \ll 1)$ the direction of the spatial variation of $J(\varepsilon, \nu, \Phi)$ plays an important part in the necessary and sufficient stability condition. Hence the direction of the curvature plays a dominant part in the necessary and sufficient stability condition over the whole range of β, and most of the marginal states are the configurations with small curvature. In this part of the marginal region the terms which do not reduce to the form of a square are thus small, and by neglecting these terms one has the following stability conditions

$$\frac{B^2}{\mu_0} - p_- > 0, \quad \frac{B^2}{\mu_0} + 2p_+ + c > 0, \quad \bar{R} \cdot (\nabla \cdot \bar{\bar{p}}) < 0 .$$

A state with quasi-rectilinear magnetic field lines (in the region occupied by the plasma $\bar{\bar{p}} \neq 0$) will result from the deformation of a magnetic well (vacuum field) by the diamagnetic field produced by the currents in the plasma

$$\bar{J} = \nabla \wedge \bar{M} \quad \text{with} \quad \bar{M} = \int \sum m \, d\varepsilon \, d\nu \, \frac{B}{u_\parallel} \, \Theta \, F \, (- \nu \, \bar{n}) .$$

A system with magnetic mirrors is such a configuration; thus, the anisotropy of the pressure is such that $p_\perp > p_\parallel$, and the condition $(B^2/\mu_0) - p_- > 0$ is always satisfied.

The other condition imposed on the anisotropy is satisfied if

$$\nabla_\parallel p_\perp \cdot \nabla_\parallel \left(\frac{B^2}{2\mu_0} + p_\perp \right) < 0 .$$

The critical anisotropy may be evaluated in the following manner:

$$\frac{|2p_+ + c\,|}{|p_-|} = \frac{|\nabla_{\|} p_\perp|}{|\nabla_{\|} p_{\|}|} \sim \frac{p_\perp}{p_{\|}} ,$$

whence

$$\beta \left(\frac{p_\perp}{p_{\|}} - I \right) \lesssim 2 .$$

The condition imposed on the curvature is satisfied if

$$\nabla_\perp p_\perp \cdot \nabla_\perp \left(\frac{B^2}{2\mu_o} + p_\perp \right) < 0 .$$

If the spatial distribution of the pressure has a single maximum the two above conditions referring respectively to the anisotropy and to the curvature, require that the "total perpendicular pressure" $[(B^2/\mu_o) + p_\perp]$ has no maximum inside the plasma. One thus has an evaluation of the critical β which can be supported by a magnetic well configuration

$$\beta_c \sim \frac{\delta_m \left(\frac{B^2}{2\mu_o} \right)}{\frac{B^2}{2\mu_o}}$$

where δ_m represents the smallest variation between the centre of the well and the edge of the plasma. In practice this smallest variation will often be the transverse variation.

Another equally interesting grouping is the following:

$$\delta^2 W = \tfrac{1}{2} \int d\tau \left\{ \frac{\left(\frac{B^2}{\mu_o} + p_\perp - p_{\|} \right)}{\frac{B^2}{\mu_o}} \bar{J}_{\|} \cdot \bar{\xi} \wedge \bar{Q}_\perp - \int F_\varepsilon + \left(\frac{B^2}{\mu_o} + p_\perp - p_{\|} \right) \left(\frac{\bar{Q}_\perp}{B} \right)^2 \right.$$

$$+ (\bar{n}\,\bar{n} : \nabla\bar{\xi})\, \bar{\xi} \cdot \left(\nabla p_{\|} - (p_{\|} - p_\perp) \frac{\nabla B}{B} \right) + \left(\frac{B^2}{\mu_o} + 2p_\perp + c \right) \left(\frac{\bar{Q}_{\|}}{B} \right)^2$$

$$\left. - \left[2\, (\bar{n}\,\bar{n} : \nabla\bar{\xi} - \nabla \cdot \bar{\xi}) - \bar{\xi} \cdot \frac{\nabla B}{B} \right] \bar{\xi} \cdot \left(\nabla p_\perp - (2p_\perp + c) \frac{\nabla B}{B} \right) \right\} .$$

Thus, when considering the states without parallel current and with a distribution independent of $J = \int u_{\|}\, d\ell$, say $F = F(\varepsilon, \nu)$, the moments - and in particular the pressure - tensor are only dependent on space through B; the general relations between the longitudinal components of the gradients of $p_\perp$, $p_{\|}$ and B thus become relations between the vector gradients themselves, and the terms which do not reduce to squares are zero. In this case the reduction is therefore complete. This result has been obtained by R.J. Hastie and J.B. Taylor(2). For a magnetic well configuration the stability condition has the following form

$$\nabla \left(\frac{B^2}{2\mu_o} \right) \cdot \nabla \left(\frac{B^2}{2\mu_o} + p_\perp \right) > 0 .$$

For these particular distributions the parallel and perpendicular components of $\nabla p_{\perp}$ and $\nabla p_{\parallel}$ are not independent; the anisotropy of the pressure and the transverse inhomogeneity are related, which accounts for the disappearance of the distinction between "mirror" instability and "interchange" instability.

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