CLM - Trans 9

United Kingdom Atomic Energy Authority

RESEARCH GROUP

Translation

H-FUNCTIONS IN RESONANCE-RADIATION TRANSFER THEORY

V. V. IVANOV
D. I. NAGIRNER

Culham Laboratory
Abingdon Berkshire

1967

Available from H. M. Stationery Office

TWO SHILLINGS AND SIXPENCE NET

U.D.C.
535.338.3

UNCLASSIFIED
(Approved for Sale)

CLM - Trans 9

H-FUNCTIONS IN RESONANCE-RADIATION TRANSFER THEORY

by

V. V. Ivanov and D. I. Nagirner

Astrofizika,
Akademiya Nauk Armyanskoi SSR
vol. 1, no. 2, June, 1965 pp. 143-166

Translation prepared by
CULHAM TRANSLATIONS OFFICE

U. K. A. E. A. Research Group,
Culham Laboratory,
Nr. Abingdon,
Berks.

May, 1966

ABSTRACT

The transfer of radiation in a Doppler broadened resonance line is investigated. A semi-infinite atmosphere is considered with negligibly small absorption in the continuous spectrum, using the approximation of complete frequency redistribution. The intensity of the emergent radiation is expressed by the corresponding H-function defined by expressions (6) and (7). Five-figure tables of $H(z, \lambda)$ are given for a wide range of values of the parameter λ. Special attention is given to values of λ close to unity. The asymptotic behaviour of $H(z, \lambda)$ for $z \gg 1$ is investigated. It is shown that for $z \gg 1$ the function $H(z, \lambda)$ does not depend on z and λ separately but only on a combination of z and λ. The accuracy of the asymptotic expressions and their ranges of validity are discussed. It is found that the region of validity is fairly wide, whilst the accuracy is sufficiently high to make their use practical.

An important part in the theory of multiple scattering of light in a spectral line is played by the function $H(z, \lambda)$ introduced by V. V. Sobolev[1, 2]. It is a generalisation of the well-known function of V. A. Ambartsumyan[3] for $\varphi(\mu, \lambda)$ to the case of scattering with complete frequency redistribution. For many forms of the dependence on depth of the source strength, the radiation intensity emerging from a semi-infinite atmosphere can be expressed simply by means of $H(z, \lambda)$. The related problems are discussed in Sobolev's book[4]. The function $H(z, \lambda)$ also enters into the integral representation, obtained by D. I. Nagirner[5, 6], of the resolvent of the equation describing the transfer of resonance radiation in a semi-infinite atmosphere.

The properties of the H-function for scattering with complete frequency redistribution were investigated by V. V. Ivanov[7, 8]. However no detailed tables of these functions are available. V. V. Sobolev[2, 4] calculated the H-function for three λ values ($\lambda = 1.00$; 0.999 ; 0.99) with an absorption coefficient due to the simultaneous action of the Doppler effect and damping. In these calculations it was assumed that the ratio of the coefficient of absorption in the line to the absorption coefficient in a continuous spectrum equals 10^4. A table of H-functions for pure scattering ($\lambda = 1$) and the Doppler coefficient of absorption is given in a paper by V. V. Ivanov[7], which also shows graphs of $H(z, \lambda)$ for $\lambda = 0.7$, 0.9 and 0.95. Values of $H(z, \lambda)$ for $\lambda = 0.4$ and 0.7 with a Doppler absorption coefficient were determined by A. M. Samson[9]. In all these papers the H-function was obtained numerically from the equation for $H(z, \lambda)$.

For practical calculations of the intensities and line profiles these data are quite insufficient. In this connection the tabulation of the H-function is being undertaken at Leningrad University. In this paper results are presented of calculations for the Doppler coefficient of absorption. It is assumed that absorption in the continuous spectrum is absent. Together with a description of the calculations, this paper also gives a list of the principal properties of the H-functions under consideration. To facilitate use of the tables, the paper gives at the outset the equations describing the transfer of resonance radiation and the expressions for the intensity of the radiation emergent from the medium in terms of the function $H(z, \lambda)$.

1. Suppose μ is the cosine of the angle of emergence of radiation from the medium calculated from the external normal, and x the dimensionless frequency

$$x = \frac{\nu - \nu_0}{\Delta\nu_D},$$

where ν_0 is the frequency of the line centre, $\Delta\nu_D$ the Doppler half-width. In the absence of absorption in the continuous spectrum, the intensity of the radiation emergent from the medium in the resonance line, with complete frequency redistribution in scattering, is given by the familiar expression

$$I(0, \mu, x) = \int_0^\infty S(\tau)\, e^{-\frac{\alpha(x)}{\mu}\tau}\, \alpha(x)\, \frac{d\tau}{\mu}, \qquad \ldots (1)$$

where $S(\tau)$ is the so-called source function, $\alpha(x)$ is the ratio of the coefficient of absorption at the frequency x to the absorption coefficient at the line centre, and τ is the optical depth at the line centre. When stimulated emission is neglected, source function is related to the occupation numbers n_1 and n_2 of the lower and upper levels by the relation

$$S(\tau) = \frac{2h\nu_0^3}{c^2}\,\frac{g_1}{g_2}\,\frac{n_2}{n_1}, \qquad \ldots (2)$$

where g_i is the statistical weight of the i-th level. $S(\tau)$ is determined by the integral equation

$$S(\tau) = \frac{\lambda}{2}\int_0^\infty K(|\tau - \tau'|)\, S(\tau')\, d\tau' + S^*(\tau). \qquad \ldots (3)$$

The function $S^*(\tau)$ represents the strength of the primary sources of radiation (excitation by electron impact and recombination to the upper level). It is considered a datum. The parameter λ is the so-called survival probability of a quantum during scattering, or the albedo for single scattering, and is determined by the relative part played by radiative transitions from the upper level compared with transitions due to collisions of the second kind and ionizations from the second level (for details see e.g. ref. 10). The values of λ lie between zero and unity, although in astrophysical problems one most frequently encounters cases in which λ is very close to unity (nearly conservative scattering).

With a Doppler coefficient of absorption $\alpha(x) = e^{-x^2}$, which alone will be considered in what follows, the kernel of the integral equation (3) has the form

$$K(\tau) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-2x^2} E_1(\tau e^{-x^2})\, dx, \qquad \ldots (4)$$

where $E_1(t)$ is the usual exponential-integral function

$$E_1(t) = \int_0^1 e^{-\frac{t}{\zeta}} \frac{d\zeta}{\zeta}. \qquad \ldots (5)$$

The function $K(\tau)$ has recently been investigated in detail and has been tabulated[11, 12]. We note that equation (3) with the kernel (4) was first considered by L. M. Biberman[13].

In a number of recently published papers[12, 14, 15] the integral equation for $S(\tau)$ with the kernel (4) has been solved numerically. The function $S^*(\tau)$ was assumed to be either constant; $S^*(\tau) = S^*_o$ [12, 14] or exponentially decreasing: $S^*(\tau) = S^*_o e^{-m\tau}$ [12, 15]. In the latter case several values were taken for m, and equation (3) was solved separately for each m. The $S(\tau)$ values so obtained were then used to calculate the intensity of the emergent radiation by means of formula (1).

It can however be easily shown that if the function $S^*(\tau)$ is given by the product of a polynomical in τ and an exponential function of τ the intensity of the emergent radiation $I(0, \mu, x)$ can be determined without the determination of source function. It is sufficient to have the values of the function $H(z, \lambda)$ satisfying the following nonlinear integral equation

$$H(z, \lambda) = 1 + \frac{\lambda}{2} z H(z, \lambda) \int_0^\infty \frac{H(z', \lambda)}{z + z'} G(z') dz', \qquad \ldots (6)$$

where, if the kernel of equation (3) has the form (4),

$$G(z) = \begin{cases} \dfrac{1}{\sqrt{2}} & \text{при } z \leqslant 1, \\ \sqrt{\dfrac{2}{\pi}} \displaystyle\int_{\sqrt{2 \ln z}}^\infty e^{-t^2} dt & \text{при } z > 1. \end{cases} \qquad \ldots (7)$$

In fact it is known (see e.g. refs. 4, 7) that if

$$S^*(\tau) = e^{-m\tau}, \qquad \ldots (8)$$

then

$$I(0, \mu, x) \equiv I_0(0, \mu, x, m) = \frac{H(\mu e^{x^2}) H\left(\frac{1}{m}\right)}{1 + m\mu e^{x^2}}. \qquad \ldots (9)$$

In particular, assuming $m = o$ in the last expressions and making use of the circumstance that $H(\infty, \lambda) = (1 - \lambda)^{-\frac{1}{2}}$ (see below, equation (25)), it is found that with a uniform distribution of the sources (take $S^*(\tau) = 1$) the intensity of the emergent radiation is equal to

$$I(0, \mu, x) = \frac{H(\mu e^{x^2})}{\sqrt{1 - \lambda}}. \qquad \ldots (10)$$

For

$$S^*(\tau) = \tau e^{-m\tau} \qquad \ldots (11)$$

the values of $I(0, \mu, x)$ are expressed in terms of $H(z)$ in the following form:

$$I(0, \mu, x) \equiv I_1(0, \mu, x, m) = \frac{H(\mu e^{x'}) H\left(\frac{1}{m}\right)}{1 + m\mu e^{x'}} \left(\frac{\mu e^{x'}}{1 + \mu m e^{x'}} + \right. \qquad \ldots (12)$$

$$\left. + \frac{\lambda}{2} H\left(\frac{1}{m}\right) \int_0^\infty \frac{z' H(z')}{(1 + mz')^2} G(z')\, dz' \right).$$

Generally speaking, if

$$S^*(\tau) = \sum_{j=0}^{n} a_j \tau^j e^{-m\tau}, \qquad \ldots (13)$$

then because of the linearity of (3) and (1) we have

$$I(0, \mu, x) = \sum_{j=0}^{n} a_j I_j(0, \mu, x, m), \qquad \ldots (14)$$

where

$$I_j(0, \mu, x, m) = \int_0^\infty S_j(\tau, m)\, e^{-\frac{\alpha(x)}{\mu}\tau} \alpha(x) \frac{d\tau}{\mu}, \qquad \ldots (15)$$

and $S_j(\tau, m)$ is the solution of equation

$$S_j(\tau, m) = \frac{\lambda}{2} \int_0^\infty K(|\tau - \tau'|) S_j(\tau', m)\, d\tau' + \tau^j e^{-m\tau}. \qquad \ldots (16)$$

From the last equation it follows that

$$\frac{\partial S_j(\tau, m)}{\partial m} = - S_{j+1}(\tau, m), \qquad \ldots (17)$$

and therefore, as is apparent from (15),

$$\frac{\partial I_j(0, \mu, x, m)}{\partial m} = - I_{j+1}(0, \mu, x, m). \qquad \ldots (18)$$

This relation gives

$$I_j(0, \mu, x, m) = (-1)^j \frac{\partial^j I_0(0, \mu, x, m)}{\partial m^j}. \qquad \ldots (19)$$

Substituting (19) in (14) we obtain finally, when $S^*(\tau)$ is of the form (13),

$$I(0, \mu, x) = \sum_{j=0}^{n} (-1)^j a_j \frac{\partial^j I_0(0, \mu, x, m)}{\partial m^j}. \qquad \ldots (20)$$

To calculate the sum entering in this expression it is necessary to find the first n derivatives with respect to m of the expression on the right-hand side of (9). To obtain $\frac{\partial^j I_0}{\partial m^j}$ it is therefore necessary to have the derivatives of the H-functions up to and including the j-th. However, with the aid of equation (6), these derivatives can easily be expressed in terms of the H-function and its integrals, so that for calculating (20) it is, in fact, necessary only to have H(z). For instance, one may show that

$$\frac{dH(z,\lambda)}{dz} = \frac{\lambda}{2} H^2(z,\lambda) \int_0^\infty \frac{z' H(z',\lambda)}{(z+z')^2} G(z')\, dz'. \qquad \ldots (21)$$

It should be noted that in the papers referred to above[12, 15] containing the numerical solution of equation (3) with $S^*(\tau)$ having the form (8) to obtain the intensity of the emergent radiation, there was no need to solve equation (3) for each value of m. When calculating the intensity of the emergent radiation for m = 0, i.e. with a uniform distribution of the sources, it follows from equation (10) that we obtain the values of H(z). Using these values and expression (9) we may then easily obtain $I_0(0, \mu, x, m)$ for any given m. Moreover it can be shown that the source function $S_0(\tau, m)$ may be expressed simply in terms of $S_0(\tau, 0)$ and the H-function, in particular

$$S_0(\tau, m) = \sqrt{1-\lambda}\, H\left(\frac{1}{m}\right)\left[S_0(\tau, 0) - m\int_0^\tau e^{-m(\tau-\tau')} S_0(\tau', 0)\, d\tau'\right]. \qquad \ldots (22)$$

The above examples illustrate the important part played by H-functions in the theory of the transfer of resonance radiation.

2. Let us now consider briefly the methods of calculation used in compiling the tables of the H-functions. We note to begin with that the infinite interval of integration in equation (6) and the complex behaviour of G(z) lead to an irregularity of $H(z, \lambda)$ at infinity and considerably increase the size of calculation compared with the calculation of Ambartsumyan's $\varphi(\mu, \lambda)$ functions.

For $H(z, \lambda)$ the following expression[7] can be obtained

$$\ln H(z,\lambda) = -\frac{1}{\pi}\int_0^\infty \ln\left[1 - \lambda V(u)\right] \frac{z\,du}{1+z^2u^2}, \qquad \ldots (23)$$

where

$$V(u) \equiv \int_0^\infty G(z)\frac{dz}{1+z^2u^2} = \frac{2}{u\sqrt{\pi}}\int_0^\infty e^{-2x^2} \operatorname{arctg} u e^{x^2}\, dx. \qquad \ldots (24)$$

The function V(u), apart from a constant factor, coincides with the Fourier transform of the kernel (4) of the principal integral equation (3).

It follows from (23) that $H(z, \lambda)$ is a strictly increasing function of z, varying from $H(0, \lambda) = 1$ to

$$H(\infty, \lambda) = \frac{1}{\sqrt{1-\lambda}}. \qquad \ldots (25)$$

Let us denote

$$H_0(\lambda) = \int_0^\infty H(z,\lambda) G(z)\, dz. \qquad \ldots (26)$$

From (6), taking into account (25), it follows that

$$H_0(\lambda) = \frac{2}{\lambda}(1 - \sqrt{1-\lambda}). \qquad \ldots(27)$$

Using the last expression, equation (6) can be rewritten in the form

$$\frac{1}{H(z,\lambda)} = \sqrt{1-\lambda} + \frac{\lambda}{2}\int_0^\infty \frac{z' H(z',\lambda)}{z+z'} G(z')\,dz'. \qquad \ldots(28)$$

The equation for H(z, λ) was solved numerically by an iterative method. For λ not very close to unity ($\lambda \lesssim 0.9$) the iterations rapidly converge and it is more or less immaterial which form of the equation - (6) or (28) - is used in the calculations. However, for λ close to unity equation (28) has considerable advantages over (6). In the solution of equation (28) the successive approximations behave in such a way that after any two iterations it is useful to take the half-sum of the iterates as the next approximation. This considerably improves the convergence. We note that in replacing the integrals in the equations for H(z, λ) by the sums, the singularity of the derivatives of H(z, λ) and G(z) for z = 0 and z = 1 + 0 respectively was taken into account, as was the peculiar behaviour of these functions at infinity.

In addition to the method just described, the H-functions were calculated from expression (23) in a way similar to that employed by Stibbs and Weir[16] in computing φ(μ, λ). We first investigated in detail the integrand of (23). The behaviour of V(u) for small and for large u differs substantially. For $0 \leqslant v \leqslant 1$ one readily obtains the series expansion of V(1/v)

$$V\left(\frac{1}{v}\right) = \frac{\pi}{2\sqrt{2}} v - \sum_{j=0}^{\infty} (-1)^j \frac{v^{2j+2}}{\sqrt{2j+3}\,(2j+1)}. \qquad \ldots(29)$$

For u close to zero the behaviour of V(u) is fairly complex. It can be shown that for $u \to 0$ the following asymptotic expansion is valid:

$$V(u) = 1 - \frac{\sqrt{\pi}}{4} \frac{u}{\sqrt{\ln \frac{1}{u}}} \left[1 + \frac{1}{4}\frac{1}{\ln u} + \left(\frac{\pi^2}{2} + 1\right)\frac{3}{16}\left(\frac{1}{\ln^2 u} + \right.\right.$$
$$\left.\left. + \frac{5}{4}\frac{1}{\ln^3 u}\right) + \left(\frac{5\pi^4}{8} + \frac{3}{2}\pi^2 + 3\right)\frac{35}{256}\left(\frac{1}{\ln^4 u} + \frac{9}{4}\frac{1}{\ln^5 u}\right) + \cdots\right]. \qquad \ldots(30)$$

It was determined from the integral representation of V(u) by investigating the behaviour of Cauchy-type integrals near the boundary points of the integration contour, as explained in the book by F. D. Gakhov[17]. Expression (30) including all of the written terms gives 1 - V(u) for $u \leqslant 10^{-5}$ to four significant figures, and for $u \leqslant 10^{-9}$ to six figures.

In calculating the integral (23) the singularity of the derivative of $H(z, \lambda)$ at $z = 0$ was isolated. Also, in order to take into account the rapid variations of the integrand for small u and large z, we made the substitution $u = e^{-t}$ and carried out the integration with a uniform step - length in t. The final expressions which were used during the calculations has the form:

for $z < 1$

$$\ln H(z, \lambda) = \frac{\lambda}{4\sqrt{2}} z \ln \frac{1+z^2}{z^2} - \frac{z}{\pi}\left\{\int_0^1 \left[\ln\left[1 - \lambda V\left(\frac{1}{v}\right)\right] + \right.\right.$$

$$\left. + \frac{\lambda\pi}{2\sqrt{2}} v\right] \frac{dv}{v^2 + z^2} + 2\int_0^1 \ln\left[1 - \lambda V(e^{-x^2})\right] \frac{xe^{-x^2}dx}{1 + z^2e^{-2x^2}} + \qquad \ldots(31)$$

$$\left. + \int_1^{20} \ln\left[1 - \lambda V(e^{-t})\right] \frac{e^{-t}dt}{1 + z^2e^{-2t}}\right\};$$

for $z > 1$

$$\ln H(z, \lambda) = -\frac{1}{\pi}\left\{\frac{1}{z}\int_0^1 \ln\left[1 - \lambda V\left(\frac{1}{v}\right)\right] \frac{z^2dv}{z^2 + v^2} + \right.$$

$$\left. + 2\int_0^1 \ln\left[1 - \lambda V(e^{-x^2})\right] \frac{zxdx}{z^2e^{-x^2} + e^{x^2}} + \int_1^{20+\ln z} \ln\left[1 - \lambda V(e^{-t})\right] \frac{zdt}{z^2e^{-t} + e^t}\right\}. \qquad \ldots(32)$$

All calculations were carried out on the IBM M-20 at the University of Leningrad Computer Centre. The $H(z, \lambda)$ values so obtained are given in the appendix, in Tables 1 (for $\lambda \leq 0.9$), $2(0.925 \leq \lambda \leq 1 - 10^{-8})$ and $3(\lambda = 1)$. In addition, Table 3 gives the values of the integral

$$G_1(z) = \int_0^\infty \frac{z' H(z', 1)}{(z + z')^2} G(z')\,dz', \qquad \ldots(33)$$

entering into the expression for the derivative of the H-functions for $\lambda = 1$ (see equation (21)). The calculation of one value of $H(z, \lambda)$ from equation (23) occupied approximately one second of computer time. The function $V(u)$ was previously tabulated in detail, which took approximately thirty minutes.

A check on the accuracy of the calculations was provided by the coincidence of the values of $H(z, \lambda)$ obtained by the iteration method with those calculated from equation(23). In addition, for some λ, the generalised moment $H_0(\lambda)$ determined by equation (26) was calculated from the calculated $H(z, \lambda)$ values. The $H_0(\lambda)$ values so obtained differ from the exact values given by equation (27) by less than one unit of the sixth digit. From all this one can assume that the error in the values of $H(z, \lambda)$ listed in Tables 1 - 3 would seem to be smaller than one unit in the last significant digit.

3. From Table 2 it is evident that when λ is close to unity the function $H(z,\lambda)$ approximates to its asymptotic value $H(\infty, \lambda) = (1 - \lambda)^{-\frac{1}{2}}$ only at very large values of z, which become larger as $1 - \lambda$ becomes smaller. It is therefore practically impossible to compile a table of $H(z,\lambda)$ from which the values of this function could be directly determined for 4 given z and λ values. This difficulty is easily overcome by the following observation. For small values of z function $H(z,\lambda)$ for $1 - \lambda \ll 1$ is close to $H(z, 1)$, as is clearly evident from Tables 2 and 3. On the other hand, for large values of z a comparatively simple asymptotic representation for $H(z, \lambda)$ can be obtained, to which we now turn our attention.

For $z \gg 1$ the main contribution to the integral (23) is provided by values of the integrand for u close to zero. Therefore by replacing $V(u)$ in the integrand by the two first terms of the series expansion (30) we obtain approximately

$$\ln H(z, \lambda) = -\frac{1}{2} \ln(1-\lambda) - -\frac{1}{\pi}\int_0^\infty \ln\left[1+\frac{\lambda}{1-\lambda}\frac{u\sqrt{\pi}}{4}\frac{1}{\sqrt{\ln\frac{1}{u}}}\right]\frac{z\,du}{1+z^2u^2}. \quad \ldots (34)$$

Effecting the substitution $zu = t$ and defining,

$$\frac{\lambda}{1-\lambda}\frac{\sqrt{\pi}}{4z}\frac{1}{\sqrt{\ln z}} = q, \quad \ldots (35)$$

we find that, with the same accuracy as in (34),

$$\ln H(z, \lambda) = -\frac{1}{2}\ln(1-\lambda) - \frac{1}{\pi}\int_0^\infty \ln(1+qt)\frac{dt}{1+t^2}. \quad \ldots (36)$$

Thus for $z \gg 1$ the function $\sqrt{1 - \lambda}\, H(z,\lambda)$ does not depend on the two arguments with z and λ but only on their combination (35). Let us investigate this dependence.

Let us define

$$\ln h(q) = -\frac{1}{\pi}\int_0^\infty \ln(1+qt)\frac{dt}{1+t^2}. \quad \ldots (37)$$

s From this formula it is easily shown that

$$h\left(\frac{1}{q}\right) = \sqrt{q}\, h(q), \quad \ldots (38)$$

so that it is sufficient to have this function for $q \leqslant 1$.

Differentiating (37) and evaluating the integral obtained on the right, we find

$$\frac{d}{dq}\ln h(q) = -\frac{1}{2}\frac{q}{1+q^2} + \frac{1}{\pi}\frac{1}{1+q^2}\ln q. \quad \ldots (39)$$

Whence

$$\ln h(q) = -\frac{1}{4}\ln(1+q^2) + \frac{1}{\pi}\ln q \operatorname{arc tg} q - \frac{1}{\pi}\int_0^q \frac{\operatorname{arc tg} x}{x}\,dx. \quad \ldots (40)$$

For small q we have the expansion

$$\ln h(q) = \frac{1}{\pi} q \ln q - \frac{q}{\pi} - \frac{1}{4} q^2 + \cdots \qquad \ldots (41)$$

With its aid we find from (36) that

$$H(z, \lambda) = \frac{1}{\sqrt{1-\lambda}} \left[1 - \frac{\lambda}{1-\lambda} \frac{\sqrt{\ln z}}{4\sqrt{\pi}\, z} \right] \qquad \ldots (42)$$

for

$$\frac{\lambda\sqrt{\pi}}{4} \frac{1}{z\sqrt{\ln z}} \ll 1 - \lambda. \qquad \ldots (43)$$

For $q >> 1$, (38) and (41) give

$$\ln h(q) = \frac{1}{2} \ln q - \frac{1}{\pi} \frac{1}{q} \ln q - \frac{1}{\pi q} + \cdots, \qquad \ldots (44)$$

and for values of $z >> 1$ satisfying the condition

$$\frac{\lambda\sqrt{\pi}}{4} \frac{1}{z\sqrt{\ln z}} \gg 1 - \lambda, \qquad \ldots (45)$$

we obtain from (36)

$$H(z, \lambda) = \sqrt{\frac{4z\sqrt{\ln z}}{\sqrt{\pi}}}. \qquad \ldots (46)$$

It is obvious that the region of validity of this expression will be wider the smaller $1 - \lambda$. For $\lambda = 1$ the condition (45) does not impose an upper limitation on z. The expressions (42) and (46) were obtained earlier by one of the authors[7, 8]. They are limiting cases of the general asymptotic expressions (equivalent to equation (36))

$$H(z, \lambda) = \frac{h(q)}{\sqrt{1-\lambda}}, \qquad \ldots (47)$$

which is valid for large z for any relation between the values of z and λ.

The values of the function h(q) for $0 \leqslant q \leqslant 1$ are given in the appendix in Table 4. For $q < 1$ they can easily be calculated from the tabulated values with the help of relation (38). As far as the accuracy provided by the asymptotic representation (47) is concerned, the following can be said. For $z \geqslant 10$ this expression gives $H(z, \lambda)$ for values of $\lambda \geqslant 0.9$ with a maximum error of about 3%. When z is larger than 1000 the accuracy is higher than 1.7% and for z = 10000 the error is smaller than 1.1%

For $\lambda = 1$ a more accurate asymptotic expansion for H(z) can be obtained, in which the principal term coincides with (46). This expansion has the form

$$H(z, 1) = \frac{2}{\pi^{1/4}} \sqrt{z\sqrt{\ln z}} \exp\left\{ \frac{1}{8} \frac{1}{\ln z} - \frac{5}{64} (\pi^2 + 1) \frac{1}{\ln^2 z} + \cdots \right\}. \qquad \ldots (48)$$

with all written terms, it gives H(z, 1) for z > 100 with an error of less than 0.05%. This expansion was obtained with the aid of the same method as was used in deducing expression (30). The argument is too cumbersome to be reproduced here.

It should be noted that the leading term in the expansion i.e. the pre-exponential factor, by itself provides an accuracy which is quite sufficient as a rule: for $z \geqslant 100$ the error is smaller than 1%, and for z from 10 to 100 it does not exceed 3%; in the latter region inclusion of the second and third terms in the expansion even leads to a reduced accuracy.

It is thus evident that the range of validity of the asymptotic expressions obtained is fairly wide. In combination with the above tables of $H(z, \lambda)$ they permit one to determine the values of the H - functions for any given z and λ with an accuracy completely sufficient for any applications of theory, since for practical purposes it is sufficient to have the H - function to 2 or 3 digits. The question may therefore arise whether in fact the tabulation of the $H(z, \lambda)$ function with as high an accuracy as has been done above is justified. The answer to this question, it seems to us, must be positive. In fact, most methods of solving the transfer equation involve certain approximations of a purely mathematical character whose accuracy it is impossible to estimate in advance. It therefore seems natural to tabulate with a high accuracy the vigorous solutions of some of the simpler problems, obtained without any approximations of unknown accuracy. They can then later be used as standards in estimating the accuracy of any given approximate method. One of these standard problems must be in our opinion, the problem of the scattering of light in a semi-infinite sphere.

As an example of the application of the tables of the $H(z, \lambda)$ functions with this object in mind we quote the following. Recently Dr. Hummer has kindly made available to us unpublished numerical solutions of equation (3) with $S^{*}(\tau)$ = const, and also the values of $I_0(0, \mu, x)$ calculated subsequently from equation (1). A comparison of the latter results with the $H(z, \lambda)$ values given in the tables revealed that the accuracy is only one unit of the last (third) digit given by Dr. Hummer. Similarly we arrived at an estimate of the accuracy of the numerical method extensively used by Avrett and Hummer[12] for calculating the radiation fields, not only in semi-infinite, but also in finite atmospheres.

In conclusion, we point out that in calculating the line profiles from the expressions given at the beginning of this paper, the radiation intensity can be conveniently calculated for those frequency values x for which the corresponding z-values are available in the tables (for instance, to $z = \mu e^{x^2} = 10$ for $\mu = 1$ corresponds $x = 1.52$ and so forth). This avoids interpolation of the tabulated values of $H(z, \lambda)$ without creating appreciable difficulties in plotting the graphs.

Above we have given the values of the H-functions only for the Doppler coefficient of absorption. At present work is in progress on the tabulation of H-functions with an absorption coefficient due to the simultaneous action of the Doppler effect and damping, i.e. the Voigt contour. The results of these calculations will be published in a later paper.

The authors are indebted to E. Dzepe and S. B. Mikhailov who took part in the separate stages of the machine computations. We are also grateful to Dr. D. Hummer and to Dr E. H. Avrett for making available unpublished results of the numerical solution of the transfer equation and preprints, and for stimulating correspondence.

REFERENCES

1. V.V. SOBOLEV, Astron, Zh. 26, 129, 1949.

2. V.V. SOBOLEV, Astron, Zh. 31, 231, 1954.

3. V.A. AMBARTSUMYAN, Astron, Zh. 19, 30, 1942;

4. V.V. SOBOLEV. Transfer of Radiant Energy in the Atmospheres at Stars and Planets, Moscow, 1956; Eng. trans. : A Treatise on Radiative Transfer, Van Nostrand, Princeton, 1963.

5. D.I. NAGIRNER, Astron. Zh. 31, 669, 1964.

6. D.I. NAGIRNER, Vestnik LGU, No.1, 142, 1964.

7. V.V. IVANOV, Astron, Zh. 39, 1020, 1962.

8. V.V. IVANOV, Uch. Zap. LGU, No. 307 (Trudi Astron. obs. LGU, 19), 52, 1962.

9. A.M. SAMSON, Izv. AN. SSSR., Ser. Fizich. 24, 496, 1960.

10. R.N. THOMAS and R.G. ATHAY, Physics of the Solar Chromosphere, Interscience Publ., New York, 1961.

11. V.V. IVANOV and V.T. SHCHERBAKOV, Astrofizika. 1, 31, 1965.

12. E.H. AVRETT and D.G. HUMMER, M.N. 130, 295, 1965.

13. L.M. BIBERMAN, Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki. 17, 416, 1947.

14. A.G. HEARN, Proceedings of the Physical Society. 81, 648, 1963.

15. E.H. AVRETT. Proceedings of the Second Harvard-Smithsonian Conference on Stellar Atmospheres, Smithsonian Institution Astrophys. Obs., Special Report, April, 1965.

16. D.W.H. STIBBS and R.E. WIER, M.N. 119, 512, 1959.

17. F.D. GAKHOV, Boundary Value Problems, Fizmatgiz, Moscow, 1963; Eng. trans: Boundary Value Problems, Pergamon Press, 1966.

TABLE 1

Values of Function H(z, λ) for λ < 0.9

z \ λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.85	0.9
0.00	1.0000	1.0000	1.00000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.05	1.0059	1.0121	1.0187	1.0256	1.0330	1.0409	1.0496	1.0543	1.0594	1.0649	1.0710
0.1	1.0096	1.0167	1.0304	1.0420	1.0545	1.0681	1.0833	1.0917	1.1008	1.1108	1.1221
0.2	1.0147	1.0304	1.0475	1.0661	1.0866	1.1095	1.1357	1.1504	1.1666	1.1848	1.2058
0.3	1.0184	1.0384	1.0602	1.0843	1.1111	1.1416	1.1771	1.1974	1.2200	1.2457	1.2759
0.4	1.0213	1.0445	1.0703	1.0989	1.1311	1.1681	1.2118	1.2370	1.2655	1.2982	1.3373
0.5	1.0237	1.0488	1.0786	1.1110	1.1478	1.1906	1.2416	1.2714	1.3053	1.3447	1.3922
0.6	1.0257	1.0541	1.0857	1.1214	1.1622	1.2101	1.2678	1.3018	1.3408	1.3864	1.4421
0.8	1.0289	1.0610	1.0971	1.1383	1.1860	1.2426	1.3120	1.3536	1.4018	1.4590	1.5303
1.0	1.0313	1.0664	1.1061	1.1516	1.2049	1.2688	1.3483	1.3965	1.4528	1.5207	1.6064
1.5	1.0356	1.0759	1.1220	1.1756	1.2394	1.3174	1.4168	1.4785	1.5520	1.6427	1.7608
2.0	1.0384	1.0821	1.1326	1.1918	1.2630	1.3513	1.4658	1.5379	1.6252	1.7347	1.8805
3.0	1.0419	1.0900	1.1461	1.2127	1.2939	1.3964	1.5323	1.6199	1.7278	1.8666	2.0578
4.0	1.0441	1.0949	1.1545	1.2258	1.3135	1.4254	1.5761	1.6746	1.7976	1.9585	2.1852
5.0	1.0455	1.0982	1.1603	1.2349	1.3272	1.4459	1.6075	1.7143	1.8488	2.0269	2.2825
7.5	1.0477	1.1033	1.1691	1.2489	1.3486	1.4783	1.6579	1.7786	1.9330	2.1417	2.4503
10	1.0490	1.1062	1.1742	1.2571	1.3611	1.4975	1.6882	1.8177	1.9849	2.2140	2.5592
15	1.0504	1.1095	1.1800	1.2662	1.3753	1.5195	1.7235	1.8636	2.0467	2.3015	2.6946
20	1.0512	1.1113	1.1832	1.2714	1.3832	1.5319	1.7436	1.8901	2.0828	2.3533	2.7768
30	1.0520	1.1132	1.1866	1.2769	1.3920	1.5456	1.7662	1.9199	2.1237	2.4130	2.8735
50	1.0528	1.1149	1.1897	1.2819	1.3998	1.5580	1.7866	1.9471	2.1614	2.4687	2.9660
75	1.0532	1.1158	1.1913	1.2846	1.4040	1.5647	1.7980	1.9624	2.1827	2.5006	3.0199
100	1.0534	1.1163	1.1922	1.2860	1.4063	1.5684	1.8040	1.9706	2.1942	2.5179	3.0496
150	1.0536	1.1168	1.1931	1.2875	1.4087	1.5722	1.8105	1.9793	2.2066	2.5366	3.0819
200	1.0537	1.1171	1.1936	1.2883	1.4099	1.5742	1.8139	1.9839	2.2131	2.5466	3.0993
300	1.0538	1.1174	1.1941	1.2891	1.4112	1.5763	1.8175	1.9888	2.2200	2.5572	3.1179
500	1.0539	1.1176	1.1945	1.2898	1.4123	1.5781	1.8205	1.9929	2.2259	2.5662	3.1339
750	1.0540	1.1178	1.1947	1.2902	1.4129	1.5790	1.8221	1.9951	2.2290	2.5710	3.1425
1000	1.0540	1.1178	1.1948	1.2904	1.4132	1.5795	1.8230	1.9962	2.2306	2.5735	3.1470
∞	1.0541	1.1180	1.1952	1.2910	1.4142	1.5811	1.8257	2.0000	2.2361	2.5820	3.1623

TABLE 2

Values of the function H(z, λ) for λ near the unity

z \ 1−λ	0.075	0.050	0.025	0.01	$7.5\cdot10^{-3}$	$5\cdot10^{-3}$	$2.5\cdot10^{-3}$	10^{-3}	$5\cdot10^{-4}$	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.05	1.0743	1.0781	1.0824	1.0856	1.0862	1.0869	1.0877	1.0882	1.0884	1.0886	1.0887	1.0887	1.0887	1.0887
0.1	1.1285	1.1356	1.1440	1.1503	1.1516	1.1529	1.1545	1.1556	1.1560	1.1565	1.1566	1.1566	1.1566	1.1566
0.2	1.2180	1.2317	1.2483	1.2611	1.2637	1.2665	1.2697	1.2721	1.2730	1.2739	1.2742	1.2742	1.2743	1.2743
0.3	1.2936	1.3140	1.3389	1.3585	1.3624	1.3668	1.3719	1.3756	1.3771	1.3785	1.3790	1.3790	1.3791	1.3791
0.4	1.3604	1.3873	1.4208	1.4474	1.4529	1.4589	1.4660	1.4711	1.4732	1.4752	1.4758	1.4759	1.4759	1.4759
0.5	1.4207	1.4542	1.4962	1.5301	1.5371	1.5450	1.5541	1.5608	1.5635	1.5662	1.5670	1.5671	1.5671	1.5671
0.6	1.4758	1.5158	1.5665	1.6079	1.6166	1.6263	1.6375	1.6459	1.6493	1.6526	1.6537	1.6538	1.6538	1.6538
0.7	1.5267	1.5731	1.6326	1.6817	1.6920	1.7036	1.7172	1.7272	1.7313	1.7354	1.7367	1.7368	1.7369	1.7369
0.8	1.5741	1.6268	1.6950	1.7520	1.7641	1.7776	1.7936	1.8054	1.8103	1.8151	1.8166	1.8168	1.8168	1.8168
0.9	1.6184	1.6774	1.7544	1.8193	1.8332	1.8488	1.8671	1.8809	1.8865	1.8921	1.8938	1.8941	1.8941	1.8941
1.0	1.6600	1.7251	1.8110	1.8849	1.8997	1.9173	1.9382	1.9539	1.9603	1.9667	1.9687	1.9690	1.9691	1.9691
1.2	1.7362	1.8136	1.9169	2.0064	2.0258	2.0478	2.0739	2.0936	2.1017	2.1099	2.1125	2.1128	2.1129	2.1129
1.4	1.8048	1.8939	2.0146	2.1208	2.1441	2.1705	2.2022	2.2262	2.2361	2.2461	2.2493	2.2497	2.2498	2.2498
1.6	1.8671	1.9676	2.1055	2.2286	2.2558	2.2869	2.3241	2.3527	2.3645	2.3765	2.3803	2.3808	2.3809	2.3809
1.8	1.9240	2.0355	2.1904	2.3305	2.3618	2.3976	2.4408	2.4739	2.4878	2.5018	2.5063	2.5069	2.5070	2.5070
2.0	1.9763	2.0986	2.2701	2.4274	2.4627	2.5034	2.5526	2.5906	2.6066	2.6227	2.6279	2.6286	2.6287	2.6288
2.2	2.0247	2.1573	2.3453	2.5197	2.5592	2.6047	2.6602	2.7032	2.7213	2.7397	2.7456	2.7465	2.7466	2.7466
2.4	2.0697	2.2123	2.4163	2.6079	2.6516	2.7022	2.7640	2.8121	2.8325	2.8532	2.8599	2.8608	2.8610	2.8610
2.6	2.1116	2.2639	2.4838	2.6924	2.7404	2.7960	2.8643	2.9177	2.9404	2.9635	2.9709	2.9720	2.9722	2.9722
2.8	2.1508	2.3125	2.5479	2.7736	2.8258	2.8866	2.9615	3.0203	3.0453	3.0709	3.0792	3.0804	3.0805	3.0805
3.0	2.1876	2.3584	2.6090	2.8517	2.9082	2.9741	3.0557	3.1200	3.1474	3.1756	3.1847	3.1861	3.1862	3.1863
3.2	2.2222	2.4018	2.6673	2.9269	2.9878	3.0589	3.1473	3.2172	3.2471	3.2779	3.2879	3.2893	3.2895	3.2896
3.4	2.2549	2.4430	2.7232	2.9995	3.0647	3.1411	3.2364	3.3120	3.3444	3.3779	3.3888	3.3904	3.3906	3.3906
3.6	2.2858	2.4822	2.7767	3.0697	3.1392	3.2209	3.3231	3.4045	3.4395	3.4758	3.4876	3.4893	3.4896	3.4896
3.8	2.3150	2.5195	2.8280	3.1376	2.2114	3.2985	3.4076	3.4949	3.5326	3.5717	3.5845	3.5863	3.5866	3.5866

z \ 1−λ	0.075	0.050	0.025	0.01	$7.5\cdot10^{-3}$	$5\cdot10^{-3}$	$2.5\cdot10^{-3}$	10^{-3}	$5\cdot10^{-4}$	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
4.0	2.7428	2.5551	2.8774	3.2034	4.2816	3.3739	3.4902	3.5834	3.6238	3.6658	3.6795	3.6815	3.6818	3.6818
4.2	2.3692	2.5891	2.9248	3.2671	3.3497	3.4474	3.5707	3.6701	3.7132	3.7581	3.7728	3.7749	3.7752	3.7753
4.4	2.3944	2.6216	2.9706	3.3290	3.4159	3.5190	3.6495	3.7550	3.8009	3.8488	3.8645	3.8668	3.8671	3.8671
4.6	2.4184	2.6527	3.0147	3.3892	3.4804	3.5888	3.7266	3.8383	3.8870	3.9379	3.9546	3.9571	3.9574	3.9575
4.8	2.4413	2.6826	3.0573	3.4476	3.5431	3.6570	3.0820	3.9200	3.9715	4.0255	4.0434	4.0460	4.0463	4.0464
5.0	2.4633	2.7113	3.0984	3.5045	3.6043	3.7236	3.8759	4.0002	4.0547	4.1118	4.1307	4.1335	4.1338	4.1339
5.5	2.5142	2.7783	3.1956	3.6403	3.7509	3.8836	4.0543	4.1947	4.2566	4.3218	4.3434	4.3467	4.3471	4.3471
6.0	2.5602	2.8394	3.2854	3.7679	3.8891	4.0353	4.2246	4.3814	4.4508	4.5244	4.5489	4.5526	4.5531	4.5531
6.5	2.6021	2.8954	3.3688	3.8881	4.0198	4.1795	4.3874	4.5609	4.6382	4.7204	4.7479	4.7520	4.7526	4.7526
7.0	2.6404	2.9470	3.4466	4.0018	4.1439	4.3170	4.5437	4.7341	4.8194	4.9104	4.9410	4.9456	4.9462	4.9463
7.5	2.6755	2.9948	3.5193	4.1096	4.2620	4.4483	4.6938	4.9014	4.9948	5.0950	5.1288	5.1338	5.1345	5.1346
8.0	2.7080	3.0391	3.5876	4.2119	4.3745	4.5740	4.8384	5.0633	5.1651	5.2745	5.3116	5.3172	5.3179	5.3180
8.5	2.7380	3.0804	3.6518	4.3094	4.4820	4.6947	4.9779	5.2203	5.3305	5.4495	5.4899	5.4960	5.4968	5.4970
9.0	2.7660	3.1189	3.7124	4.4025	4.5840	4.8106	5.1127	5.3728	5.4915	5.6202	5.6641	5.6707	5.6716	5.6717
9.5	2.7920	3.1551	3.7696	4.4914	4.6836	4.9221	4.2430	5.5209	5.6483	5.7869	5.8344	5.8416	5.8425	5.8427
10	2.8164	3.1890	3.8239	4.5765	4.7783	5.0296	5.3693	5.6651	5.8013	5.9500	6.0011	6.0088	6.0099	6.0100
11	2.8606	3.2512	3.9244	4.7364	4.9570	5.2335	5.6106	5.9426	6.0967	6.2660	6.3246	6.3334	6.3347	6.3348
12	2.8999	3.3067	4.0155	4.8843	5.1230	5.4242	5.8385	6.2068	6.3792	6.5697	6.6361	6.6462	6.6476	6.6478
13	2.9350	3.3568	4.0987	5.0215	5.2779	5.6032	6.0544	6.4593	6.6502	6.8626	6.9370	6.9484	6.9499	6.9501
14	2.9666	3.4021	4.1750	5.1494	5.4230	5.7719	6.2595	6.7012	7.9109	7.1456	7.2283	7.2410	7.2428	7.2430
15	2.9952	3.4434	4.2453	5.2691	5.5592	5.9313	6.4549	6.9334	7.1622	7.4198	7.5110	7.5251	7.5270	7.5273
16	3.0213	3.4812	4.3104	5.3813	5.6876	6.0822	6.6415	7.1569	7.4050	7.6858	7.7858	7.8013	7.8034	7.8037
17	3.0451	3.5161	4.3708	5.4870	5.8089	6.2256	6.8200	7.3723	7.6398	7.9443	8.0534	8.0703	8.0726	8.0730
18	3.0670	3.5482	4.4271	5.5867	5.9238	6.3620	6.9912	7.5803	8.8675	8.1960	8.3143	8.3327	8.3352	8.3356
19	3.0873	3.5780	4.4798	5.6809	6.0328	6.4921	7.1555	7.7814	8.0884	8.4413	8.5690	8.5889	8.5917	8.5921

z \ 1−λ	0.075	0.050	0.025	0.01	$7.5\cdot10^{-3}$	$5\cdot10^{-3}$	$2.5\cdot10^{-3}$	10^{-3}	$5\cdot10^{-4}$	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
20	3.1060	3.6057	4.5291	5.7702	6.1365	6.6164	7.3136	7.9761	8.3030	8.6807	8.8180	8.8394	8.8425	8.8429
22	3.1396	3.6557	4.6191	5.9358	6.3294	6.8492	7.6126	8.3481	8.7151	9.1432	9.3003	9.3249	9.3284	9.3289
24	3.1689	3.6996	4.6992	6.0860	6.5057	7.0636	7.8914	8.6992	9.1066	9.5861	9.7637	9.7917	9.7957	9.7962
26	3.1948	3.7386	4.7712	6.2232	6.6675	7.2619	8.1522	9.0318	9.4797	10.012	10.219	10.242	10.246	10.247
28	3.2177	3.7734	4.8361	6.3492	6.8168	7.4462	8.3973	9.3477	9.8365	10.422	10.642	10.677	10.682	10.683
30	3.2383	3.8047	4.8952	6.4654	6.9552	7.6181	8.6282	9.6487	10.178	10.818	11.060	11.099	11.104	11.105
32	3.2569	3.8331	4.9491	6.5730	7.0840	7.7790	8.8465	9.9362	10.507	11.201	11.466	11.509	11.515	11.515
34	3.2737	3.8589	4.9986	6.6730	7.2041	1.9300	9.0534	10.211	10.823	11.572	11.861	11.908	11.914	11.915
36	3.2890	3.8825	5.0443	6.7663	7.3166	8.0722	9.2498	10.475	11.127	11.933	12.245	12.296	12.303	12.304
38	3.3030	3.9042	5.0865	6.8535	7.4222	8.2064	9.4368	10.729	11.422	12.283	12.620	12.675	12.683	12.684
40	3.3159	3.9242	5.1257	6.9354	7.5217	8.3334	9.6150	10.972	11.706	12.625	12.986	13.045	13.054	13.055
42	3.3278	3.9427	5.1622	7.0123	7.6155	8.4537	9.7853	11.207	11.982	12.957	13.344	13.407	13.416	13.417
44	3.3388	3.9599	5.1962	7.0849	7.7042	8.5680	9.9482	11.434	12.249	13.282	13.694	13.762	13.771	13.773
46	3.3490	3.9759	5.2282	7.1534	7.7882	8.6767	10.104	11.653	12.508	13.599	14.037	14.109	14.119	14.121
48	3.3586	3.9909	5.2581	7.2182	7.8679	8.7803	10.254	11.865	12.759	13.909	14.372	14.449	14.460	14.462
50	3.3674	4.0048	5.2863	7.2797	7.9437	8.8792	10.398	12.069	13.004	14.212	14.702	14.784	14.795	14.797
55	3.3873	4.0362	5.3499	7.4204	8.1180	9.1080	10.734	12.555	13.588	14.942	15.500	15.594	15.607	15.609
60	3.4044	4.0633	5.4055	7.5452	8.2735	9.3138	11.042	13.005	14.136	15.638	16.265	16.371	16.387	16.389
65	3.4193	4.0870	5.4544	7.6568	8.4133	9.5004	11.324	13.426	14.652	16.303	17.001	17.120	17.138	17.140
70	3.4323	4.1079	5.4980	7.7574	8.5399	9.6705	11.584	13.820	15.140	16.940	17.711	17.843	17.863	17.865
75	3.4439	4.1265	5.5369	7.8486	8.6551	9.8263	11.825	14.190	15.603	17.552	18.397	18.543	18.565	18.567
80	3.4542	4.1431	5.5720	7.9317	8.7606	9.9698	12.050	14.539	16.043	18.141	19.062	19.222	19.245	19.249
85	3.4635	4.1581	5.6039	8.0078	8.8576	10.102	12.260	14.869	16.462	18.710	19.707	19.882	19.907	19.911
90	3.4719	4.1716	5.6329	8.0779	8.9471	10.226	12.456	15.182	16.863	19.259	20.334	20.524	20.552	20.555
95	3.4795	4.1840	5.6595	8.1425	9.0300	10.340	12.640	15.479	17.247	19.791	20.945	21.150	21.180	21.184

z \ 1−λ	0.075	0.050	0.025	0.01	$7.5\cdot10^{-3}$	$5\cdot10^{-3}$	$2.5\cdot10^{-3}$	10^{-3}	$5\cdot10^{-4}$	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-3}
100	3.4865	4.1953	5.6839	8.2025	9.1071	10.447	12.814	15.762	17.615	20.307	21.541	21.760	21.793	21.797
110	3.4988	4.2153	5.7273	8.3102	9.2461	10.641	13.133	16.290	18.307	21.294	22.690	22.941	22.979	22.984
120	3.5092	4.2324	5.7647	8.4043	9.3684	10.813	13.419	16.773	18.949	22.228	23.789	24.073	24.116	24.122
130	3.5183	4.2473	5.7974	8.4874	9.4767	10.967	13.678	17.217	19.548	23.116	24.844	25.163	25.211	25.217
140	3.5262	4.2603	5.8262	8.5615	9.5736	11.105	13.914	17.628	20.107	23.962	25.860	26.213	26.267	26.274
150	3.5331	4.2717	5.8517	8.6278	9.6608	11.230	14.130	18.010	20.632	24.770	26.841	27.230	27.289	27.297
160	3.5393	4.2820	5.8746	8.6877	9.7398	11.344	14.328	18.366	21.127	25.544	27.790	28.215	28.280	28.289
170	3.5448	4.2911	5.8952	8.7421	9.8117	11.448	14.511	18.699	21.594	26.287	28.709	29.172	29.243	29.252
180	3.5498	4.2994	5.9138	8.7917	9.8775	11.544	14.681	19.012	22.036	27.002	29.602	30.102	30.179	30.190
190	3.5543	4.3069	5.9308	8.8372	9.9380	11.632	14.839	19.306	22.456	27.691	30.469	31.009	31.092	31.104
200	3.5584	4.3137	5.9464	8.8791	9.9937	11.714	14.986	19.583	22.855	28.356	31.314	31.894	31.983	31.996
220	3.5656	4.3258	5.9738	8.9536	10.093	11.861	15.253	20.093	23.598	29.620	32.942	33.602	33.705	33.719
240	3.5718	4.3360	5.9972	9.0179	10.180	11.989	15.489	20.553	24.277	30.806	34.494	35.239	35.355	35.371
260	3.5770	4.3448	6.0176	9.0742	10.256	12.102	15.699	20.970	24.901	31.923	35.981	36.812	36.942	36.961
280	3.5816	4.3525	6.0353	9.1238	10.323	12.203	15.888	21.350	25.477	32.980	37.410	38.328	38.474	38.494
300	3.5856	4.3592	6.0510	9.1680	10.382	12.293	16.059	21.699	26.012	33.983	38.785	39.794	39.954	39.977
320	3.5892	4.3652	6.0650	9.2075	10.436	12.374	16.215	22.020	26.509	34.937	40.113	41.214	41.390	41.414
340	3.5923	4.3706	6.0776	9.2432	10.485	12.448	16.357	22.318	26.974	35.846	41.397	42.592	42.783	42.810
360	3.5952	4.3754	6.0889	9.2755	10.529	12.515	16.487	22.594	27.411	36.716	42.641	43.931	44.139	44.169
380	3.5978	4.3798	6.0991	9.3050	10.569	12.576	16.608	22.852	27.820	37.548	43.849	45.236	45.460	45.492
400	3.6001	4.3837	6.1085	9.3320	10.606	12.633	16.719	23.092	28.207	38.346	45.022	46.507	46.749	46.783
420	3.6022	4.3873	6.1170	9.3568	10.640	12.685	16.822	23.318	28.571	39.113	46.163	47.749	48.007	48.044
440	3.6042	4.3906	6.1249	9.3797	10.671	12.734	16.919	23.529	28.917	39.850	47.275	48.962	49.238	49.277
460	3.6060	4.3937	6.1321	9.4009	10.701	12.779	17.009	23.729	29.244	40.561	48.360	50.149	50.443	50.485
480	3.6077	4.3965	6.1388	9.4205	10.728	12.821	17.093	22.917	29.556	41.247	49.418	51.310	51.623	51.668

$1-\lambda$ / z	0.075	0.050	0.025	0.01	$7.5\cdot10^{-3}$	$5\cdot10^{-3}$	$2.5\cdot10^{-3}$	10^{-3}	$5\cdot10^{-4}$	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
500	3.6092	4.3991	6.1450	9.4389	10.753	12.860	17.172	24.095	29.852	41.909	50.452	52.449	52.780	52.828
550	3.6126	4.4049	6.1588	9.4796	10.810	12.948	17.350	24.501	30.534	43.470	52.938	55.202	55.581	55.636
600	3.6154	4.4098	6.1706	9.5145	10.858	13.023	17.505	24.859	31.145	44.913	55.296	57.836	58.264	58.326
650	3.6179	4.4140	6.1806	9.5447	10.900	13.089	17.641	25.178	31.695	46.254	57.549	60.364	60.843	60.912
700	3.6200	4.4176	6.1894	9.5711	10.937	13.147	17.761	25.464	32.195	47.504	59.700	62.798	63.330	63.407
750	3.6219	4.4208	6.1971	9.5944	10.969	13.198	17.869	25.723	32.651	48.675	61.762	65.148	65.733	65.819
800	3.6235	4.4236	6.2040	9.6152	10.998	13.244	17.966	25.958	33.069	49.776	63.744	67.422	68.062	68.156
850	3.6250	4.4262	6.2101	9.6338	11.024	13.285	18.053	26.172	33.455	50.813	65.652	69.626	70.322	70.424
900	3.6263	4.4284	6.2155	9.6505	11.048	13.323	18.133	26.369	33.811	51.792	67.493	71.766	72.521	72.631
950	3.6275	4.4304	6.2205	9.6657	11.069	13.357	18.206	26.550	34.142	52.720	69.272	73.848	74.661	74.781
1000	3.6286	4.4323	6.2250	9.6796	11.088	13.388	18.272	26.718	34.451	53.602	70.994	75.876	76.749	76.877
1100	3.6304	4.4355	6.2329	9.7039	11.123	13.443	18.391	27.019	35.009	55.238	74.283	79.784	80.779	80.927
1200	3.6320	4.4382	6.2395	9.7246	11.152	13.490	18.493	27.281	35.501	56.729	77.387	83.516	84.638	84.806
1300	3.6334	4.4406	6.2452	9.7424	11.177	13.530	18.582	27.511	35.938	58.096	80.328	87.094	88.347	88.534
1400	3.6345	4.4426	6.2502	9.7580	11.199	13.566	18.660	27.716	36.331	59.356	83.124	90.534	91.920	92.129
1500	3.6356	4.4443	6.2546	9.7716	11.218	13.597	18.729	27.899	36.685	60.521	85.790	93.850	95.373	95.604
1600	3.6365	4.4459	6.2584	9.7838	11.236	13.624	18.791	28.064	37.007	61.605	88.339	97.054	93.718	98.970
1700	3.6373	4.4473	6.2618	9.7946	11.251	13.649	18.846	28.213	37.301	62.615	90.782	100.16	101.96	102.24
1800	3.6380	4.4485	6.2649	9.8043	11.265	13.672	18.897	28.350	37.570	63.561	93.127	103.17	105.12	105.42
1900	3.6386	4.4496	6.2677	9.8131	11.277	13.693	18.943	28.475	37.818	64.448	95.383	106.09	108.19	108.51
2000	3.6392	4.4506	6.2702	9.8211	11.289	13.712	18.985	28.589	38.047	65.283	97.557	108.93	111.18	111.53
2200	3.6402	4.4524	6.2746	9.8351	11.309	13.744	19.058	28.793	38.458	66.815	101.68	114.40	116.96	117.36
2400	3.6411	4.4539	6.2783	9.8470	11.326	13.772	19.122	28.968	38.815	68.190	105.54	119.61	122.49	122.94
2600	3.6418	4.4552	6.2814	9.8572	11.340	13.796	19.176	29.122	39.130	69.433	109.16	124.60	127.80	128.30
2800	3.6424	4.4563	6.2842	9.8660	11.353	13.817	19.224	29.257	39.409	70.563	112.59	129.38	132.91	133.47

$1-\lambda$ / z	0.075	0.050	0.025	0.01	$7.5\cdot10^{-3}$	$5\cdot10^{-3}$	$2.5\cdot10^{-3}$	10^{-3}	$5\cdot10^{-4}$	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
3000	3.6430	4.4572	6.2866	9.8738	11.364	13.835	19.266	29.377	39.659	71.597	115.83	133.98	137.85	138.47
3200	3.6435	4.4581	6.2887	9.8807	11.374	13.851	19.304	29.484	39.883	72.547	118.90	138.42	142.63	143.31
3400	3.6439	4.4588	6.2906	9.8868	11.383	13.866	19.337	29.581	40.087	73.424	121.83	142.70	147.27	148.00
3600	3.6443	4.4595	6.2923	9.8923	11.391	13.879	19.368	29.668	40.274	74.236	124.62	146.85	151.78	152.57
3800	3.6446	4.4601	6.2938	9.8973	11.398	13.891	19.395	29.748	40.443	74.992	127.28	150.88	156.16	157.02
4000	3.6450	4.4606	6.2952	9.9018	11.404	13.902	19.420	29.821	40.600	75.697	129.84	154.78	160.44	161.35
4200	3.6452	4.4611	6.2965	9.9059	11.410	13.911	19.443	29.888	40.743	76.356	132.29	158.58	164.60	165.59
4400	3.6455	4.4616	6.2976	9.9096	11.416	13.920	19.464	29.950	40.877	76.975	134.64	162.28	168.68	169.73
4600	3.6457	4.4620	6.2987	9.9131	11.421	13.929	19.484	30.008	41.001	77.557	136.91	165.88	172.66	173.78
4800	3.6460	4.4624	6.2996	9.9163	11.425	13.936	19.501	30.061	41.116	78.105	139.09	169.39	176.56	177.74
5000	3.6462	4.4628	6.3005	9.9192	11.429	13.943	19.518	30.110	41.223	78.623	141.19	172.82	180.37	181.63
5500	3.6466	4.4635	6.3025	9.9257	11.439	13.959	19.555	30.220	41.464	79.801	146.14	181.06	189.60	191.03
6000	3.6470	4.4742	6.3042	9.9312	11.447	13.972	19.586	30.314	41.671	80.838	150.70	188.87	198.42	200.03
6500	3.6473	4.4648	6.3056	9.9359	11.454	13.984	19.613	30.396	41.851	81.759	154.92	196.31	206.88	208.68
7000	3.6476	4.4652	6.3068	9.9400	11.459	13.994	19.637	30.467	42.010	82.584	158.86	203.41	215.02	217.01
7500	3.6478	4.4657	6.3079	9.9436	11.465	14.002	19.658	30.530	42.150	83.328	162.53	210.21	222.87	225.06
8000	3.6480	4.4660	6.3088	9.9467	11.469	14.010	19.676	30.585	42.275	84.003	165.97	216.73	230.46	232.86
8500	3.6482	4.4664	6.3097	9.9495	11.473	14.017	19.692	30.636	42.388	84.618	169.21	223.01	237.82	240.42
9000	3.6484	4.4667	6.3104	9.9520	11.477	14.023	19.707	30.681	42.490	85.181	172.27	229.06	244.96	247.78
9500	3.6485	4.4669	6.3111	9.9543	11.480	14.028	19.720	30.722	42.583	85.700	175.15	234.90	251.90	254.93
10000	3.6487	4.4672	6.3117	9.9563	11.483	14.033	19.732	30.759	42.668	86.179	177.89	240.55	258.66	261.91

TABLE 3

Values of the functions $H(z, 1)$ and $G_1(z)$

z	$H(z,1)$	$G_1(z)$	z	$H(z,1)$	$G_1(z)$	z	$H(z,1)$	$G_1(z)$	z	$H(z,1)$	$G_1(z)$
0.00	1.0000	∞	8.0	5.3180	$2.5617\cdot10^{-2}$	90	20.556	$6.0244\cdot10^{-4}$	1000	76.897	$1.4023\cdot10^{-5}$
0.05	1.0887	2.4733	8.5	5.4970	$2.3406\cdot10^{-2}$	95	21.185	$5.5340\cdot10^{-4}$	1100	80.950	$1.2097\cdot10^{-5}$
0.1	1.1566	1.9065	9.0	5.6718	$2.1488\cdot10^{-2}$	100	21.798	$5.1058\cdot10^{-4}$	1200	84.832	$1.0571\cdot10^{-5}$
0.2	1.2743	1.3546	9.5	5.8427	$1.9814\cdot10^{-2}$	110	22.985	$4.3962\cdot10^{-4}$	1300	88.563	$9.3385\cdot10^{-6}$
0.3	1.3791	1.0553	10	6.0100	$1.8341\cdot10^{-2}$	120	24.123	$3.8350\cdot10^{-4}$	1400	92.161	$8.3263\cdot10^{-6}$
0.4	1.4759	$8.6085\cdot10^{-1}$	11	6.3349	$1.5880\cdot10^{-2}$	130	25.218	$3.3824\cdot10^{-4}$	1500	95.639	$7.4831\cdot10^{-6}$
0.5	1.5671	$7.2297\cdot10^{-1}$	12	6.6478	$1.3915\cdot10^{-2}$	140	26.275	$3.0112\cdot10^{-4}$	1600	99.009	$6.7721\cdot10^{-6}$
0.6	1.6538	$6.1986\cdot10^{-1}$	13	6.9502	$1.2316\cdot10^{-2}$	150	27.298	$2.7024\cdot10^{-4}$	1700	102.28	$6.1660\cdot10^{-6}$
0.7	1.7369	$5.3985\cdot10^{-1}$	14	7.2431	$1.0996\cdot10^{-2}$	160	28.290	$2.4424\cdot10^{-4}$	1800	105.46	$5.6445\cdot10^{-6}$
0.8	1.8168	$4.7604\cdot10^{-1}$	15	7.5273	$9.8916\cdot10^{-3}$	170	29.254	$2.2210\cdot10^{-4}$	1900	108.56	$5.1920\cdot10^{-6}$
0.9	1.8941	$4.2406\cdot10^{-1}$	16	7.8038	$8.9570\cdot10^{-3}$	180	30.192	$2.0308\cdot10^{-4}$	2000	111.58	$4.7964\cdot10^{-6}$
1.0	1.9691	$3.8098\cdot10^{-1}$	17	8.0730	$8.1580\cdot10^{-3}$	190	31.105	$1.8659\cdot10^{-4}$	2200	117.42	$4.1398\cdot10^{-6}$
1.2	2.1129	$3.1394\cdot10^{-1}$	18	8.3356	$7.4689\cdot10^{-3}$	200	31.997	$1.7219\cdot10^{-4}$	2400	123.01	$3.6193\cdot10^{-6}$
1.4	2.2491	$2.6446\cdot10^{-1}$	19	8.5921	$6.8697\cdot10^{-3}$	220	33.721	$1.4832\cdot10^{-4}$	2600	128.38	$3.1988\cdot10^{-6}$
1.6	2.3809	$2.2667\cdot10^{-1}$	20	8.8429	$6.3451\cdot10^{-3}$	240	35.374	$1.2945\cdot10^{-4}$	2800	133.56	$2.8532\cdot10^{-6}$
1.8	2.5070	$1.9702\cdot10^{-1}$	22	9.3289	$5.4728\cdot10^{-3}$	260	36.963	$1.1422\cdot10^{-4}$	3000	138.57	$2.5653\cdot10^{-6}$
2.0	2.6288	$1.7324\cdot10^{-1}$	24	9.7963	$4.7802\cdot10^{-3}$	280	38.497	$1.0172\cdot10^{-4}$	3200	143.42	$2.3223\cdot10^{-6}$
2.2	2.7466	$1.5383\cdot10^{-1}$	26	10.247	$4.2200\cdot10^{-3}$	300	39.980	$9.1330\cdot10^{-5}$	3400	148.12	$2.1152\cdot10^{-6}$
2.4	2.8610	$1.3774\cdot10^{-1}$	28	10.683	$3.3754\cdot10^{-3}$	320	41.418	$8.2573\cdot10^{-5}$	3600	152.70	$1.9368\cdot10^{-6}$
2.6	2.9722	$1.2423\cdot10^{-1}$	30	11.106	$3.7593\cdot10^{-3}$	340	42.814	$7.5117\cdot10^{-5}$	3800	157.16	$1.7820\cdot10^{-6}$
2.8	3.0805	$1.1275\cdot10^{-1}$	32	11.516	$3.0515\cdot10^{-3}$	360	44.173	$6.8707\cdot10^{-5}$	4000	161.50	$1.6467\cdot10^{-6}$
3.0	3.1863	$1.0291\cdot10^{-1}$	34	11.915	$2.7755\cdot10^{-3}$	380	45.497	$6.3149\cdot10^{-5}$	4200	165.75	$1.5275\cdot10^{-6}$
3.2	3.2896	$9.4397\cdot10^{-2}$	36	12.304	$2.5379\cdot10^{-3}$	400	46.788	$5.8294\cdot10^{-5}$	4400	169.90	$1.4218\cdot10^{-6}$
3.4	3.3906	$8.6973\cdot10^{-2}$	38	12.684	$2.3319\cdot10^{-3}$	420	48.050	$5.4024\cdot10^{-5}$	4600	173.96	$1.3279\cdot10^{-6}$
3.6	3.4896	$8.0455\cdot10^{-2}$	40	13.055	$2.1518\cdot10^{-3}$	440	49.283	$5.0246\cdot10^{-5}$	4800	177.94	$1.2437\cdot10^{-6}$
3.8	3.5866	$7.4696\cdot10^{-2}$	42	13.418	$1.9934\cdot10^{-3}$	460	50.491	$4.6883\cdot10^{-5}$	5000	181.83	$1.1680\cdot10^{-6}$
4.0	3.6818	$6.9579\cdot10^{-2}$	44	13.773	$1.8531\cdot10^{-3}$	480	51.674	$4.3876\cdot10^{-5}$	5500	191.27	$1.0088\cdot10^{-6}$
4.2	3.7753	$6.5009\cdot10^{-2}$	46	14.121	$1.7283\cdot10^{-3}$	500	52.835	$4.1174\cdot10^{-5}$	6000	200.30	$8.8242\cdot10^{-7}$
4.4	3.8671	$6.0908\cdot10^{-2}$	48	14.462	$1.6167\cdot10^{-3}$	550	55.644	$3.5496\cdot10^{-5}$	6500	208.98	$7.8025\cdot10^{-7}$
4.6	3.9575	$5.7211\cdot10^{-2}$	50	14.797	$1.5164\cdot10^{-3}$	600	58.335	$3.1002\cdot10^{-5}$	7000	217.35	$6.9634\cdot10^{-7}$
4.8	4.0464	$5.3866\cdot10^{-2}$	55	15.610	$1.3057\cdot10^{-3}$	650	60.923	$2.7374\cdot10^{-5}$	7500	225.43	$6.2621\cdot10^{-7}$
5.0	4.1339	$5.0828\cdot10^{-2}$	60	16.389	$1.1389\cdot10^{-3}$	700	63.419	$2.4395\cdot10^{-5}$	8000	233.26	$5.6721\cdot10^{-7}$
5.5	4.3471	$4.4340\cdot10^{-2}$	65	17.140	$1.0044\cdot10^{-3}$	750	65.832	$2.1916\cdot10^{-5}$	8500	240.86	$5.1672\cdot10^{-7}$
6.0	4.5531	$3.9099\cdot10^{-2}$	70	17.866	$8.9402\cdot10^{-4}$	800	68.170	$1.9825\cdot10^{-5}$	9000	248.25	$4.7331\cdot10^{-7}$
6.5	4.7526	$3.4794\cdot10^{-2}$	75	18.568	$8.0220\cdot10^{-4}$	850	70.440	$1.8045\cdot10^{-5}$	9500	255.45	$4.3563\cdot10^{-7}$
7.0	4.9463	$3.1210\cdot10^{-2}$	80	19.249	$7.2487\cdot10^{-4}$	900	72.648	$1.6513\cdot10^{-5}$	10000	262.47	$4.0270\cdot10^{-7}$
7.5	5.1346	$2.8190\cdot10^{-2}$	85	19.911	$6.5903\cdot10^{-4}$	950	74.799	$1.5184\cdot10^{-5}$			

TABLE 4

Values of the function h(g)

q	*h* (*q*)	*q*	*h* (*q*)	*q*	*h* (*q*)
0.00	1.000	0.34	0.781	0.68	0.687
0.01	0.982	0.35	0.778	0.69	0.685
0.02	0.969	0.36	0.774		
0.03	0.958	0.37	0.771	0.70	0.683
0.04	0.947	0.38	0.767	0.71	0.681
0.05	0.938	0.39	0.764	0.72	0.679
0.06	0.929			0.73	0.677
0.07	0.921	0.40	0.761	0.74	0.675
0.08	0.913	0.41	0.758	0.75	0.673
0.09	0.905	0.42	0.755	0.76	0.671
		0.43	0.751	0.77	0.669
0.10	0.898	0.44	0.748	0.78	0.667
0.11	0.891	0.45	0.745	0.79	0.665
0.12	0.885	0.46	0.742		
0.13	0.879	0.47	0.740	0.80	0.663
0.14	0.873	0.48	0.737	0.81	0.661
0.15	0.867	0.49	0.734	0.82	0.659
0.16	0.861			0.83	0.657
0.17	0.856	0.50	0.731	0.84	0.655
0.18	0.850	0.51	0.728	0.85	0.653
0.19	0.845	0.52	0.726	0.86	0.652
		0.53	0.723	0.87	0.650
0.20	0.840	0.54	0.720	0.88	0.648
0.21	0.835	0.55	0.718	0.89	0.646
0.22	0.830	0.56	0.715		
0.23	0.826	0.57	0.713	0.90	0.645
0.24	0.821	0.58	0.710	0.91	0.643
0.25	0.817	0.59	0.708	0.92	0.641
0.26	0.813			0.93	0.640
0.27	0.808	0.60	0.705	0.94	0.638
0.28	0.804	0.61	0.703	0.95	0.636
0.29	0.800	0.62	0.701	0.96	0.635
		0.63	0.698	0.97	0.633
0.30	0.796	0.64	0.696	0.98	0.631
0.31	0.792	0.65	0.694	0.99	0.630
0.32	0.789	0.66	0.692		
0.33	0.785	0.67	0.689	1.00	0.628

Available from
HER MAJESTY'S STATIONERY OFFICE
49 High Holborn, London, W.C.1
423 Oxford Street, London W.1
13a Castle Street, Edinburgh 2
109 St. Mary Street, Cardiff
Brazennose Street, Manchester 2
50 Fairfax Street, Bristol 1
35 Smallbrook, Ringway, Birmingham 5
80 Chichester Street, Belfast
or through any bookseller.

Printed in England

S. O. Code No. 91-3-20-30