Notes on the calculation of the JETTO current

Emmi Tholerus

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1 Introduction

This document compiles notes on the plasma current density and its individual components as defined in JETTO. The derivations present any simplifying assumptions made. Unless stated otherwise, Gaussian units are used throughout this document, which is what is used internally in JETTO and in the JETTO manual [1]. As a general flux surface coordinate, we will use the non-normalised JETTO ρ in the derivations, defined as

$$\rho = \sqrt{\Psi_{\mathrm{tor}}/(\pi B_{\mathrm{geo}})}, \tag{1}$$

where Ψ_{tor} is the total toroidal magnetic flux, and B_{geo} is the vacuum toroidal magnetic field at the centre of the midplane (at $R = R_{\mathrm{geo}} = (R_{\mathrm{max}} + R_{\mathrm{min}})/2$).

1.1 Derivations

1.2 Magnetic field

JETTO assumes that the magnetic field can be expressed as

$$\boldsymbol{B} = F\nabla\phi + \frac{1}{2\pi}\nabla\phi \times \nabla\Psi_{\mathrm{pol}}, \tag{2}$$

where $F(\rho) = RB_\phi$ is the poloidal current stream function, ϕ is the toroidal coordinate ($\nabla\phi$ is in the clock-wise direction), and Ψ_{pol} is the total poloidal magnetic flux. This exact form of the magnetic field follows from assuming an axisymmetric equilibrium in MHD force balance. To show this, we note that the MHD force balance

$$\frac{\boldsymbol{J} \times \boldsymbol{B}}{c} = \nabla p = \frac{\mathrm{d}p}{\mathrm{d}\varrho}\nabla\varrho, \tag{3}$$

for an arbitraty flux label ϱ, implies that both $\boldsymbol{J}$ and $\boldsymbol{B}$ lie in a plane perpendicular to $\nabla\varrho$, i.e., they are tangential to the flux surface. Combining $\boldsymbol{J} \cdot \nabla\varrho = 0$ with Ampère's law, neglecting the displacement current, yields

$$(\nabla \times \boldsymbol{B}) \cdot \nabla\varrho = 0. \tag{4}$$

We choose to write the magnetic field on (normalised) covariant form according to

$$\boldsymbol{B} = h_\varrho B_\varrho \nabla\varrho + h_\theta B_\theta \nabla\theta + h_\phi B_\phi \nabla\phi, \tag{5}$$

where θ is an arbitrary poloidal coordinate, and

$$h_\varrho = \left|\nabla\varrho\right|^{-1},\ h_\theta = \left|\nabla\theta\right|^{-1},\ h_\phi = \left|\nabla\phi\right|^{-1} = R. \tag{6}$$

Inserted into eq. (4) yields

$$\Big(\nabla(h_\varrho B_\varrho)\times\nabla\varrho + \nabla(h_\theta B_\theta)\times\nabla\theta + \nabla(h_\phi B_\phi)\times\nabla\phi\Big)\cdot\nabla\varrho =$$

$$(\nabla\theta\times\nabla\varrho)\cdot\nabla(h_\theta B_\theta) + (\nabla\phi\times\nabla\varrho)\cdot\nabla(h_\phi B_\phi) =$$

$$\frac{1}{\mathcal{J}}\left(-\frac{\partial}{\partial\phi}(h_\theta B_\theta) + \frac{\partial}{\partial\theta}(h_\phi B_\phi)\right) = 0, \tag{7}$$

where

$$\mathcal{J} = \left(\frac{\partial\boldsymbol{r}}{\partial\varrho}\times\frac{\partial\boldsymbol{r}}{\partial\theta}\right)\cdot\frac{\partial\boldsymbol{r}}{\partial\phi} = \Big((\nabla\varrho\times\nabla\theta)\cdot\nabla\phi\Big)^{-1} \tag{8}$$

is the Jacobian. From axisymmetry, the first term of eq. (7) vanishes, and the equation simplifies to

$$\frac{\partial}{\partial\theta}(h_\phi B_\phi) = \frac{\partial}{\partial\theta}(RB_\phi) = 0. \tag{9}$$

Since R and B_ϕ are independent of ϕ due to axisymmetry, and RB_ϕ is independent of θ, it follows that RB_ϕ is a function of ϱ only, i.e., it is a flux label. We define this flux label as $F(\varrho) = RB_\phi$.

The poloidal field has to be tangential to the flux surfaces by definition, i.e., it is perpendicular to both $\nabla\phi$ and $\nabla\varrho$. A scalar field $\psi(\varrho)$ can be chosen such that

$$\boldsymbol{B}_{\text{pol}} = \nabla\phi\times\nabla\psi, \tag{10}$$

where $\psi = 0$ on-axis. We can then compute the poloidal magnetic flux as

$$\Psi_{\text{pol}}(\varrho) = \int_{S_{\text{P}}(\varrho,\theta)} \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B}_{\text{pol}} = \int_0^{2\pi}\mathrm{d}\phi\int_0^{\varrho}\mathrm{d}\tilde{\varrho}\,\mathcal{J}\nabla\theta\cdot(\nabla\phi\times\nabla\psi) =$$

$$\int_0^{2\pi}\mathrm{d}\phi\int_0^{\varrho}\mathrm{d}\tilde{\varrho}\,\frac{\cancel{\nabla\theta\cdot(\nabla\phi\times\nabla\tilde{\varrho})}}{\cancel{(\nabla\tilde{\varrho}\times\nabla\theta)\cdot\nabla\phi}}\frac{\mathrm{d}\psi}{\mathrm{d}\tilde{\varrho}} = \int_0^{2\pi}\mathrm{d}\phi\int_0^{\varrho}\mathrm{d}\tilde{\varrho}\,\frac{\mathrm{d}\psi}{\mathrm{d}\tilde{\varrho}} = 2\pi\int_0^{\psi(\varrho)}\mathrm{d}\tilde{\psi} =$$

$$2\pi\psi(\varrho)\ \Leftrightarrow\ \psi = \frac{\Psi_{\text{pol}}}{2\pi}, \tag{11}$$

where S_{P} is the surface at a constant θ (oriented in the $\nabla\theta$ direction) spanning from the magnetic axis to the flux surface ϱ. Combining the above conclusions results in a magnetic field on the form of eq. (2).

1.3 Flux surface averaging

Many parts of the derivations deal with flux surface averaged quantities. In JETTO, it is defined as the volume average of the quantity in a shell surrounding a given flux surface, in the limit of the thickness of the shell going to zero. For an arbitrary scalar field $f(R,z)$ (independent of toroidal coordinate ϕ) the flux surface average is defined as

$$\langle f\rangle = \lim_{\delta\rho\to 0}\frac{1}{V'\delta\rho}\int_{\rho}^{\rho+\delta\rho}\mathrm{d}V\,f = \lim_{\delta\rho\to 0}\frac{1}{V'\delta\rho}\oint_{\partial S}\mathrm{d}\ell\int_{\rho}^{\rho+\delta\rho}\mathrm{d}\rho\,\frac{2\pi R}{|\nabla\rho|}f$$

$$= \frac{\Psi'_{\mathrm{pol}}}{V'}\oint_{\partial S}\mathrm{d}\ell\,\frac{2\pi R}{|\nabla\Psi_{\mathrm{pol}}|}f = \frac{\Psi'_{\mathrm{pol}}}{V'}\oint_{\partial S}\frac{\mathrm{d}\ell}{B_{\mathrm{pol}}}f, \tag{12}$$

where any primed quantities are derivatives with respect to ρ. Setting $f=1$, it can be identified that

$$\frac{\Psi'_{\mathrm{pol}}}{V'} = \left(\oint_{\partial S}\frac{\mathrm{d}\ell}{B_{\mathrm{pol}}}\right)^{-1}. \tag{13}$$

The flux surface average of the divergence of an arbitrary vector field $\boldsymbol{A}$ (independent of ϕ) can be written as

$$\langle\nabla\cdot\boldsymbol{A}\rangle = \lim_{\delta\rho\to 0}\frac{1}{V'\delta\rho}\int_{\rho}^{\rho+\delta\rho}\mathrm{d}V\,\nabla\cdot\boldsymbol{A} = \frac{1}{V'}\lim_{\delta\rho\to 0}\frac{1}{\delta\rho}\left(\oint_{\Sigma+\delta\rho}\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{A} - \oint_{\Sigma}\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{A}\right) =$$

$$\frac{1}{V'}\frac{\partial}{\partial\rho}\oint_{\Sigma}\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{A} = \frac{1}{V'}\frac{\partial}{\partial\rho}\oint_{\partial S}\mathrm{d}\ell\,\frac{2\pi R}{|\nabla V|}\boldsymbol{A}\cdot\nabla V =$$

$$\frac{1}{V'}\frac{\partial}{\partial\rho}\left(\frac{\Psi'_{\mathrm{pol}}}{V'}\oint_{\partial S}\frac{\mathrm{d}\ell}{B_{\mathrm{pol}}}\boldsymbol{A}\cdot\nabla V\right) \underset{(12)}{=} \frac{1}{V'}\frac{\partial}{\partial\rho}\langle\boldsymbol{A}\cdot\nabla V\rangle, \tag{14}$$

where Σ is the surface of the flux tube at ρ, oriented radially outwards, and $\Sigma+\delta\rho$ is the surface of the perturbed flux tube located at $\rho+\delta\rho$.

A scalar field f integrated over the cross sectional area is

$$\int_S\mathrm{d}S\,f = \int_0^{\Psi_{\mathrm{pol}}}\mathrm{d}\Psi\oint_{\partial S}\frac{\mathrm{d}\ell}{|\nabla\Psi|}f = \frac{1}{2\pi}\int_0^{\Psi_{\mathrm{pol}}}\mathrm{d}\Psi\oint_{\partial S}\frac{\mathrm{d}\ell}{B_{\mathrm{pol}}}\frac{f}{R} \underset{(12)}{=}$$

$$\frac{1}{2\pi}\int_0^{\Psi_{\mathrm{pol}}}\mathrm{d}\Psi\,\frac{V'}{\Psi'}\left\langle\frac{f}{R}\right\rangle = \frac{1}{2\pi}\int\mathrm{d}V\left\langle\frac{f}{R}\right\rangle. \tag{15}$$

Another useful relationship can be derived from the definition of the safety factor

$$q = \frac{\partial\Psi_{\mathrm{tor}}}{\partial\Psi_{\mathrm{pol}}} = \frac{\partial}{\partial\Psi_{\mathrm{pol}}}\int_S\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B} = \frac{\partial}{\partial\Psi_{\mathrm{pol}}}\int_S\mathrm{d}S\,\frac{F}{R} \underset{(15)}{=}$$

$$\frac{1}{2\pi}\frac{\partial}{\partial\Psi_{\rm pol}}\int_V \mathrm{d}V\,F\langle R^{-2}\rangle = \frac{1}{2\pi}\frac{\partial}{\partial\Psi_{\rm pol}}\int_0^{\Psi_{\rm pol}}\mathrm{d}\Psi\frac{FAV'}{\Psi'} = \frac{FAV'}{2\pi\Psi'_{\rm pol}}$$
$$\Leftrightarrow\ \frac{\Psi'_{\rm pol}}{V'} = \frac{FA}{2\pi q}, \tag{16}$$

where we have introduced the definition $A = \langle R^{-2}\rangle$. Furthermore, it can be shown that

$$\frac{FAV'}{2\pi} = \Psi'_{\rm pol}q = \frac{\partial\Psi_{\rm pol}}{\partial\rho}\frac{\partial\Psi_{\rm tor}}{\partial\Psi_{\rm pol}} = \Psi'_{\rm tor} \underset{(1)}{=} 2\pi\rho B_{\rm geo}\ \Leftrightarrow\ \frac{V'}{4\pi^2\rho} = \frac{B_{\rm geo}}{FA}. \tag{17}$$

1.4 Plasma current

The total plasma current inside a flux surface is defined as (from JETTO manual [1], eq. (2.6)):

$$I_{\rm p} = \frac{c}{32\pi^4}FAK\iota, \tag{18}$$

where c is the speed of light, $F = RB_\phi$, $A = \langle R^{-2}\rangle$, $K = \langle|\nabla V|^2/R^2\rangle$, and $\iota = 1/q$. To show this, we start from Ampère's law, without the displacement current:

$$I_{\rm p} = \frac{c}{4\pi}\oint_{\partial S}\mathrm{d}\boldsymbol{\ell}\cdot\boldsymbol{B}_{\rm pol} = \frac{c}{8\pi^2}\oint_{\partial S}\mathrm{d}\boldsymbol{\ell}\cdot(\nabla\phi\times\nabla\Psi_{\rm pol}). \tag{19}$$

Since $\mathrm{d}\boldsymbol{\ell}$ is tangential to the flux surface in the poloidal plane, it is perpendicular to both $\nabla\phi$ and $\nabla\Psi_{\rm pol}$, and eq. (19) simplifies to

$$I_{\rm p} = \frac{c}{4\pi}\oint_{\partial S}\mathrm{d}\ell\,B_{\rm pol} = \frac{c}{4\pi}\oint_{\partial S}\frac{\mathrm{d}\ell}{B_{\rm pol}}B_{\rm pol}^2 = \frac{c}{16\pi^3}\oint_{\partial S}\frac{\mathrm{d}\ell}{B_{\rm pol}}\frac{|\nabla\Psi_{\rm pol}|^2}{R^2} =$$

$$\frac{c}{16\pi^3}\left(\frac{\Psi'_{\rm pol}}{V'}\right)^2\oint_{\partial S}\frac{\mathrm{d}\ell}{B_{\rm pol}}\frac{|\nabla V|^2}{R^2} \underset{(12)}{=} \frac{c}{16\pi^3}\frac{\Psi'_{\rm pol}}{V'}\left\langle\frac{|\nabla V|^2}{R^2}\right\rangle = \frac{c}{16\pi^3}\frac{\Psi'_{\rm pol}K}{V'}. \tag{20}$$

Using this together with eq. (16) yields

$$I_{\rm p} = \frac{c}{32\pi^4}\frac{FAK}{q} = \frac{c}{32\pi^4}FAK\iota. \tag{21}$$

1.5 $\langle\boldsymbol{J}\cdot\boldsymbol{B}\rangle$

Equation (3.2) in the JETTO manual reads

$$\langle\boldsymbol{J}\cdot\boldsymbol{B}\rangle = \frac{cF^3A}{64\pi^5\rho B_{\rm geo}}\frac{\partial}{\partial\rho}\Big(AK\iota\Big). \tag{22}$$

In order to derive this, we again start from Ampère's law:

$$\frac{4\pi}{c}\boldsymbol{J} = \nabla\times\boldsymbol{B} = \nabla\times\left(F\nabla\phi + \frac{1}{2\pi}\nabla\phi\times\nabla\Psi_{\text{pol}}\right) = \nabla F\times\nabla\phi +$$

$$\frac{1}{2\pi}\Big((\nabla\Psi_{\text{pol}}\cdot\nabla)\nabla\phi + \nabla\phi\nabla^2\Psi_{\text{pol}} - (\nabla\Psi_{\text{pol}}\cdot\nabla)\nabla\phi - \nabla\Psi_{\text{pol}}\cancel{\nabla^2\phi}\Big) =$$

$$-\frac{2\pi F'}{\Psi'_{\text{pol}}}\boldsymbol{B}_{\text{pol}} + \frac{1}{2\pi}\left[\left(\nabla^2\Psi_{\text{pol}} - \frac{1}{R}\frac{\partial\Psi_{\text{pol}}}{\partial R}\right)\nabla\phi - \frac{1}{R^2}\frac{\partial\nabla\Psi_{\text{pol}}}{\partial\phi}\right] =$$

$$-\frac{2\pi F'}{\Psi'_{\text{pol}}}\boldsymbol{B}_{\text{pol}} + \frac{1}{2\pi}\left[\left(\nabla^2\Psi_{\text{pol}} - \frac{1}{R}\frac{\partial\Psi_{\text{pol}}}{\partial R}\right)\nabla\phi - \frac{1}{R^2}\frac{\partial\Psi_{\text{pol}}}{\partial R}\frac{\partial\hat{R}}{\partial\phi}\right] =$$

$$-\frac{2\pi F'}{\Psi'_{\text{pol}}}\boldsymbol{B}_{\text{pol}} + \frac{1}{2\pi}\left(\nabla^2\Psi_{\text{pol}} - \frac{2}{R}\frac{\partial\Psi_{\text{pol}}}{\partial R}\right)\nabla\phi =$$

$$-\frac{2\pi F'}{\Psi'_{\text{pol}}}\boldsymbol{B}_{\text{pol}} + \frac{R^2}{2\pi}\nabla\cdot\left(\frac{\nabla\Psi_{\text{pol}}}{R^2}\right)\nabla\phi. \tag{23}$$

It follows that

$$\boldsymbol{J}\cdot\boldsymbol{B} = -\frac{cF'B^2_{\text{pol}}}{2\Psi'_{\text{pol}}} + \frac{cF}{8\pi^2}\nabla\cdot\left(\frac{\nabla\Psi_{\text{pol}}}{R^2}\right) = \frac{c}{8\pi^2}\left[F\nabla\cdot\left(\frac{\nabla\Psi_{\text{pol}}}{R^2}\right) - \frac{F'|\nabla\Psi_{\text{pol}}|^2}{\Psi'_{\text{pol}}R^2}\right] =$$

$$\frac{c}{8\pi^2}\left[F\nabla\cdot\left(\frac{\nabla\Psi_{\text{pol}}}{R^2}\right) - \frac{\nabla F\cdot\nabla\Psi_{\text{pol}}}{R^2}\right] = \frac{cF^2}{8\pi^2}\nabla\cdot\left(\frac{\nabla\Psi_{\text{pol}}}{FR^2}\right). \tag{24}$$

Taking the flux surface average of the above expression yields

$$\langle\boldsymbol{J}\cdot\boldsymbol{B}\rangle = \frac{cF^2}{8\pi^2}\left\langle\nabla\cdot\left(\frac{\nabla\Psi_{\text{pol}}}{FR^2}\right)\right\rangle \underset{(14)}{=} \frac{cF^2}{8\pi^2V'}\frac{\partial}{\partial\rho}\left\langle\nabla V\cdot\left(\frac{\nabla\Psi_{\text{pol}}}{FR^2}\right)\right\rangle =$$

$$\frac{cF^2}{8\pi^2V'}\frac{\partial}{\partial\rho}\left(\frac{\Psi'_{\text{pol}}}{FV'}\left\langle\frac{|\nabla V|^2}{R^2}\right\rangle\right) = \frac{cF^2}{8\pi^2V'}\frac{\partial}{\partial\rho}\left(\frac{K\Psi'_{\text{pol}}}{FV'}\right) \underset{(16)}{=} \frac{cF^2}{16\pi^3V'}\frac{\partial}{\partial\rho}\Big(AK\iota\Big) \underset{(17)}{=}$$

$$\frac{cF^3A}{64\pi^5\rho B_{\text{geo}}}\frac{\partial}{\partial\rho}\Big(AK\iota\Big). \tag{25}$$

1.6 Current diffusion equation

By using Faraday's law together with the definition of the magnetic field in eq. (2), it can be shown that [3]

$$\left.\frac{\partial\Psi_{\text{pol}}}{\partial t}\right|_{\Psi_{\text{tor}}} = \frac{2\pi c}{FA}\langle\boldsymbol{E}\cdot\boldsymbol{B}\rangle. \tag{26}$$

From Ohm's law follows that

$$\boldsymbol{E} \cdot \boldsymbol{B} = \eta(\boldsymbol{J} - \boldsymbol{J}_{\mathrm{NI}}) \cdot \boldsymbol{B}, \tag{27}$$

where $\boldsymbol{J}_{\mathrm{NI}}$ is the non-inductive current density component, and $\boldsymbol{J}$ is the total current density. Inserted into eq. (26) yields

$$\left.\frac{\partial \Psi_{\mathrm{pol}}}{\partial t}\right|_{\Psi_{\mathrm{tor}}} = \frac{2\pi c\eta}{FA}\Big(\langle \boldsymbol{J} \cdot \boldsymbol{B}\rangle - \langle \boldsymbol{J}_{\mathrm{NI}} \cdot \boldsymbol{B}\rangle\Big). \tag{28}$$

The flux surface averaged $\boldsymbol{J} \cdot \boldsymbol{B}$ is

$$\langle \boldsymbol{J} \cdot \boldsymbol{B}\rangle \underset{(22)}{=} \frac{cF^3 A}{64\pi^5 \rho B_{\mathrm{geo}}} \frac{\partial}{\partial \rho}\Big(AK\iota\Big) \;\Leftrightarrow\; \frac{2\pi c}{FA}\langle \boldsymbol{J} \cdot \boldsymbol{B}\rangle = \frac{c^2 F^2}{32\pi^4 B_{\mathrm{geo}}} \frac{1}{\rho}\frac{\partial}{\partial \rho}\Big(AK\iota\Big), \tag{29}$$

where $\iota = 1/q$. Inserting eq. (29) into (28) yields

$$\left.\frac{\partial \Psi_{\mathrm{pol}}}{\partial t}\right|_{\Psi_{\mathrm{tor}}} = \frac{c^2 F^2 \eta}{32\pi^4 B_{\mathrm{geo}}} \frac{1}{\rho}\frac{\partial}{\partial \rho}\Big(AK\iota\Big) - \frac{2\pi c\eta}{FA}\langle \boldsymbol{J}_{\mathrm{NI}} \cdot \boldsymbol{B}\rangle \;\Rightarrow$$

$$\frac{\partial \iota}{\partial t} = \frac{\partial}{\partial \Psi_{\mathrm{tor}}} \left.\frac{\partial \Psi_{\mathrm{pol}}}{\partial t}\right|_{\Psi_{\mathrm{tor}}} = \frac{1}{2\pi B_{\mathrm{geo}}} \frac{1}{\rho}\frac{\partial}{\partial \rho} \left.\frac{\partial \Psi_{\mathrm{pol}}}{\partial t}\right|_{\Psi_{\mathrm{tor}}}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{c^2 F^2 \eta}{64\pi^5 B_{\mathrm{geo}}^2}\frac{1}{\rho}\frac{\partial}{\partial \rho}\Big(AK\iota\Big) - \frac{c\eta}{B_{\mathrm{geo}} FA}\langle \boldsymbol{J}_{\mathrm{NI}} \cdot \boldsymbol{B}\rangle\right). \tag{30}$$

Substituting $u_1 = B_{\mathrm{geo}} AK\iota / R_{\mathrm{geo}}$ into eq. (30) yields

$$\frac{1}{AK}\frac{\partial u_1}{\partial t} - \frac{u_1}{A^2 K^2}\frac{\partial}{\partial t}(AK)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{c^2 F^2 \eta}{64\pi^5 B_{\mathrm{geo}}^2}\frac{1}{\rho}\frac{\partial u_1}{\partial \rho} - \frac{c\eta}{R_{\mathrm{geo}} FA}\langle \boldsymbol{J}_{\mathrm{NI}} \cdot \boldsymbol{B}\rangle\right). \tag{31}$$

In JETTO, it is assumed that the second term of the left hand side is small in comparison to the first term, i.e.,

$$\frac{1}{u_1}\frac{\partial u_1}{\partial t} \gg \frac{1}{AK}\frac{\partial}{\partial t}(AK), \tag{32}$$

which yields the final expression for the current diffusion equation, as implemented in JETTO:

$$\frac{1}{AK}\frac{\partial u_1}{\partial t} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{c^2 F^2 \eta}{64\pi^5 B_{\mathrm{geo}}^2}\frac{1}{\rho}\frac{\partial u_1}{\partial \rho} - \frac{c\eta}{R_{\mathrm{geo}} FA}\langle \boldsymbol{J}_{\mathrm{NI}} \cdot \boldsymbol{B}\rangle\right). \tag{33}$$

1.7 Loop voltage

The loop voltage can be computed as

$$V_{\text{loop}} = -\oint_{C_{\text{HFS}}} \mathrm{d}\boldsymbol{\ell}\cdot\boldsymbol{E}, \tag{34}$$

where C_{HFS} is the clockwise closed curve where the midplane intersects with the last closed flux surface on the high field side. By adding a curve C_{ax} in the clockwise direction along the magnetic axis, we have

$$V_{\text{loop}} = \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma}\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B} - \oint_{C_{\text{ax}}}\mathrm{d}\boldsymbol{\ell}\cdot\boldsymbol{E}, \tag{35}$$

where Σ is the ring shaped surface bounded by C_{HFS} and C_{ax} oriented downwards. The first term is (assuming a fixed boundary shape)

$$\frac{1}{c}\left.\frac{\mathrm{d}\Psi_{\text{pol,sep}}}{\mathrm{d}t}\right|_{\Psi_{\text{tor,sep}}} \underset{(28),(29)}{=} \frac{cF^2\eta}{32\pi^4 B_{\text{geo}}}\frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(AK\iota\Big)\bigg|_{\rho_{\text{sep}}} - \frac{2\pi\eta}{FA}\langle\boldsymbol{J}_{\text{NI}}\cdot\boldsymbol{B}\rangle\bigg|_{\rho_{\text{sep}}}. \tag{36}$$

If we substitute $u_1 = B_{\text{geo}}AK\iota/R_{\text{geo}}$ into the above equation, we find that

$$\frac{1}{c}\left.\frac{\mathrm{d}\Psi_{\text{pol,sep}}}{\mathrm{d}t}\right|_{\Psi_{\text{tor,sep}}} = \frac{cF^2\eta R_{\text{geo}}}{32\pi^4 B_{\text{geo}}^2}\frac{1}{\rho}\frac{\partial u_1}{\partial\rho}\bigg|_{\rho_{\text{sep}}} - \frac{2\pi\eta}{FA}\langle\boldsymbol{J}_{\text{NI}}\cdot\boldsymbol{B}\rangle\bigg|_{\rho_{\text{sep}}}. \tag{37}$$

The second term of eq. (35) is

$$-\int\mathrm{d}\phi\, R_{\text{ax}}\eta_{\text{ax}}(J_{\phi,\text{ax}} - J_{\text{NI,ax}}) = -2\pi R_{\text{ax}}\eta_{\text{ax}}(J_{\phi,\text{ax}} - J_{\text{NI,ax}}). \tag{38}$$

The on-axis current densities can be written in terms of the JETTO definition of current density using eq. (63):

$$J_{\phi,\text{ax}} = \frac{B_{\text{ax}}}{B_{\text{geo}}}J_{z,\text{ax}},\;\; J_{\text{NI,ax}} = \frac{B_{\text{ax}}}{B_{\text{geo}}}J_{z,\text{NI,ax}}. \tag{39}$$

The total loop voltage then becomes

$$V_{\text{loop}} = \frac{cF^2\eta R_{\text{geo}}}{32\pi^4 B_{\text{geo}}^2}\frac{1}{\rho}\frac{\partial u_1}{\partial\rho}\bigg|_{\rho_{\text{sep}}} - \frac{2\pi\eta}{FA}\langle\boldsymbol{J}_{\text{NI}}\cdot\boldsymbol{B}\rangle\bigg|_{\rho_{\text{sep}}} - 2\pi R_{\text{ax}}\eta_{\text{ax}}\frac{B_{\text{ax}}}{B_{\text{geo}}}\left(J_{z,\text{ax}} - J_{z,\text{NI,ax}}\right), \tag{40}$$

or in SI units

$$V_{\text{loop}} = \frac{F^2 R_{\text{geo}}\eta}{8\pi^3\mu_0 B_{\text{geo}}^2}\frac{1}{\rho}\frac{\partial u_1}{\partial\rho}\bigg|_{\rho_{\text{sep}}} - \frac{2\pi\eta}{FA}\langle\boldsymbol{J}_{\text{NI}}\cdot\boldsymbol{B}\rangle\bigg|_{\rho_{\text{sep}}} -$$

$$2\pi R_{\mathrm{ax}}\eta_{\mathrm{ax}}\frac{B_{\mathrm{ax}}}{B_{\mathrm{geo}}}\left(J_{z,\mathrm{ax}} - J_{z,\mathrm{NI,ax}}\right). \tag{41}$$

JETTO appears to only include the terms corresponding to $\mathrm{d}\Psi_{\mathrm{pol}}/\mathrm{d}t$, see [4], so it should be checked if the on-axis integral over $\boldsymbol{E}$ also needs to be included. A more compact way to express the loop voltage is

$$V_{\mathrm{loop}} = 2\pi \left.\frac{\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle}{FA}\right|_{\rho=0}^{\rho_{\mathrm{sep}}}, \tag{42}$$

which follows from the fact that $F_{\mathrm{ax}}A_{\mathrm{ax}} = B_{\mathrm{ax}}/R_{\mathrm{ax}}$.

When the loop voltage is set to zero as a boundary condition, eq. (42) implies that $\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle/(FA)$ is forced to have the same value on-axis as on the separatrix. Since the separatrix value, being proportional to $\mathrm{d}\Psi_{\mathrm{pol}}/\mathrm{d}t$, vanishes in stationary conditions, the on-axis $E_{\|}$ has to vanish at the same rate. This implies that the on-axis inductive current goes to zero, whereas it can remain non-zero when the on-axis terms are excluded from the loop voltage definition.

2 Current conversions

This section deals with conversions between $\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$, J_z and J_{tor}, where J_z is the JETTO definition of the current density, and $J_{\mathrm{tor}}(\rho)$ is the more conventional definition of the toroidal current density profile. Conversions for the total plasma current and for microwave current drive (such as ECCD and EBWCD) are handled separately.

2.1 Conversion from J_z to $\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$

2.1.1 Total current

The JETTO definition of the current density is

$$J_z = \frac{I_{\mathrm{p}}'}{2\pi\rho} \underset{(18)}{=} \frac{c}{64\pi^5\rho}\frac{\partial}{\partial\rho}\Big(FAK\iota\Big) = \frac{c}{64\pi^5\rho}\left(F\frac{\partial}{\partial\rho}\Big(AK\iota\Big) + F'AK\iota\right). \tag{43}$$

Using $\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$ from eq. (22)

$$\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle = \frac{cF^3A}{64\pi^5\rho B_{\mathrm{geo}}}\frac{\partial}{\partial\rho}\Big(AK\iota\Big) \;\Leftrightarrow\; \frac{cF}{64\pi^5\rho}\frac{\partial}{\partial\rho}\Big(AK\iota\Big) = \frac{B_{\mathrm{geo}}}{F^2A}\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle, \tag{44}$$

inserted into eq. (43), results in

$$J_z = \frac{B_{\mathrm{geo}}}{F^2A}\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle + \frac{cF'AK\iota}{64\pi^5\rho} = \frac{B_{\mathrm{geo}}}{F^2A}\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle + \frac{F'}{2\pi\rho F}I_{\mathrm{p}} \;\Rightarrow$$

$$\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle = \frac{F^2 A}{B_{\text{geo}}}\left(J_z - \frac{F'}{2\pi\rho F} I_{\text{p}}\right) = \frac{F^2 A}{B_{\text{geo}}}\left(J_z - \frac{1}{\pi F}\frac{\partial F}{\partial \rho^2} I_{\text{p}}\right). \tag{45}$$

The last step removes the singularity at $\rho = 0$, assuming that $\mathcal{O}(F') = \rho^n$ for $n \geq 1$.

2.1.2 Microwave current drive

For EC/EBW current drive, the driven current is parallel to the magnetic field, and the quantity J/B is a flux function [2]. The second property is derived in Sec. 3.1. These properties makes a different relationship between $\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$ and J_z compared to eq. (45). Based on these assumptions, $\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$ can be written as

$$\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle = \langle JB\rangle = \left\langle \frac{J}{B}B^2\right\rangle = \frac{J}{B}\left\langle B^2\right\rangle \;\Leftrightarrow\; \frac{J}{B} = \frac{\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle}{\langle B^2\rangle}. \tag{46}$$

$\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$ can also be written as

$$\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle = \langle JB\rangle = \left\langle \frac{J_\phi B^2}{B_\phi}\right\rangle = \langle J_\phi B_\phi\rangle + \left\langle \frac{J_\phi B_{\text{pol}}^2}{B_\phi}\right\rangle$$

$$= F\left\langle \frac{J_\phi}{R}\right\rangle + \frac{J}{B}\left\langle B_{\text{pol}}^2\right\rangle \underset{(46)}{=} F\left\langle \frac{J_\phi}{R}\right\rangle + \frac{\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle}{\langle B^2\rangle}\left\langle B_{\text{pol}}^2\right\rangle \;\Leftrightarrow\; F\left\langle \frac{J_\phi}{R}\right\rangle =$$

$$\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle\left(1 - \frac{\left\langle B_{\text{pol}}^2\right\rangle}{\langle B^2\rangle}\right) = \langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle \frac{\left\langle B_\phi^2\right\rangle}{\left\langle B_\phi^2\right\rangle + \left\langle B_{\text{pol}}^2\right\rangle} = \langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle\left(1 + \frac{\left\langle B_{\text{pol}}^2\right\rangle}{\left\langle B_\phi^2\right\rangle}\right)^{-1}. \tag{47}$$

We note that

$$\langle B_\phi^2\rangle = \left\langle \frac{F^2}{R^2}\right\rangle = F^2 A, \tag{48}$$

$$\left\langle B_{\text{pol}}^2\right\rangle = \frac{1}{4\pi^2}\left\langle \frac{|\nabla\Psi_{\text{pol}}|^2}{R^2}\right\rangle = \frac{1}{4\pi^2}\left(\frac{\Psi_{\text{pol}}'}{V'}\right)^2\left\langle \frac{|\nabla V|^2}{R^2}\right\rangle = \frac{K}{4\pi^2}\left(\frac{\Psi_{\text{pol}}'}{V'}\right)^2 \underset{(16)}{=} \frac{F^2 A^2 K}{16\pi^4 q^2}. \tag{49}$$

Inserted into eq. (47) yields

$$F\left\langle \frac{J_\phi}{R}\right\rangle = \langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle\left(1 + \frac{AK}{16\pi^4 q^2}\right)^{-1} \;\Leftrightarrow$$

$$\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle = F\left\langle \frac{J_\phi}{R}\right\rangle\left(1 + \frac{AK}{16\pi^4 q^2}\right). \tag{50}$$

The JETTO current density can be written as

$$J_z = \frac{1}{2\pi\rho}\frac{\partial I}{\partial \rho} = \frac{1}{2\pi\rho}\frac{\partial}{\partial\rho}\int \mathrm{d}S\, J_\phi \underset{(15)}{=} \frac{1}{4\pi^2\rho}\frac{\partial}{\partial\rho}\int \mathrm{d}V \left\langle \frac{J_\phi}{R}\right\rangle = \frac{V'}{4\pi^2\rho}\left\langle \frac{J_\phi}{R}\right\rangle \underset{(17)}{=}$$

$$\frac{B_\mathrm{geo}}{FA}\left\langle \frac{J_\phi}{R}\right\rangle \;\Leftrightarrow\; \left\langle \frac{J_\phi}{R}\right\rangle = \frac{FA}{B_\mathrm{geo}} J_z. \tag{51}$$

Inserted into eq. (50) yields

$$\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle = \frac{F^2 A}{B_\mathrm{geo}} J_z \left(1 + \frac{AK}{16\pi^4 q^2}\right). \tag{52}$$

The first term of eq. (52) is identical to the first term of eq. (45), but the second terms are different. In JETTO, the second term is ignored when converting between $\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$ and J_z for the microwave driven current components. That assumption is equivalent to

$$\langle B_\phi^2\rangle \gg \langle B_\mathrm{pol}^2\rangle. \tag{53}$$

Using eqs. (48) and (49), eq. (52) can be simplified as

$$\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle = \frac{J_z}{B_\mathrm{geo}}\Big(\langle B_\phi^2\rangle + \langle B_\mathrm{pol}^2\rangle\Big) = \frac{J_z\langle B^2\rangle}{B_\mathrm{geo}}, \tag{54}$$

with $\langle B^2\rangle$ being replaced by $\langle B_\phi^2\rangle$ when neglecting the second term of eq. (52). Using eq. (46), the flux function J/B is simply expressed as

$$\frac{J}{B} = \frac{J_z}{B_\mathrm{geo}}. \tag{55}$$

2.2 Conversion from $\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$ to J_z

2.2.1 Total current

Equation (45) can be rewritten as

$$J_z - \frac{F'}{2\pi\rho F} I_\mathrm{p} = \frac{1}{2\pi\rho}\left(I_\mathrm{p}' - \frac{F'}{F} I_\mathrm{p}\right) = \frac{B_\mathrm{geo}}{F^2 A}\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle \;\Rightarrow$$

$$\frac{1}{F} I_\mathrm{p}' - \frac{F'}{F^2} I_\mathrm{p} = \frac{\partial}{\partial\rho}\left(\frac{I_\mathrm{p}}{F}\right) = \frac{2\pi\rho B_\mathrm{geo}}{F^3 A}\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle. \tag{56}$$

This gives the total current inside ρ

$$I_\mathrm{p} = 2\pi B_\mathrm{geo} F \int_0^\rho \mathrm{d}\tilde\rho\; \frac{\rho\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle}{F^3 A}\bigg|_{\tilde\rho}, \tag{57}$$

which results in the JETTO current density

$$J_z = \frac{B_{\rm geo}\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle}{F^2 A} + \frac{B_{\rm geo}F'}{\rho}\int_0^\rho {\rm d}\tilde{\rho}\ \frac{\rho\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle}{F^3 A}\bigg|_{\tilde{\rho}} = \frac{B_{\rm geo}\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle}{F^2 A} + 2B_{\rm geo}\frac{\partial F}{\partial \rho^2}\int_0^\rho {\rm d}\tilde{\rho}\ \frac{\rho\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle}{F^3 A}\bigg|_{\tilde{\rho}} \qquad (58)$$

An alternative form of the integral term can be derived using eq. (17) in eq (58), which yields

$$J_z = \frac{B_{\rm geo}\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle}{F^2 A} + \frac{1}{2\pi^2}\frac{\partial F}{\partial \rho^2}\int_0^\rho {\rm d}\tilde{\rho}\ \frac{V'}{F^2}\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle\bigg|_{\tilde{\rho}} = \frac{B_{\rm geo}\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle}{F^2 A} + \frac{1}{2\pi^2}\frac{\partial F}{\partial \rho^2}\int_V {\rm d}V\ \frac{\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle}{F^2}. \qquad (59)$$

2.2.2 Microwave current drive

To write J_z as a function of $\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$, we simply rewrite eq. (52) as

$$J_z = \frac{B_{\rm geo}}{F^2 A}\left(1 + \frac{AK}{16\pi^4 q^2}\right)^{-1}\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle. \qquad (60)$$

2.3 Conversions between J_z and $J_{\rm tor}$

$J_{\rm tor}$ is the toroidal current density averaged over the poloidal plane. It can be computed as

$$J_{\rm tor} = \frac{{\rm d}I_{\rm p}}{{\rm d}S} = \frac{I'_{\rm p}}{S'} = \frac{2\pi\rho}{S'}J_z, \qquad (61)$$

where $S(\rho)$ is the cross sectional area inside the flux surface ρ. S' can be written as

$$S' = \frac{\partial}{\partial\rho}\int_S {\rm d}S \underset{(15)}{=} \frac{1}{2\pi}\frac{\partial}{\partial\rho}\int_V {\rm d}V\ \langle R^{-1}\rangle = \frac{V'}{2\pi}\langle R^{-1}\rangle. \qquad (62)$$

Inserted into eq. (61) yields

$$J_{\rm tor} = \frac{4\pi^2\rho}{V'\langle R^{-1}\rangle}J_z \underset{(17)}{=} \frac{FA}{B_{\rm geo}\langle R^{-1}\rangle}J_z. \qquad (63)$$

This relationship holds both for the total plasma current and for any individual current component. The flux surface averaged R^{-1} is not available in the JSP outputs. However, it can be computed from S', where S' is evaluated with a finite difference method.

$$\frac{FA}{B_{\rm geo}\langle R^{-1}\rangle} = \frac{2\pi\rho}{S'} \ \Leftrightarrow\ \langle R^{-1}\rangle = \frac{FAS'}{2\pi\rho B_{\rm geo}} = \frac{FA}{\pi B_{\rm geo}}\frac{{\rm d}S}{{\rm d}\rho^2}. \qquad (64)$$

The final step eliminates the on-axis singularity.

3 Microwave current drive and normalised current drive efficiency

3.1 Properties of microwave current drive

Microwave current drive, such as that driven by EC and EBW systems, has the properties that the driven current is parallel to the magnetic field, and that J/B is a flux function [2]. The second property follows from the first property combined with the fact that the microwave current drive does not perturb the charge density:

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho_{\mathrm{c}}}{\partial t} = 0 \;\Rightarrow\; \nabla \cdot \boldsymbol{J} = \nabla \cdot \left(\frac{J\boldsymbol{B}}{B}\right) = \boldsymbol{B} \cdot \nabla \left(\frac{J}{B}\right) + \cancel{\frac{J}{B}\nabla \cdot \boldsymbol{B}} =$$

$$\boldsymbol{B} \cdot \nabla \left(\frac{J}{B}\right) = 0, \tag{65}$$

where ρ_{c} is the charge density. Equation (65) implies that J/B is constant along the magnetic field lines, and by extension constant along each flux surface.

3.2 Normalised current drive efficiency

3.2.1 Luce definition

Fisch [5] showed that wave driven currents typically scale with the thermal electron collision rate. A dimensionless current drive efficiency $\zeta = \tilde{J}_{\mathrm{tor}}/\tilde{Q}$ is proposed by Luce, *et al.* [6], where

$$\tilde{J}_{\mathrm{tor}} = \frac{J_{\mathrm{tor}}}{e n_{\mathrm{e}} v_{\mathrm{th}}} \tag{66}$$

and

$$\tilde{Q} = \frac{Q}{n_{\mathrm{e}} T_{\mathrm{e}} \nu_{\mathrm{e}}} \tag{67}$$

are the normalised current density and power density, respectively,

$$v_{\mathrm{th}} = \sqrt{\frac{2T_{\mathrm{e}}}{m_{\mathrm{e}}}} \tag{68}$$

is the electron thermal velocity, and

$$\nu_{\mathrm{e}} = \frac{e^4 n_{\mathrm{e}} \ln \Lambda}{4\pi \epsilon_0^2 m_{\mathrm{e}}^2 v_{\mathrm{th}}^3} \tag{69}$$

is the electron collision frequency. Note that the above equations are presented in SI units rather than Gaussian units (with T_{e} in units of energy), which is relevant for the normalisations done later. Inserting the above relationships into the definition for ζ yields

$$\zeta = \frac{T_e \nu_e}{e v_{th}} \frac{J_{tor}}{Q} = \frac{e^3 n_e T_e \ln\Lambda}{4\pi\epsilon_0^2 m_e^2 v_{th}^4} \frac{J_{tor}}{Q} = \frac{\ln\Lambda}{16\pi} \frac{e^3}{\epsilon_0^2} \frac{n_e}{T_e} \frac{J_{tor}}{Q}. \tag{70}$$

If the above formula is modified such that n_e is expressed in units of 10^{19} m^{-3}, and T_e is in keV, we find that

$$\zeta = \left(\frac{10^{16} e^2}{\epsilon_0^2}\right) \frac{\ln\Lambda}{16\pi} \frac{n_{e,19}}{T_{e,keV}} \frac{J_{tor}}{Q}. \tag{71}$$

The constant within the parentheses has a value of ≈ 3.27 in SI units. Thus, ζ can be written as

$$\zeta = 3.27 \times \frac{\ln\Lambda}{16\pi} \frac{n_{e,19}}{T_{e,keV}} \frac{J_{tor}}{Q}. \tag{72}$$

Note that the expression becomes undefined in regions with no power or current absorption, i.e., $J_{tor} = Q = 0$. In [6], it is assumed that the current drive and power absorption are well localised in a region $\delta\rho$ around a flux surface ρ_{abs}, with a total driven current $I = J_{tor}\delta S$ and total power $P = Q\delta V$. We then find that

$$\frac{J_{tor}}{Q} \approx \frac{I}{P} \frac{\delta V}{\delta S} \approx \frac{I}{P} \frac{V'}{S'} \underset{(62)}{=} \frac{I}{P} \frac{2\pi}{\langle R^{-1} \rangle}. \tag{73}$$

Inserted into eq. (72) yields

$$\zeta = 3.27 \times \frac{\ln\Lambda}{8} \frac{1}{\langle R^{-1} \rangle} \frac{n_{e,19}(\rho_{abs})}{T_{e,keV}(\rho_{abs})} \frac{I}{P}. \tag{74}$$

The profiles $n_{e,19}$ and $T_{e,keV}$ should now be evaluated at the absorption surface ρ_{abs}. Assuming a negligible Shafranov shift, $\langle R^{-1} \rangle$ can be approximated with R_{ax} for any flux surface. Furthermore, we drop the dimensionless factor $\ln\Lambda/8$ (but we keep the dimensional constant 3.27), which yields the final expression for ζ as derived in [6]

$$\zeta = 3.27 \times \frac{R_{ax} n_{e,19}(\rho_{abs})}{T_{e,keV}(\rho_{abs})} \frac{I}{P}. \tag{75}$$

3.2.2 JETTO definition

The JETTO extra namelist parameter `ECCDNORM` defines the EC current drive efficiency (and conversely `EBWCDNORM` for EBWCD). Depending on the value of `ECCDRHO`, $J_{z,EC}$ is computed differently from `ECCDNORM`.

- ECCDRHO = -1:

$$J_{\mathrm{z,EC}} = 1000 \times \texttt{ECCDNORM} \frac{T_{\mathrm{e,keV}} Q_{\mathrm{EC,norm}} P_{\mathrm{EC,MW}}}{n_{\mathrm{e,19}}} \times (1 - f_{\mathrm{t}}), \tag{76}$$

- ECCDRHO = -3:

$$J_{\mathrm{z,EC}} = 1000 \times \texttt{ECCDNORM} \frac{T_{\mathrm{e,keV}} Q_{\mathrm{EC,norm}} P_{\mathrm{EC,MW}}}{n_{\mathrm{e,19}}}, \tag{77}$$

- ECCDRHO = -4:

$$J_{\mathrm{z,EC}} = 1000 \times \texttt{ECCDNORM} \frac{T_{\mathrm{e,keV}} Q_{\mathrm{EC,norm}} P_{\mathrm{EC,MW}}}{n_{\mathrm{e,19}}} \times \frac{3(1 - f_{\mathrm{t}})}{2 + Z_{\mathrm{eff}}}, \tag{78}$$

- ECCDRHO = -5:

$$J_{\mathrm{z,EC}} = 1000 \times \texttt{ECCDNORM} \frac{T_{\mathrm{e,keV}} Q_{\mathrm{EC,norm}} P_{\mathrm{EC,MW}}}{n_{\mathrm{e,19}}} \times \frac{3}{2 + Z_{\mathrm{eff}}}, \tag{79}$$

where

$$Q_{\mathrm{EC,norm}}(\rho) = Q_{\mathrm{EC}}(\rho) \left(2\pi \int_0^{\rho_{\mathrm{sep}}} \mathrm{d}\tilde{\rho}\, Q_{\mathrm{EC}}(\tilde{\rho})\tilde{\rho} \right)^{-1} \tag{80}$$

is the normalised EC power density, and

$$f_{\mathrm{t}} = 1 - \frac{(1 - \epsilon)^2}{(1 + 1.46\sqrt{\epsilon})\sqrt{1 - \epsilon^2}} \tag{81}$$

is the trapped electron fraction, as a function of the inverse aspect ratio $\epsilon = a_{\mathrm{min}}/R_{\mathrm{geo}}$. The options ECCDRHO = -2 and ECCDRHO $\in$ [0, 1] rescales an input ECCD density profile to a given total current value (the ECCD density profile is assumed to be available in the ex-file). When ECCDRHO = -2, the ECCD current is the value of ECCDNORM (in A). When ECCDRHO $\in$ [0,1],

$$I_{\mathrm{EC}} = 1000 \times \texttt{ECCDNORM} \frac{T_{\mathrm{e,keV}}(\rho_{\mathrm{tor}} = \texttt{ECCDRHO})}{n_{\mathrm{e,19}}(\rho_{\mathrm{tor}} = \texttt{ECCDRHO})} P_{\mathrm{EC,MW}}. \tag{82}$$

Assuming that ECCDRHO is the flux surface of maximal EC absorption, eq. (82) is equivalent to the Luce definition of eq. (75), where

$$\texttt{ECCDNORM} = \frac{1000}{3.27 R_{\mathrm{ax}}} \zeta = \frac{305.4}{R_{\mathrm{ax}}} \zeta. \tag{83}$$

This relationship also holds for ECCDRHO = -3, assuming that $Q_{\mathrm{EC}} = P_{\mathrm{EC}}\delta(\rho - \rho_{\mathrm{abs}})/(\mathrm{d}V/\mathrm{d}\rho)$, and $J_{\mathrm{z,EC}} = I_{\mathrm{EC}}\delta(\rho - \rho_{\mathrm{abs}})/(2\pi\rho)$, where $\delta(\rho)$ is the Dirac delta function.

There is also the option to assume a ρ dependence to ECCDNORM using the extra namelist setting ECCDNEX = 1, which reads the ECCDN profile from the ex-file and overrides ECCDNORM for the ECCDRHO options -1, -3, -4, and -5. In addition, the option ECCDNEX = 2 assumes that ECCDN is a profile in ζ, relating to ECCDNORM according to eq. (83).

3.2.3 Wilson definition

For high-β, strongly shaped plasmas (e.g. spherical tokamaks), we have that

$$R_{\text{ax}}\langle R^{-1}\rangle \geq 1, \tag{84}$$

with the value being equal to 1 on-axis, and increasing with increasing ρ. For a given value of ζ, which is equivalent to a constant `ECCDNORM` value with `ECCDRHO` $\geq$ `0` (and equivalent to `ECCDRHO = -3` for a narrow absorption region), this means that the driven auxiliary current might be underestimated, supposing that eq. (74) is valid. This is especially true for off-axis current drive, such as that predicted for EBW. Conversely, if ζ is computed for a given predicted current density, the value of ζ would be overestimated using the Luce definition. To correct for this, Wilson suggests to replace R_{ax} with $1/\langle R^{-1}\rangle$ in eq (75). Thus, the corrected normalised current drive efficiency would be

$$\zeta^* = 3.27 \times \left.\frac{n_{\text{e},19}}{\langle R^{-1}\rangle T_{\text{e,keV}}}\right|_{\rho=\rho_{\text{abs}}} \frac{I}{P}. \tag{85}$$

Note that the constant prefactor can be any value, as long as the same factor is used consistently. Here, the prefactor $10^{16}e^2/\epsilon_0^2 \approx 3.27$ has been kept to make sure that ζ^* remains dimensionless, and that it agrees with the Luce definition on-axis. For a profile of current drive efficiency, according to eq. (72), the microwave current density can be computed as

$$J_{\text{tor}}(\rho) = \frac{2\pi}{3.27} \times \zeta^*(\rho)\frac{T_{\text{e,keV}}(\rho)Q(\rho)}{n_{\text{e},19}(\rho)}, \tag{86}$$

or, equivalently

$$J_z \underset{(61)}{=} \frac{1}{3.27} \times \zeta^* \frac{T_{\text{e,keV}}}{n_{\text{e},19}} \frac{S'Q}{\rho} \underset{(62)}{=} \frac{1}{3.27} \times \zeta^* \frac{T_{\text{e,keV}}}{n_{\text{e},19}} \frac{V'\langle R^{-1}\rangle Q}{2\pi\rho} \underset{(17)}{=} \frac{2\pi}{3.27} \times \zeta^* \frac{T_{\text{e,keV}}}{n_{\text{e},19}} \frac{B_{\text{geo}}\langle R^{-1}\rangle Q}{FA}. \tag{87}$$

4 Summary of suggested updates

4.1 Current diffusion equation

In eq. (30), it was shown that

$$\frac{\partial \iota}{\partial t} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{c^2F^2\eta}{64\pi^5 B_{\text{geo}}^2}\frac{1}{\rho}\frac{\partial}{\partial \rho}\Big(AK\iota\Big) - \frac{c\eta}{B_{\text{geo}}FA}\langle \boldsymbol{J}_{\text{NI}} \cdot \boldsymbol{B}\rangle\right). \tag{88}$$

When substituting $u_1 = B_{\rm geo} A K \iota / R_{\rm geo}$ into the above equation, you get

$$\frac{1}{AK}\frac{\partial u_1}{\partial t} - \frac{u_1}{A^2K^2}\frac{\partial}{\partial t}(AK)$$
$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{c^2F^2\eta}{64\pi^5 B_{\rm geo}^2}\frac{1}{\rho}\frac{\partial u_1}{\partial\rho} - \frac{c\eta}{R_{\rm geo}FA}\langle \boldsymbol{J}_{\rm NI}\cdot\boldsymbol{B}\rangle\right). \tag{89}$$

In JETTO, the second term of the left hand side is not included, which is based on the assumption that

$$\frac{1}{u_1}\frac{\partial u_1}{\partial t} \gg \frac{1}{AK}\frac{\partial}{\partial t}(AK). \tag{90}$$

Including the term should not impact the stationary solution, which states that the right hand side is zero, i.e., the quantity inside the $\partial/\partial\rho$ derivative is constant in ρ. But it could potentially impact the inductive current solutions in transient stages, or impact the time scales on which the current density relaxes to stationary conditions.

4.2 Loop voltage

It was found in eq. (40) that

$$V_{\rm loop} = \frac{cF^2\eta R_{\rm geo}}{32\pi^4 B_{\rm geo}^2}\frac{1}{\rho}\frac{\partial u_1}{\partial\rho}\bigg|_{\rho_{\rm sep}} - \frac{2\pi\eta}{FA}\langle \boldsymbol{J}_{\rm NI}\cdot\boldsymbol{B}\rangle\bigg|_{\rho_{\rm sep}} -$$
$$2\pi R_{\rm ax}\eta_{\rm ax}\frac{B_{\rm ax}}{B_{\rm geo}}\left(J_{z,\rm ax} - J_{z,\rm NI,ax}\right). \tag{91}$$

The JETTO definition of the loop voltage only includes the first two terms (see [4]), which is proportional to $\mathrm{d}\Psi_{\rm pol}/\mathrm{d}t$. Consequently, the first two terms will naturally cancel during stationary conditions ($\mathrm{d}\Psi_{\rm pol}/\mathrm{d}t \to 0$). When setting the loop voltage to zero as a boundary condition, it means that $J_{z,\rm ax}$ and $J_{z,\rm NI,ax}$ would simultaneously cancel in stationary conditions, i.e., the on-axis inductive current vanishes. No such cancellation occurs when the on-axis terms are not included.

4.3 Conversions between $\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$ and J_z for microwave current drive

It was found in eq. (52) that the relationship between $\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$ and J_z for microwave current drive is

$$\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle = \frac{F^2A}{B_{\rm geo}}J_z\left(1 + \frac{AK}{16\pi^4 q^2}\right). \tag{92}$$

When JETTO converts between $\langle \boldsymbol{J} \cdot \boldsymbol{B} \rangle$ and J_z for the microwave current drive (EC and EBW driven current), the term $AK/(16\pi^4 q^2)$ is ignored. Including it would increase $\langle \boldsymbol{J} \cdot \boldsymbol{B} \rangle$ by a factor $1 + \langle B_{\mathrm{pol}}^2 \rangle / \langle B_\phi^2 \rangle$ relative to J_z. While this correction factor is typically negligible, it could in principle perturb the stationary solution of the current diffusion equation, including the inductive current component.

4.4 Normalised current drive efficiency

The JETTO definition of the current drive efficiency agrees with the standard definition as derived in [6] for well-localised EC and EBW absorption. However, the derivation of this current drive efficiency relies on the approximation that $R_{\mathrm{ax}} \langle R^{-1} \rangle = 1$, which is a good approximation for large aspect ratio plasmas, but generally not for spherical tokamaks. A modified rescaling of the microwave driven current I [A] relative to the power P [W] is suggested according to

$$I = \frac{1}{3.27} \times \zeta^* P \left. \frac{\langle R^{-1} \rangle T_{\mathrm{e,keV}}}{n_{\mathrm{e,19}}} \right|_{\rho_{\mathrm{abs}}}, \tag{93}$$

where ζ^* is the normalised current drive efficiency at the absorption surface ρ_{abs}. The corresponding relationship between (the JETTO definition of) the current density J_z [A/m^2] and the power density Q [W/m^3] is

$$J_z = \frac{2\pi}{3.27} \times \zeta^* \frac{T_{\mathrm{e,keV}}}{n_{\mathrm{e,19}}} \frac{B_{\mathrm{geo}} \langle R^{-1} \rangle Q}{FA}. \tag{94}$$

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