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Adjoint Variational Principles for Regularised Conservative Systems

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Abstract. Variational principles are powerful tools in many branches of theoretical physics. Certain conservative systems which do not admit of a traditional Euler-Lagrange variational formulation are given a novel generalization. Illustrative examples, including the recently discovered scale-invariant analogue of the Korteweg-de Vries equation are presented. The new "adjoint variational method" is applied to regularized, incompressible, conservative hydrodynamics expressed in Eulerian variables, as opposed to the usual Lagrangian variables. The regularized, two-fluid, non-dissipative, quasi-neutral, incompressible plasma equations [known as "Hall MHD"] and the electromagnetic field equations are derived from the new formulation. It turns out that the associated adjoint equations are precisely the two-fluid "cross-helicity/frozen-field" theorems pertaining to these regularized systems which have no standard variational formulation). The adjoint equations also provide a direct route to the integral invariants of the system and suggest new analytical and numerical approaches to the dynamics.

Keywords: Variational principles, regularized fluid and plasma dynamics, generalized enstrophies and invariants
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1. INTRODUCTION

Hamilton's Action principle is a powerful tool in both classical physics and quantum field theories. In Mechanics, for example, we have a Lagrangian L, a function of a generalized coordinate q, a generalized velocity, $\dot{q} = \frac{dq}{dt}$, and possibly also of the time for a system with one-degree of freedom. Varying the action integral, $S = \int_{t_1}^{t_2} L(q,\dot{q},t)dt$, over a suitable "space of paths", in the usual manner, we get the familiar Euler-Lagrange equation of motion of the system:

$$\frac{d}{dt}[\frac{\partial L}{\partial \dot{q}}] = \frac{\partial L}{\partial q} \tag{1}$$

If $L = T - V$, where T, V are the kinetic and potential energies of the system respectively, the above equation is exactly equivalent to Lagrange's equation, ie Newton's Laws. The power of the Action Principle derives from its invariant formulation, numerical tractability and the direct connection with Feynman's path integral quantization. It is a well-known fact that variational principles underly many classical field equations: eg. Laplace, Poisson, D'Alembert, Maxwell and Einstein. In this paper, certain conservative continuum field theories which are not described by the simple variational formulation above but seemingly require an "extended" approach are explored.

Lagrange's equations of evolution are second-order ordinary or partial differential equations. For example, the D'Alembert wave equation is derivable from: $L = \frac{1}{2}[\phi_t^2 -$

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$c^2\phi_x^2]$. We investigate the interesting special case of Hamilton's equations of mechanics.

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$

The variables, q, p, unlike $q, \dot{q}$ are to be considered on the same footing and to be regarded as independent *generalized coordinates/conjugate variables* in a phase space, (q, p). This is different from the Lagrangian generalized coordinate space spanned by q. Hamilton himself introduced the *Hamiltonian*, $H(p,q)$, defined in terms of Lagrangian by, $L = p\dot{q} - H(p,q)$, and formally treated p, q as independent variables. Then, by varying the action integral, he obtained two first order ordinary differential equations named after him.

In a similar manner, by introducing the "conjugate field momentum", $\Pi(x,t)$ and $H = \frac{1}{2}[\Pi^2 + c^2\phi_x^2]$, the Lagrangian $L = \Pi\phi_t - H$ leads to the field equations which are clearly equivalent to the D'Alembert wave equation in one-dimension:

$$\Pi = \phi_t \tag{2}$$
$$\Pi_t = c^2\phi_{xx} \tag{3}$$
$$\phi_{tt} = c^2\phi_{xx} \tag{4}$$

A classic first-order (in time) differential equation is the Schrödinger equation of non-relativistic quantum theory:

$$\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \tag{5}$$

To derive this from a Lagrangian, we essentially have to "double" the equation by writing down the adjoint of this equation, satisfied by $\bar{\Psi}$, the complex conjugate of Ψ.

$$-\frac{\hbar}{i}\frac{\partial\bar{\Psi}}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2\bar{\Psi}}{\partial x^2} \tag{6}$$

The Lagrangian density is now a function of both Ψ and $\bar{\Psi}$, and takes the following form: $L[\Psi, \Psi_t, \Psi_x, \bar{\Psi}, \bar{\Psi}_t, ..] = \frac{\hbar}{2i}(\bar{\Psi}\Psi_t - \Psi\bar{\Psi}_t) + \frac{\hbar^2}{2m}\Psi_x\bar{\Psi}_x$. Upon independent variation of this Lagrangian with respect to $\Psi, \bar{\Psi}$, we obtain the two equations above. This example will serve as a model for more general applications of "adjoint variational principles". Note that $\bar{\Psi}$ is the solution of the "time-reversed" form of Eq.(2), namely, the adjoint equation.

2. TWO EXAMPLES OF ADJOINT VARIATIONAL PRINCIPLES

Using the "genetic programming" technique of finding equations which are satisfied by specified solutions, Sen and Ahalpara [1] found a remarkable "scale-invariant" relative of the famous Korteweg-de Vries (KdV) equation. Many interesting properties of a generalized form of this equation ("SIdV") can be found in the recent paper published by Sen *et al* [2]. The equation takes, in a special case, the form:

$$u_t + (\frac{2u_{xx}}{u})u_x \quad = \quad u_{xxx} \tag{7}$$

It has the same solitary wave solution as KdV. Unlike KdV which has a simple advection term, the above equation has a *scale-invariant* advecting velocity [analogous to the $\frac{\mathbf{E}\times\mathbf{B}}{B^2}$ velocity of ideal MHD], $c_{\text{ad}} = \frac{2u_{xx}}{u}$. The equation is invariant when $u \to C.u$, C being an arbitrary non-zero constant. This scale-invariance implies that this *nonlinear* partial differential equation shares with linear equations the property that an arbitrary constant multiple of a solution is also a solution. Furthermore, the equation admits genuine "plane-wave" solutions of arbitrary amplitude. The equation is time-reversible and has only two conservation laws/integral invariants. It is not derivable from any polynomial Lagrangian (quite unlike KdV, which is).

We consider the following "polynomial" form of the SIdV equation, setting $u = e^w$:

$$w_t + [w_x^2 - w_{xx}]w_x \quad = \quad w_{xxx} \tag{8}$$

We introduce a "conjugate" field $\theta(x,t)$ and the adjoint Lagrangian density which is a local function of both w and θ and their derivatives.

$$\mathscr{L} \quad = \quad \theta w_t - \theta_{xx} w_x + \theta w_x^3 + \theta_x(\frac{1}{2}w_x^2) \tag{9}$$

$$\frac{\delta \mathscr{L}}{\delta \theta} = 0 \rightarrow \qquad w_t + [w_x^2 - w_{xx}]w_x - w_{xxx} = 0 \tag{10}$$

$$\frac{\delta \mathscr{L}}{\delta w} = 0 \rightarrow \qquad \theta_t + [3\theta w_x^2 + \theta_x w_x - \theta_{xx}]_x = 0 \tag{11}$$

We note some key points arising from these results: a) the "adjoint equation" is not the same as (nor obviously related to) SIdV b) The conjugate field $\theta(x,t)$ satisfies a *linear* partial differential equation involving the original function. c) While Eq.(10) is not in conservation form, Eq.(11) is! d) The zero value for the conjugate field is a trivial solution of the adjoint equation.

As a second nontrivial example, the inviscid, incompressible Euler equations of hydrodynamics are considered next:

$$\begin{aligned} \nabla.\mathbf{U} \quad &= \quad 0 \\ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{W}\times\mathbf{U} \quad &= \quad -\nabla H; \qquad [\mathbf{W} = \nabla\times\mathbf{U}] \end{aligned}$$

where $\mathbf{U}, H$ are the "standard" Euler fields and $\mathbf{g}, G$ are their conjugate fields. We introduce the (adjoint) Lagrangian density function:

$$\mathscr{L}_* \quad = \quad \mathbf{g}.\mathbf{U}_t + \mathbf{W}.(\mathbf{U}\times\mathbf{g}) + \mathbf{g}.\nabla H - \mathbf{U}.\nabla G \tag{12}$$

Variation with respect to G [using the appropriate functional and suitable (unvaried) boundary data], we obtain the scalar continuity equation. Variation with respect to $\mathbf{g}$

yields the Eulerian momentum equation of motion:

$$\nabla.\mathbf{U} = 0 \tag{13}$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{W} \times \mathbf{U} = -\nabla H \tag{14}$$

Variations with respect to the standard fields yield the adjoint equations:

$$\nabla.\mathbf{g} = 0 \tag{15}$$

$$\frac{\partial \mathbf{g}}{\partial t} + \mathbf{W} \times \mathbf{g} = -\nabla G + \nabla \times (\mathbf{U} \times \mathbf{g}) \tag{16}$$

The adjoint vector field is solenoidal. However, the adjoint equations are formally linear in the conjugate variables. Note that we may, for appropriate initial data, always find exact solutions to the adjoint equations. Thus, it will be seen that if μ is an arbitrary constant, $\mathbf{g} = \mu \mathbf{U}; G = \mu H$ satisfy the adjoint equation whenever the standard fields satisfy the Euler equation. Thus, in this sense, the Eulerian fields are proportional to their conjugates and could be called self-conjugate.

A remarkable particular solution of the adjoint equations is given by, $G = 0; \mathbf{g} = \lambda \mathbf{W}$, where the standard fields satisfy the Euler equations and λ is an arbitrary constant. Indeed, more generally, we see that,$\mathbf{g} = \lambda \mathbf{W} + \mu \mathbf{U}; G = \mu H$ satisfy the equations for arbitrary constants, λ, μ.

Thus, we find that the **Eulerian vorticity** is a field conjugate to the velocity field and the adjoint equation is none other than the Kelvin-Helmholtz vorticity equation of Eulerian hydrodynamics. It immediately follows that the adjoint variational principle implies the standard momentum equations and, simultaneously, the vorticity equation. It is an open problem to characterize the space of all solenoidal vector fields **g** together with the associated scalar fields G which satisfy the adjoint equation.

3. ADJOINT VARIATIONAL PRINCIPLE FOR THE REGULARIZED HYDRODYNAMICS

It is well-known that incompressible rotational Euler flows in 3-d are subject to a powerful direct cascade to short length-scales driven by "vortex stretching". In particular, Taylor and Green[3] found that enstrophy can evolve unboundedly, although it has not yet been proved [4] if there is (or is not) a "finite-time blow up" of the solution. It has not even been shown yet if adding viscosity to the equations (making it Navier-Stokes) regularizes the solution, although higher order dissipative regularizations are known (due to Ladyzhenskaya [5]). The problem of finding a *conservative regularization* of the Euler system was solved by Thyagaraja [6] who proposed the following equations:

$$\nabla.\mathbf{U} = 0 \tag{17}$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{W} \times \mathbf{U} = -\nabla H - \lambda^2 [\mathbf{W} \times (\nabla \times \mathbf{W})] \tag{18}$$

where λ is a "regularization constant" with dimensions of length [not to be confused with previously introduced multiplicative constants]. It represents the "cut-off" wave length (cf. *op. cit*). This system, with suitable boundary data, has two quadratic integral invariants:

$$\mathscr{E} = \int [\mathbf{U}.\mathbf{U} + \lambda^2 \mathbf{W}.\mathbf{W}] dV \tag{19}$$

$$\mathscr{H} = \int \mathbf{U}.\mathbf{W} dV \tag{20}$$

It will now be shown that this system can be derived from an adjoint variational principle. The corresponding Lagrangian density is:

$$\mathscr{L}_* = \mathbf{g}.\mathbf{U}_t + \mathbf{W}.(\mathbf{U} \times \mathbf{g}) + \mathbf{g}.\nabla H - \mathbf{U}.\nabla G + \lambda^2 \mathbf{g}.[\mathbf{W} \times (\nabla \times \mathbf{W})] \tag{21}$$

The variations with respect to $\mathbf{g}, \mathbf{U}$ respectively, lead to the standard and adjoint (regularized) equations:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{W} \times \mathbf{U} + \nabla H = -\lambda^2 [\mathbf{W} \times (\nabla \times \mathbf{W})] \tag{22}$$

$$\begin{aligned} \frac{\partial \mathbf{g}}{\partial t} + \mathbf{W} \times \mathbf{g} + \nabla G &= \nabla \times (\mathbf{U} \times \mathbf{g}) - \lambda^2 \nabla \times [\mathbf{g} \times (\nabla \times \mathbf{W}] \\ &\quad + \lambda^2 \nabla \times \nabla \times (\mathbf{W} \times \mathbf{g}) \end{aligned} \tag{23}$$

Variations with respect to the corresponding scalar fields give the respective continuity equations. As before, we find that $[\mathbf{W}, G = 0]$ is seen to be a conjugate field. However, the standard field does not satisfy adjoint equations (as is the case in real, self-adjoint equations).

4. ADJOINT VARIATIONAL PRINCIPLE FOR INCOMPRESSIBLE, DISSIPATIONLESS TWO-FLUID PLASMAS

The incompressible, quasi-neutral, dissipationless two-fluid plasma equations with Maxwell's equations present an interesting variational problem involving Lagrange multipliers. We consider for two fully ionized charge species [electron-ion or electron-positron plasma] $s(= i, e)$, vector fields designated by $\mathbf{v}_s, \mathbf{g}_s, \mathbf{P}_s$ and scalar fields σ_s, χ_s. The electromagnetic field is represented by the vector potential, $\mathbf{A}$. We define: $\mathbf{W}_s = \nabla \times \mathbf{v}_s; \mathbf{B} = \nabla \times \mathbf{A}; \rho_{ms} = m_s n$, where m_s are the species masses and

n is the common, quasi-neutral, constant, uniform, particle density. We introduce two "Lagrangian densities": $I[\mathbf{v}_{i,e},\mathbf{g}_{i,e},\mathbf{P}_{i,e},\mathbf{A},\sigma_{i,e},\chi_{i,e}]$ and, $J[\mathbf{v}_{i,e},\mathbf{g}_{i,e},\mathbf{P}_{i,e},\mathbf{A},\sigma_{i,e},\chi_{i,e}]$. The corresponding "action integrals" are:

$$I^* = \int (\mathbf{g}_i.[\frac{\partial \mathbf{P}_i}{\partial t} + (\nabla\times\mathbf{P}_i)\times\mathbf{v}_i + \nabla\sigma_i] + \mathbf{g}_e.[\frac{\partial \mathbf{P}_e}{\partial t} + (\nabla\times\mathbf{P}_e)\times\mathbf{v}_e + \nabla\sigma_e])dV \tag{24}$$

$$J^* = \int ([\mathbf{P}_i.\mathbf{v}_i - \frac{1}{2}\rho_{mi}\mathbf{v}_i^2 - en\mathbf{A}.\mathbf{v}_i + \mathbf{v}_i.\nabla\chi_i] + [\mathbf{P}_e.\mathbf{v}_e - \frac{1}{2}\rho_{me}\mathbf{v}_e^2 + en\mathbf{A}.\mathbf{v}_e + \mathbf{v}_e.\nabla\chi_e] - \frac{1}{2\mu_0}(\nabla\times\mathbf{A})^2)dV \tag{25}$$

The extremals of I^* subject to the constancy of J^* are obtained using a Lagrange multiplier, varying the functional, $I^* + \alpha J^*$: The Euler-Lagrange equations resulting from varying the four scalar fields imply the "incompressibility" relations for the four "matter" vector fields:

$$\nabla.\mathbf{g}_i = 0; \nabla.\mathbf{g}_e = 0; \nabla.\mathbf{v}_i = 0; \nabla.\mathbf{v}_e = 0$$

Varying the $\mathbf{g}_s$ fields, we obtain the evolution equations:

$$\frac{\partial \mathbf{P}_i}{\partial t} + (\nabla\times\mathbf{P}_i)\times\mathbf{v}_i + \nabla\sigma_i = 0 \tag{26}$$

$$\frac{\partial \mathbf{P}_e}{\partial t} + (\nabla\times\mathbf{P}_e)\times\mathbf{v}_e + \nabla\sigma_e = 0 \tag{27}$$

Variation of $\mathbf{v}_s$ yields the relations providing the usual canonical momenta if we choose $\mathbf{g}_s = \kappa_s\nabla\times\mathbf{P}_s; \chi_{i,e} = 0$; $\nabla.\mathbf{A} = 0$, where κ_s are arbitrary constants:

$$\mathbf{P}_i = \rho_{mi}\mathbf{v}_i + en\mathbf{A} + \nabla\chi_i + \frac{1}{\alpha}(\nabla\times\mathbf{P}_i)\times\mathbf{g}_i \tag{28}$$

$$\mathbf{P}_e = \rho_{me}\mathbf{v}_e - en\mathbf{A} + \nabla\chi_e + \frac{1}{\alpha}(\nabla\times\mathbf{P}_e)\times\mathbf{g}_e \tag{29}$$

The variation with $\mathbf{A}$ gives Ampère's Law with the current density,$\mathbf{j} = en(\mathbf{v}_i - \mathbf{v}_e)$:

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j} \tag{30}$$

Note that $\mathbf{j}$ is automatically solenoidal. Finally, two more equations are obtained upon varying $\mathbf{P}_s$. These equations are formally evolution equations for the fields $\mathbf{g}_s$ and are the only ones which involve the Lagrange parameter α:

$$\frac{\partial \mathbf{g}_i}{\partial t} + \nabla\times[\mathbf{g}_i\times\mathbf{v}_i] - \alpha\mathbf{v}_i = 0 \tag{31}$$

$$\frac{\partial \mathbf{g}_e}{\partial t} + \nabla\times[\mathbf{g}_e\times\mathbf{v}_e] - \alpha\mathbf{v}_e = 0 \tag{32}$$

The curl of the two "equations of motion" (satisfied by the $\mathbf{P}_s$), give,

$$\frac{\partial}{\partial t}[\nabla \times \mathbf{P}_s] + \nabla \times [(\nabla \times \mathbf{P}_s) \times \mathbf{v}_s] \quad = \quad 0, \qquad s = i, e \tag{33}$$

The regularity of the equations as $\alpha \to 0$, implies the consistency relations:

$$\mathbf{g}_s = \kappa_s \nabla \times \mathbf{P}_s$$

Using gauge invariance, it can be shown that we may self-consistently set $\chi_s = 0$, without loss of generality.

5. ADJOINT VARIATIONAL PRINCIPLE FOR REGULARIZED TWO-FLUID PLASMA DYNAMICS

To obtain the "regularized" incompressible two-fluid equations, we replace the functional I^* by K^* (retaining J^* as before)thus:

$$K^*(\lambda^2) \quad = \quad I^* + \lambda^2 \mathbf{g}_i.[(\nabla \times \mathbf{P}_i) \times (\nabla \times \mathbf{W}_i)] + \lambda^2 \mathbf{g}_e.[(\nabla \times \mathbf{P}_e) \times (\nabla \times \mathbf{W}_e)] \tag{34}$$

We hold the functional J^* constant by introducing the Lagrange multiplier α and varying $K^* + \alpha J^*]$ as before. Varying the $\mathbf{V}_s$, (having obtained the scalar variational equations as before) we find:

$$\frac{\partial \mathbf{P}_s}{\partial t} + (\nabla \times \mathbf{P}_s) \times \mathbf{v}_i + \nabla \sigma_s \quad = \quad -\lambda^2 [(\nabla \times \mathbf{P}_s) \times (\nabla \times \mathbf{W}_s)] \tag{35}$$

Variation with respect to $\mathbf{v}_s$ yields (Nb. $q_i = +1, q_e = -1$):

$$[\mathbf{P}_s - \rho_{ms}\mathbf{v}_s - enq_s\mathbf{A} + \nabla \chi_s] \quad = \quad -\frac{1}{\alpha}[\mathbf{g}_s \times (\nabla \times \mathbf{P}_s)] + \lambda^2 (\nabla \times)^2 [(\nabla \times \mathbf{P}_s) \times \mathbf{g}_s] \tag{36}$$

Varying with respect to $\mathbf{P}_s$ leads to, [with, $\mathbf{g}_s = \kappa_s \nabla \times \mathbf{P}_s; \alpha = 0$]:

$$\frac{\partial \mathbf{g}_s}{\partial t} + \nabla \times [\mathbf{g}_s \times \mathbf{v}_s] \quad = \quad -\lambda^2 \nabla \times [\mathbf{g}_s \times (\nabla \times \mathbf{W}_s)] \tag{37}$$

As before, the field equations and $\chi_s = 0$, follow from gauge invariance and variation with respect to $\mathbf{A}$. The equations derived above possess the following properties: The fields $\mathbf{v}_s$ correspond to the (Eulerian) velocity fields of the two species. The fields $\mathbf{P}_s$ are the canonical electromagnetic momenta. $\mathbf{g}_s$ are the curls of the canonical momenta. These vector fields satisfy, $\nabla.\mathbf{v}_s = 0; \nabla.\mathbf{g}_s = 0; \nabla.\mathbf{A} = 0; \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j}; \mathbf{j} = en(\mathbf{v}_i - \mathbf{v}_e)$.

$$\mathbf{P}_s \quad = \quad \rho_{ms}\mathbf{v}_s + q_s n \mathbf{A} \tag{38}$$

$$\mathbf{g}_s \quad = \quad \nabla \times \mathbf{P}_s \tag{39}$$

$$\frac{\partial \mathbf{P}_s}{\partial t} + (\nabla \times \mathbf{P}_s) \times \mathbf{v}_s + \nabla \sigma_s \quad = \quad -\lambda^2 [(\nabla \times \mathbf{P}_s) \times (\nabla \times \mathbf{W}_s)] \tag{40}$$

$$\frac{\partial \mathbf{g}_s}{\partial t} + \nabla \times [\mathbf{g}_s \times \mathbf{v}_s] \quad = \quad -\lambda^2 \nabla \times [\mathbf{g}_s \times (\nabla \times \mathbf{W}_s)] \tag{41}$$

The last equation is evidently the curl of the previous one: a remarkable property of the Eulerian adjoint system. Note that, we have when n is a constant: $\mathbf{g}_s = \rho_{ms}\mathbf{W}_s + q_s n\mathbf{B}$ Defining the energies in the ion and electron fluids and in the magnetic field we can easily show that:

$$\mathscr{E}^*_{i,e} = \int \rho_{mi,e}\frac{1}{2}[\mathbf{v}^2_{i,e} + \lambda^2\mathbf{W}^2_{i,e}]dV \tag{42}$$

$$\mathscr{E}^*_{\text{mag}} = \int \frac{1}{2\mu_0}[\mathbf{B}^2 + \lambda^2(\nabla\times\mathbf{B})^2)]dV \tag{43}$$

$$\frac{d}{dt}[\mathscr{E}^*_i + \mathscr{E}^*_e] = \int \mathbf{E}.\mathbf{j}dV - \lambda^2\int(\nabla\times\mathbf{j}).\frac{\partial\mathbf{B}}{\partial t}dV = -\frac{d}{dt}[\mathscr{E}^*_{\text{mag}}]$$

$$\frac{d}{dt}[\mathscr{E}^*_i + \mathscr{E}^*_e + \mathscr{E}^*_{\text{mag}}] = 0 \tag{44}$$

Defining the **the generalized cross-helicities**,

$$\mathscr{H}^*_s = \int \mathbf{P}_s.\mathbf{V}_s dV$$

we readily obtain:

$$\frac{d}{dt}[\mathscr{H}^*_s] = 0 \tag{45}$$

6. DISCUSSION AND CONCLUSIONS

Adjoint variational principles introduced in this work are generalizations [in this context, it is interesting to note the discussion by Moiseiwitsch [7], p. 112 *et seq.*] of the classical Hamilton's principle in canonical variables. They can be applied to conservative, regularized, conservative systems in *Eulerian field variables* which do not possess a standard variational principle (eg. SIdV). Hitherto, variational formulations of hydrodynamics and ideal MHD exist only in numerically unsuitable *Lagrangian variables* [8, 9]. It is well=known that no such "classical" formulations exist for ideal MHD with equilibrium flows. Adjoint principles for (regularized) hydrodynamics and quasi-neutral two-fluid-Maxwell systems lead to the corresponding equations of motion for the standard Eulerian fields, and remarkably, also to the associated "vorticity/magnetic field/frozen-in" equations for the conjugate fields. This unique feature and the invariants of the system recover standard classical results [cf. [10]]. Conjugate fields satisfy formally linear adjoint equations associated with the standard fields. They are analogous to the conjugate complex wave amplitude field in quantum theory. There, it has important physical significance: time-reversal, probability conservation and currents. The adjoint variational principles are believed to be powerful new ways to approach numerical simulations [via Galerkin approximations, for example] of regularized conservative fluid and plasma physics which are guaranteed to have bounded integral invariants unlike their classical "non-regularized" counterparts.

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