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STRUCTURE OF SHORT-WAVELENGTH MODES IN A TOROIDAL PLASMA

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Abstract:

Short wavelength fluctuations may be a source of anomalous transport in toroidal plasmas. Early investigations concerned electron and ion modes that occur only at a particular radius and have a localised eigenfunction: such modes do not seem important for transport. Recently, Connor, Taylor and Wilson [1] described electron modes that occur at all radii and have extended eigenfunctions: these could lead to large transport. Similar ion modes were described by Romanelli and Zonca [2]. All these modes have discrete frequency spectra. Now, Kim and Wakatani [3] have described electron modes that occur at all radii but are localised and, it is claimed, have a continuous spectrum. We re-examine this problem of mode structure and location. We find that there are just two basic forms: 'isolated' modes that occur only at a particular radius and 'general' modes that occur at all radii. Isolated modes are always localised. In special circumstances general modes may be extended, but in typical cases they are also localised. All modes have a discrete spectrum. These results are consistent with a diffusion coefficient having the Bohm scaling and decreasing with plasma rotational velocity shear.

1. INTRODUCTION

Short wavelength fluctuations may be a source of anomalous transport in magnetically confined, toroidal, plasmas. Prominent among these fluctuations are the electron drift (ED) mode and the ion temperature gradient (ITG) mode - which have been known for more than 30 years. They were first studied in a plane slab, but it was shown by Taylor [4] that their properties may be entirely different in a torus. This led to the development of the, so-called, 'ballooning representations' which were used to investigate the ED [5] and ITG [6] modes in a torus. Early investigations naturally concentrated on the most unstable forms of the ED and ITG modes. However, these occur only at isolated radii and have a highly localised eigenfunction. Consequently they may not be the most important for transport.

Recently [1], Connor, Taylor and Wilson pointed out that there could be another, more general, class of ED modes in a torus. These have a higher stability threshold than the 'isolated' modes, but can occur at all radii. Similar ITG modes were described by Romanelli and Zonca [2]. At first sight, these general modes appear to have an extended eigenfunction, spanning a large fraction of the plasma radius. If so they might lead to large plasma transport[1]. However, we will show that this large radial extension occurs, if at all, only for a very narrow range of plasma parameters.

In the latest development, Kim and Wakatani [3] introduced an apparently different class of ED mode. Although these are not restricted to isolated radii, they resemble the isolated modes in that they have a localised eigenfunction. It is also claimed that they have a continuous spectrum. However, we will show that these modes are, in fact, a manifestation of the modes discussed by Connor, Taylor and Wilson - indeed they are the most typical form. As such they have a discrete, not continuous, spectrum.

2. THE MODEL

A standard model [7] for short wavelength plasma fluctuations with large toroidal mode number $n \gg 1$ in a large aspect ratio torus is described by

$$\left[\frac{1}{(nq')^2}\frac{\partial^2}{\partial x^2} - \frac{\sigma^2}{\Omega^2}\left(\frac{\partial}{\partial\theta} + inq'x\right)^2 - \frac{\alpha}{\Omega}\left(\cos\theta + \frac{is}{nq'}\sin\theta\frac{\partial}{\partial x}\right) - \Lambda\right]\phi(x,\theta) = 0 \qquad (1)$$

where x is the distance from some magnetic surface and θ is the poloidal angle. The first term arises from ion larmor radius, the second from ion sound and the third is the effect of toroidal coupling. The parameters of the model are: $s = rq'/q, \sigma = \varepsilon/bqs, \Omega = \omega/\omega_{*e}$ and $\alpha = 2\varepsilon/bs^2$, where $\varepsilon = L_n/R\tau, \tau = T_e/T_i$ and $b = k_\theta^2\rho_i^2/2$. (L_n is the density scale length.)

The parameter

$$\Lambda = \frac{1}{bs^2}\left[\frac{\Omega - 1}{\tau\Omega + \eta_i + 1} - b\right] \equiv \frac{\hat{\Lambda}}{bs^2} \qquad (2)$$

embodies the frequency shift $(\Omega - 1)$ (important for ED modes) and the temperature gradient $\eta_i = L_n/L_T$ (important for ITG modes).

Schlüter [8] has criticised Eq (1), suggesting that it is not based on a systematic expansion. This criticism is unfounded: Eq (1) can be derived from an asymptotic expansion with the ordering

$$b \sim \rho_i\frac{\partial^2}{\partial x^2} \sim \varepsilon \sim \hat{\Lambda} \ll 1 \qquad (3)$$

(Formally, this restricts the model to $\eta_i \gg 1$ for ITG modes.) With this ordering the parameters σ, α, and Ω are order unity and $1/n$ is the small expansion parameter. The theory below is then asymptotically correct for $1/n \to 0, n\varepsilon > 1$.

[1] Of course, anomalous transport is a non-linear effect and we are discussing only linear modes, but in a weakly turbulent state some influence of linear modes should persist.

3. ANALYSIS

A standard method for solving Eq (1) is by introducing a ballooning transformation,

$$\phi(x,\theta) = \sum e^{im\theta} \int e^{-imn}\hat{\phi}(x,\eta)d\eta \tag{4}$$

which automatically ensures periodicity in θ, and writing

$$\hat{\phi}(x,\eta) = \xi(x,\eta)\exp[-inq'(x\eta - S(x))] \tag{5}$$

Then for large n

$$\left[\frac{\sigma^2}{\Omega^2}\frac{d^2}{d\eta^2} + (\eta-k)^2 + \frac{\alpha}{\Omega}(\cos\eta + s(\eta-k)\sin\eta) + \Lambda\right]\xi(\eta) = \lambda\xi(\eta) \tag{6}$$

where $k = dS/dx$. Here we have introduced an intermediate eigenvalue $\lambda(\omega,k,x)$ defined by the appropriate boundary conditions at $|\ \eta\ |\rightarrow\infty$. Note that $\lambda(k) = \lambda(k+2\pi)$ and $\xi(k+2\pi,\eta) = \xi(k,\eta-2\pi)$.

The plasma eigenvalue ω satisfies

$$\lambda(\omega,k,x) = 0 \tag{7}$$

but at this point ω is undetermined. Nevertheless, all the information needed to calculate ω and the mode structure is embodied in $\lambda(\omega,k,x)$. Within this description we can identify two classes of eigenmode: isolated (occurring only at special radii), and general (occuring at any radius).

i) Isolated modes

These arise when $\lambda(\omega,k,x)$ has a stationary point near which Eq (7) becomes

$$\lambda_0(\omega) + \frac{1}{2}[\lambda_{xx}(x-x_0)^2 + \lambda_{kk}(k-k_0)^2] = 0 \tag{8}$$

This defines two branches $k_+(\omega,x)$ and $k_-(\omega,x)$ which correspond to two WKB solutions. Matching these solutions at the turning points x_1, x_2 requires

$$\frac{nq'}{2\pi}\int_{x_1}^{x_2}(k_+(\omega,x) - k_-(\omega,x))dx = \text{integer } + \frac{1}{2} \tag{9}$$

which defines the frequency of all isolated modes. Such modes are essentially confined between the WKB turning points and have a width

$$x_1 - x_2 = \left(\frac{\pi}{nq'}\right)^{\frac{1}{2}}\left(\frac{\lambda_{kk}}{\lambda_{xx}}\right)^{\frac{1}{4}} \tag{10}$$

(λ_0 has been eliminated using Eq (9)).

ii) General modes

Otherwise than at stationary points, Eq (7) becomes

$$\lambda \sim \lambda_0(\omega, k) + \lambda_x(\omega, k)(x - x_0) = 0 \tag{11}$$

Since λ is periodic in k, this defines an infinity of branches $k_i(\omega, x)$ and the WKB solution involves all these branches. This WKB solution is discussed in [9], but it is more convenient to use an alternative method.

Instead of using the ballooning transformation we write the perturbation as

$$\phi(x, \theta) = \int \xi(p, \theta) \exp[inq'(x(p - \theta) + \hat{S}(p))]dp \tag{12}$$

Note that this is not automatically periodic in θ. Then $\xi(p, \theta)$ satisfies Eq(6) with the substitutions $k \rightarrow p, x \rightarrow -d\hat{S}/dp$. Consequently, Eq(11) becomes

$$\lambda_0(\omega, p) - \lambda_x(\omega, p)\left(x_0 + \frac{d\hat{S}}{dp}\right) = 0 \tag{13}$$

We now note that, because $\xi(p, \theta + 2\pi) = \xi(p - 2\pi, \theta)$, the expression (12) will be periodic in θ if exp $i(nq'\hat{S}(p))$ is periodic in p. This requires

$$\frac{nq'}{2\pi} \oint \left(\frac{\lambda_0(x_0, p)}{\lambda_x(x_0, p)} - x_0\right) dp = \text{integer} \tag{14}$$

which defines the frequency of all general modes. (More precisely [1,9], it defines the growth rate and relates the real frequency to the arbitrary mode centre x_0.)

The structure of the general modes can be illustrated by retaining only the first terms in the fourier expansion of $\lambda(p)$,so that Eq(13) becomes

$$\lambda_0 + \lambda_c \cos p - \lambda_x \left(x_0 + \frac{d\hat{S}}{dp}\right) = 0 \tag{15}$$

(We will see later that this is adequate for the present discussion of ED and ITG modes.) Then, using Eq(14),

$$\hat{S}(p) = \frac{\lambda_c}{\lambda_x} \sin p \tag{16}$$

Using a saddle-point/stationary phase approximation for the integral (12), the structure of a general mode is then

$$\phi \sim \exp\left[inq'\left(x \cos^{-1}\left(\frac{x}{g}\right) \pm g\left(1 - \frac{x^2}{g^2}\right)^{\frac{1}{2}} - x\theta\right)\right] \tag{17}$$

with $g = -\lambda_c/\lambda_x$, and the sign is chosen so that ϕ decays at large x.

Eq (17) brings out the important point that the width of a general mode may be determined in two distinct ways. If g is real, the width is determined by the 'turning points' $X = \pm g$ and is independent of the small parameter $1/n$. Within the turning points ϕ is oscillatory, beyond them it decays - eventually as $\exp[-n \mid q'x \mid \log \mid x \mid]$.

However, if g is complex, and we expand Eq(17) for small x, we find

$$\mid \phi \mid \sim \exp[-nx^2 \mid q'g_i \mid /2 \mid g \mid^2] \tag{18}$$

where $g_i = Im(g)$. In this case, therefore, the mode decays in a distance

$$\Delta \sim \mid g \mid /(n \mid q'g_i \mid)^{\frac{1}{2}} \tag{19}$$

Note that, because n is large, this width is much less than the distance between turning points unless $Im(g)$ is close to zero. This effect of $Im(g)$ is much greater than that due to the fact that the turning points are no longer on the real axis. In fact the turning points play no part in determining the mode width, but they are important in determining the frequency, Eq(14).

4. APPLICATION TO ED AND ITG MODES

The discussion in Sec (3) applies to any high n mode. In this section we consider specifically the application to ED and ITG modes.

As we have emphasised, all the properties of modes are contained in the auxiliary quantity $\lambda(\omega, k, x)$ calculated from the ordinary differential Eq (6). Specific cases must be calculated numerically [10] but an approximate solution can be found using a quadratic approximation for the trigonometric terms. Then Eq (6) becomes a Weber equation which is readily solved to give

$$\lambda = \Lambda + \frac{\alpha \cos k}{\Omega} - \frac{\left(\frac{\alpha}{2\Omega}(1-s)\sin k\right)^2}{\left(1 - \frac{\alpha}{2\Omega}(1-2s)\cos k\right)} \pm \frac{i\sigma}{\Omega}\left(1 - \frac{\alpha}{2\Omega}(1-2s)\cos k\right)^{\frac{1}{2}} \tag{20}$$

This expression is somewhat unwieldy, but the main features can be captured if we restrict ourselves to $\alpha/\Omega \ll 1, \sigma/\Omega \ll 1$. Then

$$\lambda = \Lambda \pm \frac{i\sigma}{\Omega} + \frac{\alpha}{\Omega}\left(1 \mp \frac{i\sigma(1-2s)}{4\Omega}\right)\cos k \tag{21}$$

which is of the form of Eq(15). The last term $\sim (1-2s)$ is small in α/Ω but, as explained above, it must be retained when it is the only contribution to $Im(g)$.

i) ED Modes

Electron Drift modes have a frequency $\omega \sim \omega_{*e}$, so that (putting $\eta_i = 0, \tau = 1$ for brevity)

$$\lambda \simeq \frac{1}{bs^2}\left[\frac{\omega - \omega_{xe}}{2\omega_{*e}} - b \pm \frac{i\varepsilon s}{q} + 2\varepsilon\left(1 \mp \frac{i\varepsilon}{4bqs}(1-2s)\right)\cos k\right] \tag{22}$$

In this case the dominant x-dependence of λ arises from ω_{*e}.

Isolated ED modes arise where ω_{*e} has a maximum and are determined by $\lambda_{xx}/\lambda_{kk}$. From Eq(10) their width is

$$\Delta \sim \varepsilon^{\frac{1}{4}}(rL_2/nqs)^{\frac{1}{2}} \tag{23}$$

with $L_2 = (\omega_*''/\omega_*)^{1/2}$. This is typically a small fraction of the plasma radius - though it extends over many mode-rational surfaces.

Elsewhere than at the maximum of ω_* there will be general ED modes determined by the parameter

$$g = \frac{-\lambda_c}{\lambda_x} \sim 4\varepsilon\left(1 - \frac{i\varepsilon}{4qbs}(1-2s)\right) L_1 \tag{24}$$

where $L_1 = (\omega_*'/\omega_*)^{-1}$. If g were real the width of these modes would be $\sim 8\varepsilon L_1$ - which is the large width first calculated by Connor, Taylor and Wilson[1].

However, except for special values of the parameters (represented here by $(1-2s) = 0$) g is not real. Although $Im(g)$ is small, it is nevertheless important and the mode width is given by Eq(19),

$$\Delta \sim \left(\frac{brL_1}{n(1-2s)}\right)^{\frac{1}{2}} \tag{25}$$

Typically this is again a small fraction of the plasma radius. It is essentially the width of the modes described by Kim and Wakatani [3]. (But note that the frequency spectrum is not continuous as they claimed.)

ii ITG modes

To illustrate the nature of ITG modes in a torus we consider η_i large, formally $\sim 1/\varepsilon$. Then the frequency of the toroidal ITG mode is of order $(\varepsilon\eta_i)^{\frac{1}{2}}\omega_{*i}$ and

$$\lambda = \frac{1}{bs^2}\left[\frac{\omega}{\eta_i\omega_*} - b + \frac{i\varepsilon s\omega_*}{q\omega} + \frac{2\varepsilon\omega_*}{\omega}\left(1 - \frac{i\varepsilon\omega_*}{4qbs\omega}(1-2s)\cos k\right)\right] \tag{26}$$

Several parameters now contribute to the x-variation of λ and we assume these have a similar scale L. Isolated ITG modes arise if λ_x vanishes at some radius and their width (Eq 10) is

$$\Delta \sim (rL/nqs)^{\frac{1}{2}} \tag{27}$$

Once again this is small compared to the plasma radius (though formally $\varepsilon^{-1/4}$ larger than the width of the isolated ED modes.)

Elsewhere than at $\lambda_x = 0$ there are general ITG modes determined by the parameter $g = -\lambda_c/\lambda_x$. For ITG modes λ_x and λ_c have comparable real and imaginary parts, with

$$| \lambda_x | \sim \frac{1}{bs^2}\left(\frac{\varepsilon}{\eta_i}\right)^{\frac{1}{2}}\frac{1}{L_1} \qquad | \lambda_c | \sim \frac{1}{bs^2}\left(\frac{\varepsilon}{\eta_i}\right)^{\frac{1}{2}} \tag{28}$$

Consequently, from Eq(19) the width of a general ITG mode is

$$\Delta \sim (rL_1/nqs)^{\frac{1}{2}} \tag{29}$$

This is also small compared to the plasma radius (though formally $b^{-\frac{1}{2}}$ larger than the general ED modes).

If g had been real, as assumed by Romanelli and Zonca [2], the width of a general ITG mode would have been $\sim L_1$. However, bearing in mind that several plasma parameters contribute to λ_x, and that $| \lambda_x | \sim | \lambda_c |$ the conditions for $Im(g)$ to vanish seem unrealistic.

5. SUMMARY AND CONCLUSIONS

Short wavelength electron drift (ED) and ion temperature gradient (ITG) modes have two basic forms in a toroidal plasma. At particular, isolated, radii there may be ED and ITG modes whose widths are a fraction $\sim \varepsilon^{\frac{1}{4}}(1/nq)^{\frac{1}{2}}$ and $\sim (1/nq)^{\frac{1}{2}}$ of the plasma radius respectively. Because of their restriction to particular radii these modes do not seem important for transport.

At all other radii there can be ED and ITG modes that potentially have a large radial width, - comparable to the plasma radius. This would imply that they have an overwhelming effect on transport and would force the plasma to adopt a marginally stable profile, as suggested by Connor, Taylor and Wilson. However, we now see that, except for a very special circumstances, this large radial width is not realised. Instead these general ED and ITG modes have typical widths $\sim (b/n)^{\frac{1}{2}}$ and $\sim (1/nq)^{\frac{1}{2}}$ of the radius respectively. Nevertheless they remain the most important for transport because they occur at all plasma radii.

In this connection it is worth noting that, for $b = constant$ ($b = 0.1$ is a typical value for maximum instability) the width of the general ED and ITG modes can be expressed as $(\rho_i L)^{\frac{1}{2}}$. If this is taken as a step length in conjunction with the natural correlation time L/v_i, one obtains a diffusion coefficient having Bohm scaling with magnetic field and temperature.

As we mentioned in our introduction, Kim and Wakatani [3] claimed there was a third class of ED modes, also with a width $\sim (1/nq)^{\frac{1}{2}}$ but with a continuous frequency spectrum. Indeed it was this claim which stimulated the present investigation. However, we now see that this third class is included in the general modes described above and as such has a discrete spectrum. This arises from the need, overlooked by Kim and Wakatani, to ensure that the amplitude tends to zero at large distances, greater than the mode width and comparable to the distance beween turning points. As shown in Sec (3), when g is complex the amplitude decays at small distances whatever the frequency - but for it to continue to decay at large distances, beyond the turning points, the frequency must satisfy Eq (14).

Finally we would like to comment on the effect of plasma rotation due to radial electric fields. If the rotation frequency $\omega_E = kv_E$ is comparable with ω_{*e} its effect can be incorporated by replacing ω_* by $(\omega_* - \omega_E)$. Then as ω_E' increases, its first effect may be to increase or decrease the width of general modes, depending on the sign of ω_E', but as it exceeds ω_*' the mode width is reduced by a factor ω_E'/ω_*', – with a corresponding reduction in the estimated plasma transport. However, it must be remembered that the discussion here, and elsewhere, is based on a Ballooning approximation, or its equivalent Eq (12). This is valid only so long as the resulting mode width is much greater than the separation between mode rational surfaces. Formally, this requires $\omega_E'/\omega_*' \leq 1$, that is $v_E/v_i \leq 1/n$ (for $k\rho_i \sim 1$). When $v_E/v_i \geq 1$ the perturbations should be described in terms of Fourier modes [7,10] rather than in terms of Ballooning modes and the problem once again becomes similar to that in a plane slab, - but with the velocity shear predominant.

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