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A spectral approach to ballooning theory. Part 1

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This paper and a forthcoming one develop the spectral theory of ballooning transformations relevant to tokamak physics from first principles in a rigorous and yet intuitively clear manner. The power of the ballooning representation to throw light on the spectral characteristics of the plasma problems to which it is applicable is emphasized, and examples are given to illustrate the general notions. The ballooning representation is shown to be essentially a method to separate variables and reduce two-dimensional partial differential equations with periodic coefficients to infinite sets of soluble ordinary differential equations. This paper is concerned with an elementary approach to the techniques in the context of nearly exactly soluble problems involving the anisotropic diffusion operator in toroidal geometry. Two different perturbation methods are discussed. Applications to plasma instability problems and the subtleties involving the continuous spectra of ballooning operators will be taken up in Part 2.

1. Introduction

The idea of solving linear time-evolution problems associated with stability of physical systems by suitable expansions in terms of known analytic functions is as old as the principle of superposition itself, and goes back to Daniel Bernoulli and Fourier. This method of 'eigenfunction expansion' is extremely well developed, especially since the early works of Sturm, Liouville and Green in the 19th century and up to the modern foundations laid by Hilbert, Von Neumann, Weyl and their successors following the advent of quantum mechanics. Plasma physics is no exception in being a field in which the techniques of spectral theory are remarkably fruitful. Partial differential equations of plasma physics, when linearized, admit rather simple representations involving ordinary Fourier series in the cylindrical approximation. In toroidal geometry, while azimuthal symmetry persists (in tokamaks), the poloidal variation of equilibrium quantities renders Fourier expansions somewhat unmanageable. The 'ballooning representation' (Connor *et al.* 1978, 1979; Lee and Van Dam 1979; Dewar and Glasser 1983; Zhang *et al.* 1994, and references therein) was invented to overcome this difficulty – at least in certain asymptotic limits of high toroidal wavenumber. The idea of this representation is as old as Fourier's representation of periodic functions, and seems to have been invented by Poisson and Jacobi (at various levels of generality) in connection with the so-called 'theta functions' of classical analysis. The idea resurfaced again in solid-state theory, and is known in that field through 'Wannier' and 'tight-binding' representations, complementary to the more usual Bloch representation (Ziman 1965).

The purpose of the present paper is to introduce the basic concepts involved in ballooning and present some new insights offered by examining the spectral properties of typical plasma physics operators revealed by the ballooning transformation.

2. Basic ideas of ballooning transformations

We begin with a discussion of the basic ideas of ballooning theory (Connor *et al.* 1978, 1979). Consider the one-dimensional interval $(-\pi, \pi)$. If $f(\theta)$ is any 2π-periodic function in this interval that is sufficiently 'well behaved' (throughout this paper, we shall assume that the functions studied are as reasonable as required for applications, except where special considerations are necessary) then we have Fourier's theorem, which asserts that the function may be expanded as follows:

$$f(\theta) = \sum_{m=-\infty}^{\infty} \hat{f}_m \exp(im\theta), \tag{1}$$

where

$$\hat{f}_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \exp(-im\theta)\, d\theta. \tag{2}$$

This traditional 'Fourier representation' forms the backbone of the subject of mathematical analysis and its applications, but has certain drawbacks in plasma applications, which we shall discuss later. It is less well known that there is another, related, representation of periodic functions called the 'Poisson' or 'shifted-sum' representation, which arises as follows. Consider now the *infinite* interval $(-\infty, +\infty)$ in the variable η. Suppose we consider a function $F(\eta)$ that is defined in this interval and is as smooth as desired. We suppose further that $F(\eta)$ tends to zero sufficiently fast as $\mid \eta \mid \rightarrow \infty$ that it is absolutely integrable. We then form the 'shifted sum' constructed as in the theory of groups:

$$f^*(\eta) = \sum_{n=-\infty}^{\infty} F(\eta - 2n\pi). \tag{3}$$

It is now easy to prove (Bellman 1961) that, under very mild conditions on F, the infinite series over all integers n converges uniformly and absolutely and defines a function $f^*(\eta)$ over the same infinite interval. Furthermore, exactly as in group theory, we verify that the sum is invariant under the transformation $\eta \rightarrow \eta \pm 2\pi$, and hence infer that the function $f^*(\eta)$ is a *periodic* function of its argument with period 2π (i.e. $f^*(\eta) = f^*(\eta + 2\pi)$). This trick of 'manufacturing' periodic functions on a finite interval from suitable functions on an infinite 'extended' interval is precisely one form of the celebrated 'ballooning transformation'. The reason for adopting such a procedure is that it is often possible to use asymptotic methods such as eikonals (i.e. WKB functions) to construct solutions to linear differential equations in the infinite domain without necessarily satisfying periodic boundary conditions, and use the above shifted sum trick to define solutions *of the same equations* in periodic domains. We shall consider cases where such strategies *fail*, but fortunately, in plasma applications, the problems that can be solved using the technique are extremely important.

A classical example of the above transformation (of period π) is provided when we choose $F(\eta) = 1/\eta^2$, leading to

$$\frac{1}{\sin^2 \eta} = \sum_{n=-\infty}^{\infty} \frac{1}{(\eta - n\pi)^2}. \tag{4}$$

A corresponding construction in the complex plane yields most of the important elliptic functions as well (i.e. double-shifted sums giving rise to doubly periodic functions). Another famous example is furnished by choosing F to be Gaussian:

$$\Theta(\eta, \tau) = \frac{1}{\tau^{1/2}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{(\eta - n\pi)^2}{\pi\tau}\right]. \tag{5}$$

Here τ is real and positive. It can be shown (Bellman 1961; Whittaker and Watson 1965) that the Θ function satisfies the heat-conduction equation with respect to the 'time variable' τ and the spatial variable η in the π-periodic domain:

$$\frac{\partial \Theta}{\partial \tau} = \frac{\pi}{4} \frac{\partial^2 \Theta}{\partial \eta^2}. \tag{6}$$

In fact, it is the *fundamental* solution of the equation, and can be used to solve the initial-value problem. This example provides an illustration of the fact that the Gaussians on the right in (5) are fundamental solutions of the same equation in the *infinite domain*. The fact that the left-hand side can be obtained by standard separation of variables of the heat equation in the periodic domain directly leads to a remarkable identity, which is sometimes referred to as the 'theta function transformation' formula of Jacobi:

$$\sum_{n=-\infty}^{\infty} \exp(-\pi\tau n^2 + 2ni\eta) = \frac{1}{\tau^{1/2}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{(\eta - n\pi)^2}{\pi\tau}\right]. \tag{7}$$

Note, in particular, the remarkable feature that, on the left, the time parameter τ enters the exponentials *linearly* while the dependence on it on the right is anything but simple. As we shall see, this type of identity is characteristic of ballooning theory, and can be manufactured at will using a fundamental result due to Poisson. There are many ways of expressing this basic theorem of Fourier analysis. We choose a form that appears easiest to use in applications and depends upon the simplest properties of the Dirac delta function. Recall the formula that expresses Fourier's integral theorem in complex form in terms of the delta function:

$$\delta(x - y) = \int_{-\infty}^{\infty} \exp[ik(x - y)] \frac{dk}{2\pi}. \tag{8}$$

To obtain the Poisson formula, we replace $x - y$ by $\eta - 2n\pi$ and sum over all positive and negative values of the integer n. This sum, denoted by $\Delta(\eta)$, is evidently a periodic generalized function over $(-\pi, \pi)$. Applying Fourier's series theorem to it, we obtain the rule

$$\sum_{n=-\infty}^{\infty} \delta(\eta - 2n\pi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \exp(im\eta). \tag{9}$$

The more usual, 'classical' form is obtained by multiplying both sides by $g(\eta)$ (an arbitrary function that has a Fourier transform f) and integrating over the infinite domain to get

$$\sum_{n=-\infty}^{\infty} g(2n\pi) = \sum_{m=-\infty}^{\infty} f(m), \tag{10}$$

where

$$f(m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\eta)\exp(im\eta)\,d\eta,$$

$$g(\eta) = \int_{-\infty}^{\infty} f(x)\exp(-i\eta x)\,dx.$$

Evidently, we can readily obtain (7) and many other identities like it from (10). More importantly, we also derive the following fact. If the periodic function $f^*(\eta)$ is constructed using (3) from the 'generating function' $F(\eta)$ defined on the infinite domain, the Fourier coefficients g_m^* of $f^*(\eta)$ are simply expressible in terms of $G(k)$, the Fourier transform of F. We have, quite simply,

$$g_m^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(\eta)\exp(-im\eta)\,d\eta = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\eta)\exp(-im\eta)\,d\eta = G(m).$$

Thus the Fourier coefficients of f^* are just the values of the Fourier transform of F at integral values of its argument. This statement is the content of the identity

$$f^*(\eta) = \sum_{n=-\infty}^{\infty} F(\eta + 2n\pi) = \sum_{m=-\infty}^{\infty} \exp(im\eta)\frac{1}{2\pi}\int_{-\infty}^{\infty} F(y)\exp(-imy)\,dy, \tag{11}$$

which will be used frequently and generalizes the preceding one.

This idea of 'extending' the definition of the Fourier coefficients of a periodic function defined on the set of integers to a function defined for all real values (of the wavenumber) played a basic role in the considerations of Connor *et al.* (1978, 1979). They suggested that, given a periodic function f^*, its Fourier coefficients g_m could be 'interpolated' by a suitable $G(k) \equiv g_m$ (for $k = m$). Then, one can define F in the infinite interval using the Fourier transform of G. This procedure 'extends' the definition of f^* from its periodic domain onto the infinite interval. Such an extension is, of course, not unique, since many functions can be defined that will interpolate the Fourier coefficients of f^* in this way. Connor *et al.* proposed the definition

$$G(k) = \sum_{m=-\infty}^{\infty} g_m \frac{\sin[\pi(m-k)]}{\pi(m-k)}. \tag{12}$$

For suitably smooth f^*, the g_m die away rapidly for large $|m|$, and the series for G is convergent and defines a function G on the infinite interval with a well-defined Fourier transform F. Furthermore, it is clear, by construction, that $G(m) = g_m$. Hence, as claimed by Connor *et al.*, $F(\eta)$ is indeed an extension of f^* to the infinite interval and would satisfy formally the same differential equation as f^* with respect to a translation-invariant operator. However, as we shall shortly demonstrate, *this particular extension* (i.e. defined by the specific interpolation formula (12)) is a 'trivial' extension that is valueless in actual applications of the ballooning theory.

To see this, we explicitly calculate F from G. Introducing the characteristic function $\Xi_{-\pi,\pi}(\eta) = 1$ for $-\pi \leqslant \eta \leqslant \pi$ and zero otherwise, we note its Fourier transform

$$X(k) = \int_{-\infty}^{\infty} \Xi_{-\pi,\pi}(\eta)\exp(ik\eta)\,d\eta = \int_{-\pi}^{\pi} \exp(ik\eta)\,d\eta = 2\pi\frac{\sin(\pi k)}{\pi k}.$$

It is then straightforward to verify that $F(\eta) = f^*(\eta)\Xi_{-\pi,\pi}(\eta)$. Thus, $F(\eta)$ is identical to f^* in the interval $(-\pi, \pi)$ and *identically zero outside it*. Furthermore,

$$G(k) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(\eta)\exp(-ik\eta)\,d\eta$$

can be obtained by simply substituting the real variable k in place of the integer m in Fourier's formula for the coefficients of the periodic function f^*.

It is elementary to verify that this 'trivial extension' leads back to f^* periodic for all η if used in Poisson's shifted sum, as it is guaranteed to do anyway. The reason for categorizing this extension as trivial is that F is simply the restriction of f^* to the fundamental interval. It is not in general a continuous or smooth extension of f^* unless f^* and its derivatives vanish identically at the endpoints of the fundamental interval. While it certainly formally satisfies the same differential equation as f^* within the fundamental interval, outside, it is a trivial solution that does not generally match with the interior solution. This fact has apparently not been noticed in the literature (Connor *et al.* 1978, 1979). As we shall see, most useful extensions of f^* do not vanish identically outside the fundamental interval, unlike the Fourier transform of G defined by (12).

It should also be noted that all extensions G^* of g_m are simply related to $G(k)$ by the formula $G^*(k) = G(k) + \sin(\pi k)\,H(k)$, where $H(k)$ is an arbitrary continuous and integrable function over the infinite k domain. In interesting cases, the determination of G^* (or its Fourier transform) constitutes the problem, and H itself is never identically zero unless homogeneous boundary conditions are imposed.

With these preliminaries out of the way, let us consider the following problem. Suppose we are required to solve a linear eigenvalue problem defined by the partial differential equation

$$L\left(x, \frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial \theta}\right)\phi = \lambda\phi, \tag{13}$$

where we assume that the x domain is (say) $(-\infty, \infty)$ and the θ domain is periodic, $(-\pi, \pi)$. We assume further that θ enters the operator L periodically, i.e. it is 'translation-invariant'. Thus, in principle, the operator is defined in the 'extended' θ domain,$(-\infty, \infty)$. Connor *et al.* (1978, 1979) had the idea of considering a related eigenvalue problem in the extended θ domain for $F(x, \eta)$. Thus, they proposed to consider the equation

$$L\left(x, \frac{\partial}{\partial x}, \eta, \frac{\partial}{\partial \eta}\right)F = \lambda F \qquad (-\infty < \eta < \infty) \tag{14}$$

and impose the condition that F should go to zero as $|\ \eta\ | \to \infty$, apart from the standard ones relating to x. If this eigenvalue problem has a nontrivial solution $F(x, \eta)$, the periodicity of the operator guarantees that $F(x, \eta + 2n\pi)$ is a nontrivial, linearly independent solution for any integer n and the *same* eigenvalue, λ. Thus, for the proposition to succeed, it is *essential* that in the extended domain, nontrivial eigensolutions of infinite multiplicity must exist. Supposing that they do, Connor *et al.* were able to prove that $\phi(x, \theta) = \sum_{m=-\infty}^{\infty} F(x, \theta + 2n\pi)$ is a solution to the original eigenvalue problem, but now satisfying the imposed periodic boundary conditions. Crucial to the success of the method is the question, 'What properties must the operator L have for the eigenvalue problem in the extended domain to have a suitable solution F?'

Unfortunately, not all operators L periodic in θ admit of this approach since infinite multiplicity of eigenvalues in the extended domain is a very special property. We shall consider a typical counterexample.

Example 1

Let the operator L be defined by

$$L \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2} + \mu \cos\theta, \tag{15}$$

where μ is a real parameter. Evidently the eigenvalue problem in a finite x domain and periodic θ domain can be solved explicitly by separation of variables, and the exact problem has a well-defined discrete spectrum of eigenvalues that depend analytically on the parameter μ. If the 'ballooning approach' is attempted, it is easy to show that there are no solutions to the extended eigenvalue problem. In the infinite domain, the nature of the spectrum is totally different to that in the periodic domain. This is because, in the infinite domain, there are no solutions to the Mathieu equation that go to zero at infinity. Thus the ballooning transformation fails in this case to generate the true solution in the periodic domain. The same fate awaits any operator L that is separable or 'close' to a separable one. This negative result shows that the ballooning transformation is not the appropriate tool for certain problems. It can be (rightly) argued that if the problem is separable, it is not necessary to use ballooning in the first place! It must be stated that, while nonseparability is a necessary condition for the applicability of the ballooning transformation, there appear to be no simple necessary and sufficient conditions.

Our next example deals with a problem which is much closer in spirit to plasma applications, and in particular to the problems solved by Connor *et al*. It illustrates, in a technically manageable model, all the relevant aspects of ballooning theory, and presents at the same time, a physically nontrivial equation of considerable importance to plasma theory.

Example 2

We begin by considering the heat-diffusion equation in a slab in the presence of a sheared magnetic field. Thus consider the geometry where x, θ and ζ correspond respectively to the 'radial', 'poloidal' and 'toroidal' directions. We assume that the domain is periodic in the θ and ζ directions and infinite in x. In addition to a uniform B_ζ field, we consider a sheared 'poloidal' field $B_\theta(x)$. We are interested in temperature *fluctuations*, and consider the linearized, anisotropic heat-conduction equation for the temperature perturbation Θ, which is assumed to take the following form:

$$\begin{aligned} \frac{\partial \Theta}{\partial t} &= \chi_\perp \nabla_\perp^2 \Theta + \chi_\parallel \nabla_\parallel^2 \Theta + \mu \cos\theta\, \Theta, \\ \nabla_\perp^2 &= \frac{\partial^2}{\partial x^2}, \\ \nabla_\parallel &= \frac{1}{L_\parallel}\left(\frac{\partial}{\partial \theta} + \frac{x}{L_s}\frac{\partial}{\partial \zeta}\right). \end{aligned} \tag{16}$$

In the above equation, we note that the thermal diffusivities $\chi_\perp$ and $\chi_\parallel$ are taken to be constant, as are the two lengths $L_\parallel(\approx qR$ in tokamaks) and $L_s(\approx q/q'$ in tokamaks). We have also modelled the perpendicular transport using the 'radial' Laplacian, neglecting the 'poloidal' term. The parallel gradient operator is taken with respect to a particular origin in x, and is the simplest generic approximation to a typical tokamak problem, as the reader can verify by considering the single-temperature Braginskii energy equation with constant density in a cylindrical-tokamak approximation. Finally, the $\mu\cos\theta$ term (it can be easily generalized to an arbitrary periodic bounded function of θ if required) with (μ real) simulates a poloidally modulated source/sink proportional to the temperature perturbation. This is included to demonstrate the effects of 'toroidicity'. We require the solution to be 2π-periodic in θ and ζ and to vanish at infinity with respect to x. The most general problem that one can consider with a parabolic equation of this type is the initial-value problem, in which one imposes an initial distribution $\Theta(x,\theta,\zeta,0)$ satisfying the boundary conditions and square-integrable over the solution domain with respect to x,θ and ζ, and seeks the solution for $t>0$. It should be noted that in this model, time-reversal symmetry is violated (i.e. heat diffusion is dissipative). The closely related problem of the Schrödinger equation is obtained from the above by merely changing $t\to i\tau$. The resulting equation for the complex wave function $\Theta(x,\theta,\zeta,\tau)$ is, of course, conservative (i.e. $\int |\Theta|^2\,dV=\text{const}$) and gives a better motivation for the square-integrability condition on the solution. This problem is less close to plasma physics than the heat-diffusion equation, but, as we shall see later, has a more subtle analogy with shear Alfvén and drift dynamics.

If we wish to solve the initial-value problem for the heat equation, it is clearly expedient to calculate 'eigensolutions' of the form $\Theta=\Phi_n(x,\theta)\exp(in\zeta-\lambda t)$. Apart from the trivial case of $n=0$, where separation of variables works, the equation with suitable redefinitions of variables x and t can be brought to the following general form. Thus, letting

$$\xi=\frac{nx}{L_s},\qquad \tau_\parallel=\frac{L_\parallel^2}{\chi_\parallel},\qquad \tau_\perp=\frac{L_s^2}{\chi_\perp},$$

$$t^*=\frac{t}{\tau_\parallel},\qquad \mu^*=\mu\tau_\parallel,\qquad \epsilon=\frac{\chi_\perp L_\parallel^2}{\chi_\parallel L_s^2}n^2=\frac{\tau_\parallel}{\tau_\perp}n^2,$$

and redefining the eigenvalue appropriately, we see that the eigenvalue problem reduces to the solution of the equation

$$-\epsilon\frac{\partial^2}{\partial\xi^2}\Phi_n-\left(\frac{\partial}{\partial\theta}+i\xi\right)^2\Phi_n-\mu^*\cos\theta\,\Phi_n=\lambda\Phi_n. \tag{17}$$

This two-dimensional partial differential equation is to be solved for the eigenvalue λ subject to periodicity in θ and square-integrability in ξ over $-\infty<\xi<\infty$, for given ϵ and μ^*. Obviously, the n dependence of the equation is through the parameter ϵ, which measures the ratio of the parallel and perpendicular heat-diffusion times.

The operator is not separable as it stands. Following the philosophy of ballooning as proposed by Connor *et al.* we consider it in the extended (i.e. infinite) θ domain. We recognize that the operator $\partial/\partial\theta+i\xi$ annihilates the 'eikonal' function (aperiodic in θ), $\exp[-i\xi(\theta-\theta_0)]$, where θ_0 is an arbitrary constant. Thus we try a solution of the form $\Phi=\exp[-i\xi(\theta-\theta_0)]W(\theta)$. We note two features of this trial function. It is not periodic in θ, since the exponential factor has a period varying with ξ. More disturbingly, it is certainly not square-integrable in ξ over the infinite domain. However, we proceed formally. The transformation is effectively a 'separation of variables'. Thus we find that W satisfies, upon setting $\theta = y + \theta_0$, the so-called 'ballooning equation'

$$-\frac{\partial^2 W}{\partial y^2} + \epsilon y^2 W - \mu^* \cos(y + \theta_0)\, W = \lambda(\mu^*, \epsilon, \theta_0) W, \tag{18}$$

where we emphasize the fact that the eigenvalue λ is a function both of μ^*, ϵ and of the 'separation' parameter θ_0. We note the crucial feature of the ballooning ansatz of representing the solution in the extended domain with an aperiodic eikonal factor: the partial differential equation (17) in ξ and θ has been reduced to an *ordinary* differential equation in which θ_0 enters through a periodic function but y has a periodic as well as 'secular' dependence via the second term on the left.

The equation for W (i.e. (18)) can be solved in principle in the infinite y domain for a complete orthonormal set of eigenfunctions $W_p(y, \mu^*, \epsilon, \theta_0)$ and eigenvalues $\lambda_{n,\theta_0,p}(p = 0, 1, 2 \ldots)$. This is best seen in the case when μ^* is small (corresponding physically to a source rate small compared with the parallel diffusion rate). Then we can carry out a standard quantum-mechanical ('Rayleigh–Schrödinger') perturbation solution based on the Weber–Hermite/harmonic-oscillator functions, which evidently satisfy the $\mu^* = 0$ equation. Since θ_0 enters the equation periodically, both the eigenfunctions and the eigenvalues are periodic, entire transcendental functions of this parameter. Evidently, θ_0 is a continuous 'eigenlabel', just as the discrete index $p = 0, 1, 2, \ldots$ is an eigenlabel. Thus we see that, in this case, the approach of Connor *et al.* apparently leads to an exact solution for the eigenvalue problem in the extended θ (or y) domain, with the eigenvalues labelled by the discrete indices n and p and the continuous parameter θ_0, which can, without loss of generality, be taken to vary in $(-\pi, \pi)$.

Since, by construction, the functions W_p decay rapidly at infinity, the shifted sum can now be constructed to give the solution to (17) as a function of ξ and θ in the 'physical' domain:

$$\Phi_{n,p,\theta_0}(\xi, \theta) = \sum_{m=-\infty}^{\infty} \exp[-i\xi(\theta - \theta_0 + 2m\pi)]\; W_p(\theta - \theta_0 + 2m\pi, \theta_0). \tag{19}$$

This is a *formal* solution of the eigenvalue problem, corresponding to the eigenvalue $\lambda(\mu^*, \epsilon, p, \theta_0)$. The ballooning transformation has enabled us to effectively 'reduce' a two-dimensional, nonseparable equation in a periodic domain to a one-dimensional separable equation in an infinite domain.

The eigenvalue λ is obviously an entire function of μ^*. It is also clearly a function of the three eigenlabels, n, p and θ_0, as is required of a fully three-dimensional eigenvalue problem. Its most important property is that, while the labels n and p are manifestly discrete integers, the 'poloidal' parameter θ_0 introduced during the course of the ballooning transformation is a *continuous* eigenlabel. Thus the present problem has a continuous eigenvalue spectrum as it stands.

However, the continuity of the spectrum must give us pause. Let us examine in somewhat greater detail, the eigenfunction $\Phi_{n,p,\theta_0}(\xi, \theta)$ obtained using the ballooning transformation. By construction, it is periodic in θ, thanks to the shifted sum. By changing $\xi \to \xi + 1$, we immediately find the relation

$$\Phi_{n,p,\theta_0}(\xi + 1, \theta) = \exp[-i(\theta - \theta_0)]\; \Phi_{n,p,\theta_0}(\xi, \theta). \tag{20}$$

Clearly, this proves that the eigenfunction behaves in the classic Floquet–Bloch manner (i.e. quasiperiodically in ξ), with Floquet exponent dependent on $\theta - \theta_0$. This immediately shows that Φ *cannot be localized* in ξ. Therefore it is *not* normalizable in the infinite ξ domain. Indeed, it is clear that the absolute square of the eigenfunction is actually a periodic function of ξ with unit period. This 'translational invariance' in the radial direction indicates that all rational surfaces are 'equivalent'. As we shall see later, this invariance is at the heart of the ballooning representation. In the present problem, the invariance is exact, whereas in applications, it tends to be a 'broken symmetry'. It is a consequence of Hilbert space theory (Riesz and Nagy 1971) that the solution constructed is not a proper eigenfunction and the 'eigenvalue' λ_{n,p,θ_0} is not a proper eigenvalue either. According to the spectral theory of self-adjoint differential operators (Riesz and Nagy 1971) (the present one falls within this class), an 'eigenfunction' associated with a real eigenvalue must be *square-integrable* with respect to the independent variables in its domain of definition. Bounded formal solutions that are not square-integrable are related to a continuously varying 'eigenvalue' and define the so-called continuous spectrum (a typical example is furnished by the one-dimensional free-particle Schrödinger equation and plane-wave solutions). In the present problem, there are no true eigenvalues and only a continuous spectrum as determined above. This is all the more remarkable when it is recalled that the $\mu^* = 0$ problem in a slab has a pure discrete spectrum labelled by n, m and p, where m is the 'poloidal' wavenumber. Thus toroidicity is a 'singular perturbation' in the sense that the eigenvalue spectrum is *qualitatively* different with and without it, however small the coupling constant may be. The genesis of the continuous spectrum due to toroidicity will be explained later.

To appreciate the significance of this result and the full implications for the initial-value problem, it is necessary to consider a seemingly totally different approach to ballooning theory – this time not based on the 'shifted-sum' representation but on the familiar Fourier representation of a periodic function. It is therefore convenient to leave the discussion of our model at this point to consider the alternative approach to ballooning due to Dewar, Mahajan and co-workers (see Dewar and Glasser 1983; Zhang *et al.* 1994, and references therein).

3. An alternative approach to ballooning theory

Further insight into ballooning transformations is afforded by an alternative route, also considered by Connor *et al.* (1978, 1979), but developed to its full potential by Dewar and Glasser (1983) and Zhang *et al.* (1994). To motivate this, we start with (17) for Φ_n. Since the solution is required to be periodic in θ, we may always use the Fourier series representation

$$\Phi_n(\xi, \theta) = \sum_{l=-\infty}^{\infty} \exp(-il\theta)\, \phi_{n,l}(\xi). \tag{21}$$

Substitution into (15) gives the following infinite system of difference–differential equations for the Fourier components, $\phi_{n,l}(\xi), l = 0, \pm 1, \pm 2, \ldots$:

$$-\epsilon \frac{\partial^2 \phi_{n,l}}{\partial \xi^2} + (\xi - l)^2 \phi_{n,l} - \frac{\mu^*}{2}(\phi_{n,l+1} + \phi_{n,l-1}) = \lambda \phi_{n,l}. \tag{22}$$

We note that the differential operator on $\phi_{n,l}$ is invariant under the simultaneous translations, $\xi \to \xi + 1, l \to l + 1$. The finite-difference operator acts on exponentials involving l in a simple way. This 'invariance' (called 'ballooning symmetry' by Zhang *et al.* 1994) suggests that the $\phi_{n,l}$ should be represented as follows:

$$\phi_{n,l} = \exp(il\theta_0) \int_{-\infty}^{\infty} \exp[-iy(\xi - l)]\, W(y, \theta_0)\, dy. \tag{23}$$

An arbitrary constant θ_0 (chosen real to ensure boundedness with respect to l) has been introduced. The l dependence is exponential and ξ enters only in the combination $\xi - l$. Of course, the function W depends upon n (via ϵ) and also on μ^*; this dependence is not explicitly indicated to avoid needless subscripts. It is now straightforward to substitute into (22) and obtain the following ordinary differential equation for $W(y, \theta_0)$, in which n and θ_0 appear only parametrically, the former via ϵ:

$$-\frac{\partial^2 W}{\partial y^2} + \epsilon y^2 W - \mu^* \cos(y + \theta_0)\, W = \lambda W. \tag{24}$$

It is evident that W satisfies (18). Thus, in attempting to represent the Fourier coefficient $\phi_{n,l}(\xi)$ as the Fourier transform, we are following exactly the procedure suggested by the Poisson summation formula, and end up with the same equation. Note that in this representation, y appears as the Fourier conjugate of the 'twisted' *radial* variable $\xi - l$, whilst in the shifted-sum approach, it was related to the poloidal angle variable in the extended domain. Clearly, θ_0 is an eigenlabel analogous to n (conjugate to the toroidal coordinate ζ) and p (labelling the eigenfunctions relative to y). The eigenvalues depend continuously on θ_0. Suppose we have found the solutions labelled by the discrete index p. Substitution into the Fourier series, (21) leads to

$$\Phi_{n,p,\theta_0}(\xi, \theta) = \sum_{l=-\infty}^{\infty} \exp[-il(\theta - \theta_0)] \int_{-\infty}^{\infty} \exp[-iy(\xi - l)]\, W_p(y, \theta_0)\, dy. \tag{25}$$

This looks very different from (19). Yet, we expect the two functions to be identical apart possibly from a constant multiplier. Indeed, the general Poisson summation formula (11) shows that the two results are identical. Using the results obtained, we can now establish a 'completeness' relation for the continuum eigenfunctions $\Phi_{n,p,\theta_0} \exp(in\zeta)$. Thus, multiplying this function by the complex conjugate of a similar function labelled by n', p' and θ_0', and integrating over ζ, θ and the infinite x domain, we obtain the following 'resolution of the identity' or expression of completeness:

$$\frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} d\zeta\, d\theta\, dx\, \Phi^*_{n',p',\theta_0'} \Phi_{n,p,\theta_0} \exp[i\zeta(n - n')] = \delta_{n',n} \delta_{p',p} \delta(\theta_0' - \theta_0). \tag{26}$$

In deriving this important relation, we have made use of the orthonormality of trigonometric functions and of the W_ps. As is well known, this shows that an 'arbitrary' function of x, ζ and θ can be expanded in terms of the complete set $\Phi_{n,p,\theta_0} \exp(in\zeta)$ labelled by the discrete parameters n and p and the continuous one θ_0 (called λ by Zhang *et al.* 1994). The latter is, without loss of generality, restricted to the fundamental interval $(-\pi, \pi)$, since the expansion functions are manifestly periodic in it. We have also proved that the operator on the left in (17), when acting on these functions, reduces to the generalized 'multiplier' (not a proper eigenvalue, since the functions are not square-integrable) λ_{n,p,θ_0}. The solution to the initial-value problem of the heat equation can be readily constructed by superposition of the above 'continuum modes'. Indeed, the Green function of the problem can be obtained explicitly.

The above spectral representation of the problem shows the following features that have already been noted.

(1) The basic operator is not separable, but the 'ballooning symmetry' is exact and the transformation leads to one-dimensional equations for the eigenfunctions.

(2) The solutions are periodic in ζ and θ, but are *quasiperiodic* in x, with the Floquet (stable) exponent depending upon the angle. This inevitably results in a continuous spectrum with respect to the conjugate variable θ_0, which varies continuously in the fundamental interval.

(3) The generalized eigenvalue λ is labelled by the discrete indices n and p and by the continuum parameter θ_0. It is perfectly 'respectable' as a continuum eigenvalue in the sense that it occurs in the solution of the initial-value problem exactly as does any ordinary eigenvalue.

(4) The Fourier coefficients $\phi_{n,l}$ are smooth at specific rational surfaces for given n and l, and their Fourier conjugates W_p are thus square-integrable and hence localized in k. The fact that they are localized implies that the shifted sum converges. Conversely, if the $\phi_{n,l}$ are sharply localized in $\xi - l$, the label p would be continuous and there is a serious problem with the shifted sum since convergence in the ordinary sense cannot be guaranteed.

(5) The eigenfunctions determined as above are complete and may be used to solve the initial-value problem and the inhomogeneous problem where a time-dependent source is prescribed and 'drives' the linear response. (This is relevant to plasma transport applications in which the temperature fluctuations associated with a given electromagnetic fluctuation spectrum may be required, as discussed for example by Haas and Thyagaraja (1984). Further elementary discussions of turbulence and transport can be found in the review by Haas and Thyagaraja (1986)). It is remarkable that the spectrum, which is discrete (but infinitely degenerate) in the cylindrical problem $\mu^* = 0$, becomes a continuous spectrum for arbitrarily small toroidicity.

(6) If one deals with a problem in which the partial differential operator discussed constitutes a 'principal term' and perturbations about its solutions are appropriate, the exact eigenfunctions can always be expanded in terms of the complete set. The equations for the expansion coefficients can be solved perturbatively to determine the 'true' spectrum. In this sense, the exact ballooning continuum eigenfunctions can be thought of as a useful expansion set, just as plane waves in quantum mechanics (i.e. the 'momentum representation' associated with the free-particle continuum resulting from translational invariance) are useful in the solution of problems – whether or not the exact problem admits translational symmetry and irrespective of the nature of the exact spectrum. For example, the momentum representation can be used to solve the harmonic oscillator (discrete spectrum, no translational invariance) as well as the Bloch

problem of a periodic potential (continuous spectrum with 'gaps', full translational invariance 'broken' but periodicity obtains).

In the rest of this paper, we shall systematically exploit this spectral perspective.

4. Broken ballooning symmetry

In realistic problems, the ballooning invariance exhibited by plasma equations such as (16) is only ever manifested (if at all) in the leading order of some parameter. When the symmetry is broken, interesting effects arise that can be tackled using the leading order spectral representations. A typical example illustrating this arises when we consider a sheared toroidal *advective flow* in addition to the diffusion. Toroidal and poloidal sheared flows are thought to be of considerable importance in stability theory (Taylor and Wilson 1996). The present model offers a concrete and relatively elementary introduction to this topic. Interestingly, the problem actually becomes non-self-adjoint, implying a complex spectrum in general. We consider a new term on the left of (16) of the form $\Omega(x/L_s)\partial\Theta/\partial\zeta$, where Ω is a measure of the vorticity (i.e. flow shear) in the advecting velocity. Accordingly, we are now required to solve

$$-\epsilon\frac{\partial^2}{\partial\xi^2}\Phi_n-\left(\frac{\partial}{\partial\theta}+i\xi\right)^2\Phi_n-\mu^*\cos\theta\,\Phi_n+i\kappa\xi\Phi_n=\lambda\Phi_n, \tag{27}$$

where the flow-shear effects are measured by $\kappa=\Omega\tau_{\|}$. Adopting the approach of Mahajan and co-workers (see Zhang *et al.* 1994), we substitute the Fourier expansion for the eigenfunction and derive the following equation for the $\phi_{n,l}(\xi)$:

$$-\epsilon\frac{\partial^2\phi_{n,l}}{\partial\xi^2}+(\xi-l)^2\phi_{n,l}+i\kappa(\xi-l)\phi_{n,l}+i\kappa l\phi_{n,l}-\frac{\mu^*}{2}(\phi_{n,l+1}+\phi_{n,l-1})=\lambda\phi_{n,l}. \tag{28}$$

In this equation, it is clear that the $i\kappa l$ term violates the ballooning symmetry. Following the principle of the method of variation of constants, we expect the solutions to be superpositions of the previously constructed continuum modes. Accordingly, we express the Fourier coefficient as follows:

$$\phi_{n,l}=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta_0\exp(il\theta_0)\int_{-\infty}^{\infty}\exp[-iy(\xi-l)]\,W(y,\theta_0)\,dy. \tag{29}$$

Substitution gives the following *partial* differential equation for W, in contrast to (24):

$$-\frac{\partial^2W}{\partial y^2}+\kappa\frac{\partial W}{\partial y}-\kappa\frac{\partial W}{\partial\theta_0}+\epsilon y^2W-\mu^*\cos(y+\theta_0)\,W=\lambda W. \tag{30}$$

The term involving the first partial derivative with respect to y is easily handled by the substitution $W=Z\exp(\frac{1}{2}\kappa y),\Lambda=\lambda+\frac{1}{4}\kappa^2$. We then find that Z satisfies the equation

$$-\frac{\partial^2Z}{\partial y^2}+\kappa\frac{\partial Z}{\partial\theta_0}+\epsilon y^2Z-\mu^*\cos(y+\theta_0)\,Z=\Lambda Z. \tag{31}$$

We solve this equation following the standard Rayleigh–Schrödinger perturbation theory of quantum mechanics. To keep the notation as perspicuous as possible, in the following work, we set $\theta_0\equiv u$, and drop the asterisk on μ^* and assume $\mu\ll 1$. Although κ is essentially unrestricted, it is convenient, but not necessary, to write

$\kappa = \alpha\mu$ and formally equi-order κ and μ. We shall see that the results apply for arbitrary real $\alpha \neq 0$.

We first consider the eigenfunctions of the operator, $L^0 \equiv -\partial^2/\partial y^2 + \epsilon y^2$. It is easily seen that these are expressible in terms of the Weber–Hermite harmonic oscillator functions denoted by $\boldsymbol{\Psi}_p(y)$ with eigenvalues ϵ_p (linearly proportional to $2p+1$). We now expand the solution Z and the eigenvalue in powers of μ:

$$Z(y,u) = \sum_{\nu=0}^{\infty} \mu^\nu Z^{(\nu)}, \tag{32}$$

$$\Lambda = \sum_{\nu=0}^{\infty} \mu^\nu \Lambda^{(\nu)}. \tag{33}$$

The above expansions are substituted in (31), and the following equations are generated.

Zeroth order

$$L^0 Z^{(0)} = \Lambda^{(0)} Z^{(0)}. \tag{34}$$

We now solve this trivially, with the proviso that interest is focused on a specific eigenfunction. For definiteness, we label this by the suffix j. Thus $Z^{(0)} \equiv f^{(0)}(u)\boldsymbol{\Psi}_j(y)$ and $\Lambda^{(0)} = \epsilon_j$. The arbitrary function $f^{(0)}(u)$ is required to be periodic in u, but is undetermined at this order.

First order

$$L^0 Z^{(1)} - \Lambda^{(0)} Z^{(1)} = -\alpha \frac{df^{(0)}}{du} \boldsymbol{\Psi}_j(y) + \cos(y+u)\, f^{(0)} \boldsymbol{\Psi}_j(y) + \Lambda^{(1)} f^{(0)} \boldsymbol{\Psi}_j(y). \tag{35}$$

This equation determines, in principle, the unknown function $f^{(0)}$, the correction to the eigenvalue, $\Lambda^{(1)}$, and the first-order perturbation, $Z^{(1)}$. From the Fredholm alternative, the equation is soluble for $Z^{(1)}$ if and only if the right-hand side is orthogonal to $\boldsymbol{\Psi}_j(y)$. Multiplying by $\boldsymbol{\Psi}_j$ and integrating over all y, we obtain the ordinary differential equation

$$\alpha \frac{df^{(0)}}{du} = \Delta_j(u) f^{(0)} + \Lambda^{(1)} f^{(0)}, \tag{36}$$

where

$$\Delta_j(u) = \int_{-\infty}^{\infty} \cos(y+u)\, \boldsymbol{\Psi}_j^2(y)\, dy. \tag{37}$$

The function $\Delta_j(u)$ is obtainable explicitly using the generating function for the harmonic-oscillator functions, and is manifestly a periodic function of u such that $\int_{-\pi}^{\pi} \Delta_j(u)\, du = 0$. Note that this is a general fact, applicable to arbitrary periodic perturbations in the original eigenvalue problem. It is plain that the periodic solution of (36), with $f^0(0) = 1$, is

$$f^{(0)}(u) = \exp\left\{ \left[\Lambda^{(1)} u + \int_0^u \Delta_j(s)\, ds \right] \Big/ \alpha \right\}, \tag{38}$$

where

$$\Lambda^{(1)} = im\alpha, \tag{39}$$

$$m = 0, \pm 1, \pm 2, \ldots\ . \tag{40}$$

We see that the integers j and m are eigenlabels in this problem. We recall that ϵ already includes n, the 'toroidal' label. It is interesting to note that, at this order, the eigenvalue Λ becomes complex. This is due to the fact that the advection–diffusion equation is not self-adjoint. Note also that, at this order, the real part remains unaffected. Thus the sheared flow does not affect stability. However, it clearly makes the spectrum discrete, in contrast to the preceding problem, where the spectrum was shown to be continuous for arbitrarily small μ.

Having thus self-consistently determined $\Lambda^{(1)}$ and $f^{(0)}$, we turn to the calculation of $Z^{(1)}$. Observing that the right-hand side of (35) is now completely determined, and also is orthogonal to Ψ_j, we expand $Z^{(1)}$ in eigenfunctions orthogonal to Ψ_j:

$$Z^{(1)}(y,u) = \sum_{p\neq j} f_p^{(1)}(u)\Psi_p(y). \tag{41}$$

In this expansion, the index p runs over the complete orthonormal set Ψ_p with the exception of $p = j$. Substituting into the first-order equations and taking the projections with respect to Ψ_p, we get a system of first-order ordinary differential equations for the functions $f_p^{(1)}$ with simple 'particular integrals' that may be written as follows:

$$f_p^{(1)}(u) = \frac{V_{pj}(u) f^{(0)}(u)}{\epsilon_p - \Lambda^0}, \tag{42}$$

with

$$V_{pj}(u) = \int_{-\infty}^{\infty} \Psi_p(y)\cos(y+u)\,\Psi_j(y)\,dy. \tag{43}$$

The result demonstrates that $f_p^{(1)}$ (and hence $Z^{(1)}$) are bounded, periodic functions of u. The denominators cannot vanish, since $\Lambda^{(0)}$ can never equal any of the ϵ_ps. It is important to observe that this is only the 'particular integral' of (35). We must add the 'complementary function' to get the complete solution. The complementary function is of the form $f_j^{(1)}(u)\Psi_j(u)$, where the function $f_j^{(1)}$ is to be determined (at the next order).

We proceed to the second order to demonstrate that the expansion is really consistent as claimed.

Second order

$$L^0 Z^{(2)} - \Lambda^{(0)} Z^{(2)} = -\alpha\frac{dZ^{(1)}}{du} + \cos(y+u)\, Z^{(1)} + \Lambda^{(2)} Z^{(0)} + \Lambda^{(1)} Z^{(1)}. \tag{44}$$

We recall that neither the complementary function in $Z^{(1)}$ (i.e. $f_j^{(1)}$) nor the second-order correction to the eigenvalue, $\Lambda^{(2)}$, are yet determined. The Fredholm alternative requires the equation

$$\alpha\frac{df_j^{(1)}}{du} = \Delta_j(u) f_j^{(1)} + \Lambda^{(2)} f^{(0)} + \Lambda^{(1)} f_j^{(1)} + \sum_{p\neq j}\frac{V_{jp}V_{pj}}{\epsilon_p - \Lambda^{(0)}} f^{(0)}. \tag{45}$$

In deriving this relation, one must take account of the fact that the particular integral in $Z^{(1)}$ is orthogonal to Ψ_j, as well as the definitions of Δ_j and $V_{pj}(u)$. Evidently, the substitution $f_j^{(1)} \equiv f^{(0)}(u) C_j^{(1)}(u)$ reduces (45) to

$$\alpha\frac{dC_j^{(1)}}{du} = \Lambda^{(2)} + \sum_{p\neq j}\frac{V_{jp}(u)V_{pj}(u)}{\epsilon_p - \Lambda^{(0)}}. \tag{46}$$

The periodicity condition on $f_j^{(1)}$ requires the periodicity of $C_j^{(1)}$, which in turn implies that

$$\Lambda^{(2)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{p \neq j} \left[\frac{V_{jp}(u) V_{pj}(u)}{\Lambda^{(0)} - \epsilon_p} \right] du. \tag{47}$$

This condition determines the second-order correction to the eigenvalue. Moreover, the particular integral determines $f_j^{(1)}$ completely. The complementary function is denoted by $f_j^{(2)}$ and can only be found at the next order. If required, we may now construct the particular integral of (44) as before. It is obvious by now how the successive terms of the power series are generated. It seems highly likely that these series expansions actually converge – at least for sufficiently small μ.

It is evident that α need not be small, and can even be large (i.e. $\kappa = O(1)$). The matrix elements are all readily calculated, bounded periodic functions of u, and decay quite rapidly with respect to their suffixes. There are no resonant denominators, and the solution seems valid for all μ and nonzero κ. The real key to the success of the method is the fact that the operator L^0 has a discrete spectrum and the non-self-adjointness of the full problem avoids 'resonances' in the perturbation series. It is equally clear that the method applies virtually unchanged to *any* problem involving a one-dimensional operator L^0 generated by the ballooning transformation with a discrete spectrum in the extended angle variable (i.e. the Fourier conjugate of the 'twisted radial' variable $\xi - l$) and an arbitrary (not necessarily $\cos\theta$) periodic perturbation in the presence of a sheared advection. Let us recall that the 'true' eigenvalue of the equation is $\lambda = \Lambda - \frac{1}{4}\kappa^2$. The complete correction to second order must take this into account in estimating effects of advection. The eigenfunctions are different from those of the $\kappa = 0$ case, since the latter effectively have a $\delta(\theta_0 - \theta_0')$ variation in the θ_0 space and are associated with quasiperiodicity in the ξ direction. The introduction of sheared flow delocalizes the solution relative to $\theta_0 = u$, but destroys the quasiperiodicity with respect to the radial coordinate. The solutions are now radially localized. The eigenfunctions are not orthonormal, owing to the non-self-adjointness, but the Green function for the initial-value problem can be constructed using the solutions found by the perturbation expansion. A completeness relation using the solutions of the adjoint equation can also be derived.

The method suggested above enables the solution to be obtained as power series in μ. Zhang *et al.* (1994) used an apparently different method. This will now be explained with reference to the present problem. The key idea here is essentially that of the famous Born–Oppenheimer method in atomic physics (Davydov 1965), and parallels in some respects the original scheme due to Connor *et al.* (1978, 1979). Recall that in the Born–Oppenheimer treatment of a molecule, the ionic motions are 'frozen' at leading order in $(m_e/M_i)^{1/4}$, and one solves for the electronic eigenfunctions. The electronic levels serve as effective potentials in which the ions move. This solution gives the true eigenvalues. The method is really a systematic multiple-timescale technique, and uses the method of variation of constants systematically. To formally implement it, we regard μ (dropping the asterisk!) in (27) as not small but treat κ as small in some sense. We are therefore faced with solving

$$-\frac{\partial^2 Z}{\partial y^2} + \kappa \frac{\partial Z}{\partial u} + \epsilon y^2 Z - \mu \cos(y+u)\, Z = \Lambda Z. \tag{48}$$

We now choose the leading-order operator to be

$$L^*\left(y, \frac{\partial}{\partial y}, u\right) \equiv -\frac{\partial^2}{\partial y^2} + \epsilon y^2 - \mu \cos(y+u).$$

This too is self-adjoint and has a discrete spectrum of Weber–Mathieu eigenfunctions $\Psi_p^*(y,u)$ and eigenvalues $E_p^*(u)$, which are entire, periodic functions of u. We assume that these are chosen real and orthonormal. To proceed further, we formally expand the solution Z:

$$Z(y,u) = \sum_{p=0}^{\infty} f_p(u)\Psi_p^*(y,u). \tag{49}$$

Substitution gives the following infinite matrix differential system for the coefficients $f_p(u)$:

$$\kappa\frac{df_p}{du} = (\Lambda - E_p^*)f_p - \sum_q V_{pq}(u)f_q(u), \tag{50}$$

$$V_{pq} = \int_{-\infty}^{\infty} \Psi_p^*(y,u)\kappa\frac{\partial}{\partial u}\Psi_q^*(y,u)\,dy. \tag{51}$$

It is useful to note two simplifying relations. First, since the Ψ_p^* are real and orthonormal, it follows that $V_{pp} \equiv 0$. There is therefore no 'diagonal' term in the sum. Secondly, the periodic functions $E_p^*(u)$ may always be written in the form $E_p^*(u) = \bar{E}_p^* + \Delta_p^*(u)$, where $\bar{E}_p^*$ are the average values over a single period and $\Delta_p^*(u)$ are the residual parts, which are periodic with zero mean. Observe that, for $\mu \approx 1$, the functions Δ^* and V_{pq} vary on an $O(1)$ scale with respect to u. The f_ps can vary more rapidly with respect to u. In order to solve this system, we *formally* expand in powers of V. Thus we introduce a 'bookkeeping' parameter δ on the right and consider the system in the form

$$\kappa\frac{df_p}{du} = (\Lambda - \bar{E}_p^*)f_p - \Delta_p^*(u)f_p - \delta\sum_q V_{pq}(u)f_q(u). \tag{52}$$

At the end of the calculations, we set $\delta = 1$.

Choosing a specific mode j as before, the zeroth-order equations are as follows.

Zeroth order

$$\kappa\frac{df_p^{(0)}}{du} = (\Lambda^{(0)} - \bar{E}_p^*)f_p^{(0)} - \Delta_p^*(u)f_p^{(0)}. \tag{53}$$

Clearly, periodicity of the zeroth-order solutions requires us to set,

$$f_p^{(0)} = \delta_{p,j}\exp\left[imu - \frac{1}{\kappa}\int_0^u \Delta_p^*(s)\,ds\right], \qquad \Lambda_{j,m}^0 = \bar{E}_j^* + im\kappa,$$

where $m = 0, \pm 1, \ldots$ runs over all integers. Thus the modes are labelled by j and m. The necessity for setting $f_p^{(0)}$ to zero for $p \neq j$ is obvious, since these functions cannot be periodic in u once we choose $\Lambda_{j,m}^{(0)}$, unless $\bar{E}_p^*$ happens to be equal to $\bar{E}_j^*$ for some p. In this case of degeneracy, the eigenvalue has a multiplicity greater than unity. This can only happen if the operator L^* has a rather strong periodic part (i.e. μ large or ϵ small). Excluding this degeneracy, which requires a special treatment, we proceed to the next order (for given j and m).

First order

For $p = j$,

$$\kappa \frac{df_j^{(1)}}{du} = (\Lambda_{j,m}^{(0)} - \bar{E}_j^*) f_j^{(1)} - \Delta_j^*(u) f_j^{(1)} + \Lambda^{(1)} f_j^{(0)}. \tag{54}$$

For $p \neq j$,

$$\kappa \frac{df_p^{(1)}}{du} = (\Lambda_{j,m}^{(0)} - \bar{E}_p^*) f_p^{(1)} - \Delta_p^*(u) f_p^{(1)} - V_{pj}(u) f_j^{(0)}. \tag{55}$$

Without loss of generality we have $f_j^{(1)} \equiv 0$ and $\Lambda^{(1)} = 0$. Turning to the $p \neq j$ equations, we must find periodic solutions for this inhomogeneous set. Since we assume nondegeneracy, we immediately find that the 'complementary function' must be identically zero in order to satisfy periodicity. To find the particular integral, we write

$$f_p^{(1)} \equiv \exp\left[-\frac{1}{\kappa}\int_0^u \Delta_p^*(s)\, ds\right] C_p^{(1)}(u).$$

Substitution shows that $C_p^{(1)}(u)$ must satisfy (for the case $p \neq j$)

$$\kappa \frac{dC_p^{(1)}}{du} = (\Lambda_{jm}^{(0)} - \bar{E}_p^*) C_p^{(1)} - V_{pj}(u) f_j^{(0)}(u) \exp\left[\frac{1}{\kappa}\int_0^u \Delta_p^*(s)\, ds\right]. \tag{56}$$

This forced equation with constant coefficients has a periodic forcing function and obviously has a periodic particular integral, which can be obtained readily by Fourier expanding the forcing function and $C_p^{(1)}$. Moreover, the solution is bounded and periodic for all p considered, since there cannot be any 'resonant' denominators. Thus the solution has been completed to first order. We shall demonstrate that the procedure works at second and all subsequent orders.

Second order

For $p = j$,

$$\kappa \frac{df_j^{(2)}}{du} = (\Lambda_{j,m}^{(0)} - \bar{E}_j^*) f_j^{(2)} - \Delta_j^*(u) f_j^{(2)} + \Lambda^{(2)} f_j^{(0)} - \sum_p V_{jp} f_p^{(1)}. \tag{57}$$

For $p \neq j$,

$$\kappa \frac{df_p^{(2)}}{du} = (\Lambda_{j,m}^{(0)} - \bar{E}_p^*) f_p^{(2)} - \Delta_p^*(u) f_p^{(2)} - \sum_q V_{pq}(u) f_q^{(0)}. \tag{58}$$

The first equation is an inhomogeneous one with a free parameter $\Lambda^{(2)}$ in it. This must be chosen so that the particular integral (the complementary function is identically zero without loss of generality) is periodic in u. Making the familiar transformation $f_j^{(2)} = f_j^{(0)} C_j^{(2)}$ and requiring periodicity of $C_j^{(2)}$, we get the equation

$$\Lambda^{(2)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_p V_{jp}(u) \frac{f_p^{(1)}(u)}{f_j^{(0)}(u)} du, \tag{59}$$

where the integrand is a periodic function determined by functions calculated previously. This condition 'suppresses the secular term' in the equation and enables the periodic function $f_j^{(2)}$ to be calculated (using Fourier series). Nor is there any difficulty in solving the $p \neq j$ equations. The process can obviously be carried out to any desired order. Note that we again obtain a discrete spectrum, the eigenvalues being related to the 'averaged eigenvalues' (i.e. $\bar{E}_p^*$) of the unperturbed problem. The solutions are well-defined, ordinary functions in physical space.

The expansion is clearly not a power series in either μ or κ, and it is not clear under what conditions the formal expansion converges. It is more likely to be asymptotic. Unlike the previous Rayleigh–Schrödinger expansion, the dependence of the solutions and the eigenvalues on μ and κ is not through a simple power series. The results are likely to be valid even when μ is not small. The success of both expansion methods is crucially dependent upon the spectrum of L^0 or L^* being discrete. If this is not so, the eigenfunctions $\boldsymbol{\Psi}$ will not be square-integrable over k and there is effectively a non-integrable singularity in $\phi_{nl}(x)$. Furthermore, it is questionable if the shifted sum converges to an ordinary function of x and θ in the physical domain, since the terms of the series do not go to zero.

The case when κ is purely imaginary is of interest. This happens when we consider (48) to be a Schrödinger-like equation. When we come to consider the solutions using the Born–Oppenheimer technique sketched above, everything goes through as before, except that we have, instead of (56), the $C_p^{(1)}(u)$ satisfying (we set $i\kappa$ for κ and consider the $p \neq j$ case)

$$i\kappa \frac{dC_p^{(1)}}{du} = (\Lambda_{jm}^{(0)} - \bar{E}_p^*)C_p^{(1)} - V_{pj}(u) f_j^{(0)}(u) \exp\left[\frac{1}{\kappa}\int_0^u \Delta_p^*(s)\, ds\right]. \tag{60}$$

Since $\Lambda_{j,m}^0 = \bar{E}_j^* + m\kappa$ in this case, we shall encounter resonant denominators unless the averaged eigenvalues $\bar{E}_p^*$ satisfy a 'nondegeneracy' condition. Thus, for the above equation to have a periodic solution, it is necessary and sufficient that no two of these averaged eigenvalues differ by an integer multiple of κ. Whenever the nondegeneracy condition is satisfied, the solution can be carried to all orders. The eigenvalues in this model are always real, since we are solving a Schrödinger equation with a self-adjoint spatial operator. When only a finite number of eigenvalues are degenerate in the above sense, it is possible to formulate a problem involving a finite Floquet system and carry out the perturbation expansion. This is analogous to degenerate-state perturbation theory in quantum mechanics. If, however, an infinite number of the unperturbed mean eigenvalues differ by integer multiples of κ, the perturbed system has a continuous spectrum. The entire perturbation scheme breaks down if the unperturbed system has a continuous spectrum (this does not obviously occur in the present examples).

As we shall show in Part 2, this is precisely the situation that occurs in applications. The treatment of continuous spectra in ballooning problems will be taken up there.

5. Conclusions

In this expository paper, the principal ideas of ballooning theory have been discussed making use of a model derived from the anisotropic diffusion equation in a slab model of toroidal geometry with a minimum of technical complications that tend to detract from the main lines of the argument. Two complementary approaches to ballooning exist. These are shown to be mathematically equivalent using the Poisson transformation formula. The model problems chosen do not explicitly require large n, but do imply rather simple equilibrium structure in the radial variable. Many of the special properties of the ballooning representation are most readily understood in this context in which WKB ideas play no fundamental role. The ability of the method to provide useful basis functions to expand the complete solution is thereby emphasized. The radial localization effects of sheared flow are simply illustrated in the model, showing how it is an instance of 'broken ballooning symmetry'. The model captures the essence of more realistic problems where shear flow plays a crucial role in stabilizing certain instabilities (Taylor and Wilson 1996).

Two different perturbation techniques, each with its own strengths and limitations, have been explained in full, illustrating the singular nature of toroidal perturbations when the unperturbed system has infinite degeneracy. The fact that an arbitrarily 'mild' perturbation of a system with infinite degeneracy leads to a continuous spectrum is explicitly illustrated by the example given. The phenomenon is discussed in the literature (Riesz and Nagy 1971) and is well known in quantum mechanics: thus an infinite periodic array of atoms tend to form the Bloch spectrum from the original degenerate atomic levels when they are brought closer together, starting from a very large lattice distance. Perturbation methods of ballooning theory require care when the relevant operators lead to extended eigenfunctions. The treatment of such problems will be presented in Part 2.

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