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Citation: Physics of Plasmas **13**, 072108 (2006); doi: 10.1063/1.2219428
View online: https://doi.org/10.1063/1.2219428
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Relativistic electron distribution function of a plasma in a near-critical electric field

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(Received 19 May 2006; accepted 12 June 2006; published online 28 July 2006)

A corrected relativistic collision operator is used to derive a Fokker-Planck equation for the distribution function of relativistic suprathermal electrons in a weakly relativistic plasma, which is then solved by a procedure similar to that employed in Connor and Hastie [Nucl. Fusion **15**, 415 (1975)]. Analytical expressions are derived for the electron distribution function in plasmas with the electric field close to critical, which is typical of plasmas with grassy sawteeth on the Joint European Torus. A numerical solution is used for determining the normalization constant, which matches the relativistic region onto the weakly relativistic region. It is found that the scaling of the runaway rate with the electric field obtained by Connor and Hastie is a good approximation in spite of their use of an incomplete form of the collision operator not conserving number of particles. The present analysis determines the proportionality constant and introduces corrections to the earlier scaling of the runaway rate with respect to the electric field. The results obtained for the electron distribution function constitute a basis for studies of experimentally observed phenomena in near-threshold electric field plasmas with a significant suprathermal electron population. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2219428]

I. INTRODUCTION

It has long been recognized that electric fields generated during reconnection events in magnetized plasmas may accelerate electrons and ions of the plasma. In tokamaks, the disruptive instability[1] that sometimes terminates the discharge often induces electric fields E well above the critical value E_c corresponding to the minimum of the friction force on a relativistic electron. Thus, the critical electric field parameter α, defined by[2]

$$\alpha = \frac{E}{E_c} = \frac{4\pi\epsilon_0^2 m_e c^2}{n_e e^3 \ln\Lambda} E, \tag{1}$$

satisfies $\alpha \gg 1$, where n_e is the thermal electron density, m_e the electron rest mass, e the absolute value of the electron charge, $\ln\Lambda$ the Coulomb logarithm, ϵ_0 the dielectric constant, and c the speed of light. In Joint European Torus (JET) plasmas, such fields accelerate electrons up to relativistic energies generating a runaway electron current exceeding 1 MA.[3] These postdisruption relativistic electrons have a number of remarkable properties among which are runaway avalanching[4] and the production of positrons.[5]

Another type of reconnection event, the sawtooth instability,[1] does not terminate the plasma, but affects significantly the central part of the plasma by redistributing particle and energy densities and relaxing the central electron temperature. Although the electric fields induced during sawteeth, E_{saw}, may not be as high as those caused by disruptions, they can nevertheless accelerate electrons repeatedly at each sawtooth crash throughout the discharge.[6] Recent observations of suprathermal electrons during magnetic reconnection at sawtooth crashes in the T-10 tokamak showed the appearance of beams of suprathermal electrons with energies up to 20–100 keV localized around the X point of the $m=1$ and $m=2$ magnetic islands accompanying the sawteeth.[7] Fast electron bremsstrahlung was also found to be significantly enhanced[8,9] in low-density JET plasmas with short-period, chaotic sawteeth, so-called grassy sawteeth.[10] Estimates show that, throughout these discharges, electric fields induced by each sawtooth crash satisfy $\alpha \gg 1$, while the estimated on-axis inductive electric field was close to the critical field, $\alpha \sim 1$,[8,9] at all times between sawtooth crashes. Under such conditions on JET, a reconnection electric field generates a suprathermal electron population repeatedly during each sawtooth crash, while the presence of a steady-state, near-threshold inductive electric field $E \sim E_c$ prevents a rapid deceleration of the suprathermal electrons due to Coulomb collisions between the sawtooth crashes.

To accurately describe suprathermal electrons in such plasmas, one would need to derive and solve the time-dependent Fokker-Planck equation for the electron distribution function at $\alpha \sim 1$, with $\alpha \gg 1$ at the times of the sawtooth crashes. As a step in this direction the present paper focuses on a local analysis of the steady-state electron veloc-

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ity distribution function at $\alpha \sim 1$. A theory of the suprathermal electron velocity distribution function in a weakly relativistic plasma was developed by Connor and Hastie in Ref. 2. However, it was pointed out by Karney and Fisch[11] that the collision operator employed in Ref. 2 does not conserve the number of particles. Since the number of fast electrons is exponentially sensitive to the electric field and to plasma parameters, we have employed the correct relativistic collision operator in this problem and reassessed the theory presented in Ref. 2. In the current paper, an approach similar to that used by Connor and Hastie is employed for describing a weakly relativistic plasma with a small, dimensionless expansion parameter quantifying the ratio between critical and Dreicer electric fields, E_D, according to

$$\epsilon = \frac{E_c}{E_D} = \frac{T_e}{m_e c^2} \ll 1. \tag{2}$$

Special attention is paid to the case of the grassy sawtooth discharges on JET where the inductive electric field of the tokamak is close to the critical electric field, $\alpha \sim 1$, and the weakly relativistic parameter is $\epsilon \gtrsim 10^{-2}$. An exact evaluation of the deceleration/acceleration of suprathermal electrons generated by sawtooth crashes crucially depends on an accurate description of the cases $\alpha > 1$, $\alpha < 1$, and $\alpha - 1 \ll 1$, which is the focus of the present paper.

The paper is organized as follows. In Sec. II, a relativistic collision operator, derived in Appendix A, is used to obtain the governing Fokker-Planck equation for relativistic electrons in a weakly relativistic plasma. The solution to this equation is found analytically, by asymptotic techniques applied to five separate regions in momentum space, in Sec. III and in Appendix B. In Sec. IV, the results of the runaway region are used to calculate the runaway rate, i.e. the number of runaway electrons generated per unit volume and unit time. Finally, our conclusions are presented in Sec. V.

II. FOKKER-PLANCK EQUATION FOR RELATIVISTIC ELECTRONS IN A WEAKLY RELATIVISTIC PLASMA

In the presence of an external electric field $\mathbf{E}$, the steady-state electron velocity distribution function f satisfies the equation

$$-e\mathbf{E}\cdot\nabla_p f = C(f), \tag{3}$$

where p is the momentum related to the velocity v through $v = pc/\sqrt{p^2 + m_e^2 c^2}$ and $C(f)$ is the collision operator for a weakly relativistic plasma derived in Appendix A.

By introducing spherical coordinates (p, θ, φ) in velocity space with momentum $\mathbf{p}$ directed along $-\mathbf{E}$, the direction of electron acceleration, Eq. (3) can be gyro-averaged to yield

$$eE\left(\mu\frac{\partial f}{\partial p} + \frac{1-\mu^2}{p}\frac{\partial f}{\partial \mu}\right) = C(f), \tag{4}$$

where $\mu = \cos\theta = p_\parallel/p$ and $p_\parallel = -\mathbf{p}\cdot\mathbf{E}/|\mathbf{E}|$. Upon using the collision operator $C(f)$ from Appendix A, Eq. (A24), and a critical electric field parameter α defined by Eq. (1), Eq. (4) becomes

$$\begin{aligned}\alpha&\left(\mu\frac{\partial f}{\partial q} + \frac{1-\mu^2}{q}\frac{\partial f}{\partial \mu}\right)\\ &= \frac{\sqrt{q^2+1}}{2q^3}\left(1 + Z - \epsilon\frac{2q^2+1}{q^2(q^2+1)}\right)\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}\\ &\quad + \frac{1}{q^2}\left(q^2 + 1 - \epsilon\frac{(1-2q^2)\sqrt{q^2+1}}{q^2}\right)\frac{\partial f}{\partial q}\\ &\quad + \epsilon\frac{(q^2+1)^{3/2}}{q^3}\frac{\partial^2 f}{\partial q^2} + \frac{2f}{q}.\end{aligned} \tag{5}$$

Here, ϵ is the small expansion parameter defined by Eq. (2), $q = p/m_e c$ is the normalized momentum, Z is the effective charge number of the plasma ions, and terms of order $\mathcal{O}(\epsilon^2)$ have been neglected. Here, the total particle density has been normalized to unity in order to be consistent with Ref. 2. This Fokker-Planck equation is the governing equation for the electron distribution function, from which its dependence on momentum and parallel electric field is obtained. The analysis below will show how a suprathermal electron tail, peaked around $\mu = 1$, i.e., parallel to the direction of electron acceleration, is formed.

III. SOLUTION OF THE FOKKER-PLANCK EQUATION

In a weakly relativistic plasma with $\epsilon \ll 1$ Eq. (5) can be solved using asymptotic techniques devised by Kruskal and Bernstein, who first solved the corresponding nonrelativistic problem.[12] As in their analysis as well as in Ref. 2, different asymptotic expansions must be used in five different regions of velocity space.

A. Region I: Nonrelativistic region

In region I, we consider two small quantities, ϵ and q, satisfying the ordering $v = q/\sqrt{\epsilon} \sim \mathcal{O}(1)$. Since $q^2 \ll 1$ the nonrelativistic limit of the collision operator can be used. In this limit the Fokker-Planck Eq. (5) takes the form

$$\begin{aligned}\frac{\alpha}{\sqrt{\epsilon}}&\left(\mu\frac{\partial f}{\partial v} + \frac{1-\mu^2}{v}\frac{\partial f}{\partial \mu}\right)\\ &= \frac{1}{2\epsilon^{3/2}v^3}\left(1 + Z - \frac{1}{v^2}\right)\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}\\ &\quad + \frac{1}{\epsilon^{3/2}}\frac{1}{v^2}\frac{\partial f}{\partial v} - \frac{1}{\epsilon^{3/2}}\frac{1}{v^4}\frac{\partial f}{\partial v} + \frac{1}{\epsilon^{3/2}}\frac{1}{v^3}\frac{\partial^2 f}{\partial v^2}.\end{aligned} \tag{6}$$

Equation (6) is solved by expanding the solution f_I in the small parameter,

$$f_I = f_0 + \epsilon f_1 + \cdots. \tag{7}$$

Using this expansion in Eq. (6), and requiring $f_0 = f_0(v)$ to be isotropic, gives to order $\mathcal{O}(\epsilon^{-3/2})$ the equation

$$\frac{\partial}{\partial v}\left(f_0 + \frac{1}{v}\frac{\partial f_0}{\partial v}\right) = 0, \tag{8}$$

and the solution $f_0(q) = \exp(C_I - q^2/2\epsilon)$ is Maxwellian, where C_I is a normalization constant determined by

$$n_e = \int_0^\infty f_I \, dq = \int_0^\infty \exp(C_I - q^2/2\epsilon) dq + \mathcal{O}(\epsilon). \tag{9}$$

An equation for f_1, obtained from the $\mathcal{O}(\epsilon^{-1/2})$ terms of Eq. (6), constitutes the Spitzer problem[13] and is given by

$$\alpha\mu \frac{\partial f_0}{\partial v} = \frac{1}{2v^3}\left(1 + Z - \frac{1}{v^2}\right)\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f_1}{\partial \mu} + \frac{1}{v^2}\frac{\partial f_1}{\partial v} - \frac{1}{v^4}\frac{\partial f_1}{\partial v} + \frac{1}{v^3}\frac{\partial^2 f_1}{\partial v^2}, \tag{10}$$

and the solution to this equation was solved for large momenta by Cohen.[14] However, for the present analysis, which is taken to order $\mathcal{O}(1)$, the first-order term f_1 is not needed.

The expansion (7) in region I breaks down when $\epsilon f_1 \sim f_0$, i.e., when $q^2 \sim \epsilon^{1/2}$, and a new region has to be considered.

B. Region II: Weakly relativistic region

Since the ordering used in region I breaks down at $q^2 \sim \epsilon^{1/2}$, it is convenient to introduce a new variable $u = \epsilon^{-1/4} q \sim \mathcal{O}(1)$ for analyzing this higher momentum region. In addition to this new variable, one should also use a relativistic expression for the collision operator. Since $f_0(u) \propto \exp(-u^2/2\sqrt{\epsilon})$ cannot be expanded as a power series in $\sqrt{\epsilon}$, the perturbation expansion of Eq. (7) cannot be employed. Instead, an expansion in $\sqrt{\epsilon}$ is applied to the exponent $F = \ln f_{II}$ according to

$$F = \epsilon^{-1/2} F^{(0)} + F^{(1)} + \epsilon^{1/2} F^{(2)} + \cdots + C_{II}. \tag{11}$$

Substitution of F and u into the Fokker-Planck Eq. (5) gives, to leading order, $\mathcal{O}(\epsilon^{-1})$,

$$\frac{1}{2u^3}(1+Z)\left(\frac{\partial F^{(0)}}{\partial \mu}\right)^2 = 0 \Rightarrow \frac{\partial F^{(0)}}{\partial \mu} = 0 \tag{12}$$

and $F^{(0)} = F^{(0)}(u)$. To next order, $\mathcal{O}(\epsilon^{-1/2})$, one finds

$$\frac{\partial F^{(0)}}{\partial u} = -u \Rightarrow F^{(0)} = -u^2/2. \tag{13}$$

The equation of the $\mathcal{O}(\epsilon^0)$ order can be reduced by the substitution $F^{(1)} = u^4/8 + F'$ to give

$$\frac{1+Z}{2}\left\{(1-\mu^2)\left[\frac{\partial^2 F'}{\partial \mu^2} + \left(\frac{\partial F'}{\partial \mu}\right)^2\right] - 2\mu\frac{\partial F'}{\partial \mu}\right\} - u\frac{\partial F'}{\partial u} + \alpha\mu u^4 = 0, \tag{14}$$

where a misprint of the first derivative in the μ term in Ref. 2 has been corrected. Assuming a series expansion for F', the following solution, valid for all μ except in a boundary layer near $\mu = -1$, is obtained to order $\mathcal{O}(u^{-2})$ for large u:

$$\begin{aligned} F'(u,\mu) &= a_4 u^4 + a_2(\mu)u^2 + \tilde{a}\ln u + a_0(\mu) + \mathcal{O}(u^{-2}) \\ &= \frac{\alpha}{4}u^4 + \sqrt{\frac{2\alpha}{1+Z}}\left(2\sqrt{1+\mu} - \frac{9+Z}{2\sqrt{2}}\right)u^2 \\ &\quad + \frac{5+Z}{4}\ln u + \frac{1}{4}\ln(1+\mu) \\ &\quad - \frac{1}{1+Z}\int_\mu^1 \frac{d\mu}{1-\mu^2}\left\{4 + \mu(5+Z) - (9+Z)\sqrt{\frac{1+\mu}{2}}\right\} + b(\alpha,Z) + \mathcal{O}(u^{-2}). \end{aligned} \tag{15}$$

The logarithmic term has been incorporated in order to have consistency between the $\mathcal{O}(u^2)$ and $\mathcal{O}(u^0)$ equations at $\mu = 1$. The minus sign in front of the integral corrects misprints in Refs. 12 and 2.

In summary, the large u solution in region II (neglecting terms of order $\mathcal{O}(u^{-2})$) reads

$$\begin{aligned} \ln f_{II} = F &= -u^2/2\sqrt{\epsilon} + u^4/8 + \frac{\alpha u^4}{4} + \sqrt{\frac{2\alpha}{1+Z}}u^2 \\ &\quad \times\left(2\sqrt{1+\mu} - \frac{9+Z}{2\sqrt{2}}\right) + \frac{5+Z}{4}\ln u + \frac{1}{4}\ln(1+\mu) \\ &\quad - \frac{1}{1+Z}\int_\mu^1 \frac{d\mu}{1-\mu^2}\left\{4 + \mu(5+Z) - (9+Z)\sqrt{\frac{1+\mu}{2}}\right\} + b(\alpha,Z) + C_I, \end{aligned} \tag{16}$$

where $b(\alpha,Z)$ is a constant of order $\mathcal{O}(1)$. Here, we have chosen $C_{II} = C_I$ (see also Appendix B). The constant $b(\alpha,Z)$ is obtained by matching the general solution in region II (i.e., for $u \sim \mathcal{O}(1)$) onto the solution in region I as follows. We introduce $h = \exp(F')$ so that Eq. (14) can be rewritten as a linear equation,

$$\frac{1+Z}{2}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial h}{\partial \mu} - u\frac{\partial h}{\partial u} + \alpha\mu u^4 h = 0. \tag{17}$$

This parabolic, diffusion-type equation, with an anisotropic term proportional to μu^4, can be solved numerically by the decomposition

$$h(u,\mu) = \sum_n g_n(u) P_n(\mu), \tag{18}$$

where $P_n(\mu)$ are the Legendre polynomials. A numerical solution, presented in Appendix B, shows that the asymptotic analytical solution, Eq. (15), does not deviate from the numerical solution by more than ~20% in the relevant parameter range $\alpha = 0.8-1.2$, $Z = 1-3$ in the most important region at $\mu = 1$ for a matching constant b equal to 0.69. The matching of the high-momentum asymptote in Eq. (15) to the numerical solution at $\mu = 1$ for $u = 7$ is illustrated in Fig. 1.

Equation (16) shows that the region II ordering breaks down at $u^2 \sim \epsilon^{-1/2}$ and a new region must be introduced for normalized momenta $q^2 \sim \mathcal{O}(1)$.

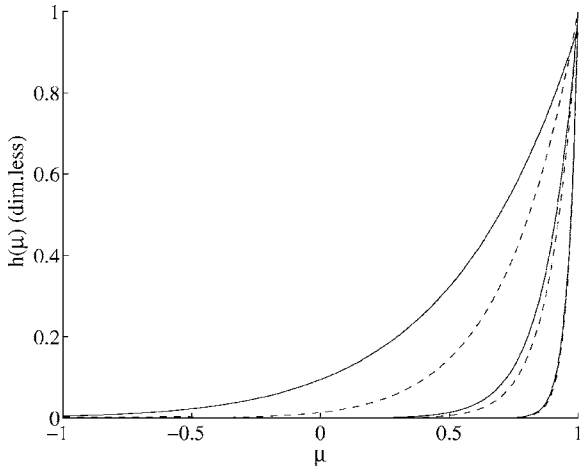

FIG. 1. Numerical solutions of Eq. (17) (solid lines) and high momenta analytic asymptotes in Eq. (15) (broken lines) as functions of pitch angle μ for normalized momenta $u=[2.25,4,7]$. The plasma parameters are $\alpha=0.9$, $Z=2.1$. The numerical solution agrees well with the high momentum asymptote for $u=7$, and over a wide range of typical plasma parameters, the match at $\mu=1$ for $u=7$ does not deviate more than ~20% for a matching constant $b=0.69$.

C. Region III: Relativistic region

In region III, $q^2\sim\mathcal{O}(1)$, the electrons are fully relativistic. It can be seen from comparing the first and third terms of Eq. (16) that one needs $q^2\sim\alpha^{-1}$ for the critical electric field to cause a strong deviation of the electron distribution function from the Maxwellian. This is possible in region III, which manifests itself as the most sensitive region where a near-critical electric field plays a crucial role. Region III either extends to infinity in q space ($\alpha\leq 1$) in which case only exponentially few energetic electrons are generated, or its solution breaks down for $\alpha>1$, and a region of runaway acceleration appears for $q>1/\sqrt{\alpha-1}$.

The appropriate expansion for $F=\ln f_{\mathrm{III}}$ in region III has the form

$$F=\epsilon^{-1}F^{(0)}+\epsilon^{-1/2}F^{(1)}+F^{(2)}+\cdots+\mathrm{C}_{\mathrm{III}}, \tag{19}$$

and substitution into the Fokker-Planck Eq. (5) yields, to leading order, $\mathcal{O}(\epsilon^{-2})$,

$$\frac{\partial F^{(0)}}{\partial\mu}=0. \tag{20}$$

The equation of order $\mathcal{O}(\epsilon^{-3/2})$,

$$\frac{\partial F^{(0)}}{\partial\mu}\frac{\partial F^{(1)}}{\partial\mu}=0, \tag{21}$$

is trivially satisfied. Evaluating the next-order equation, $\mathcal{O}(\epsilon^{-1})$,

$$\alpha\mu\frac{\partial F^{(0)}}{\partial q}=\frac{\sqrt{1+q^2}}{q^3}\left(\frac{1+Z}{2}\right)(1-\mu^2)\left(\frac{\partial F^{(1)}}{\partial\mu}\right)^2+\frac{(q^2+1)}{q^2}\frac{\partial F^{(0)}}{\partial q}+\frac{(q^2+1)^{3/2}}{q^3}\left(\frac{\partial F^{(0)}}{\partial q}\right)^2 \tag{22}$$

at $\mu=1$ gives

$$F^{(0)}=\alpha\left(\sqrt{q^2+1}+\frac{1}{\sqrt{q^2+1}}\right)-\sqrt{q^2+1}, \tag{23}$$

in agreement with Ref. 2. The second term on the right-hand side of Eq. (23) represents the Maxwellian distribution function $F_{\mathrm{M}}^{(0)}=-\sqrt{q^2+1}$, while the first term gives the deviation caused by the electric field.

A suitable equation for $\partial F^{(1)}/\partial\mu$ is obtained by subtracting Eq. (22) evaluated at $\mu=1$ from itself, leading to

$$F^{(1)}(q,\mu)=\tilde{F}^{(1)}(q,\mu)+\bar{F}^{(1)}(q), \tag{24}$$

where $\tilde{F}^{(1)}(q,\mu)$ is given by

$$\tilde{F}^{(1)}(q,\mu)=2\sqrt{\frac{2\alpha}{1+Z}}\frac{q^2}{q^2+1}\sqrt{1-(\alpha-1)q^2}\sqrt{1+\mu} \tag{25}$$

and $\bar{F}^{(1)}(q)$ is determined by the next-order equation, $\mathcal{O}(\epsilon^{-1/2})$,

$$\begin{aligned}\alpha\left(\mu\frac{\partial F^{(1)}}{\partial q}+\frac{(1-\mu^2)}{q}\frac{\partial F^{(1)}}{\partial\mu}\right)&=\frac{\sqrt{q^2+1}}{q^3}\left(\frac{1+Z}{2}\right)\times\left[-2\mu\frac{\partial F^{(1)}}{\partial\mu}+(1-\mu^2)\frac{\partial^2F^{(1)}}{\partial\mu^2}\right]\\&+\frac{\sqrt{q^2+1}}{q^3}\left(\frac{1+Z}{2}\right)(1-\mu^2)2\frac{\partial F^{(1)}}{\partial\mu}\frac{\partial F^{(2)}}{\partial\mu}\\&+\frac{q^2+1}{q^2}\frac{\partial F^{(1)}}{\partial q}+\frac{(q^2+1)^{3/2}}{q^3}2\frac{\partial F^{(0)}}{\partial q}\frac{\partial F^{(1)}}{\partial q}.\end{aligned} \tag{26}$$

At $\mu=1$, this equation relates $\tilde{F}^{(1)}(q,\mu)$ and $\bar{F}^{(1)}(q)$ as follows:

$$\left(\frac{\partial\bar{F}^{(1)}}{\partial q}\right)_{\mu=1}=-\left(\frac{\partial\tilde{F}^{(1)}}{\partial q}\right)_{\mu=1}-\sqrt{\alpha(1+Z)}\times\frac{q}{\sqrt{1+q^2}\sqrt{1-(\alpha-1)q^2}}. \tag{27}$$

The general solution of $\bar{F}(q)^{(1)}$, for arbitrary values of α, is obtained by integrating Eq. (27). We now consider explicit integrations for the three different cases corresponding to supercritical, subcritical, and critical electric fields. We start with the supercritical case $\alpha>1$.

1. Case $\alpha>1$

For this case, the electric field is greater than the critical electric field for runaway electron production. From Eq. (24) we obtain

$$\left(\frac{\partial F^{(1)}}{\partial q}\right)_{\mu=1}=\left(\frac{\partial\tilde{F}^{(1)}}{\partial q}\right)_{\mu=1}+\left(\frac{\partial\bar{F}^{(1)}}{\partial q}\right)_{\mu=1}, \tag{28}$$

and Eq. (25) at $\mu=1$ gives

$$\left(\frac{\partial \tilde{F}^{(1)}}{\partial \mu}\right)_{\mu=1} = \sqrt{\frac{\alpha}{1+Z}}\frac{q^2}{1+q^2}\sqrt{1-(\alpha-1)q^2}. \tag{29}$$

Substituting Eqs. (28) and (29) into Eq. (27) gives

$$\bar{F}^{(1)}(q) = -\tilde{F}^{(1)}(q,\mu=1) - \frac{1}{2}\sqrt{\frac{\alpha(1+Z)}{\alpha-1}} \times \sin^{-1}\left[1-\frac{2}{\alpha}+2q^2\left(1-\frac{1}{\alpha}\right)\right]. \tag{30}$$

We finally obtain $F^{(1)}(q,\mu)$ by inserting Eq. (30) into Eq. (24), and, using Eq. (25), we find

$$F^{(1)}(q,\mu) = 2(\sqrt{1+\mu}-\sqrt{2})\sqrt{\frac{2\alpha}{1+Z}}\frac{q^2}{q^2+1}\sqrt{1-(\alpha-1)q^2} - \frac{1}{2}\sqrt{\frac{\alpha(1+Z)}{\alpha-1}}\sin^{-1}\left[1-\frac{2}{\alpha}+2q^2\left(1-\frac{1}{\alpha}\right)\right], \tag{31}$$

which is in agreement with the result obtained in Ref. 2.

We now use the $\mathcal{O}(\epsilon^{-1/2})$ equation, Eq. (26), to obtain $\partial F^{(2)}/\partial\mu$. Taking the derivatives of $F^{(0)}$ and $F^{(1)}$, respectively, with respect to μ and q, and performing a great deal of algebra, we find

$$F^{(2)}(q,\mu) = \frac{\alpha q^2}{1+Z}\frac{(\mu-1)}{\sqrt{1+q^2}} + \frac{1}{4}\ln(1+\mu) - \int_\mu^1 \frac{d\mu}{1-\mu^2}\left\{\mu + \frac{2}{1+Z}\cdot\frac{[\alpha(\mu-2)q^2+1+q^2]}{[1-(\alpha-1)q^2]} \times\left[[(1+\mu)-\sqrt{2(1+\mu)}]\frac{[(q^2+1)(q^2+2)-\alpha q^2(q^2+3)]}{(1+q^2)^{3/2}} - \frac{(1+Z)}{2}\frac{\sqrt{1+\mu}}{\sqrt{2}}\right]\right\} + \bar{F}^{(2)}(q). \tag{32}$$

It is easy to check, using l'Hospital's rule, that no singularity exists at $\mu=1$. Moreover, note that the integration constant has been chosen such that $F^{(2)}(q,\mu=1)=\frac{1}{4}\ln 2+\bar{F}^{(2)}(q)$. In order to find $\bar{F}^{(2)}(q)$, we turn to the $\mathcal{O}(\epsilon^0)$ equation of the Fokker-Planck equation in region III, evaluated at $\mu=1$. After an even greater amount of algebra, one obtains

$$\bar{F}^{(2)}(q) = \frac{1}{4}\left(\frac{1+Z}{2}\right)\left\{\ln\left(\frac{\sqrt{1+q^2}-1}{\sqrt{1+q^2}+1}\right) + \frac{(\alpha-2)\sqrt{\alpha}}{(\alpha-1)^{3/2}}\ln\left(\frac{\sqrt{\alpha}+\sqrt{\alpha-1}\sqrt{1+q^2}}{\sqrt{\alpha}-\sqrt{\alpha-1}\sqrt{1+q^2}}\right) + \frac{2\alpha\sqrt{1+q^2}}{(\alpha-1)[1-(\alpha-1)q^2]}\right\} + \ln\frac{q}{\sqrt{1+q^2}} - \frac{1}{4}\frac{\alpha+1}{\alpha-1}\ln[1-(\alpha-1)q^2]. \tag{33}$$

Substituting Eq. (33) into Eq. (32) finally gives

$$F^{(2)}(q,\mu) = \frac{1}{4}\left(\frac{1+Z}{2}\right)\left\{\ln\left(\frac{\sqrt{1+q^2}-1}{\sqrt{1+q^2}+1}\right) + \frac{(\alpha-2)\sqrt{\alpha}}{(\alpha-1)^{3/2}}\ln\left(\frac{\sqrt{\alpha}+\sqrt{\alpha-1}\sqrt{1+q^2}}{\sqrt{\alpha}-\sqrt{\alpha-1}\sqrt{1+q^2}}\right) + \frac{2\alpha\sqrt{1+q^2}}{(\alpha-1)[1-(\alpha-1)q^2]}\right\} + \ln\frac{q}{\sqrt{1+q^2}} - \frac{1}{4}\frac{\alpha+1}{\alpha-1}\ln[1-(\alpha-1)q^2] + \frac{\alpha q^2}{1+Z}\frac{(\mu-1)}{\sqrt{1+q^2}} + \frac{1}{4}\ln(1+\mu) - \int_\mu^1 \frac{d\mu}{1-\mu^2}\left\{\mu + \frac{2}{1+Z}\frac{[\alpha(\mu-2)q^2+1+q^2]}{[1-(\alpha-1)q^2]} \times\left[[(1+\mu)-\sqrt{2(1+\mu)}]\frac{[(q^2+1)(q^2+2)-\alpha q^2(q^2+3)]}{(1+q^2)^{3/2}} - \frac{(1+Z)}{2}\frac{\sqrt{1+\mu}}{\sqrt{2}}\right]\right\}. \tag{34}$$

This expression differs from the corresponding Eq. 39 in Ref. 2 by a factor $(1+Z)$ in the last term and by the fifth term in Ref. 2 which resulted from the use of an incorrect collision operator and is consequently not present here.

The C_{III} constant in Eq. (19) is now obtained by asymptotic matching of the solution in region III, Eqs. (23), (31), and (34), for $q^2 \ll 1$, to the large u solution in region II, Eq. (16), for $u \to \infty$. The three different contributions from $\mathcal{O}(\epsilon^{-1})$, $\mathcal{O}(\epsilon^{-1/2})$, and $\mathcal{O}(\epsilon^0)$, respectively, result in a matching constant

$$C_{III} = C_I + b(\alpha,Z) - \frac{(2\alpha-1)}{\epsilon} + \frac{1}{2}\sqrt{\frac{(1+Z)\alpha}{\epsilon(\alpha-1)}} \times\sin^{-1}\left[1-\frac{2}{\alpha}\right] - \frac{(5+Z)}{16}\ln\epsilon - \frac{(1+Z)}{4}\frac{\alpha}{\alpha-1} + \frac{(1+Z)}{4}\ln 2 - \frac{(1+Z)}{4}\frac{(\alpha-2)\sqrt{\alpha}}{(\alpha-1)^{3/2}}\ln(\sqrt{\alpha}+\sqrt{\alpha-1}). \tag{35}$$

From Eq. (34), it is clear that a singularity exists when

$$q \to q_c = \frac{1}{\sqrt{\alpha - 1}}, \tag{36}$$

and it becomes necessary to consider the solution in more detail in a layer at $q=q_c$ (region IV). Before analyzing the runaway case $\alpha>1$ and region IV, we consider the solution in region III for the subcritical case $\alpha<1$ and critical case $\alpha=1$.

2. Case $\alpha<1$

For this case, the electric field is below the critical electric field that determines the runaway production threshold. However, a suprathermal, but not runaway, tail in the electron distribution function still develops as was shown in Ref. 2, creating a source for fast electron phenomena, e.g., enhanced bremsstrahlung.

The function $F^{(0)}$ is found equal to Eq. (23) and the equation for $F^{(1)}$ is obtained by changing the sign of $(\alpha-1)$ in Eqs. (25) and (27). By performing an analysis similar to Eqs. (28)–(30), one obtains

$$F^{(1)}(q,\mu) = 2(\sqrt{1+\mu} - \sqrt{2}) \sqrt{\frac{2\alpha}{1+Z}} \frac{q^2}{q^2+1} \sqrt{1+(1-\alpha)q^2} - \frac{1}{2} \sqrt{\frac{\alpha(1+Z)}{1-\alpha}} \times \cosh^{-1}\left[\frac{2}{\alpha} - 1 + 2q^2\left(\frac{1}{\alpha} - 1\right)\right], \quad \alpha < 1. \tag{37}$$

In order to find $F^{(2)}$ for the $\alpha<1$ case, one performs the same analysis as in Eqs. (32)–(34), and observes when the analysis is sensitive to sign$(\alpha-1)$. The resulting $F^{(2)}$ is

$$F^{(2)}(q,\mu) = \frac{1}{4}\left(\frac{1+Z}{2}\right)\left\{ \ln\left(\frac{\sqrt{1+q^2}-1}{\sqrt{1+q^2}+1}\right) + \frac{(4-2\alpha)\sqrt{\alpha}}{(1-\alpha)^{3/2}} \tan^{-1}\left(\frac{\sqrt{1-\alpha}\sqrt{1+q^2}}{\sqrt{\alpha}}\right) - \frac{2\alpha\sqrt{1+q^2}}{(1-\alpha)[1+(1-\alpha)q^2]} \right\} + \ln\frac{q}{\sqrt{1+q^2}} + \frac{1}{4}\frac{1+\alpha}{1-\alpha}\ln[1+(1-\alpha)q^2] + \frac{\alpha q^2}{1+Z}\frac{(\mu-1)}{\sqrt{1+q^2}} + \frac{1}{4}\ln(1+\mu) - \int_\mu^1 \frac{d\mu}{1-\mu^2}\left\{ \mu + \frac{2}{(1+Z)}\frac{[\alpha(\mu-2)q^2+1+q^2]}{[1+(1-\alpha)q^2]} \times \left[[(1+\mu) - \sqrt{2(1+\mu)}]\frac{[(q^2+1)(q^2+2) - \alpha q^2(q^2+3)]}{(1+q^2)^{3/2}} - \frac{(1+Z)}{2}\frac{\sqrt{1+\mu}}{\sqrt{2}}\right]\right\}, \quad \alpha < 1. \tag{38}$$

This new result was not considered in Ref. 2, which can be explained by the difference in the aims of the analyses; previous authors were mostly interested in the runaway beam, while the experimental conditions discussed in the current paper satisfy the near-threshold subcritical conditions. The C_{III} constant is obtained, once again, by asymptotic matching of the region III solution for low momenta to the region II solution for momenta going to infinity. It is found that

$$C_{III} = C_I + b(\alpha,Z) - \frac{(2\alpha-1)}{\epsilon} + \frac{1}{2}\sqrt{\frac{(1+Z)\alpha}{\epsilon(1-\alpha)}} \times \cosh^{-1}\left[\frac{2}{\alpha} - 1\right] - \frac{(5+Z)}{16}\ln\epsilon + \frac{(1+Z)}{4}\frac{\alpha}{1-\alpha} + \frac{(1+Z)}{4}\ln 2 - \frac{(1+Z)}{4}\frac{(2-\alpha)\sqrt{\alpha}}{(1-\alpha)^{3/2}} \times \tan^{-1}\left[\sqrt{\frac{1}{\alpha} - 1}\right], \quad \alpha < 1. \tag{39}$$

Note that the solution in this case, $\alpha<1$, possesses no singularity in q, i.e., for $\alpha<1$ region III extends to infinity and no runaway region exists.

3. Perturbation solution close to $\alpha=1$

The analysis for $\alpha>1$ breaks down in an α boundary layer close to $\alpha=1$, which can be observed as a rapid qualitative change of the solution for $\alpha>1$ in the limit $\alpha\to 1^+$. More quantitatively, by comparing terms in the $\alpha>1$ solution, the width of this boundary layer, $\delta\alpha=\alpha-1>0$ and $\delta\alpha \ll 1$, can be estimated. The ordering of the zeroth and first-order terms breaks down when [see Eqs. (23) and (30)]

$$\epsilon^{-1}F^{(0)} \sim \epsilon^{-1/2}F^{(1)} \Rightarrow \epsilon^{-1} \sim \frac{\epsilon^{-1/2}}{\sqrt{\alpha-1}} \Leftrightarrow \delta\alpha \sim \epsilon, \tag{40}$$

where the effective charge $Z\sim\mathcal{O}(1)$ for a tokamak plasma and $q\sim\mathcal{O}(1)$ in region III have been used, whereas the ordering between the first and second-order terms breaks down when [see Eq. (34)]

$$\epsilon^{-1/2}F^{(1)} \sim F^{(2)} \Rightarrow \frac{\epsilon^{-1/2}}{\sqrt{\alpha-1}} \sim \frac{1}{(\alpha-1)^{3/2}} \Leftrightarrow \delta\alpha \sim \epsilon^{1/2}. \tag{41}$$

In order to guarantee the validity of the expansion technique, the most restrictive width must be used, i.e., $\delta\alpha\sim\epsilon^{1/2}$ is where the solution in region III for $\alpha>1$ breaks down. This breakdown will also appear as an unphysical qualitative change in the runaway rate calculation below.

A first-order perturbation solution is obtained by making an ansatz, $\tilde{F}=F_0+\delta\alpha F_1$, and using the same expansion for the two unknown zeroth-order (unperturbed) and first-order (perturbed) terms, F_0 and F_1, respectively, as for F in the $\alpha>1$ analysis; see Eq. (19). The unperturbed solution is obtained by setting $\delta\alpha=0$ and performing the same analysis as was previously outlined for the $\alpha>1$ case. One finds, to order $\mathcal{O}(1)$ in ϵ, that

$$F_0=\epsilon^{-1}F_0^{(0)}+\epsilon^{-1/2}F_0^{(1)}+F_0^{(2)}+\mathrm{C}_{\mathrm{III},0}, \tag{42}$$

where

$$F_0^{(0)}=\frac{1}{\sqrt{1+q^2}}, \tag{43}$$

$$F_0^{(1)}=2(\sqrt{1+\mu}-\sqrt{2})\sqrt{\frac{2}{1+Z}}\frac{q^2}{1+q^2}-\sqrt{1+Z}\sqrt{1+q^2}, \tag{44}$$

$$\begin{aligned}F_0^{(2)}=&\left(\frac{1+Z}{4}\right)\left\{\frac{5+2q^2}{3}\sqrt{1+q^2}+\ln\left(\frac{q}{1+\sqrt{1+q^2}}\right)\right\}+\frac{1}{2}q^2\\&+\ln\frac{q}{\sqrt{1+q^2}}+\frac{q^2}{1+Z}\frac{(\mu-1)}{\sqrt{1+q^2}}+\frac{1}{4}\ln(1+\mu)\\&-\int_\mu^1\frac{\mathrm{d}\mu}{1-\mu^2}\left\{\mu+\frac{2}{(1+Z)}(1+(\mu-1)q^2)\right.\\&\times\left[[(1+\mu)-\sqrt{2(1+\mu)}]\frac{2}{(1+q^2)^{3/2}}\right.\\&\left.\left.-\frac{(1+Z)}{2}\frac{\sqrt{1+\mu}}{\sqrt{2}}\right]\right\},\end{aligned} \tag{45}$$

and where the unperturbed matching constant is

$$\begin{aligned}\mathrm{C}_{\mathrm{III},0}=&\mathrm{C}_\mathrm{I}+b(\alpha,Z)-\frac{1}{\epsilon}+\sqrt{\frac{1+Z}{\epsilon}}-\frac{(5+Z)}{16}\ln\epsilon\\&-\frac{(1+Z)}{4}\frac{5}{3}+\frac{(1+Z)}{4}\ln 2.\end{aligned} \tag{46}$$

The first-order perturbed solution is obtained by linearizing the collision operator in $\delta\alpha$ and then solving for F_1. One finds, to order $\mathcal{O}(1)$ in ϵ, that

$$F_1=\epsilon^{-1}F_1^{(0)}+\epsilon^{-1/2}F_1^{(1)}+\mathrm{C}_{\mathrm{III},1}, \tag{47}$$

where

$$F_1^{(0)}=\sqrt{1+q^2}+\frac{1}{\sqrt{1+q^2}}, \tag{48}$$

$$\begin{aligned}F_1^{(1)}=&\sqrt{\frac{2}{1+Z}}\frac{q^2(1-q^2)}{1+q^2}\sqrt{1+\mu}+\frac{2}{\sqrt{1+Z}}\frac{3+2q^2+q^4}{1+q^2}\\&-\frac{\sqrt{1+Z}}{6}(1+q^2)^{3/2},\end{aligned} \tag{49}$$

and where the first-order correction in the matching constant is

$$\mathrm{C}_{\mathrm{III},1}=-\frac{2}{\epsilon}+\frac{\sqrt{1+Z}}{6\sqrt{\epsilon}}. \tag{50}$$

The correctness of the $F_0^{(0)}$ and $F_1^{(0)}$ terms is easily verified by taking $\alpha=1+\delta\alpha$ in Eq. (23). A term $F_1^{(2)}$ was omitted in Eq. (47) since it would be of order $\mathcal{O}(\delta\alpha)$, $\delta\alpha\ll 1$, and therefore would not contribute to the $\mathcal{O}(1)$ analysis.

By comparing terms in Eqs. (43) and (44), one finds that the ordering of terms in Eq. (42) breaks down for $q\gtrsim\epsilon^{-1/4}$, and a new boundary layer should be considered. Therefore, the treatment of the runaway region below is only valid for critical field parameters satisfying $\epsilon^{1/2}\ll\alpha-1\ll 1$.

The perturbed solution, together with the $\alpha>1$ and $\alpha<1$ solutions, covers the entire α parameter space and the analysis of region III is thus complete.

Returning to the case $\alpha>1$, one considers a boundary layer at $q=q_\mathrm{c}=1/\sqrt{\alpha-1}$, region IV.

D. Region IV: Runaway source region

Inspecting the solutions of different orders in region III, one observes that $F^{(0)}$ is the only function that is well-behaved as $q\to\infty$, while $F^{(1)}$ and $F^{(2)}$ possess singular behavior for q exceeding q_c for the supercritical case $\alpha>1$. In order to adjust the analysis for $q>q_\mathrm{c}$, one considers a boundary layer at $q=q_\mathrm{c}$ by introducing a new variable

$$q=q_\mathrm{c}(1+\epsilon^{1/3}x) \tag{51}$$

and using the expansion

$$\ln f_\mathrm{IV}=F=\epsilon^{1/3}F^{(0)}+F^{(1)}+\epsilon^{1/3}F^{(2)}+\cdots+\mathrm{C}_\mathrm{IV}. \tag{52}$$

The new independent variable reflects the characteristic width of the boundary layer determined by

$$\left(\frac{\partial F^{(1)}}{\partial q}\right)\Big/\left(\frac{\partial F^{(0)}}{\partial q}\right)\sim\frac{1}{[1-(\alpha-1)q^2]^{3/2}}\sim\epsilon^{-1/2} \tag{53}$$

for $q\sim\mathcal{O}(1)$. The Fokker-Planck Eq. (5) gives to order $\mathcal{O}(\epsilon^{-2/3})$

$$\left(\frac{1+Z}{2}\right)\sqrt{\frac{\alpha-1}{\alpha}}(1+\mu)\left(\frac{\partial F^{(0)}}{\partial\mu}\right)^2+\frac{\partial F^{(0)}}{\partial x}=0. \tag{54}$$

This is a nonlinear PDE, to be solved in terms of its boundary values on $\mu=1$ by the method of characteristics.[15]

In order to find the boundary values for the general solution in region IV, Eq. (52), one considers a subregion IV′ close to $\mu=1$, where

$$\mu=1-\epsilon^{1/3}\nu,\quad \nu>0 \text{ and finite}, \tag{55}$$

and uses again the variable $q=q_\mathrm{c}(1+\epsilon^{1/3}x)$. A simultaneous application of Eqs. (51) and (55) to the Fokker-Planck Eq. (5) gives, to leading order, $\mathcal{O}(\epsilon^{-1})$,

$$\frac{\partial F^{(0)}}{\partial\nu}=0, \tag{56}$$

whereas the next-order equation, $\mathcal{O}(\epsilon^{-2/3})$, is trivially satisfied, and the order $\mathcal{O}(\epsilon^{-1/3})$ equation is

$$(1+Z)\left\{\nu\left[\frac{\partial^2 F^{(1)}}{\partial \nu^2}+\left(\frac{\partial F^{(1)}}{\partial \nu}\right)^2\right]+\frac{\partial F^{(1)}}{\partial \nu}\right\}+\alpha\left(\frac{\partial F^{(0)}}{\partial x}\right)^2 = \sqrt{\frac{\alpha}{\alpha-1}}\left\{-\nu+2x\left(1-\frac{1}{\alpha}\right)\right\}\frac{\partial F^{(0)}}{\partial x}. \tag{57}$$

Since the inhomogeneous part has only powers of ν^0 and ν^1, one makes the ansatz[12]

$$F^{(1)} = P(x) + \nu Q(x). \tag{58}$$

Equating the coefficients in front of equal powers gives

$$\frac{\partial F^{(0)}}{\partial x} = -\sqrt{\frac{\alpha-1}{\alpha}}(1+Z)Q^2, \tag{59}$$

$$Q^3 + \frac{2x}{\alpha(1+Z)}Q + \frac{1}{(1+Z)(\alpha-1)} = 0. \tag{60}$$

Differentiating Eq. (60) with respect to x and using the result of a parametric integration of Eq. (59), one obtains

$$F^{(0)} = \sqrt{\alpha(\alpha-1)}(1+Z)^2\frac{Q^4}{4} - \frac{1+Z}{2}\sqrt{\frac{\alpha}{\alpha-1}}Q. \tag{61}$$

In order to find $P(x)$, one goes to the next-order equation, $\mathcal{O}(\epsilon^0)$, of the Fokker-Planck equation and uses the ansatz

$$F^{(2)} = A(x) + \nu B(x) + \nu^2 C(x). \tag{62}$$

After some algebra, it is found that

$$P = -\frac{(1+Z)^3}{8}\sqrt{\alpha(\alpha-1)}(\alpha-2)Q^6 - \frac{(1+Z)^2}{4}\sqrt{\frac{\alpha}{\alpha-1}}\left(\alpha-\frac{7}{3}\right)Q^3 - \frac{1}{2}\ln\left|2\alpha(1+Z)Q^3 - \frac{\alpha}{\alpha-1}\right| + \left(\frac{\alpha-2}{\alpha-1}\right)\left[1-\frac{(1+Z)}{4}\sqrt{\frac{\alpha}{\alpha-1}}\right]\ln|Q|. \tag{63}$$

Before matching the solution in subregion IV′ to the solution in region III at $\mu=1$, a comment on the signs of x and Q has to be made. From Eq. (60), it is observed that as $x\to+\infty$, $Q\to 0$ is the only root. Root tracing then shows that, as $x\to-\infty$, $Q\to-\infty$.

The numerical scheme for determining the matching constant $b=b(\alpha,Z)$ between regions II and III enables a calculation of the runaway rate to order $\mathcal{O}(1)$. However, the matching of region IV onto region III then has to be taken to order $\mathcal{O}(1)$, whereas previous authors[2] only took the matching to order $\mathcal{O}(\ln\epsilon)$ [since previously, $b\sim\mathcal{O}(1)$ was unknown]. Matching the above solution, in subregion IV′, as $x\to-\infty$, i.e., when $Q\to-\infty$, onto the solution in region III as $q\to q_c$, $\mu\to 1$ determines the constant $C_{IV'}$. To $\mathcal{O}(\epsilon^{-1})$, $\mathcal{O}(\epsilon^{-1/2})$, $\mathcal{O}(\ln\epsilon)$ as well as to $\mathcal{O}(1)$, it is found that

$$\begin{aligned} C_{IV'} = {} & \frac{2}{\epsilon}\left(\sqrt{\alpha(\alpha-1)}-\alpha+\frac{1}{2}\right) - \frac{1}{2}\sqrt{\frac{\alpha(1+Z)}{\epsilon(\alpha-1)}}\left[\frac{\pi}{2}-\sin^{-1}\left(1-\frac{2}{\alpha}\right)\right] - \frac{\ln\epsilon}{48(\alpha-1)} \\ & \times\left\{\alpha(3Z+19)-11-3Z+2(1+Z)(\alpha-2)\sqrt{\frac{\alpha}{\alpha-1}}\right\} + C_I + b(\alpha,Z) - \frac{(1+Z)}{4}\frac{\alpha}{\alpha-1} + \frac{(1+Z)}{4}\ln 2 \\ & + \frac{1}{4}\left(\frac{\alpha-3}{\alpha-1} - \frac{(1+Z)}{2}\frac{\sqrt{\alpha}(\alpha-2)}{(\alpha-1)^{3/2}}\right)\ln(1+Z) - \frac{(1+Z)}{4}\ln\left(1+\sqrt{\frac{\alpha-1}{\alpha}}\right) + \frac{1}{4}\left(3+(1+Z)\frac{\sqrt{\alpha}(\alpha-2)}{(\alpha-1)^{3/2}}\right)\ln 2 \\ & + \frac{(1+Z)}{8}\frac{\sqrt{\alpha}(3-\alpha)}{(\alpha-1)^{3/2}} - \frac{1}{4}\left(2\frac{(\alpha+1)}{(\alpha-1)}+\frac{(1+Z)}{2}\right)\ln\alpha - \frac{(1+Z)}{4}\frac{\sqrt{\alpha}(\alpha-2)}{(\alpha-1)^{3/2}}\ln(\sqrt{\alpha}+\sqrt{\alpha-1}). \end{aligned} \tag{64}$$

It is now appropriate to return to the analysis of the full solution in region IV, away from $\mu=1$. From Eq. (54) one obtains the first-order equation

$$A(1+\mu)\left(\frac{\partial F^{(0)}}{\partial \mu}\right)^2 + \frac{\partial F^{(0)}}{\partial x} = 0, \tag{65}$$

where

$$A = \frac{1+Z}{2}\sqrt{\frac{\alpha-1}{\alpha}}. \tag{66}$$

One introduces $y=2\sqrt{1+\mu}/\sqrt{A}$, $p=\partial F^{(0)}/\partial x$, and $q=\partial F^{(0)}/\partial y$. Equation (65) then takes the form

$$F(p,q) = p + q^2 = 0. \tag{67}$$

By writing down Charpit's equations[15] for this PDE, with Cauchy data obtained from the subregion IV′ solution at $\mu=1$, the characteristic curves of the PDE are found to satisfy

$$Q^3 + \frac{2xQ}{\alpha(1+Z)} + \frac{4}{(1+Z)^2}\frac{1}{\sqrt{\alpha(\alpha-1)}} \times\left[\sqrt{\frac{1+\mu}{2}}-1+\frac{(1+Z)}{4}\sqrt{\frac{\alpha}{\alpha-1}}\right] = 0. \tag{68}$$

The PDE analysis also finds that

$$F^{(0)} = \frac{3}{4}\sqrt{\alpha(\alpha-1)}(1+Z)^2Q^4 + (1+Z)\sqrt{\frac{\alpha-1}{\alpha}}xQ^2, \tag{69}$$

which reduces to Eq. (61) for $\mu=1$ using Eq. (60). Continuing the expansion of the Fokker-Planck equation in the general case (μ arbitrary), $F^{(1)}$ could be determined by the $\mathcal{O}(\epsilon^{-1/3})$ equation. However, in order to calculate the runaway rate, the $F^{(1)}$ term in region IV need not be known. How the characteristics cover parts of the regions in (x,μ) space doubly, while some regions are inaccessible to the characteristics, is discussed further in Refs. 12 and 14.

This ends the treatment of the boundary layer region, region IV. The solution for $q>q_c$, which will be used to obtain the runaway flux, is considered next.

E. Region V: Runaway region in the near-critical limit

In order to find the runaway flux one considers the Fokker-Planck equation, Eq. (5), to leading order, i.e.,

$$\left(\alpha\mu - \frac{(q^2+1)}{q^2}\right)\frac{\partial f}{\partial q} + \left[\alpha(1-\mu^2) + \mu\frac{(1+Z)\sqrt{q^2+1}}{q^2}\right]\frac{1}{q}\frac{\partial f}{\partial \mu} - \frac{2f}{q} = \frac{(1+Z)}{2}\frac{\sqrt{q^2+1}}{q^3}(1-\mu^2)\frac{\partial^2 f}{\partial \mu^2}, \tag{70}$$

where the ϵ terms have been neglected. This equation is a "two-way" parabolic equation, discussed in Ref. 12. The sign of $[\alpha\mu-(q^2+1)/q^2]$ determines the "direction" of the solution. For $[\alpha\mu-(q^2+1)/q^2]<0$ the solution is determined "from where it came from," i.e., in this case from $q\to\infty$, where f has to vanish (no sources of runaways there), whereas for the opposite sign $[\alpha\mu-(q^2+1)/q^2]>0$, a specification of the source strength at the singularity ($q=q_c$, $\mu=1$) is necessary.[12,16]

In a plasma with a slightly supercritical electric field, $0<\alpha-1\ll 1$, the characteristic value of q in the runaway region satisfies $q>q_c=1/\sqrt{\alpha-1}\gg 1$. Further simplification of Eq. (70) is possible under such conditions as follows:

$$\alpha\left(\mu\frac{\partial f}{\partial q} + \frac{1-\mu^2}{q}\frac{\partial f}{\partial \mu}\right) = \frac{1+Z}{2}\frac{1}{q^2}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} + \frac{\partial f}{\partial q} + \frac{2f}{q}, \tag{71}$$

and an exact analytical solution f_V of this equation can then be found. In order to have a beam-like solution peaked near $\mu=1$, one introduces cylindrical coordinates $q_\perp$ and $q_\parallel$, where

$$q = \sqrt{q_\perp^2 + q_\parallel^2},$$

$$\mu = q_\parallel/\sqrt{q_\perp^2 + q_\parallel^2}, \tag{72}$$

with the two conditions $q_\parallel/q_\perp \gg 1$ and $(\partial/\partial q_\perp)/(\partial/\partial q_\parallel)\gg 1$ for the beam being valid. Thus, Eq. (71) can be written as

$$(\alpha-1)\frac{\partial f_V}{\partial q_\parallel} = \frac{(1+Z)}{2}\frac{1}{q_\perp}\frac{\partial}{\partial q_\perp}q_\perp\frac{\partial f_V}{\partial q_\perp} + \frac{q_\perp}{q_\parallel}\frac{\partial f_V}{\partial q_\perp} + \frac{2f_V}{q_\parallel}. \tag{73}$$

In terms of the new variables

$$\xi = \frac{q_\perp}{\sqrt{q_\parallel(1+Z)/2}}; \quad \eta = \frac{q_\parallel(1+Z)}{2}, \tag{74}$$

one obtains

$$\frac{\partial^2 f_V}{\partial \xi^2} + \left(\frac{1}{\xi} + \frac{\alpha+1}{2}\xi\right)\frac{\partial f_V}{\partial \xi} - (\alpha-1)\eta\frac{\partial f_V}{\partial \eta} + 2f_V = 0. \tag{75}$$

Using separation of variables,

$$f_V = \Phi(\xi)\Psi(\eta), \tag{76}$$

leads to

$$\frac{\partial^2 \Phi}{\partial \xi^2} + \left(\frac{1}{\xi} + \frac{\alpha+1}{2}\xi\right)\frac{\partial \Phi}{\partial \xi} + C_s\Phi = 0, \tag{77}$$

$$\frac{1}{\Psi}\frac{\partial \Psi}{\partial \eta} + \frac{C_s-2}{\alpha-1}\frac{1}{\eta} = 0, \tag{78}$$

where C_s is a separation constant which is determined below. Equation (78) is easily solved, and

$$\Psi(q_\parallel) = C_\Psi q_\parallel^{(2-C_s)/(\alpha-1)}, \tag{79}$$

where C_Ψ is an integration constant.

To solve Eq. (77), one makes use of the variable transformation

$$y = -\frac{(\alpha+1)}{4}\xi^2; \quad \Phi(\xi) = U(y) \tag{80}$$

and obtains the confluent hypergeometric equation (the Kummer equation),

$$y\frac{d^2U}{dy^2} + (1-y)\frac{dU}{dy} - \frac{C_s}{\alpha+1}U = 0. \tag{81}$$

The solution, which is bounded when $|y|\to\infty$, is given by the confluent hypergeometric function,[17]

$$U(\xi) = B\,{}_1F_1\left(\frac{C_s}{\alpha+1}, 1; -\frac{(\alpha+1)}{4}\xi^2\right), \tag{82}$$

where B is an integration constant. Connor and Hastie in Ref. 2 took the special case $C_s=\alpha+1$. However, this choice of the separation constant precludes asymptotic matching and is therefore incorrect, as will be obvious from the analysis
below.

By combining Eqs. (79) and (82), one obtains the solution in region V,

$$f_V = \frac{A}{q_\parallel^{(C_s-2)/(\alpha-1)}} \exp\left(-\frac{(\alpha+1)q_\perp^2}{2(1+Z)q_\parallel}\right) {}_1F_1 \times\left(1-\frac{C_s}{\alpha+1},1;\frac{(\alpha+1)q_\perp^2}{2(1+Z)q_\parallel}\right), \quad (83)$$

where A is a source strength constant, related to the runaway electron source at the boundary layer, $q=q_c$. Equation (83) is a generalization of Eq. 60 in Ref. 2, and is valid for all q in this case of a near-critical electric field in the plasma. A condition for $f_V \to 0$ as $q_\parallel \to \infty$ is that

$$C_s > 2. \quad (84)$$

That this condition holds in our near-critical electric field case is verified below.

The constants A and C_s are determined by matching the solution in region V to the region IV solution in the limit $x \to \infty$. The region V solution is expressed in the original parameters $\mu = q_\parallel/\sqrt{q_\parallel^2+q_\perp^2}$ and $q=\sqrt{q_\parallel^2+q_\perp^2}$, and matching is performed at $\mu=1$. The region V solution at $\mu=1$ can be written

$$F_V = \ln(f_V)_{\mu=1} = \ln A - \left(\frac{C_s-2}{\alpha-1}\right)\ln q. \quad (85)$$

In region V,

$$q = q_c(1+\epsilon^{1/3}x) \approx q_c\epsilon^{1/3}x, \quad (86)$$

and Eq. (85) can be approximated to

$$F_V \approx \ln A - \left(\frac{C_s-2}{\alpha-1}\right)\left(\frac{1}{3}\ln\epsilon - \frac{1}{2}\ln(\alpha-1) + \ln x\right), \quad x \to \infty. \quad (87)$$

In the asymptotic matching limit, the solution in region IV (see Sec. III D) can be approximated to

$$F_{IV} \approx -\frac{1}{2}\ln\left|\frac{\alpha}{\alpha-1}\right| + \frac{(\alpha-2)}{(\alpha-1)} \times\left(1-\frac{(1+Z)}{4}\sqrt{\frac{\alpha}{\alpha-1}}\right)\ln\left|\frac{\alpha}{2(\alpha-1)x}\right| + C_{IV}, \quad x \to \infty. \quad (88)$$

Matching now the $\ln x$ terms determines the separation constant as

$$C_s = \alpha - \frac{(1+Z)}{4}(\alpha-2)\sqrt{\frac{\alpha}{\alpha-1}}, \quad (89)$$

whereas the constant terms determine A,

TABLE I. The runaway rate proportionality constant $C_R(\alpha,Z)$ for plasmas with critical field parameters $\alpha=1.3$, 1.4, and 1.5, and with effective charge numbers $Z=1$, 2, and 3.

α	$Z=1$	$Z=2$	$Z=3$
1.3	11.2	1.67	0.400
1.4	12.2	4.01	1.67
1.5	10.8	5.74	3.47

$$A = \exp\left(C_{IV} + \frac{\alpha-2}{3(\alpha-1)}\left(1-\frac{(1+Z)}{4}\sqrt{\frac{\alpha}{\alpha-1}}\right)\ln\epsilon\right) \times\left(\frac{\alpha}{\alpha-1}\right)^{\frac{(\alpha-3)}{2(\alpha-1)}-\frac{(1+Z)}{4}\frac{(\alpha-2)}{(\alpha-1)}\sqrt{\frac{\alpha}{\alpha-1}}} \times\left(\frac{1}{\alpha-1}\right)^{\frac{(\alpha-2)}{2(\alpha-1)}\left(1-\frac{(1+Z)}{4}\sqrt{\frac{\alpha}{\alpha-1}}\right)} \times\left(\frac{1}{2}\right)^{\frac{(\alpha-2)}{(\alpha-1)}\left(1-\frac{(1+Z)}{4}\sqrt{\frac{\alpha}{\alpha-1}}\right)}. \quad (90)$$

The requirement $C_s > 2$, Eq. (84), is equivalent to requiring the coefficient of the $\ln|x|$ term in Eq. (88) to be positive, i.e.,

$$\frac{\alpha-2}{\alpha-1}\left(1-\frac{(1+Z)}{4}\sqrt{\frac{\alpha}{\alpha-1}}\right) > 0. \quad (91)$$

In the case of a near-critical electric field, $\alpha-1 \ll 1$, the condition (91) is satisfied for any effective charge number $Z>1$.

IV. RUNAWAY RATE

The runaway rate S_R, the number of runaway electrons generated per unit volume and unit time, is given by[18,19]

$$S_R \equiv \frac{\partial n_r}{\partial t} = -\frac{\partial n_b}{\partial t} = n_e\nu_{rel}(-2\pi)\int_{-1}^{1}d\mu\int_{q_c}^{\infty}dq q^2 \times\left\{\alpha\left(\mu\frac{\partial f_V}{\partial q}+\frac{1-\mu^2}{q}\frac{\partial f_V}{\partial\mu}\right)\right\} = n_e\nu_{rel}(-2\pi\alpha)\int_{-1}^{1}d\mu\mu(q^2 f_V)\Big|_{q_c}^{\infty}, \quad (92)$$

where the Fokker-Planck equation has been used. The region V solution from Sec. III E, which is valid for $\epsilon^{1/2} \ll \alpha-1 \ll 1$, gives the final expression for the runaway rate,

$$S_{\mathrm{R}} = n_{\mathrm{e}}\nu_{\mathrm{th}}\left(\frac{v_{\mathrm{th}}}{c}\right)^3 \pi\left(\frac{\alpha}{\alpha-1}\right)^{1+\frac{(\alpha-3)}{2(\alpha-1)}-\frac{(1+Z)}{4}\frac{(\alpha-2)}{(\alpha-1)}\sqrt{\frac{\alpha}{\alpha-1}}} \exp\left(\frac{2}{\epsilon}\left\{\sqrt{\alpha(\alpha-1)}-\alpha+\frac{1}{2}\right\}-\frac{1}{2}\sqrt{\frac{\alpha(1+Z)}{\epsilon(\alpha-1)}}\left\{\frac{\pi}{2}-\sin^{-1}\left[1-\frac{2}{\alpha}\right]\right\}\right.$$
$$\left.-\frac{\ln\epsilon}{16(\alpha-1)}\left\{\alpha(1+Z)-Z+7+2(1+Z)(\alpha-2)\sqrt{\frac{\alpha}{\alpha-1}}\right\}\right)$$
$$\times\exp\left(\mathrm{C_I}+b(\alpha,Z)+\frac{(1+Z)}{4}\left[-\frac{1}{2}\frac{\sqrt{\alpha}(\alpha-2)}{(\alpha-1)^{3/2}}\ln\left(\frac{1+Z}{16}\right)+\frac{1}{2}\frac{\sqrt{\alpha}(3-\alpha)}{(\alpha-1)^{3/2}}-\frac{1}{2}\ln\alpha-\frac{\sqrt{\alpha}(\alpha-2)}{(\alpha-1)^{3/2}}\ln(\sqrt{\alpha}+\sqrt{\alpha-1})-\frac{\alpha}{\alpha-1}\right.\right.$$
$$\left.\left.+\ln\left(\frac{2}{1+\sqrt{\frac{\alpha-1}{\alpha}}}\right)\right]+\frac{\alpha-3}{4(\alpha-1)}\ln(1+Z)+\frac{3\alpha+1}{4(\alpha-1)}\ln 2\ -\frac{1}{2}\frac{\alpha+1}{\alpha-1}\ln\alpha\right)$$
$$\times\int_{0^+}^{1}\mathrm{d}\mu\frac{{}_1F_1\left(\frac{\alpha}{\alpha+1}-\frac{(1+Z)}{4}\frac{(\alpha-2)}{(\alpha+1)}\sqrt{\frac{\alpha}{\alpha-1}},1;-\frac{(\alpha+1)}{2(1+Z)\sqrt{(\alpha-1)}}\frac{(1-\mu^2)}{\mu}\right)}{\mu^{\left(-1-\frac{(1+Z)}{4}(\alpha-2)\sqrt{\frac{\alpha}{\alpha-1}}\right)/(\alpha-1)}}, \tag{93}$$

where the thermal collision frequency, $\nu_{\mathrm{th}} = n_{\mathrm{e}}e^4\ln\Lambda/4\pi\epsilon_0^2 m_{\mathrm{e}}^2 v_{\mathrm{th}}^3$ (v_{th} is the speed of thermal electrons), is introduced for comparison to earlier results. The scaling of the runaway rate with the electric field, given by Connor and Hastie in Eq. 61 of Ref. 2, is here given by lines three and four of Eq. (93),

$$S_{\mathrm{R}} \propto \exp\left(\frac{2}{\epsilon}\left\{\sqrt{\alpha(\alpha-1)}-\alpha+\frac{1}{2}\right\}\right.$$
$$-\frac{1}{2}\sqrt{\frac{\alpha(1+Z)}{\epsilon(\alpha-1)}}\left\{\frac{\pi}{2}-\sin^{-1}\left[1-\frac{2}{\alpha}\right]\right\}$$
$$-\frac{\ln\epsilon}{16(\alpha-1)}\left\{\alpha(1+Z)-Z+7+2(1+Z)\right.$$
$$\left.\left.\times(\alpha-2)\sqrt{\frac{\alpha}{\alpha-1}}\right\}\right). \tag{94}$$

Connor and Hastie, who obtained the runaway rate scaling in Eq. (94) by taking the limit $x\to\infty$ in the inner variable of the region IV′ solution, had a remaining unknown proportionality constant, $\mathrm{C_R}(\alpha,Z)\sim\mathcal{O}(1)$. The present analysis, which is based on a near-critical electric field case for which asymptotic matching between region IV and region V is possible, facilitates a determination of this proportionality constant in a critical field parameter range $\epsilon^{1/2}\ll\alpha-1\ll 1$. The proportionality constant $\mathrm{C_R}(\alpha,Z)$ was determined for three different critical field parameters and with three different effective charge numbers, and the results are presented in Table I. The algorithm outlined in Appendix B was once again used to determine the matching constant $b(\alpha,Z)$. From Table I, one can deduce that the constant $\mathrm{C_R}(\alpha,Z)$ is indeed of order $\mathcal{O}(1)$. However, its value is quite sensitive to plasma parameters.

As a final remark, we want to point out that α has to be kept of the order of unity in order to ensure the validity of the analytical expansion technique here employed. For a much higher critical field parameter α (one or two orders of magnitude higher), the analytical procedure developed here for matching region IV and region V as well as the runaway rate expression in region V are no longer valid and the relevant equations should be solved within some different approach.

V. CONCLUSIONS

In this paper, a relativistic collision operator conserving the number of particles is derived and used for obtaining the Fokker-Planck equation that governs the electron velocity distribution function in the presence of a steady-state electric field in a weakly relativistic plasma with a finite value of $T_{\mathrm{e}}/m_{\mathrm{e}}c^2$. Modifications of the electron velocity distribution function were found in higher order expansions in the relativistic parameter ϵ. It is also found that, in spite of an incomplete form of the collision operator not conserving number of particles in Ref. 2, the main result concerning the runaway rate [Eqs. 61 and 62 in Ref. 2 and Eq. (94) in the present paper] is still valid. Moreover, the previously unknown proportionality constant has been calculated.

In addition to the analysis carried out in Ref. 2, new regions $E\lesssim E_{\mathrm{c}}$ and $E\gtrsim E_{\mathrm{c}}$ are considered in the present paper and the relevant expressions describing the electron velocity distribution function for a near-threshold electric field case are obtained. In the weakly relativistic region a numerical solution enables matching to the high momentum analytical asymptote. Thus, the form of the electron velocity distribution function up to the relativistic region, where suprathermal electrons are present, is accurately described.

A restriction of the present analysis comes from assuming a critical electric field parameter α and an effective

charge number Z of order unity. The restriction on the effective charge number is not limiting in the case of tokamak plasmas, where the effective charge number seldom exceeds values of about 3. However, for dusty plasmas, e.g., in astrophysics, the effective charge number does not need to be of order unity, and therefore one would like to extend the analysis to high values of Z. In any case, critical electric field parameters larger than unity are of importance in all runaway production cases.

The results obtained here can be applied to phenomena associated with suprathermal electrons in hot tokamak plasmas, e.g., to the grassy sawteeth experiments on JET as outlined in the Introduction of this paper.

ACKNOWLEDGMENTS

Fruitful discussions with H. Smith and D. Anderson are highly appreciated. The authors would also like to acknowledge comments by J. W. Connor.

This work has been conducted under association contracts between EURATOM, Sweden and UK, and was partly funded by the Swedish Research Council and the UK Engineering and Physical Sciences Research Council.

APPENDIX A: RELATIVISTIC COLLISION OPERATOR

In this appendix the relativistic collision operator for electrons with velocities $\mathbf{v}$ colliding off a Maxwellian electron background with velocities $\mathbf{v}'$ is derived. The results are used in the subsequent analysis of the electron distribution function presented in the paper.

The collision operator in a relativistic plasma is given by Beliaev and Budker[11] as

$$C(f)=\frac{\Gamma}{2n_e}\frac{\partial}{\partial \mathbf{p}}\int \overleftrightarrow{\mathbf{U}}\left(\frac{\partial f(\mathbf{p})}{\partial \mathbf{p}}f'(\mathbf{p}')-f(\mathbf{p})\frac{\partial f'(\mathbf{p}')}{\partial \mathbf{p}'}\right)\mathrm{d}^3p' \tag{A1}$$

with $\Gamma=n_e e^4 \ln\Lambda/4\pi\epsilon_0^2$ and the momentum $\mathbf{p}=\gamma m_e c\mathbf{v}$. Here, all the primed quantities are related to background plasma particles while the unprimed quantities correspond to test electrons. The kernel $\overleftrightarrow{\mathbf{U}}$ is given by[11]

$$c\overleftrightarrow{\mathbf{U}}=D\overleftrightarrow{\mathbf{E}}, \tag{A2}$$

where

$$D=\frac{\gamma\gamma'(1-\mathbf{v}\cdot\mathbf{v}')^2}{[\gamma^2\gamma'^2(1-\mathbf{v}\cdot\mathbf{v}')^2-1]^{3/2}} \tag{A3}$$

and

$$\overleftrightarrow{\mathbf{E}}=[\gamma^2\gamma'^2(1-\mathbf{v}\cdot\mathbf{v}')^2-1]\overleftrightarrow{\mathbf{I}}-\gamma^2\mathbf{vv}-\gamma'^2\mathbf{v}'\mathbf{v}'+\gamma^2\gamma'^2\times(1-\mathbf{v}\cdot\mathbf{v}')(\mathbf{vv}'+\mathbf{v}'\mathbf{v}), \tag{A4}$$

where the velocity has been normalized to c, so that $\gamma^2=(1-v^2)^{-1}$, and $\overleftrightarrow{\mathbf{I}}$ is the unitary tensor.

We take $f'=f'(\mathbf{p}')$ to be Maxwellian,

$$f'\propto e^{-\gamma m_e c^2/T_e}=e^{-c\sqrt{p^2+m_e^2c^2}/T_e},$$

with a temperature much smaller than the electron rest mass, $\epsilon=T_e/m_e c^2\ll 1$, corresponding to a weakly relativistic background plasma. The kernel $\overleftrightarrow{\mathbf{U}}$ can be expanded according to

$$\overleftrightarrow{\mathbf{U}}=\overleftrightarrow{\mathbf{U}}^{(0)}+\frac{\partial\overleftrightarrow{\mathbf{U}}^{(0)}}{\partial\mathbf{v}'}\cdot\mathbf{v}'+\frac{1}{2}\frac{\partial^2\overleftrightarrow{\mathbf{U}}^{(0)}}{\partial\mathbf{v}'\,\partial\mathbf{v}'}:\mathbf{v}'\mathbf{v}' \tag{A5}$$

around $\mathbf{v}'=\mathbf{0}$, valid for stationary background particles. One then obtains

$$D\Big|_{\mathbf{v}'=\mathbf{0}}=\frac{\gamma}{(\gamma^2-1)^{3/2}}=\frac{1}{\gamma^2v^3}, \tag{A6}$$

$$\overleftrightarrow{\mathbf{E}}\Big|_{\mathbf{v}'=\mathbf{0}}=(\gamma^2-1)\overleftrightarrow{\mathbf{I}}-\gamma^2\mathbf{vv}=\gamma^2(v^2\overleftrightarrow{\mathbf{I}}-\mathbf{vv}), \tag{A7}$$

so that

$$c\overleftrightarrow{\mathbf{U}}^{(0)}=\frac{v^2\overleftrightarrow{\mathbf{I}}-\mathbf{vv}}{v^3}. \tag{A8}$$

In order to determine the first-order corrections for finite $\mathbf{v}'$, one needs to evaluate

$$\nabla'\cdot(D\overleftrightarrow{\mathbf{E}})\Big|_{\mathbf{v}'=\mathbf{0}}=\nabla' D\cdot\overleftrightarrow{\mathbf{E}}\Big|_{\mathbf{v}'=\mathbf{0}}+D\nabla'\cdot\overleftrightarrow{\mathbf{E}}\Big|_{\mathbf{v}'=\mathbf{0}}, \tag{A9}$$

where $\nabla'=\partial/\partial\mathbf{v}'$. From Eq. (A7), the tensor $\overleftrightarrow{\mathbf{E}}$ has no component parallel to $\mathbf{v}$ and the first term on the right-hand side of Eq. (A9) vanishes. Using vector algebra, the second term is equal to

$$\nabla'\cdot(D\overleftrightarrow{\mathbf{E}})\Big|_{\mathbf{v}'=\mathbf{0}}=D\nabla'\cdot\overleftrightarrow{\mathbf{E}}\Big|_{\mathbf{v}'=\mathbf{0}}=2\mathbf{v}/v^3. \tag{A10}$$

Finally, for the second-order terms in $\mathbf{v}'$, one also needs

$$\nabla'^2(D\overleftrightarrow{\mathbf{E}})\Big|_{\mathbf{v}'=\mathbf{0}}=(\nabla'^2D)\overleftrightarrow{\mathbf{E}}\Big|_{\mathbf{v}'=\mathbf{0}}+2\nabla' D\cdot\nabla'\overleftrightarrow{\mathbf{E}}\Big|_{\mathbf{v}'=\mathbf{0}}+D\nabla'^2\overleftrightarrow{\mathbf{E}}\Big|_{\mathbf{v}'=\mathbf{0}}. \tag{A11}$$

After performing some vector algebra, one finds that

$$\nabla'^2(D\overleftrightarrow{\mathbf{E}})\Big|_{\mathbf{v}'=\mathbf{0}}=\frac{2}{v^2}(v^4-1)\frac{v^2\overleftrightarrow{\mathbf{I}}-\mathbf{vv}}{v^3}+\frac{4\mathbf{vv}}{v^5}. \tag{A12}$$

Returning to Eq. (A1), with the kernel now approximated by the Taylor expansion around $\mathbf{v}'=\mathbf{0}$, one obtains

$$C(f)\approx\frac{\Gamma}{2n_e}\frac{\partial}{\partial\mathbf{p}}\cdot\left[\int\left(\overleftrightarrow{\mathbf{U}}^{(0)}+\frac{1}{2}\frac{\partial^2\overleftrightarrow{\mathbf{U}}^{(0)}}{\partial\mathbf{v}'\,\partial\mathbf{v}'}:\mathbf{v}'\mathbf{v}'\right)\times f'd^3p'\cdot\frac{\partial f}{\partial\mathbf{p}}\;-\;f\int\frac{\partial\overleftrightarrow{\mathbf{U}}^{(0)}}{\partial\mathbf{v}'}\cdot\mathbf{v}'\frac{\partial f'}{\partial\mathbf{p}'}d^3p'\right]. \tag{A13}$$

Here, the normalization is given by

$$\int f' d^3p' = n_e \tag{A14}$$

and, since f' is isotropic (i.e., only diagonal elements contribute), one obtains

$$\int \mathbf{v}'\mathbf{v}' f' d^3p' = n_e v_t^2 \overleftrightarrow{\mathbf{I}}, \tag{A15}$$

with

$$v_t^2 = \frac{1}{3n_e}\int v'^2 f' d^3p', \tag{A16}$$

and

$$\int \mathbf{v}' \frac{\partial f'}{\partial \mathbf{p}'} d^3p' = -\int \frac{c}{T_e}\mathbf{v}'\mathbf{v}' f' d^3p' = -\frac{n_e c v_t^2}{T_e}\overleftrightarrow{\mathbf{I}}. \tag{A17}$$

After gyro-averaging, one can introduce the pitch-angle operator

$$\mathcal{L}(f) = \frac{1}{2}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu},$$

where $\mu = v_\parallel / v$ is the cosine of the pitch angle, and use the Taylor coefficients, Eqs. (A8), (A10), and (A12), so that the collision operator can be written in the form

$$C(f) = \frac{1}{p^2}\frac{\partial}{\partial p}p^2\left(A(p)\frac{\partial f}{\partial p} + F(p) f\right) + \frac{2B(p)}{p^2}\mathcal{L}(f), \tag{A18}$$

with

$$B(p) = \frac{\Gamma p^2}{2\gamma^2 m_e^2 c^3 v^3}\left(1 + v_t^2 \frac{v^4 - 1}{v^2}\right) = \frac{\Gamma}{2cv}\left(1 + v_t^2 \frac{v^4 - 1}{v^2}\right), \tag{A19}$$

$$A(p) = \frac{\Gamma m_e^3 c^2 \gamma^3}{p^3} = \frac{\Gamma v_t^2}{cv^3}, \tag{A20}$$

$$F(p) = \frac{\Gamma m_e^2 v_t^2 c^2}{T_e}\frac{\gamma^2}{p^2} = \frac{\Gamma v_t^2}{T_e v^2} = \frac{cv}{T_e}A(p). \tag{A21}$$

This form of the collision operator is in agreement with Karney and Fisch,[11] including Eq. (A19) for $B(p)$, which was obtained by them with a symbolic algebra package. It is useful to write $C(f)$ in the notation of Connor and Hastie,[2] i.e., in terms of the normalized momentum

$$q = \frac{p}{m_e c} = \gamma v = \frac{v}{\sqrt{1-v^2}} \Leftrightarrow v = \frac{q}{\sqrt{1+q^2}} \tag{A22}$$

and the relativistic collision time

$$\tau^{-1} = \frac{n_e e^4 \ln\Lambda}{4\pi\epsilon_0^2 m_e^2 c^3} = \frac{\Gamma}{m_e^2 c^3}. \tag{A23}$$

By writing the functions in (A19)–(A21) in terms of normalized momentum q, and using $v_t^2 \approx T_e/m_e c^2 = \epsilon$, one arrives at the final expression for the collision operator that manifestly conserves the number of particles,

$$C(f) = \frac{\sqrt{1+q^2}}{\tau q^3}\left[1 + Z - \epsilon\frac{1+2q^2}{q^2(1+q^2)}\right]\mathcal{L}(f) + \frac{1}{\tau q^2}\frac{\partial}{\partial q}\left[(1+q^2)f + \epsilon\frac{(1+q^2)^{3/2}}{q}\frac{\partial f}{\partial q}\right]. \tag{A24}$$

The effects of ions, with an effective charge number Z and assumed to be infinitely massive (i.e., contributing only to the pitch-angle scattering term), have also been included.

APPENDIX B: NUMERICAL SOLUTION OF REGION II

The numerical scheme for calculating the first-order correction F' in Eq. (14) is outlined here.

Instead of calculating F' directly, $h = \exp(F')$ is introduced as in Sec. III B. Equation (14) is transformed into a linear one, Eq. (17), which suggests a decomposition as in Eq. (18). The numerical scheme is obtained by using this decomposition in Eq. (17). Using recursion relations and the orthogonality property of the Legendre polynomials $P_n(\mu)$, one obtains for $g_n(u)$ the equation

$$u g_n' + \frac{1+Z}{2}n(n+1)g_n = \alpha u^4\left(\frac{n}{2n-1}g_{n-1} + \frac{n+1}{2n+3}g_{n+1}\right). \tag{B1}$$

In order to solve this equation as an initial value problem, with u acting as a time variable, an initial condition is needed. Choosing $C_{II} = C_I$, the initial condition becomes $h(0,\mu) = 1$, which in terms of the decomposition can be written

$$g_0(0) = 1,$$

$$g_i(0) = 0, \quad i = 1,2,\ldots .$$

Equation (B1) is discretized according to

$$u^{(j)}\frac{g_n^{(j)} - g_n^{(j-1)}}{\Delta u} + \frac{1+Z}{2}n(n+1)g_n^{(j)} = \alpha u^4\left(\frac{n}{2n-1}g_{n-1}^{(j)} + \frac{n+1}{2n+3}g_{n+1}^{(j)}\right), \tag{B2}$$

where $\bullet^{(j)}$ denotes quantities at time step j. This equation couples different g_n's. However, for a small step size Δu, the decoupled equation

$$\frac{u^{(j)}}{\Delta u}g_n^{(j)} + \frac{1+Z}{2}n(n+1)g_n^{(j)} = \frac{u^{(j)}}{\Delta u}g_n^{(j-1)} + \alpha u^4\left(\frac{n}{2n-1}g_{n-1}^{(j-1)} + \frac{n+1}{2n+3}g_{n+1}^{(j-1)}\right) \tag{B3}$$

can be used for the numerical scheme. The justification for this decoupling is easily verified by taking many g_n's (so that the last ones do not contribute, i.e., remain zero) and noting that an even smaller step size Δu leaves the set of g_n unchanged. $\Delta u = 10^{-5}$ has been used in the present paper.

As was stated in Sec. III B, region II ends where $u^2 \sim \epsilon^{-1/2}$. However, since the high momenta asymptote of the numerical solution has to coincide with the analytical

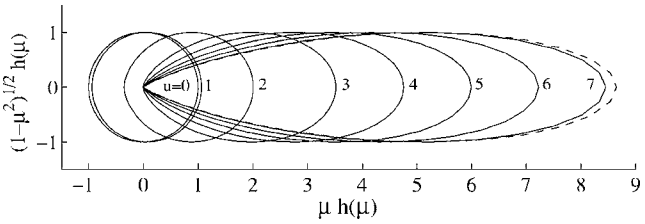

FIG. 2. A polar plot of normalized function $h(u,\mu)$ for normalized momenta $u=[0,1,2,3,4,5,6,7]$ (solid lines) and the high momentum analytic asymptote, Eq. (15), for $u=7$ (broken line). The plasma parameters are $\alpha=0.9$, $Z=2.1$. The normalized magnitude of $h(u,\mu)$ in a direction ($\mu=\cos\theta$, $\sqrt{1-\mu^2}=\sin\theta$) is given by the distance between the origin and the plotted line. The plot shows how the low momenta solutions are isotropic as compared to the high directionality of the high momenta solutions and the analytic asymptote.

solution, Eq. (16), the numerical solution was pushed to high values of u. The requirement that $h(u,\mu)>0$ for any μ broke down for $u\gtrsim 2.5$, but this numerical error was still only of the order of 10^{-4} for $u=7$. Hence, the matching constant b could be determined by matching the high momentum asymptote, Eq. (15), to the numerical solution at $u=7$—see Fig. 1.

In order to illustrate the effect on the solution of the anisotropic term proportional to μu^4, it is useful to plot the directionality of $h(u,\mu)$ for different momenta. The results are presented in Fig. 2. For momenta below unity in the weakly relativistic region (region II), $h(u,\mu)$ is still nearly isotropic in μ space. However, for u above unity, the solution becomes strongly peaked in the direction of electron acceleration and approaches the high momenta analytic asymptote.

The numerical scheme presented here can be used for general plasma parameters and the matching constant b could be determined for different scenarios. With this matching constant determined, the form of the electron distribution function up to the relativistic region, region III, is accurately described.

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