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Resonant magnetohydrodynamic modes with toroidal coupling. Part I: Tearing modes

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In a cylindrical plasma, tearing modes can be calculated by asymptotic matching of ideal magnetohydrodynamic (MHD) solutions across a critical layer. This requires a quantity Δ' that represents the "discontinuity" in the ideal solution across the layer. In a torus, poloidal harmonics are coupled and there are many critical surfaces for each toroidal mode number, and correspondingly many discontinuities Δ'_m. The ideal MHD solutions do not then determine the Δ'_m but only a relation between them—described by an "E matrix." The calculation of the E matrix for a large-aspect-ratio tokamak is discussed. In a weak-coupling approximation, it is tridiagonal and can be computed from integrals over the uncoupled eigenfunctions or from simple "basis functions" comprising triplets of coupled poloidal harmonics. This weak-coupling approximation fails if Δ'_m is already small for an uncoupled harmonic. An alternative strong-coupling approximation is developed for this case.

I. INTRODUCTION

In the cylindrical limit of a circular cross-section tokamak, linear perturbations can be described by independent Fourier harmonics with poloidal and toroidal mode numbers m and n. Then, if $q(r)$ is the "safety factor," a singularity of the ideal magnetohydrodynamic (MHD) equations occurs at $nq(r_0) = m$, where the perturbation is resonant with the field line rotation. There may then be instabilities in which nonideal effects (such as resistivity) are important only in a critical layer[1] around r_0; elsewhere, the perturbation is still described by ideal MHD. These instabilities can be calculated by asymptotic matching of a solution of the full equations in the critical layer to solutions of the marginal ideal MHD equations elsewhere. The MHD solutions satisfy the boundary conditions at the magnetic axis $r = 0$ and at the plasma boundary $r = a$.

The marginal ideal MHD solutions can be represented near $r = r_0$ as

$$\psi_{L,R} = A_{L,R}\left(|x|^{\nu} + \Delta_{L,R}|x|^{\nu_+}\right), \tag{1}$$

where ψ is the perturbed radial magnetic field, $x = r - r_0$ and L,R refer to left and right of the critical layer. The indices ν relate to the "large" and "small" solutions in the sense of Newcomb[2] and

$$\nu_{\pm} = \tfrac{1}{2} \pm (\tfrac{1}{4} + D)^{1/2}. \tag{2}$$

The Mercier[3] stability criterion is $(D + \frac{1}{4}) > 0$. At zero pressure, $\nu_+ = 1$, $\nu_- = 0$, and $\Delta_{L,R}$ are then related to the logarithmic derivative of ψ as $|x| \to 0$.

The solutions inside a symmetric critical layer have even (tearing) or odd (twisting) parity and asymptotically can be written

$$\psi_{\pm} = |X|^{\nu_-} + \Delta^{\pm}(\omega)|X|^{\nu_+} \tag{3}$$

(where X is a "stretched" coordinate x/σ with σ being the layer width). The quantities $\Delta^{\pm}(\omega)$ depend on the details of the plasma model used in the layer. Matching Eq. (3) to Eq. (1) yields[4]

$$\Delta^+(\omega)\Delta^-(\omega) - (\sigma^{\nu_+ - \nu_-}/2)(\Delta_R + \Delta_L)[\Delta^+(\omega) + \Delta^-(\omega)] + \sigma^{2(\nu_+ - \nu_-)}\Delta_R\Delta_L = 0. \tag{4}$$

If we suppose that $\Delta^+(\omega)$ and $\Delta^-(\omega)$ do not vanish simultaneously (as a function of ω), then there are two distinct solutions to Eq. (4) as $\sigma \to 0$. In one of these, $\Delta^-(\omega) \ll \Delta^+(\omega)$ and the eigenvalue is determined by

$$\Delta^-(\omega) = \sigma^{\nu_+ - \nu_-}\Delta', \tag{5}$$

with $\Delta' = (\Delta_R + \Delta_L)/2$. This is the "tearing mode." In the other solution, $\Delta^+(\omega) \ll \Delta^-(\omega)$ and

$$\Delta^+(\omega) = \sigma^{\nu_+ - \nu_-}\Delta'. \tag{6}$$

This is the "twisting mode." Note that Δ' embodies all the information needed from the ideal MHD solution. Henceforth we will refer to Δ' simply as the "discontinuity" in the MHD solution and if $\Delta' = 0$, we will refer to the solution as "continuous."

In an axisymmetric torus, the individual poloidal harmonics m are coupled together, although toroidal harmonics n remain independent. Consequently, for each toroidal mode number n there can be many critical surfaces [wherever $nq(r) = \text{integer}$]. Except in the vicinity of these critical surfaces, the perturbation is still given by ideal MHD equations. Our objective is to describe a quantity that summarizes all the information required from the ideal MHD equations in a torus, in the same way that Δ' summarizes all the information necessary in a cylinder.

In the present paper, we consider only the toroidal tearing mode. This will be seen to be a natural extension of the cylinder tearing mode. In Part II,[5] we will examine the toroidal twisting mode. Unlike the tearing mode, this is an *intrinsically* toroidal perturbation, not directly related to the cylinder twisting mode.

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The toroidal tearing mode problem was first addressed by Grimm *et al.*[6] who constructed a 2-D computer code to calculate the "outer region" solutions. In a later paper, Connor *et al.*[7] described the MHD solutions by a set of M poloidal harmonics. Then if there are N resonant surfaces among these harmonics, a set of $(M+N)$ basis functions (solutions of the marginal MHD equation each consisting of M harmonics) was introduced. These consist of M solutions regular at $r=0$ and continuous (i.e., no change in large or small solution) across the critical surfaces, and N solutions constructed by starting with (only) the small solution at each critical surface and continued outward (continuously) to $r=a$. The full solution is then written as a linear combination of these $(M+N)$ basis functions, with M coefficients α_i and N coefficients α_i'. By construction, this satisfies the boundary condition at $r=0$. The requirement that it also satisfy the boundary condition at $r=a$ yields M conditions through which the M coefficients α_i can be expressed in terms of the N coefficients α_i'. The introduction of N quantities Δ_m describing the discontinuity in the small component (relative to the large component) at each resonant surface then leads to a solubility condition that can be written

$$|E-\Delta|=0, \tag{7}$$

where the E matrix is calculated entirely from the values of the basis functions at the critical surfaces and at $r=a$ (i.e., from ideal MHD) and $\Delta=$ diagonal $\{\Delta_m\}$. The dispersion equation for the toroidal tearing mode frequency follows by writing $\Delta_m=\Delta_m(\omega)$. Clearly, the E matrix is a quantity, corresponding to Δ' in the cylinder, that summarizes all the information needed from the ideal MHD solution in order to determine an eigenvalue of the full problem. (Strictly, the matrix E described here is the inverse of that in Ref. 7.)

In the rest of this paper, we discuss the calculation of the E matrix for a large-aspect-ratio tokamak. The basic equations are briefly described in Sec. II. Then, in Sec. III, we show that a systematic treatment in powers of ϵ leads to somewhat different basis functions to those of Ref. 7. These consist only of triplets of harmonics irrespective of how many resonant surfaces there may be. Each triplet consists of a central harmonic with a discontinuity in its small solution at its resonant surface plus small sidebands that are continuous at their resonant surfaces. Furthermore, unlike the basis functions of Ref. 7, each triplet individually satisfies the boundary conditions at *both* $r=0$ and $r=a$. With these basis functions, the matrix E has a simple tridiagonal form. This development is formally a weak-coupling theory with coupling parameter ϵ/Δ_m^0 (where Δ_m^0 is the discontinuity in the mth harmonic in the absence of coupling).

"Strong coupling" occurs, even when $\epsilon \ll 1$, if one or more of the Δ_m^0 are small; this situation is discussed in Sec. IV. There we find that the most convenient description is in terms of overlap integrals between single harmonics rather than in terms of basis functions. This development also leads to a tridiagonal E matrix. An alternative description of the strong coupling situation, using basis functions, is also discussed in Sec. IV. In this case, the basis functions are multiplets of coupled harmonics (not generally just triplets) and they do not lead to a tridiagonal E matrix.

Finally, in Sec. V we mention some consequences of the E matrix for the toroidal tearing mode frequency and structure. A fuller discussion of this will be given elsewhere.

II. MARGINAL IDEAL MHD EQUATIONS

The marginal ideal MHD equations were derived in Appendix B of Ref. 7 and are summarized here. The coordinate system r,θ,ϕ (where ϕ is the toroidal angle, θ is an anglelike poloidal coordinate, and r is a flux-surface label) is chosen so that the magnetic field lines are straight and the Jacobian is $J=R^2 r/R_0$. The axisymmetric field can be written as

$$\mathbf{B}=B_0 R_0[\,f(r)\nabla\phi\times\nabla r+g(r)\nabla\phi\,] \tag{8}$$

and the safety factor is $q=rg(r)/R_0 f(r)$.

The linearized marginal MHD equations for a perturbation with toroidal mode number n can be expressed in the form

$$\frac{d}{dr}[\,(m-nq)y_m\,]=\sum_k (B^k_m z_k + C^k_m y_k), \tag{9}$$

$$(m-nq)\frac{dz_m}{dr}=\sum_k (D^k_m z_k + E^k_m y_k), \tag{10}$$

where $y=R_0 f\boldsymbol{\xi}\cdot\nabla r$ is effectively the radial component of the displacement $\boldsymbol{\xi}$ and $z=R^2\delta\mathbf{B}\cdot\nabla\phi/B_0$ is effectively the perturbed toroidal magnetic field. [Note that $\psi_m=(m-nq)y_m$.]

III. WEAK-COUPLING THEORY

In a large aspect ratio tokamak with circular cross section and $\beta\sim\epsilon^2$, the coefficients B^k_m, etc. are $\sim\epsilon^{|k-m|}$ and, in particular, the coupling between neighboring harmonics $m,m\pm1$, is $\sim\epsilon$. In order to develop a systematic expansion correct to $O(\epsilon^2)$, it is necessary to retain diagonal coefficients to order ϵ^2 and off-diagonal coefficients to order ϵ. The coefficients that remain can then all be expressed in terms of the Shafranov shift, the pressure parameter $\alpha=-(2R_0q^2/B_0^2)dp/dr$ and the shear parameter $s=rd(\log q)/dr$. Finally, we then have as our basic equations correct to $O(\epsilon^2)$

$$r\frac{d}{dr}[\,(m-nq)y_m\,]=L^m_m\hat{z}_m+\sum_{\pm}(L^{m\pm1}_m\hat{z}_{m\pm1}+M^{m\pm1}_m y_{m\pm1}), \tag{11}$$

$$r(m-nq)\frac{d\hat{z}_m}{dr}=P^m_m y_m+\sum_{\pm}(N^{m\pm1}_m\hat{z}_{m\pm1}+P^{m\pm1}_m y_{m\pm1}), \tag{12}$$

where we have also introduced $\hat{z}_m=z_m+(C^m_m/B^m_m)y_m$. The coefficients L,M,N,P are given in Appendix B of this paper.

If all coupling between different harmonics is ignored, Eqs. (11) and (12) reduce to

$$\mathscr{L}_m y_m\equiv r(m-nq)\frac{d}{dr}\frac{r}{L^m_m}\frac{d}{dr}(m-nq)y_m - P^m_m y_m=0 \tag{13}$$

and if we also retained only the zeroth-order (in ϵ) parts of P^m_m and L^m_m, this would be the usual cylinder equation

$$\left(\frac{d}{dr} r \frac{d}{dr} (m-nq) y_m - \frac{m^2}{r} (m-nq) y_m - mq\sigma' y_m\right) = 0, \tag{14}$$

with $\sigma' = d\sigma/dr$, where

$$\sigma = \frac{Rj_\parallel}{B} = \frac{1}{r} \frac{d}{dr} \left(\frac{r^2}{q}\right). \tag{15}$$

Now, Eq. (14) has Mercier indices $\nu_+ = 1$, $\nu_- = 0$, but the correct indices are

$$\nu_\pm = \tfrac{1}{2} \pm (\tfrac{1}{4} + D)^{1/2}, \quad D = (2rp'/s^2 B_0^2)(1-q^2). \tag{16}$$

Consequently, Eq. (14) cannot be used as the zeroth-order starting point for a development in powers of ϵ. To avoid singularities in higher order we must include the most singular part of P_m^m (which controls the indices $\nu_\pm$) in the lowest-order equation, even though it is formally small in ϵ. Then the lowest-order equation becomes

$$\begin{aligned} \mathscr{L}_m^{(0)} y_m \equiv (m-nq)\Big(&\frac{d}{dr} r \frac{d}{dr} (m-nq) y_m \\ &- \frac{m^2}{r} (m-nq) y_m - mq\sigma' y_m\Big) \\ &- \frac{2p'}{B_0^2} (1-q^2) m^2 y_m = 0, \end{aligned} \tag{17}$$

which has the correct Mercier indices.

As the starting point of our development in powers of ϵ, we take a particular solution $y_m^{(0)}(r)$ of Eq. (17), which satisfies the boundary conditions at $r=0$ and $r=a$ and has a discontinuity Δ_m^0 in its small solution across the singularity at $nq(r) = m$. Then, returning to Eqs. (11) and (12), one can see that $y_m^{(0)}$ induces sidebands $m \pm 1$ that are given by

$$\begin{aligned} \mathscr{L}_{m\pm1}^{(0)} y_{m\pm1} = &(N_{m\pm1}^m \hat{z}_m^{(0)} + P_{m\pm1}^m y_m^{(0)}) \frac{(m\pm1)^2}{r} \\ &+ (m \pm 1 - nq) \frac{d}{dr} (L_{m\pm1}^m \hat{z}_m^{(0)} \\ &+ M_{m\pm1}^m y_m^{(0)}), \end{aligned} \tag{18}$$

where $\hat{z}_m^{(0)} = (r/m^2)(d/dr)[(m-nq)y_m^{(0)}]$, and we have used the result $L_m^{m(0)} \equiv m^2$.

As the first-order contribution, we take solutions $y_{m\pm1}^{(1)}$ of Eq. (18) that satisfy the boundary conditions at $r=0$ and $r=a$ and that are *continuous* (in the sense described in the Introduction) across the singularities at $nq = m \pm 1$.

We also require the $O(\epsilon^2)$ contribution to y_m. This is given by

$$\begin{aligned} \mathscr{L}_m^{(0)} y_m^{(2)} = &\frac{m^2}{r} P_m^{m(2)} y_m^{(0)} - \frac{L_m^{m(2)}}{m^2} \mathscr{L}_m^{(0)} y_m^{(0)} + (m-nq) \\ &\times \frac{d}{dr}\left(\frac{L_m^{m(2)}}{L_m^{m(0)}}\right) r \frac{d}{dr} (m-nq) y_m^{(0)} + (m-nq) \\ &\times \frac{d}{dr} \sum_\pm (L_m^{m\pm1} \hat{z}_{m\pm1}^{(1)} + M_m^{m\pm1} y_{m\pm1}^{(1)}) + \frac{m^2}{r} \\ &\times \sum_\pm (N_m^{m\pm1} \hat{z}_{m\pm1}^{(1)} + P_m^{m\pm1} y_{m\pm1}^{(1)}) \end{aligned} \tag{19}$$

and we again take the continuous solution satisfying both boundary conditions.

Thus we have constructed a particular solution to the MHD equations consisting of a triplet of harmonics $[y_{m-1}^{(1)}, (y_m^{(0)} + y_m^{(2)}), y_{m+1}^{(1)}]$ all of which satisfy both boundary conditions. The central harmonic has a discontinuity in its small component at its singular surface and is correct to order ϵ^2. The sidebands are continuous at their singular surfaces and correct to order ϵ. [It is important that this triplet satisfies both boundary conditions. If, like the basis functions of Ref. 7, it satisfied only that at $r=0$, then instead of $y_{m\pm1}^{(1)}$ being $O(\epsilon)$, it would be $O(m\epsilon)$, and the theory could not accommodate large m.]

The triplet described above may be computed by integrating the coupled equations for three adjacent coupled harmonics (ignoring coupling to all other harmonics) from $r=0$ to $r=a$. A discontinuity Δ_m^0 in the central harmonic at its critical surface, and the three initial boundary conditions at $r=0$, are iterated until all three harmonics satisfy the boundary conditions at $r=a$. A computer code for constructing triplets in this way has been written and will be reported on elsewhere.

Note that a triplet does *not* itself describe a general tearing mode; this requires a solution with a discontinuity Δ_m in every harmonic. Nevertheless, a set of these triplets forms a basis for constructing a general tearing mode.

IV. THE *E* MATRIX

A general tearing mode can be constructed from a superposition of the triplets defined in the preceding section. As we will need to identify both harmonic and triplet we now write the triplet as

$$Y_m = \{y_{m,m-1}^{(1)}, y_{m,m}^{(0)} + y_{mm}^{(2)}, y_{m,m+1}^{(1)}\} \tag{20}$$

(the first subscript identifies the triplet, the second the harmonic). In the superposition

$$Y = \sum_m \alpha_m Y_m, \tag{21}$$

three triplets contribute to any harmonic m and the discontinuity Δ_m in its small component at the singular surface $[nq(r_m) = m]$ is given by

$$\begin{aligned} \Delta_m [&\alpha_{m-1} C_{m-1,m}^{(1)} + \alpha_m (C_{m,m}^{(0)} + C_{mm}^{(2)}) \\ &+ \alpha_{m+1} C_{m+1,m}^{(1)}] \\ &= \Delta_m^0 \alpha_m C_{m,m}^{(0)}, \end{aligned} \tag{22}$$

where the $C_{j,m}$ are the coefficients of the large component of the corresponding $y_{j,m}$ at the resonance r_m. This recurrence relation leads to a solubility condition

$$|F - \Delta^{-1}| = 0, \tag{23}$$

where Δ = diagonal $\{\Delta_m\}$ and F is tridiagonal. Bearing in mind that F is calculated only as an expansion in ϵ, Eq. (23) can equally be written

$$|E - \Delta| = 0, \tag{24}$$

where E is also tridiagonal. The elements of E are given by

$$E_{mm} = \Delta_m^0 [(C_{m,m}^{(0)} - C_{m,m}^{(2)})/C_{m,m}^{(0)}], \tag{25}$$

$$E_{m,m\pm1} = -\Delta_m^0 C^{(1)}_{m\pm1,m}/C^{(0)}_{m,m}. \tag{26}$$

These elements can be expressed entirely in terms of the zeroth-order function $y_m^{(0)}$. If Eq. (18) is multiplied by $y^{(0)}_{m\pm1}$ and the left-hand side integrated by parts, one obtains an expression for $C^{(1)}_{m\pm1,m}$

$$\Delta_m^{(0)}(C^{(1)}_{m\pm1,m}/C^{(0)}_{m,m}) = I_{m,m\pm1}/(\nu_+ - \nu_-)_{r_m}, \tag{27}$$

where I is a bilinear integral of $y^{(0)}_{m\pm1}$ and $y_m^{(0)}$ (of order ϵ) defined in Appendix B. A similar procedure applied to Eq. (19) allows one to express $C^{(2)}_{mm}$ in terms of integrals of $y_m^{(0)}$ and $y^{(0)}_{m\pm1}$. Another derivation of the E matrix is given in Appendix A.

V. STRONG-COUPLING THEORY

The expressions for $C^{(1)}$ and $C^{(2)}$ in the preceding section highlight the fact that the coupling parameter is really ϵ/Δ_m^0. Consequently, even when $\epsilon \ll 1$, the coupling becomes large if Δ_m^0 is small. A different treatment is then required.

We write the underlying equations in the form

$$\hat{\mathscr{L}}_m \psi_m = \sum_{\pm} K_m^{m\pm1}\psi_{m\pm1}, \tag{28}$$

where $\hat{\mathscr{L}}$ and K are defined through Eqs. (11) and (12) (see Appendix A) and $\psi_m = (m - nq)y_m$ is the mth harmonic of the perturbed radial magnetic field. We again start with the uncoupled function ψ_m^0, satisfying $\hat{\mathscr{L}}_m\psi_m = 0$ and the boundary conditions, with a discontinuity Δ_m^0 in its small component at its singularity. We then introduce a set of *radial* harmonics of the uncoupled equation, such that

$$(\hat{\mathscr{L}}_m + \lambda_m^p)\psi_m^p = 0 \tag{29}$$

and ψ_m^p has the *same* discontinuity Δ_m^0 and satisfies the same boundary conditions as ψ_m^0. Assuming that the ψ_m^p (which are orthogonal in p) form a complete set, we can expand ψ_m in them and obtain

$$\psi_m = \sum_{p,\pm} \frac{\psi_m^p \langle \psi_m^p | K_m^{m\pm1} | \psi_{m\pm1}\rangle}{(\bar{\lambda}_m^p + \bar{\Delta}_m)(C_m^p)^2(\nu_+ - \nu_-)_{r_m}}, \tag{30}$$

where $\bar{\Delta}_m = \Delta_m - \Delta_m^0$, C_m^p is the coefficient of the large component of ψ_m^p at its singularity, and

$$\bar{\lambda}_m^p = \lambda_m^p \langle (\psi_m^p)^2\rangle/(C_m^p)^2. \tag{31}$$

Now $|K_m^{m\pm1}|$ is a small quantity, of order ϵ, while the $\bar{\lambda}_m^p$ $(p \neq 0)$ are of order unity. Consequently, for each m we need only retain the $p = 0$ term in Eq. (30) (unless $\bar{\Delta}_m$ were also of order unity, but then ψ_m is, itself, negligible). Then we have, in the lowest approximation,

$$\psi_m = \alpha_m \psi_m^0. \tag{32}$$

Inserting this in Eq. (30) again yields a three-term recurrence relation

$$\alpha_m(\nu_+ - \nu_-)_{r_m}(\Delta_m^0 - \Delta_m) = \sum_{\pm} I_{m,m\pm1}\alpha_{m\pm1} \tag{33}$$

and a solubility condition

$$|E - \Delta| = 0, \tag{34}$$

with

$$E_{m,m\pm1} = -I_{m,m\pm1}/(\nu_+ - \nu_-)_{r_m} = -\frac{\langle \psi_m^0 | K_m^{m\pm1} | \psi_{m\pm1}^0\rangle}{[(C_m^0)^2(\nu_+ - \nu_-)_{r_m}]}, \tag{35}$$

$$E_{m,m} = \Delta_m^0.$$

Thus the E matrix for strong coupling is again tridiagonal and similar to that in weak-coupling theory. Indeed, as shown in Appendix A, the off-diagonal elements of the strong- and weak-coupling forms of E matrix are identical to $O(\epsilon)$. Nevertheless, the two forms are formally based on quite different approximations. The weak-coupling result depends on (ϵ/Δ_m^0) being small and includes order $(\epsilon/\Delta_m^0)^2$ contributions, whereas the strong-coupling result depends on (ϵ/λ_m^p) being small and includes only lowest-order contributions. One expects the strong-coupling form to be more generally applicable.

Although we have constructed the strong-coupling E matrix from overlap integrals, it can also be computed from basis functions $\psi^j(r)$. Thus one sets up a set of basis functions $\psi^j(r)$, each of which is a multiplet containing N-coupled harmonics (where N is the number of critical surfaces) launched from $r = 0$. In each multiplet, one harmonic (only) has a discontinuity Δ_j' at its critical surface r_j, the others are continuous. For each multiplet basis function, the discontinuity and the boundary conditions at $r = 0$ are iterated until all harmonics satisfy the boundary condition at $r = a$. When a tearing mode is constructed from these multiplet basis functions with coefficients α_j, the discontinuity Δ_m' at the mth singularity is given by

$$\Delta_m \sum_{j=1}^{N} \alpha_j \psi^j{}_m(r_m) = \Delta_m' \alpha_m \psi_m^m(r_m), \tag{36}$$

with a solubility condition

$$|F - \Delta^{-1}| = 0. \tag{37}$$

In this case, F need not be tridiagonal, since each multiplet basis function may contain several "strong" harmonics if several Δ_m^0 are small.

Returning to Eq. (34), it is clear that a self-consistent solution exists when $(\Delta_m - \Delta_m^0)$ is order ϵ for all relevant m. However, a self-consistent solution also exists if $(\Delta_m - \Delta_m^0)$ is order ϵ^2 for a single m and $O(1)$ for the remainder. This special case can be computed using just three basis functions, each of which contains the same triplet of coupled harmonics, centered on the harmonic with near vanishing $\bar{\Delta}_m$. In each triplet, one (only) harmonic has a discontinuity at its critical surface. This discontinuity and the boundary conditions at $r = 0$ are iterated until the triplet satisfies the boundary conditions at $r = a$.

VI. SUMMARY AND CONCLUSIONS

A. The *E* matrix

In a cylinder, tearing modes can be calculated through an asymptotic matching procedure, in which the full plasma physics is needed only in a critical layer where $nq(r) = m$. Elsewhere, the plasma is described by marginal ideal MHD. At the critical surface, the ideal MHD solution allows a discontinuity Δ' in its small component, which is to be equated

to the corresponding $\Delta^-(\omega)$ from the solution to the equations with full physics in the layer. This gives the tearing mode dispersion equation $\Delta^-(\omega) = \Delta'$. Thus Δ' contains all the information needed from the ideal MHD "external" part of the problem.

In a torus, the various poloidal harmonics m are coupled, so that for each toroidal number n there are many critical surfaces $nq =$ integer and correspondingly many discontinuities Δ_m in the ideal MHD solutions. Each of the Δ_m must be matched to its corresponding $\Delta_m(\omega)$. However, the MHD equations do not specify any particular set of Δ_m, *only a relation between them*. This relation is given by the E matrix discussed in this paper. Just as the single quantity Δ' contains all the ideal MHD information needed for the dispersion equation in a cylinder, so the E matrix contains all the information needed in a torus.

We have discussed the structure and calculation of the E matrix in two large-aspect-ratio limits. In the weak-coupling theory, $\epsilon \ll \Delta_m^0$ (Δ_m^0 is the discontinuity for an uncoupled harmonic) and the coupling between adjacent harmonics is $O(\epsilon/\Delta_m^0)$. The E matrix is tridiagonal and can be calculated by perturbation or by computing a set of triplet basis functions. In each triplet, the central harmonic has a discontinuity at its critical surface but the sidebands are continuous. Note that only triplets of coupled harmonics need be computed irrespective of the number of critical surfaces. The number of triplets required equals the number of critical surfaces.

However, even at large aspect ratio ($\epsilon \ll 1$), the weak-coupling approximation breaks down when Δ_m^0 is small. An alternative strong-coupling approximation requires that $\epsilon \ll \lambda_m^p$, where λ_m^p is an eigenvalue associated with a higher *radial* mode of the uncoupled mth poloidal harmonic—and is of order unity. This approximation again leads to a tridiagonal form of E matrix when this is computed from overlap integrals of uncoupled harmonics. It may also be computed from multiplet basis functions but it is then not necessarily in tridiagonal form. A special case of the strong-coupling approximation involves only three harmonics and can be computed from triplet basis functions as in the weak-coupling approximation.

B. The dispersion equation

An important feature of the asymptotic approach to tearing modes is that the E matrix calculated from ideal MHD alone, can be used in conjunction with many different models for the critical layers. Each such application must be discussed individually, but some general observations may be made.

The dispersion equation is obtained by substituting $\Delta_m^-(\omega)$ from the layer model for the Δ_m in the E matrix:

$$D(\omega) = |E - \Delta_m^-(\omega)| = 0. \tag{38}$$

Now, in both weak and strong coupling approximations, the off-diagonal elements of the E matrix are proportional to a small coupling parameter. Consequently, the toroidal dispersion equation (38) will have solutions only when at least one of the diagonal elements is small. There seem to be two generic cases: Either $(\Delta_m - \Delta_m^0)$ may be $O(\epsilon)$ for several harmonics, or it may be $O(\epsilon^2)$ for one harmonic and $O(1)$ for the remainder. In either event, the frequency of a toroidal tearing mode will be close to that of an uncoupled harmonic. Thus the toroidal tearing mode is a natural extension of the cylinder tearing mode. (However, this is not the case for the twisting mode that we discuss in Part II.[5] The toroidal twisting modes are *intrinsically* toroidal and are essentially unrelated to the cylinder modes.)

As far as mode structure is concerned, if $[\Delta_m^0 - \Delta_m^-(\omega)]$ is small for only one harmonic, then the toroidal eigenmode will comprise only that principal harmonic and weak sidebands. If $[\Delta_m^0 - \Delta_m^-(\omega)]$ is simultaneously small for several harmonics, then the toroidal eigenmodes will consist of strong admixture of these harmonics as well as many weaker sidebands.

Two important models for plasma within the critical layer are the low-beta resistive MHD and the Rutherford[8] nonlinear resistive MHD models. For these models, $\Delta_m^-(\omega)$ is proportional to some power of ω and vanishes at marginal stability for all harmonics simultaneously. In the cylindrical case, marginal stability is then described by $\Delta' = 0$ and stability by $\Delta' > 0$. In the same way, marginal stability of toroidal tearing modes, for these models, is given by $|E| = 0$ and stability by $\lambda_E < 0$, where λ_E is the greatest eigenvalue of the E matrix.

Another important layer model is that of a kinetic plasma with diamagnetic drifts.[9] Then $\Delta_m^-(\omega)$ will be *large* (in the reciprocal layer width δ), unless ω is close to the local diamagnetic frequency ω_*. In this case, the sidebands will be small both in the toroidicity ϵ and the layer width δ.

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APPENDIX A: FORMAL THEORY

The weak- and strong-coupling approximations can be considered as formal perturbation schemes. The basic coupled MHD equations can be written schematically as

$$(\hat{\mathscr{L}}_m + \hat{\Delta}_m)\psi_m = \sum_{\pm} K_m^{m\pm1}\psi_{m\pm1}, \tag{A1}$$

where $\hat{\Delta}_m$ represents both the magnitude of the discontinuity at the singular surface and an appropriate operator [for a pressureless plasma, this operator is simply $\delta(r - r_m)$]. Then the triplet introduced in Sec. III satisfies

$$(\hat{\mathscr{L}}_m + \Delta_m^T)\psi_m^T = \sum_{\pm} K_m^{m\pm1}\psi_{m\pm1}^T, \tag{A2a}$$

$$\hat{\mathscr{L}}_{m\pm1}\psi_{m\pm1}^T = K_{m\pm1}^m\psi_m^T \tag{A2b}$$

[note carefully the difference between Eqs. (A2) for a triplet, which refer only to a *single* m, and Eq. (A1), which refers to all m].

If we formally solve Eq. (A2b) for $\psi_{m\pm1}^T$, we can obtain an *uncoupled* equation for the central harmonic of the mth triplet, i.e.,

$$\left[\hat{\mathscr{L}}_m + \Delta_m^T - \sum_{\pm}\left(K_m^{m\pm1}\frac{1}{\hat{\mathscr{L}}_{m\pm1}}K_{m\pm1}^{m}\right)\right]\psi_m^T = 0. \tag{A3}$$

It is convenient to denote this as

$$H_m\psi_m^T = (\hat{\mathscr{L}}_m + \Delta_m^T - G_m)\psi_m^T = 0, \tag{A4}$$

then we can write the full set of Eq. (A1) (for all m) as

$$H_m\psi_m = (\Delta_m^T - \Delta_m - G_m)\psi_m + \sum_{\pm} K_m^{m\pm1}\psi_{m\pm1}. \tag{A5}$$

The right-hand side of Eq. (A5) is a perturbation of $O(\epsilon^2)$ (the off-diagonal term $\sim\epsilon$ is equivalent to a diagonal term $\sim\epsilon^2$). Thus we can write

$$\psi_m = \alpha_m\psi_m^T + \tilde{\psi}_m, \tag{A6}$$

where

$$H_m\tilde{\psi}_m = (\Delta_m^T - \Delta_m - G_m)\alpha_m\psi_m^T + \sum_{\pm} K_m^{m\pm1}\alpha_{m\pm1}\psi_{m\pm1}^T. \tag{A7}$$

The left side of this equation can be annihilated in the usual way. Then

$$(\Delta_m^T - \Delta_m - \langle\psi_m^T|G_m|\psi_m^T\rangle)\alpha_m + \sum_{\pm}\langle\psi_m^T|K_m^{m\pm1}|\psi_{m\pm1}^T\rangle\alpha_{m\pm1} = 0 \tag{A8}$$

and the solubility condition is

$$|E - \Delta| = 0, \tag{A9}$$

with

$$E_{mm} = \Delta_m^T - \left\langle\psi_m^T\left|\sum K_m^{m\pm1}\frac{1}{\hat{\mathscr{L}}_{m\pm1}}K_{m\pm1}^{m}\right|\psi_m^T\right\rangle, \tag{A10}$$

$$E_{m,m\pm1} = \langle\psi_m^T|K_m^{m\pm1}|\psi_{m\pm1}^T\rangle,$$

which is equivalent to the weak-coupling results derived in Sec. III.

The strong-coupling approximation can be considered in a similar way. The basic equations are again represented by Eq. (A1). Now, however, we assume that several poloidal harmonics have Δ_m small in the cylinder limit and a zeroth-order solution is therefore constructed from a linear combination of these M "degenerate" harmonics. Thus ψ_m is written

$$\psi_m = \alpha_m\psi_m^0 + \tilde{\psi}_m. \tag{A11}$$

The coefficients α_m are undetermined in lowest order. When Eq. (A11) is inserted in Eq. (A1), the solubility condition for $\tilde{\psi}_m$ in first order yields

$$(\Delta_m^0 - \Delta_m)\alpha_m + \sum_{\pm}\langle\psi_m^0|K_m^{m\pm1}|\psi_{m\pm1}^0\rangle\alpha_{m\pm1} = 0, \tag{A12}$$

with solubility condition

$$|E - \Delta_m| = 0, \tag{A13}$$

where $E_{m,m} = \Delta_m^0$ and $E_{m,m\pm1} = \langle\psi_m^0|K_m^{m\pm1}|\psi_{m\pm1}^0\rangle$, which is equivalent to the results of Sec. V.

Note that in this strong-coupling approximation, the elements of E are described in terms of ψ_m^0 and the off-diagonal elements are $O(\epsilon)$ while the diagonal elements are $O(1)$. In the weak-coupling approximation, the off-diagonal elements are also $O(\epsilon)$, and, to this order, are identical with those of strong-coupling, but the diagonal elements contain contributions of both $O(1)$ and $O(\epsilon^2/\Delta_{m\pm1})$.

APPENDIX B: SOME DEFINITIONS

The coefficients L,M,N,P of Eqs. (11) and (12) are given by the following expressions:

$$L_m^m = m^2\left[1 + \frac{3}{2}\Delta_s'^2 - \frac{\Delta_s}{R} + \frac{r^2}{R^2}\left(\frac{n^2}{m^2} - \frac{3}{4}\right)\right], \tag{B1}$$

$$L_m^{m\pm1} = m(m\pm1)\Delta_s', \tag{B2}$$

$$M_m^m = N_m^m \equiv 0, \tag{B3}$$

$$M_m^{m\pm1} = \pm m\{(\alpha/2)(m - nq) + (m\pm1 - nq)\times[r/R - \Delta_s'(1-s)]\}, \tag{B4}$$

$$N_m^{m\pm1} = \pm(m\pm1)\{(\alpha/2)(m\pm1 - nq) + (m - nq)[r/R - \Delta_s'(1-s)]\}, \tag{B5}$$

$$P_m^m = (m - nq)^2 + (r\sigma')q\frac{(m-nq)}{m} + \frac{2rp'}{B_0^2}(1 - q^2) - \frac{2rp'}{B_0^2}\frac{(m-nq)}{m}(2 - s) - q\frac{(m-nq)}{m}r\frac{d}{dr}\left[\frac{r^2}{R^2q^3}(2-s) + \frac{s}{q}\left(\frac{3}{4}\frac{r^2}{R^2} - \frac{3}{2}\Delta_s'^2 + \frac{\Delta_s}{R}\right) - \frac{1}{q}\left(3\frac{r^2}{R^2} + 2\frac{r}{R}\Delta_s' - 2\Delta_s'^2 + 2\frac{\Delta_s}{R}\right)\right] + \frac{(m-nq)^2}{m^2}\left\{\frac{n}{m}r\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{r^2}{R^2q}\right)\right] - \left[\frac{d}{dr}\left(\frac{r^2}{Rq}\right)\right]^2 - r\frac{d}{dr}\left(\frac{rp'}{B_0^2}\right) + m^2\left(\frac{3}{2}\Delta_s'^2 + \frac{7}{4}\frac{r^2}{R^2} + 3\frac{r}{R}\Delta_s'\right)\right\} \tag{B6}$$

$$P_m^{m\pm1} = (\alpha/2)(1+s) + (m - nq)(m\pm1 - nq)(\Delta_s' + r/R), \tag{B7}$$

where $\alpha = -2Rp'q^2/B_0^2$, $s = rq'/q$, and Δ_s is the Shafranov shift of the magnetic axis.

The coupling integral $I_{m,m\pm1}$ that appears in Eq. (27) is a generalization to finite pressure equilibria of the coupling integrals discussed by Edery *et al.*[10] It is defined by

$$I_{m,m\pm1} = \left(\frac{m}{m\pm1}\right)\left(\frac{1}{C_m^2}\right)\mathscr{P}\int_0^a dr\left[m(m\pm1)\frac{\psi_m\psi_{m\pm1}}{r}\left(\Delta_s' + \frac{r}{R} + \frac{\alpha(1+s)}{2(m-nq)(m\pm1-nq)}\right)\right.$$
$$- r\Delta_s'\frac{d\psi_m}{dr}\frac{d\psi_{m\pm1}}{dr} \pm m\psi_m\frac{d\psi_{m\pm1}}{dr}\left(\Delta_s'(s-1) + \frac{r}{R} + \frac{\alpha}{2}\frac{(m\pm1-nq)}{(m-nq)}\right)$$
$$\left.\mp(m\pm1)\psi_{m\pm1}\frac{d\psi_m}{dr}\left(\Delta_s'(s-1) + \frac{r}{R} + \frac{\alpha}{2}\frac{(m-nq)}{(m\pm1-nq)}\right)\right], \tag{B8}$$

where the integration is to be interpreted as a principal part integration at each singular surface in the sense that

$$\mathscr{P}\int_0^a X\,dr \equiv \int_0^{r_m-\delta} X\,dr + \int_{r_m+\delta}^{r_{m+1}-\delta'} X\,dr + \int_{r_{m+1}+\delta'}^a X\,dr \tag{B9}$$

in the limit $\delta, \delta' \to 0$. In Eq. (B8), C_m denotes the coefficient of the large asymptotic solution of ψ_m as $r \to r_m$ (i.e., $\psi_m \simeq C_m|r-r_m|^{\nu}$ near r_m).

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