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Citation: *Phys. Fluids B* **3**, 1644 (1991); doi: 10.1063/1.859684

View online: <http://dx.doi.org/10.1063/1.859684>

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# Lagrangian dynamics of a charged particle in a tokamak magnetic field

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(Received 26 June 1990; accepted 22 March 1991)

The Euler–Lagrange equations for a charged particle in a tokamak magnetic field, in the limit of large aspect ratio, are obtained in terms of toroidal coordinates  $(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi})$  from the exact Lagrangian, which is expressed in terms of the toroidal and poloidal magnetic fluxes. It differs fundamentally from the standard guiding-center approach, where the local magnetic-field direction defines a spatial axis, and where there is no direct link between the velocity coordinates and the time derivatives of the spatial coordinates, so that application to the particular magnetic geometry of the tokamak is often difficult. In contrast, the tokamak magnetic field is specified at the outset in the present approach. For example, cyclotron motion in the combined poloidal and toroidal fields appears as a libration in real space, superimposed on motion composed of linear nonoscillatory and averaged quadratic oscillatory terms. The primary application that is considered concerns the plasma response to high-frequency waves, using both cold plasma and kinetic treatments. A kinetic expression for cyclotron resonance is obtained that agrees with the gyrokinetic result [see, for example, C. N. Lashmore-Davies and R. O. Dendy, *Phys. Fluids B* **1**, 1565 (1989) and references therein], which differs from the standard expression.

## I. INTRODUCTION

There is at present considerable interest in the development of Lagrangian and Hamiltonian theories of charged-particle motion in spatially inhomogeneous magnetic fields; see, for example, Refs. 1–10. The majority of these approaches share two common features. First, the local direction of the magnetic field defines one of the spatial coordinate axes. Second, in the guiding-center system of coordinates that is usually employed, there is a dichotomy between the spatial and velocity coordinates. The three spatial coordinates are used to describe the position of the guiding center, while two of the velocity coordinates are used to describe the rapid cyclotron gyration; thus there is no close link between the velocity coordinates and the time derivatives of the spatial coordinates. The guiding-center approach permits a generalized treatment of charged-particle dynamics that is not specific to a particular magnetic-field geometry. Conversely, the ultimate application of such theories to, for example, tokamak geometry is often a complex exercise.

In this paper, we shall explore a different approach. The study of tokamak plasmas takes place within the framework of a magnetic field whose lines of force are helices wound around toroidal surfaces and, with this application in mind, we shall specialize to tokamak field geometry from the outset. Particle motion will be described in terms of the laboratory toroidal coordinates  $(r, \theta, \phi)$ :  $r$  denotes the distance from the magnetic axis, which is located at a distance  $R_0$  from the axis of symmetry,  $\theta$  denotes poloidal angle, and  $\phi$  denotes toroidal angle. The corresponding element of length  $ds$  satisfies

$$ds^2 = dr^2 + r^2 d\theta^2 + (R_0 + r \cos \theta)^2 d\phi^2. \quad (1)$$

We note that while the coordinates  $(r, \theta, \phi)$  are appropriate to the laboratory, in a tokamak the magnetic field is nowhere parallel to any of these coordinates. Furthermore, we shall

adopt a Lagrangian approach so that the velocity coordinates employed are simply the time derivatives of the spatial coordinates. Thus, for example, the cyclotron gyration of the charged particle will be described entirely in terms of the laboratory coordinates. It appears as a fast time scale libration in  $(r, \theta, \phi)$ , superimposed on the nonoscillatory motion. We shall consider the tokamak magnetic field in the limit of large aspect ratio, so that the poloidal cross section of the magnetic-flux surfaces is circular; see, for example, Ref. 11. This limit is taken in the interests of analytical tractability, but it should be noted that features such as “dee-ness,” triangularity, and the Shafranov shift are omitted.

The plan of the paper is as follows. In Sec. II, we write down the Lagrangian of a charged particle in a tokamak magnetic field using toroidal coordinates, and the resulting Euler–Lagrange equations are divided into oscillatory and nonoscillatory parts, where the latter include averages of quadratic oscillatory terms. The effect of perturbing wave fields is described in terms of the cold plasma dielectric tensor in toroidal geometry, which is calculated in Sec. III. In Sec. IV, we exploit the simple formulation of the Vlasov equation in Lagrangian coordinates to examine the kinetic cyclotron resonance condition without the explicit use of guiding-center coordinates. We find agreement with the gyrokinetic resonance condition obtained in Refs. 12–15, which differs from the “standard” condition. Our conclusions are given in Sec. V.

## II. CHARGED-PARTICLE DYNAMICS IN TOROIDAL GEOMETRY

The motion of a particle of unit mass and charge  $-e$  in a tokamak magnetic field, in the limit of large aspect ratio, with an electrostatic potential  $V$  also present, is governed by the toroidal coordinate Lagrangian

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + \frac{1}{2}(R_0 + r \cos \theta)^2\dot{\phi}^2 - (e/c) [\dot{\theta}\Psi_T(r, \theta) - \dot{\phi}\Psi_p(r)] + eV. \quad (2)$$

In Eq. (2), the first three terms describe the particle kinetic energy. As we shall see from the Euler-Lagrange equations, the term proportional to  $\dot{\theta}\Psi_T(r, \theta) - \dot{\phi}\Psi_p(r)$  gives rise to the Lorentz force, and the functions  $\Psi_p(r)$  and  $\Psi_T(r, \theta)$  can be related to the poloidal and toroidal magnetic fluxes, respectively. The Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad (3)$$

when applied to the Lagrangian of Eq. (2), yield the evolution equations

$$\ddot{r} - r\dot{\theta}^2 - \cos \theta (R_0 + r \cos \theta)\dot{\phi}^2 + \frac{e}{c} \left( \dot{\theta} \frac{\partial \Psi_T}{\partial r} - \dot{\phi} \frac{d\Psi_p}{dr} \right) - e \frac{\partial V}{\partial r} = 0, \quad (4)$$

$$\frac{d}{dt} (r^2\dot{\theta}) - \frac{e}{c} \dot{r} \frac{\partial \Psi_T}{\partial r} + r \sin \theta (R_0 + r \cos \theta)\dot{\phi}^2 - e \frac{\partial V}{\partial \theta} = 0, \quad (5)$$

$$\frac{d}{dt} [(R_0 + r \cos \theta)^2\dot{\phi}] + \frac{e}{c} \dot{r} \frac{d\Psi_p}{dr} - e \frac{\partial V}{\partial \phi} = 0. \quad (6)$$

In Eqs. (4)–(6), the terms involving the radial gradients of  $\Psi_p(r)$  and  $\Psi_T(r, \theta)$  describe the Lorentz force if we define the poloidal and toroidal components of the magnetic field to be

$$B_p(r, \theta) = \frac{1}{R_0 + r \cos \theta} \frac{d\Psi_p}{dr}, \quad (7)$$

$$B_T(r, \theta) = \frac{1}{r} \frac{\partial}{\partial r} \Psi_T(r, \theta). \quad (8)$$

Equation (7) is identical to the definition of the poloidal field in the large-aspect-ratio limit given by Eq. (3.3.2) of Ref. 11. Generalized dependence of toroidal field on poloidal angle, typically  $\sim (R_0 + r \cos \theta)^{-1}$ , is included in Eq. (8). Once we have made the identifications Eqs. (7) and (8), the functional form of  $\Psi_p(r)$  and  $\Psi_T(r, \theta)$  does not matter for our purposes, beyond the requirement that  $\Psi_T$  be periodic in  $\theta$ . The poloidal and toroidal magnetic fluxes depend only on  $r$  in the present approximation, which involves circular flux surfaces. By definition, the magnetic fluxes are

$$\Phi_T(r) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^r (R_0 + r' \cos \theta) d\phi dr' B_p(r', \theta), \quad (9)$$

$$\Phi_p(r) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^r r' d\theta dr' B_T(r', \theta). \quad (10)$$

Substituting Eqs. (7) and (8) into Eqs. (9) and (10), respectively,

$$\Phi_p(r) = \Psi_p(r), \quad (11)$$

$$\Phi_T(r) = \frac{1}{2\pi} \int_0^{2\pi} \Psi_T(r, \theta) d\theta. \quad (12)$$

Returning to, for example, Eq. (4), the remaining terms can be identified physically as follows:  $r\dot{\theta}^2$  is the radial centrifugal force associated with poloidal motion;  $\cos \theta (R_0 + r \cos \theta)\dot{\phi}^2$  is the radial component of the centrifugal force associated with toroidal motion; and the final term describes the radial electric-field force.

Because of the wide disparity in characteristic time scale between cyclotron motion and drift motion, we shall adopt a two-time scale approach. We write

$$r = r_0 + r_s + r_f, \quad \theta = \theta_s + \theta_f, \quad \phi = \phi_s + \phi_f. \quad (13)$$

Here,  $r_f, r_s \ll r_0$ , which is a constant corresponding to the average distance of the particle from the magnetic axis, and the subscripts  $f$  and  $s$  refer to rapidly oscillatory and nonoscillatory terms, respectively. Thus  $\langle \dot{\phi}_f \rangle = 0$  but  $\langle \dot{\phi}_s \rangle \neq 0$ , where  $\langle \rangle$  denotes the slow time scale average. For convenience, we shall assume that the electrostatic potential  $V$  is sufficiently weak that it does not perturb the fast time scale motion; that is, recognizable cyclotron orbits exist. Let us now consider the oscillatory motion, which is described by the terms in Eqs. (4)–(6) that are linear in the oscillatory quantities and otherwise involve only the zeroth-order quantities  $r_0$ ,  $\Psi_T$ , and  $\Psi_p$ :

$$\ddot{r}_f + \frac{e}{c} \dot{\theta}_f \frac{\partial \Psi_T}{\partial r} \Big|_{r_0, \theta_s} - \frac{e}{c} \dot{\phi}_f \frac{d\Psi_p}{dr} \Big|_{r_0} = 0, \quad (14)$$

$$r_0^2 \ddot{\theta}_f - \frac{e}{c} \dot{r}_f \frac{\partial \Psi_T}{\partial r} \Big|_{r_0, \theta_s} = 0, \quad (15)$$

$$(R_0 + r_0 \cos \theta_s)^2 \ddot{\phi}_f + \frac{e}{c} \dot{r}_f \frac{d\Psi_p}{dr} \Big|_{r_0} = 0. \quad (16)$$

Referring to Eqs. (7) and (8), we define cyclotron frequencies characteristic of the poloidal and toroidal magnetic-field components:

$$\Omega_p(r_0, \theta_s) = \frac{e}{c} B_p(r_0, \theta_s) = \frac{e}{c} \frac{1}{R_0 + r_0 \cos \theta_s} \frac{d\Psi_p}{dr} \Big|_{r_0}, \quad (17)$$

$$\Omega_T(r_0, \theta_s) = \frac{e}{c} B_T(r_0, \theta_s) = \frac{e}{c} \frac{1}{r_0} \frac{\partial \Psi_T}{\partial r} \Big|_{r_0, \theta_s}. \quad (18)$$

Then Eqs. (8)–(10) become

$$\ddot{r}_f + \Omega_T r_0 \dot{\theta}_f - \Omega_p (R_0 + r_0 \cos \theta_s) \dot{\phi}_f = 0, \quad (19)$$

$$r_0 \ddot{\theta}_f - \Omega_T \dot{r}_f = 0, \quad (20)$$

$$(R_0 + r_0 \cos \theta_s) \ddot{\phi}_f + \Omega_p \dot{r}_f = 0. \quad (21)$$

This system of equations can be integrated, noting that the secular terms generated by constants of integration (which of course describe the thermal motion parallel to the magnetic field) have nonzero slow time scale averages, and therefore belong in  $\theta_s$  and  $\phi_s$  rather than in  $\theta_f$  and  $\phi_f$ . It is also convenient to define

$$\Omega(r_0, \theta_s) = [\Omega_p^2(r_0, \theta_s) + \Omega_T^2(r_0, \theta_s)]^{1/2}, \quad (22)$$

which is the cyclotron frequency proportional to the total magnetic-field strength. We then have the following solutions to Eqs. (19)–(21):

$$r_f = \frac{1}{2} a_r \exp(-i\Omega t) + \text{c.c.} = |a_r| \sin(\Omega t + \alpha), \quad (23)$$

$$\theta_f = i \frac{\Omega_T}{\Omega} \frac{1}{2} \frac{a_r}{r_0} \exp(-i\Omega t) + \text{c.c.}, \quad (24)$$

$$\phi_f = -i \frac{\Omega_p}{\Omega} \frac{1}{2} \frac{a_r}{R_0 + r_0 \cos \theta_s} \exp(-i\Omega t) + \text{c.c.}, \quad (25)$$

where  $a_r$  is an undetermined slowly varying quantity ( $\dot{a}_r \ll \Omega a_r$ ) and c.c. denotes the complex conjugate. Equations (23)–(25) describe the cyclotron motion of the charged particle as a fast time scale libration in the laboratory coordinates  $(r, \theta, \phi)$ . Physically,  $|a_r|$  is the Larmor radius in the total magnetic field. By Eqs. (18) and (19),  $|a_r|^2$  is equal to the sum of the squares of the maximum poloidal displacement  $r_0 |\theta_f(\text{max})|$  and the maximum toroidal displacement  $(R_0 + r_0 \cos \theta_s) |\phi_f(\text{max})|$ . We note also that the poloidal and toroidal librations are out of phase with the radial libration by  $\pi/2$  and  $-\pi/2$ , respectively.

Next, we consider the slow time scale equations of motion that follow from Eqs. (4)–(6). These include both the derivatives of slow time scale quantities, and the slow time scale averages (denoted by  $\langle \rangle$ ) of quadratic oscillatory terms. From Eqs. (23)–(25), we note that

$$\langle r_f \dot{\phi}_f \rangle = -\langle \dot{r}_f \theta_f \rangle = \Omega_T |a_r|^2 / 2r_0, \quad (26)$$

$$\langle r_f \dot{\phi}_f \rangle = -\langle \dot{r}_f \phi_f \rangle = -\Omega_p |a_r|^2 / 2(R_0 + r_0 \cos \theta_s), \quad (27)$$

$$\langle \dot{\theta}_f^2 \rangle = \Omega_T^2 |a_r|^2 / 2r_0^2, \quad (28)$$

$$\langle \dot{\phi}_f^2 \rangle = \Omega_p^2 |a_r|^2 / (R_0 + r_0 \cos \theta_s)^2. \quad (29)$$

Physically, it is convenient to define quantities

$$\mu_T = \Omega_T |a_r|^2 / 2, \quad \mu_p = \Omega_p |a_r|^2 / 2, \quad (30)$$

which are proportional to the toroidal and poloidal magnetic fluxes through the fast time scale orbit of the particle.

The slow time scale radial equation of motion that follows from Eq. (4) is

$$\begin{aligned} \ddot{r}_s - r_0 \dot{\theta}_s^2 - r_0 \langle \dot{\theta}_f^2 \rangle \theta_s - \cos \theta_s (R_0 + r_0 \cos \theta_s) \dot{\phi}_s^2 \\ - \cos \theta_s (R_0 + r_0 \cos \theta_s) \langle \dot{\phi}_f^2 \rangle + \frac{e}{c} \dot{\theta}_s \frac{\partial \Psi_T}{\partial r} \Big|_{r_0, \theta_s} \\ + \frac{e}{c} \langle r_f \dot{\theta}_f \rangle \frac{\partial^2 \Psi_T}{\partial r^2} \Big|_{r_0, \theta_s} - \frac{e}{c} \dot{\phi}_s \frac{d\Psi_p}{dr} \Big|_{r_0} \\ - \frac{e}{c} \langle r_f \dot{\phi}_f \rangle \frac{d^2 \Psi_p}{dr^2} \Big|_{r_0} - e \frac{\partial V}{\partial r} \Big|_{r_0} = 0. \end{aligned} \quad (31)$$

The terms that we have neglected, for example  $-2\dot{\theta}_s \langle r_f \dot{\theta}_f \rangle$ , are small compared to the terms that we have retained by the ratio of the fast to the slow time scale, or of  $r_0$  to  $r_s$ . Using Eqs. (26) and (28), then Eq. (18), and then Eq. (8), we have

$$\begin{aligned} -r_0 \langle \dot{\theta}_f^2 \rangle + \frac{e}{c} \langle r_f \dot{\theta}_f \rangle \frac{\partial^2 \Psi_T}{\partial r^2} \Big|_{r_0, \theta_s} \\ = \frac{e}{c} \mu_T \left( \frac{1}{r_0} \frac{\partial^2 \Psi_T}{\partial r^2} \Big|_{r_0, \theta_s} - \frac{c}{e} \frac{\Omega_T}{r_0} \right) \\ = \frac{e}{c} \mu_T \frac{\partial B_T}{\partial r} \Big|_{r_0, \theta_s}. \end{aligned} \quad (32)$$

Similarly, using Eqs. (27), (29), (17), and (7),

$$-\cos \theta_s (R_0 + r_0 \cos \theta_s) \langle \dot{\phi}_f^2 \rangle - \frac{e}{c} \langle r_f \dot{\phi}_f \rangle \frac{d^2 \Psi_p}{dr^2} \Big|_{r_0}$$

$$\begin{aligned} = \frac{e}{c} \mu_p \left( \frac{1}{R_0 + r_0 \cos \theta_s} \frac{d^2 \Psi_p}{dr^2} \Big|_{r_0} \right. \\ \left. - \cos \theta_s \frac{c}{e} \frac{\Omega_p}{R_0 + r_0 \cos \theta_s} \right) \\ = \frac{e}{c} \mu_p \frac{\partial B_p}{\partial r} \Big|_{r_0, \theta_s}. \end{aligned} \quad (33)$$

Substituting from Eqs. (32) and (33) in Eq. (31), and using Eq. (13), we obtain

$$\begin{aligned} \ddot{r}_s - r_0 \dot{\theta}_s^2 - \cos \theta_s (R_0 + r_0 \cos \theta_s) \dot{\phi}_s^2 \\ + \frac{e}{c} \left( \mu_T \frac{\partial B_T}{\partial r} \Big|_{r_0, \theta_s} + \mu_p \frac{\partial B_p}{\partial r} \Big|_{r_0, \theta_s} \right) + \frac{e}{c} r_0 \dot{\theta}_s B_T(r_0, \theta_s) \\ - \frac{e}{c} (R_0 + r_0 \cos \theta_s) \dot{\phi}_s B_p(r_0, \theta_s) - e \frac{\partial V}{\partial r} \Big|_{r_0} = 0. \end{aligned} \quad (34)$$

It is now appropriate to define the nonoscillatory poloidal and toroidal velocities

$$v_\theta = r_0 \dot{\theta}_s, \quad v_\phi = (R_0 + r_0 \cos \theta_s) \dot{\phi}_s, \quad (35)$$

so that Eq. (34) can be written in the form

$$\begin{aligned} \ddot{r}_s - \frac{v_\theta^2}{r_0} - \cos \theta_s \frac{v_\phi^2}{R_0 + r_0 \cos \theta_s} \\ + \frac{e}{c} \left( \mu_T \frac{\partial B_T}{\partial r} \Big|_{r_0, \theta_s} + \mu_p \frac{\partial B_p}{\partial r} \Big|_{r_0, \theta_s} \right) \\ + \frac{e}{c} [v_\theta B_T(r_0, \theta_s) - v_\phi B_p(r_0, \theta_s)] - e \frac{\partial V}{\partial r} \Big|_{r_0, \theta_s, \phi_s} = 0. \end{aligned} \quad (36)$$

The terms appearing here are the radial components of the following forces: the centrifugal forces associated with combined drift and parallel thermal motion; the  $\mu \nabla B$  force, which here appears as  $\mu_T \nabla B_T + \mu_p \nabla B_p$  since  $\mathbf{B}$  does not define a coordinate axis; the Lorentz force associated with drift motion; and the electric field. We note that by Eqs. (32) and (33), the  $\mu \nabla B$  force includes two distinct physical effects. First, there is the change in magnetic-field strength experienced by the particle as it undergoes fast time scale libration in  $(r, \theta, \phi)$ . Second, there is the slow time scale-averaged effect of the centrifugal forces that arise from the fact that this libration occurs in a curved, tokamak magnetic field.

The equilibrium drift velocities (discussed, for example, in Refs. 16 and 17) are obtained from the radial force relation (36) when we require the radial acceleration away from the magnetic axis to vanish. Multiplying terms in Eq. (36) by unity in the form  $(B_p^2 + B_T^2)/B^2$  where appropriate, the equation can be written

$$\begin{aligned} B_T(r_0, \theta_s) \left[ v_\theta - \frac{c}{eB^2} \frac{B_T v_\theta^2}{r_0} + \frac{B_T}{B^2} \left( \mu_T \frac{\partial B_T}{\partial r} \Big|_{r_0, \theta_s} \right. \right. \\ \left. \left. + \mu_p \frac{\partial B_p}{\partial r} \Big|_{r_0, \theta_s} \right) - \frac{c}{B^2} B_T \frac{\partial V}{\partial r} \Big|_{r_0, \theta_s, \phi_s} \right] \\ - B_p(r_0, \theta_s) \left[ v_\phi + \frac{c}{eB^2} \frac{\cos \theta_s B_p v_\phi^2}{R_0 + r_0 \cos \theta_s} \right] \end{aligned}$$

$$-\frac{B_p}{B^2} \left( \mu_r \frac{\partial B_T}{\partial r} \Big|_{r_0, \theta_s} + \mu_p \frac{\partial B_p}{\partial r} \Big|_{r_0, \theta_s} \right) + \frac{c}{B^2} B_p \frac{\partial V}{\partial r} \Big|_{r_0, \theta_s, \phi_s} = -\frac{c}{e} \dot{r}_s. \quad (37)$$

Then, setting  $\dot{r}_s = 0$ , the vanishing of the coefficients in curly brackets gives the formulas for the equilibrium poloidal and toroidal  $\mu \nabla B \times \mathbf{B}$ ,  $\mathbf{E} \times \mathbf{B}$ , and curvature drift velocities. These are expressed in toroidal  $(r, \theta, \phi)$  coordinates for a tokamak magnetic field. To the extent that  $v_\theta$  and  $v_\phi$  differ from their equilibrium values, the associated Lorentz force gives rise to radial acceleration.

Now let us consider the slow time scale poloidal equation of motion that follows from Eq. (5). We have, to leading order

$$\begin{aligned} \frac{d}{dt} (r_0^2 \dot{\theta}_s + 2r_0 \langle r_f \dot{\theta}_f \rangle) - \frac{e}{c} \dot{r}_s \frac{\partial \Psi_T}{\partial r} \Big|_{r_0, \theta_s} \\ - \frac{e}{c} \langle \dot{r}_f \theta_f \rangle \frac{\partial^2 \Psi_T}{\partial r \partial \theta} \Big|_{r_0, \theta_s} + r_0 \sin \theta_s (R_0 + r_0 \cos \theta_s) \\ \times (\dot{\phi}_s^2 + \langle \dot{\phi}_f^2 \rangle) - e \frac{\partial V}{\partial \theta} \Big|_{r_0, \theta_s, \phi_s} = 0. \end{aligned} \quad (38)$$

Using Eqs. (7), (8), (17), (18), (26), (29), (30), and (35), this becomes

$$\begin{aligned} \frac{d}{dt} \left( v_\theta + \frac{2\mu_T}{r_0} \right) - \frac{e}{c} \dot{r}_s B_T(r_0, \theta_s) + \sin \theta_s \frac{v_\phi^2}{R_0 + r_0 \cos \theta_s} \\ + \frac{e}{c} \mu_T \frac{1}{r_0} \frac{\partial B_T}{\partial \theta} \Big|_{r_0, \theta_s} + \frac{e}{c} \mu_p \frac{1}{r_0} \frac{\partial B_p}{\partial \theta} \Big|_{r_0, \theta_s} \\ - e \frac{1}{r_0} \frac{\partial V}{\partial \theta} \Big|_{r_0, \theta_s, \phi_s} = 0. \end{aligned} \quad (39)$$

Similarly, the slow time scale toroidal equation of motion that follows from Eq. (6) is

$$\begin{aligned} \frac{d}{dt} [(R_0 + r_0 \cos \theta_s)^2 \dot{\phi}_s \\ + 2 \cos \theta_s (R_0 + r_0 \cos \theta_s) \langle r_f \dot{\phi}_f \rangle] + \frac{e}{c} \dot{r}_s \frac{d\Psi_p}{dr} \Big|_{r_0} \\ - e \frac{\partial V}{\partial \phi} \Big|_{r_0, \theta_s, \phi_s} = 0. \end{aligned} \quad (40)$$

Using Eqs. (17), (27), (30), and (35), this gives to leading order

$$\begin{aligned} \frac{d}{dt} \left( v_\phi - \frac{2 \cos \theta_s}{R_0 + r_0 \cos \theta_s} \mu_p \right) + \frac{e}{c} \dot{r}_s B_p(r_0, \theta_s) \\ - e \frac{1}{R_0 + r_0 \cos \theta_s} \frac{\partial V}{\partial \phi} \Big|_{r_0, \theta_s, \phi_s} = 0. \end{aligned} \quad (41)$$

Equations (39) and (41) illustrate the relation between slow time poloidal and toroidal acceleration, radial drift, and the forces that produce them. First, we may consistently set

$$\frac{d\mu_T}{dt} = \frac{d\mu_p}{dt} = 0, \quad (42)$$

since the component terms involve factors such as  $\dot{\theta}_s \langle r_f \dot{\theta}_f \rangle$  that have been neglected elsewhere; see, for example, the discussion of Eq. (31). Next, we turn to the case where the charged particle remains on the magnetic surface, with

$\dot{r}_s = 0$  since the poloidal cross section of the surface is circular in the limit of large aspect ratio considered here. Then the terms in Eq. (33) are, respectively, the poloidal components of the centrifugal force associated with toroidal drift motion; the  $\mu \nabla B$  force, which involves  $\mu_T \nabla B_T$  and  $\mu_p \nabla B_p$ , as we discussed after Eq. (30); and the electric field. Equation (35) is simpler, as neither the centrifugal nor the  $\mu \nabla B$  forces possess a toroidal component. When  $\dot{r}_s \neq 0$ , so that the particle drifts radially across magnetic surfaces, the terms in Eqs. (39) and (41) proportional to  $\dot{r}_s B_T$  and  $\dot{r}_s B_p$  are due to the Lorentz force. In particular, Eq. (41) embodies the fact (see, for example, Refs. 18 and 19) that radial motion involves a change in canonical toroidal momentum.

### III. COLD PLASMA DIELECTRIC TENSOR IN TOROIDAL GEOMETRY

Let us now continue our direct approach to toroidal geometry by calculating the local cold plasma dielectric tensor elements. These can be obtained very simply, with respect to laboratory  $(r, \theta, \phi)$  toroidal coordinates, using the fast time scale equations of motion that follow from Eqs. (4) and (6). A fast time scale electric field gives rise to terms on the right-hand side of Eqs. (19)–(21) as follows:

$$\ddot{r}_f + \Omega_T r_0 \dot{\theta}_f - \Omega_p (R_0 + r_0 \cos \theta_s) \dot{\phi}_f = -eE_r, \quad (43)$$

$$r_0 \ddot{\theta}_f - \Omega_T \dot{r}_f = -eE_\theta, \quad (44)$$

$$(R_0 + r_0 \cos \theta_s) \ddot{\phi}_f + \Omega_p \dot{r}_f = -eE_\phi. \quad (45)$$

We recall that these equations describe small-amplitude excursions in real laboratory space, and we shall neglect any spatial variation of the electric field on this length scale. It is convenient to write

$$(E_r, E_\theta, E_\phi) = \frac{1}{2} (E_{r0}, E_{\theta0}, E_{\phi0}) \exp(-i\omega t) + \text{c.c.}, \quad (46)$$

and the particular integral of Eqs. (43)–(45) in the form

$$\begin{aligned} [r_f, r_0 \theta_f, (R_0 + r_0 \cos \theta_s) \phi_f] \\ = \frac{1}{2} (a_r, a_\theta, a_\phi) \exp(-i\omega t) + \text{c.c.} \end{aligned} \quad (47)$$

Here c.c. again denotes the complex conjugate and  $\omega$  is the fixed frequency of the applied fast time scale electric field.

In general, the solutions of Eqs. (43)–(45) are a combination of the general solution of the homogeneous system—which, we have already seen, corresponds to cyclotron motion, with an undetermined coefficient corresponding to the Larmor radius—and the particular integral. In the cold plasma limit, the Larmor radius tends to zero, and we need only consider the particular integral. Thus, substituting Eqs. (46) and (47) into Eqs. (43)–(45), we obtain

$$\begin{bmatrix} \omega^2 & i\omega\Omega_T & -i\omega\Omega_p \\ -i\omega\Omega_T & \omega^2 & 0 \\ i\omega\Omega_p & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix} = e \begin{bmatrix} E_r \\ E_\theta \\ E_\phi \end{bmatrix}. \quad (48)$$

Now for particles with equilibrium number density  $n_0$ , the fast time scale current corresponding to this displacement has amplitude

$$(j_r, j_\theta, j_\phi) = i\omega n_0 e (a_r, a_\theta, a_\phi). \quad (49)$$

Denoting by  $\mathbf{M}$ , the inverse of the matrix that appears on the

left-hand side of Eq. (48), and defining the plasma frequency  $\omega_p = (4\pi n_0 e^2)^{1/2}$ , it follows that

$$(j_r, j_\theta, j_\phi) = (i\dot{\omega}/4\pi)\omega_p^2 \mathbf{M} \cdot \mathbf{E}. \quad (50)$$

Maxwell's equation  $\text{curl } \mathbf{B} = (4\pi/c)\mathbf{J} + (1/c)\partial \mathbf{E}/\partial t$  becomes

$$\epsilon = \begin{bmatrix} 1 - \frac{\omega_p^2}{\omega^2 - \Omega^2} & i \frac{\Omega_r}{\omega} \frac{\omega_p^2}{\omega^2 - \Omega^2} & -i \frac{\Omega_p}{\omega} \frac{\omega_p^2}{\omega^2 - \Omega^2} \\ -i \frac{\Omega_r}{\omega} \frac{\omega_p^2}{\omega^2 - \Omega^2} & 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \Omega_p^2}{\omega^2 - \Omega^2} & \frac{\Omega_r \Omega_p}{\omega^2} \frac{\omega_p^2}{\omega^2 - \Omega^2} \\ i \frac{\Omega_p}{\omega} \frac{\omega_p^2}{\omega^2 - \Omega^2} & \frac{\Omega_r \Omega_p}{\omega^2} \frac{\omega_p^2}{\omega^2 - \Omega^2} & 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \Omega_r^2}{\omega^2 - \Omega^2} \end{bmatrix} \quad (52)$$

for the local cold plasma dielectric tensor in laboratory  $(r, \theta, \phi)$  coordinates. The cyclotron frequencies  $\Omega_p, \Omega_r,$  and  $\Omega$  are given by Eqs. (17), (18), and (22).

We note that the derivation of  $\epsilon$  in toroidal coordinates from the toroidal fast time scale equations of motion, Eqs. (43)–(45), is very rapid. Using standard guiding-center theory, the derivation would be somewhat different. First, the dielectric tensor would be calculated from the fast time scale equations of motion with respect to local Cartesian coordinates, with the z-coordinate axis oriented along the magnetic-field direction. Next, the dielectric tensor would be pre- and postmultiplied by  $\mathbf{P}^{-1}$  and  $\mathbf{P}$ , respectively, where

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Omega_r/\Omega & -\Omega_p/\Omega \\ 0 & \Omega_p/\Omega & \Omega_r/\Omega \end{pmatrix}, \quad (53)$$

to give the result of Eq. (52).

#### IV. THE KINETIC CYCLOTRON RESONANCE CONDITION

It was noted in the Introduction that we have chosen a set of coordinates that differs from those usually employed in plasma theory, in that the velocity coordinates are simply the time derivatives of the position coordinates. The Vlasov equation for a distribution function  $f(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}, t)$  can thus be written

$$\frac{\partial f}{\partial t} + \dot{r} \frac{\partial f}{\partial r} + \dot{\theta} \frac{\partial f}{\partial \theta} + \dot{\phi} \frac{\partial f}{\partial \phi} + \ddot{r} \frac{\partial f}{\partial \dot{r}} + \ddot{\theta} \frac{\partial f}{\partial \dot{\theta}} + \ddot{\phi} \frac{\partial f}{\partial \dot{\phi}} = 0. \quad (54)$$

In the presence of a wave field, we may as usual divide  $f$  into an equilibrium part  $f_0$  and a component  $f_1$  which is due to wave-driven perturbations. Linearizing Eq. (54), we obtain

$$\frac{df_1}{dt'} = \ddot{r}_w \frac{\partial f_0}{\partial r} + \ddot{\theta}_w \frac{\partial f_0}{\partial \theta} + \ddot{\phi}_w \frac{\partial f_0}{\partial \phi}, \quad (55)$$

where  $d/dt'$  denotes the time derivative following the equilibrium particle motion, and  $(\ddot{r}_w, \ddot{\theta}_w, \ddot{\phi}_w)$  denote wave-driven accelerations. We assume that the equilibrium is sufficiently homogeneous in space that terms such as  $\dot{r}_w \partial f_0/\partial r$

$$\text{curl } \mathbf{B} = (-i\omega/c)\epsilon \cdot \mathbf{E}, \quad \epsilon = \mathbf{I} - \omega_p^2 \mathbf{M}, \quad (51)$$

where  $\mathbf{I}$  is the identity matrix. Determination of the cold plasma dielectric tensor  $\epsilon$  therefore requires simply the inversion of the matrix of Eq. (48) followed by substitution in Eq. (51). We obtain

can be neglected in comparison to the terms appearing on the right-hand side of Eq. (55). It follows from Eqs. (43)–(45) that, on the right-hand side of Eq. (55), we shall need to calculate the phase of the wave field at points on the equilibrium particle orbit. For a single-mode approach, with wave vector  $(k_r, k_\theta, k_\phi)$ , this quantity is defined by

$$\Psi(t') = k_r r(t') + k_\theta r_0 \theta(t') + k_\phi (R_0 + r_0 \cos \theta_s) \phi(t') - \omega t', \quad (56)$$

where the equilibrium orbit follows from Eqs. (23)–(25):

$$r = r_0 + \dot{r}_s(t' - t) + |a_r| \sin(\Omega t' + \alpha), \quad (57)$$

$$\theta = \theta_s + \dot{\theta}_s(t' - t) + \frac{\Omega_r}{\Omega} \frac{|a_r|}{r_0} \cos(\Omega t' + \alpha), \quad (58)$$

$$\phi = \phi_s + \dot{\phi}_s(t' - t) - \frac{\Omega_p}{\Omega} \frac{|a_r|}{(R_0 + r_0 \cos \theta_s)} \cos(\Omega t' + \alpha). \quad (59)$$

Our aim in the remainder of this section is to deduce the kinetic cyclotron resonance condition from the expressions above. In doing this, we shall not encounter one of the main features of most treatments of cyclotron resonance, namely the transformation between guiding-center coordinates and real space, which remains an active topic of research in this area. Indeed, Littlejohn<sup>5</sup> has pointed out that the principal difficulty in gyrokinetic theory is in transforming the kinetic equation to phase-space coordinates. The present approach sidesteps this problem, since we have remained in real-space coordinates throughout, and the equilibrium cyclotron motion is expressed in these coordinates, rather than in guiding-center coordinates. It follows from Eqs. (56)–(59) that the wave phase can be written

$$\Psi(t') = \Psi_0 + \Psi_1(t') + \Psi_2(t'), \quad (60)$$

where

$$\Psi_0 = k_r r_0 + k_\theta r_0 \theta_s + k_\phi (R_0 + r_0 \cos \theta_s) \phi_s - \omega t, \quad (61)$$

$$\Psi_1(t') = [k_r \dot{r}_s + k_\theta r_0 \dot{\theta}_s + k_\phi (R_0 + r_0 \cos \theta_s) \dot{\phi}_s - \omega](t' - t), \quad (62)$$

$$\Psi_2(t') = k_r |a_r| \sin(\Omega t' + \alpha) + k_T |a_r| \cos(\Omega t' + \alpha), \quad (63)$$

$$k_T = (k_\theta \Omega_T - k_\phi \Omega_p) / \Omega. \quad (64)$$

Note that the rapidly oscillating term  $\Psi_2(t')$  involves the two components of the wave vector that are perpendicular to the magnetic field, which lies in the  $(\theta, \phi)$  plane: these are the radial component  $k_r$ , and the transverse component  $k_T$  which lies in the  $(\theta, \phi)$  plane perpendicular to the field. It is convenient to define

$$k_\perp = (k_r^2 + k_T^2)^{1/2}, \quad (65)$$

$$\mu = \tan^{-1}(k_T/k_r), \quad (66)$$

so that  $\mu = 0$  corresponds to a purely radial wave. Next, we define

$$\tau = t' - t, \quad (67)$$

and recall the standard identity

$$\exp(ia \sin \delta) = \sum_{n=-\infty}^{\infty} J_n(a) \exp(in\delta). \quad (68)$$

Combining Eqs. (63), (64), and (66), we may write

$$\Psi_2(t') = k_\perp |a_r| \sin(\Omega t' + \alpha + \mu), \quad (69)$$

and using Eq. (68) it follows that

$$\exp[i\Psi_2(t')] = \sum_{n=-\infty}^{\infty} J_n(k_\perp |a_r|) \exp[in(\Omega t' + \alpha + \mu)]. \quad (70)$$

We can now obtain an expression for the full wave phase term. Using Eqs. (61), (62), and (70) in Eq. (60), and recalling Eq. (67), we may write

$$\begin{aligned} \exp[i\Psi(t')] = & \exp(i\Psi_0) \sum_{n=-\infty}^{\infty} \{J_n(k_\perp |a_r|) \\ & \times \exp[in(\Omega t + \alpha + \mu)] \\ & \times \exp[i(k_r \dot{r}_s + k_\theta r_0 \dot{\theta}_s \\ & + k_\phi (R_0 + r_0 \cos \theta_s) \dot{\phi}_s + n\Omega - \omega)\tau]\}. \end{aligned} \quad (71)$$

This is the wave phase term that appears on the right-hand side of Eq. (55). The value of  $f_1(t)$  will depend on the integral of this expression from  $\tau = -\infty$  to  $\tau = 0$ , and it follows from Eq. (71) that the condition for stationary phase in this integral is

$$k_r \dot{r}_s + k_\theta r_0 \dot{\theta}_s + k_\phi (R_0 + r_0 \cos \theta_s) \dot{\phi}_s + n\Omega - \omega = 0. \quad (72)$$

This is the cyclotron resonance condition. As we have seen in Sec. II, the velocities that multiply the components of the wave vector describe the combined drift and parallel thermal motion. In Eq. (72), the value of  $\Omega$  is that defined by Eq. (22) for the equilibrium particle orbit, given by Eqs. (23)–(25), that we have used. Thus  $\Omega = \Omega(r_0, \theta_s)$  is the cyclotron frequency evaluated at the point about which fast time scale

libration occurs, which is of course the guiding center. The remaining terms in the resonant denominator of Eq. (72) are the wave frequency  $\omega$ , and terms associated with nonoscillatory velocities, corresponding to Doppler and drift effects.

We conclude that the cyclotron resonance condition in a tokamak magnetic field given by the present Lagrangian kinetic approach agrees with the condition<sup>12</sup> that is obtained using gyrokinetic theory.<sup>13–15</sup> This condition differs from the “standard” version in that the cyclotron frequency is evaluated at the guiding-center position, rather than the particle position. In the gyrokinetic approach, the linearized perturbations are calculated in guiding-center coordinates, and there is a subsequent transformation back to real space. The present treatment involves no such transformation, since the particle motion in the inhomogeneous magnetic field is followed in real space throughout. It is encouraging that these distinct approaches agree on the cyclotron resonance condition.

## V. CONCLUSIONS

The Lagrangian approach described in this paper permits direct analytical particle-following in real space for a tokamak magnetic field in the limit of large aspect ratio. It differs from the usual guiding-center approach in two ways. First, because of the specific choice of magnetic field and coordinate system, it is possible to deal explicitly with the tokamak field from the outset. Second, because Lagrangian velocity coordinates—that is, time derivatives of spatial coordinates—are used, the Vlasov equation takes a particularly simple form. As a result, it has been possible to examine the kinetic cyclotron resonance condition from a direct particle-following point of view, and obtain confirmation of the gyrokinetic condition previously obtained in Refs. 12–15.

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