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Wave propagation near a cyclotron resonance in a nonuniform equilibrium magnetic field

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The inclusion of the variation of the equilibrium magnetic field across the Larmor orbits of the resonant particles is crucial for a self-consistent treatment of cyclotron resonance in plasmas. Two contrasting nonrelativistic self-consistent calculations [T. M. Antonsen and W. M. Manheimer, Phys. Fluids **21**, 2295 (1978); C. N. Lashmore-Davies and R. O. Dendy, Phys. Fluids B **1**, 1565 (1989)] which analyze perpendicular propagation in the same nonuniform magnetic field are compared. It is shown that the first of these, which is a full wave calculation, makes an approximation that eliminates the damping found in the second, which calculates optical depth via a Wentzel–Kramers–Brillouin (WKB) approximation. A new expansion of the exact integral equation describing the problem is given, producing full wave equations which incorporate the perpendicular damping. The equations are of the correct form to ensure energy conservation and can easily be obtained to any order in an expansion in terms of the ratio of Larmor radius to perpendicular wavelength.

I. INTRODUCTION

In this paper we shall develop a theory of wave propagation through a cyclotron resonance in a nonuniform magnetic field, with the effect of the inhomogeneous field included self-consistently. By this we mean that the variation of the equilibrium magnetic field across the Larmor orbit is included in the particle response to perturbations. We first discuss two previous theories which incorporate this effect. The first, developed by Antonsen and Manheimer[1] some years ago, gave a full wave solution of the problem but did not include damping which occurs, even at perpendicular incidence, as a result of the magnetic field variations across the Larmor orbit. The second, presented more recently by Lashmore-Davies and Dendy,[2] points out the existence of this damping mechanism but applications of it have, so far, been confined to calculations of optical depth using a local dispersion relation to obtain the imaginary part of the wave number. Calculations of this type generally agree with the optical depth obtained from full wave calculations, but do not, of course, give any information on reflection or mode conversion coefficients.

In Ref. 1 the Vlasov equation was integrated along the unperturbed orbits in the nonuniform field, while Ref. 2 obtained the particle response using the general gyrokinetic theory of Chen and Tsai.[3,4] In both cases the same finite Larmor radius correction appears in the resonance condition, so the first problem is to explain why one theory gives damping near the resonance while the other does not. An examination of Antonsen and Manheimer's procedure will be presented, showing that an approximation which they introduce, in order to make an integral equation analytically soluble, amounts to an asymptotic expansion of the plasma dispersion function. As a result, a damping profile with finite width is replaced by a resonant pole. By converting their solution, presented in terms of the Fourier transform of the field, into a differential equation in space, it will be shown that the finite Larmor radius correction plays a vital role in maintaining the correct energy conservation properties of the wave.[5]

We then turn to the main new result of this paper, namely a method of obtaining full wave equations which incorporate the effect of the finite Larmor radius in a nonuniform field, and which combine the energy conservation properties noted by Antonsen and Manheimer with the finite damping width obtained by Lashmore-Davies and Dendy. This method yields differential equations, to any order in an expansion in the ratio of Larmor radius to perpendicular wavelength, in a simple way.

To illustrate the theory we shall look at electron cyclotron absorption, in a nonrelativistic approximation, at perpendicular incidence, concentrating mainly on the O mode at the fundamental. It is, of course, well known that in reality relativistic effects are very important in this case.[6] However, it is useful for our purposes in that it is the simplest problem involving only one electric field component and one dielectric tensor element. It allows us, therefore, to exhibit the essentials of our theory with a minimum of extraneous detail. The general method can then be applied to problems in the ion cyclotron range of frequencies and to nonperpendicular propagation.

The structure of the paper is as follows. In Sec. II we obtain the integral equation which describes the problem exactly. In Sec. III we describe the approximate solutions obtained in Refs. 1 and 2 and discuss the differences between them. Section IV is concerned with the full wave equations corresponding to the approximation of Antonsen and Manheimer, and Sec. V with the more exact full wave equations that include perpendicular cyclotron damping. Finally, Sec. VI gives our conclusions.

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II. CYCLOTRON RESONANCE IN A NONUNIFORM MAGNETIC FIELD

We shall restrict the analysis in this paper to the propagation of electromagnetic waves perpendicular to the equilibrium magnetic field in the plasma. The magnetic field will be taken as

$$\mathbf{B} = \mathbf{e}_z B(1 + x/L_B), \tag{1}$$

where the scale length L_B may be identified with the major radius of a tokamak. In both Refs. 1 and 2, where the above model magnetic field was employed ($L_B \to -L$ in Ref. 1), the condition for electron cyclotron resonance was found to be

$$\omega - l\Omega_e(x) + l(v_\perp/L_B)\sin\alpha = 0; \tag{2}$$

here l is the harmonic number, $\Omega_e(x) = (|e|B/m_e c) \times (1 + x/L_B)$, $v_\perp$ is the perpendicular velocity of an electron, and α is the gyrophase angle. Since the resonance condition occurs in a velocity integral, cyclotron resonance is a wave–particle resonance, even for propagation perpendicular to the magnetic field with relativistic effects ignored. Such a resonance is expected to lead to wave damping with a smooth absorption profile. The finite Larmor radius correction to the resonance which arises from the variation of the magnetic field across the Larmor orbit will be shown below to have important consequences. Since cyclotron resonance is a localized phenomenon, the effect of the magnetic field nonuniformity need only be included in the resonant terms. After making this approximation, the problem can be formulated exactly; for this, it is convenient to use Fourier transforms in the direction of the inhomogeneity.

Let us now consider the case of the ordinary electromagnetic wave propagating across the fundamental electron cyclotron resonance. We may follow either the orbit method used by Antonsen and Manheimer[1] or the gyrokinetic method of Chen and Tsai,[4] representing the wave field as a Fourier transform:

$$\delta\mathbf{A}(x) = \int dk\, \overline{\delta\mathbf{A}}(k) e^{ikx} \tag{3}$$

and

$$\overline{\delta\mathbf{A}}(k) = \frac{1}{2\pi}\int dx\, \delta\mathbf{A}(x) e^{-ikx}. \tag{4}$$

The amplitude of the O mode is represented by the vector potential $\delta\mathbf{A} = \mathbf{e}_z \delta A$, so that

$$\delta E_z(x) = -\left(\frac{\partial}{\partial z}\delta\phi + \frac{1}{c}\frac{\partial}{\partial t}\delta A\right). \tag{5}$$

Since we have assumed $\partial/\partial z = 0$, it follows that

$$\delta E_z(x) = i(\omega/c)\delta A(x), \tag{6}$$

where we assume an $\exp(-i\omega t)$ dependence. Retaining the finite Larmor radius correction only in the resonant term, both methods[1,3] lead to an integral equation for $\overline{\delta A}(k)$ given by

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\overline{\delta A}(k) + \frac{\omega_p^2}{c^2} e^{-k^2\rho^2/2} I_0\left(\frac{k^2\rho^2}{2}\right)\overline{\delta A}(k)$$
$$= -\sum_{l\neq 0}\frac{iL_B\omega}{l\Omega_0}\frac{\omega_p^2}{c^2}\int_{-\infty}^{k} e^{-(k^2+k'^2)\rho^2/4} I_l\left(\frac{kk'\rho^2}{2}\right)$$
$$\times e^{i(k'-k)(\omega - l\Omega_0)L_B/l\Omega_0}\,\overline{\delta A}(k')dk', \tag{7}$$

where $\Omega_0 = \Omega(0)$ and $\rho = v_T/\Omega_0$ is the electron Larmor radius. In Eq. (7), one term is resonant, say $\omega = m\Omega_0$. For the nonresonant terms, the exponential $\exp i[(k'-k)(\omega - l\Omega_0)L_B/l\Omega_0]$ will be rapidly oscillating; it therefore averages to zero except for $k' = k$. Since we are considering the fundamental resonance we take $m = 1$. Separating the nonresonant terms from the resonant one, Eq. (7) becomes

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\overline{\delta A}(k) + \frac{\omega_p^2}{c^2} e^{-k^2\rho^2/2} I_0\left(\frac{k^2\rho^2}{2}\right)\overline{\delta A}(k)$$
$$= -\frac{\omega_p^2}{c^2}\sum_{l\neq 0,1} e^{-k^2\rho^2/2} I_l\left(\frac{k^2\rho^2}{2}\right)\frac{\omega}{(\omega - l\Omega_0)}\,\overline{\delta A}(k)$$
$$-\frac{i\omega_p^2}{c^2} L_B e^{-k^2\rho^2/4}$$
$$\times\int_{-\infty}^{k} e^{-k'^2\rho^2/4} I_1\left(\frac{kk'\rho^2}{2}\right)\overline{\delta A}(k')dk'. \tag{8}$$

Equation (8) describes the O mode propagating at right angles to the nonuniform magnetic field, given by Eq. (1), in the vicinity of the fundamental electron cyclotron resonance. The wave propagates in the direction of the nonuniformity and the equation is valid for large Larmor radius $k\rho > 1$, limited only by the constraint $k\rho \ll L_B/\rho$. A general solution of Eq. (8) has not yet been given.

III. THE OPTICAL DEPTH

We shall now consider the solution of Eq. (8) with the simplifying assumption $k\rho \ll 1$. Let us first review the technique used by Antonsen and Manheimer.[1] Assuming $k^2\rho^2 \ll 1$, Antonsen and Manheimer[1] were able to make the following approximations to simplify Eq. (8):

$$I_0(k^2\rho^2/2) \simeq 1, \quad e^{-k^2\rho^2/4} \simeq 1.$$

Furthermore, the integral in Eq. (8) is approximated by taking

$$e^{-k'^2\rho^2/4} I_1[(kk'\rho^2)/2] \simeq kk'\rho^2/4.$$

Neglecting the nonresonant terms on the right-hand side of Eq. (8), Antonsen and Manheimer[1] obtained

$$\left(k^2 - \frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2}\right)\overline{\delta A}(k)$$
$$= -iL_B\frac{\omega_p^2}{c^2}\frac{\rho^2}{4}k\int_{-\infty}^{k} k'\,\overline{\delta A}(k')dk'. \tag{9}$$

Now differentiate Eq. (9) with respect to k, giving

$$\frac{d}{dk}\left[\left(k^2-\frac{\omega^2}{c^2}+\frac{\omega_p^2}{c^2}\right)\overline{\delta E}_z(k)\right]$$
$$+iL_B\frac{\omega_p^2}{c^2}\frac{k^2v_T^2}{4\Omega_0^2}\overline{\delta E}_z(k)$$
$$+iL_B\frac{\omega_p^2}{c^2}\frac{v_T^2}{4\Omega_0^2}\int_{-\infty}^{k}k'\,\overline{\delta E}_z(k')dk'=0. \qquad (10)$$

Note that we have used Eq. (6) to replace $\overline{\delta A}(k)$ with $\overline{\delta E}_z(k)$ in conformity with Ref. 1 and that L_B in Eq. (10) corresponds to $-L$ in Ref. 1. Equations (9) and (10) may be solved for $\overline{\delta E}_z(k)$, which on inverting the Fourier transform yields

$$\delta E_z(x)=\int_L\frac{k\,dk}{2\pi}\frac{1}{\epsilon(k)}\times\exp\left(ikx-iL_B\int_0^k dk'\frac{G(k')}{\epsilon(k')}\right), \qquad (11)$$

where

$$\epsilon(k)=(c^2k^2/\omega^2)-1+(\omega_p^2/\omega^2), \qquad (12)$$
$$G(k)=\tfrac{1}{4}(\omega_p^2/\omega^2)k^2\rho^2. \qquad (13)$$

The electron cyclotron second harmonic resonance was also described in a similar manner. Equations analogous to Eq. (9) were obtained for $\overline{\delta E}_x(k)$ and $\overline{\delta E}_y(k)$, which were expressed in terms of $\overline{\delta E}_\pm(k)$ defined by

$$\overline{\delta E}_\pm(k)=\overline{\delta E}_x(k)\pm i\,\overline{\delta E}_y(k). \qquad (14)$$

The equations obtained by Antonsen and Manheimer[1] in this case were

$$\left(\frac{k^2c^2}{2\omega^2}-1-\sum_{r0}\right)\overline{\delta E_+}(k)-\frac{k^2c^2}{2\omega^2}\overline{\delta E_-}(k)$$
$$-2iL\alpha^2\delta\int_{-\infty}^{k}\frac{kk'}{\omega^2}c^2\delta E_+(k')dk'=0, \qquad (15)$$
$$-\frac{k^2c^2}{2\omega^2}\overline{\delta E_+}(k)+\left(\frac{k^2c^2}{2\omega^2}-1-\sum_{l0}\right)\overline{\delta E_-}(k)=0, \qquad (16)$$

where we use the notation of Ref. 1. Equations (15) and (16) may be solved by differentiating Eq. (15) with respect to k, giving

$$\frac{d}{dk}\left[\left(\frac{k^2c^2}{2\omega^2}-1-\sum_{r0}\right)\overline{\delta E_+}(k)\right]$$
$$-\frac{c^2}{2\omega^2}\frac{d}{dk}\left[k^2\,\overline{\delta E_-}(k)\right]$$
$$-2iL\alpha^2\delta\frac{k^2c^2}{\omega^2}\overline{\delta E_+}(k)$$
$$-2iL\alpha^2\delta\int_{-\infty}^{k}dk'\frac{k'c^2}{\omega^2}\overline{\delta E_+}(k')=0 \qquad (17)$$

and eliminating $\overline{\delta E_-}(k)$ using Eq. (16). This procedure yields $\delta E_+(x)$, after Fourier inversion, in a form corresponding to Eq. (11). The explicit result is given by Eq. (20) of Ref. 1. These expressions for $\delta E_z(x)$ and $\delta E_+(x)$ were evaluated asymptotically by Antonsen and Manheimer with the aid of contour integration, and the various wave coefficients were deduced. In particular, the standard[6] transmission coefficients for the O mode crossing the fundamental resonance and the X mode crossing the second harmonic resonance were obtained. For the second harmonic resonance, Antonsen and Manheimer obtained the result that the energy lost by the incident X mode was mode converted to the electron Bernstein wave. No energy was dissipated by the electrons.

We now summarize the results of a similar calculation by Lashmore-Davies and Dendy.[2] The approach of Ref. 2 differs from Ref. 1, but the same model magnetic field [Eq. (1)] is assumed and the same resonance condition, given in Eq. (2), is obtained. The following, self-consistent, local dispersion relation is obtained for the O mode crossing the fundamental resonance:

$$k^2=\frac{\omega^2}{c^2}-\frac{\omega_p^2}{c^2}-\frac{\omega_p^2}{c^2}\frac{i\omega L_B}{2v_T}k\rho[\zeta_1Z(\zeta_1)+1]$$
$$+\frac{\omega_p^2}{c^2}\frac{\omega L_B}{2v_T}k^2\rho^2\zeta_1[\zeta_1Z(\zeta_1)+1]. \qquad (18)$$

Equation (18) can be solved perturbatively for k, assuming $k=k_0+\delta k$ where k_0 is the cold plasma solution.[2] The optical depth is then obtained by carrying out the usual integration across the resonance layer

$$\tau=2\int_{-\infty}^{\infty}\mathrm{Im}\,\delta k\,dx. \qquad (19)$$

Again, the standard result[6] is obtained for the optical depth, in agreement with Ref. 1. The same procedure was also carried for the X-mode second harmonic. The dispersion relation is given by Eq. (123) of Ref. 2 and the standard result[6] is obtained, in agreement with Ref. 1.

However, there is a significant difference in the interpretation of the results obtained in Refs. 1 and 2. This difference is most clearly illustrated in the case of the second harmonic. Antonsen and Manheimer[1] found that the energy lost by the X mode, when incident from the high-field side, was mode converted to the electron Bernstein wave. For the X mode incident from the low-field side the energy is divided between transmitted, mode converted, and reflected channels. Energy is conserved amongst these three channels, and no energy is dissipated.

By contrast, it is clear from the calculation of Lashmore-Davies and Dendy[2] [cf. Eq. (123) of Ref. 2] that dissipation will occur as the X mode crosses the second harmonic resonance from either the high- or low-field side. This is because the calculation of the resonant term using the resonance condition, Eq. (2), gives rise to the plasma dispersion function.[2,7] As a result, energy is dissipated by the electrons with a well-defined absorption profile.

Since both calculations start from the same resonance condition, it is of interest to discover the source of this difference. Both calculations have approximated the exact problem illustrated by the integral equation given in Eq. (8). One possible source for the difference is that in Ref. 2 only the X mode is included in the calculation and mode conversion is excluded by the single-mode approximation. However, even if mode conversion was included, it is clear from the resonance term in Ref. 2 that dissipation would still be present, which would affect both the X mode and the electron Bernstein wave. Since Antonsen and Manheimer carried out a full wave calculation, their result may appear to be more general. However, the treatment of the resonance term by Antonsen and Manheimer is less accurate than that of Lashmore-Davies and Dendy. It is this feature that we now discuss.

Following the method described in Ref. 1, the following resonant term for the O mode at the fundamental is obtained:

$$\hat{\sum_p} = \frac{\omega_p^2}{\omega^2} L \int u v_\perp \, dv_\perp \, du \, d\theta \frac{\partial f_0}{\partial u} \times \frac{J_1(\hat{b}) e^{i(\hat{b}\sin\theta - \theta)}}{(x + (v_\perp/\Omega_0)\sin\theta + i\epsilon L)}. \tag{20}$$

Here ϵ is a positive infinitesimal introduced to satisfy causality requirements, $\hat{b} = kv_\perp/\Omega_0$, k_z has been taken to be zero and the other quantities have their usual meaning and are defined in Ref. 1. In order to include the x dependence of $\hat{\Sigma}_p$, which arises from the assumed linear magnetic field variation, Antonsen and Manheimer introduced a further Fourier transform

$$\hat{\sum_p} = \int_0^\infty dk_1 \exp(ik_1 x) \sum_p (k,k_1), \tag{21}$$

where

$$\sum_p = \frac{i\omega_p^2}{\omega^2} \int v_\perp u \, dv_\perp \, du \, d\theta \frac{\partial f_0}{\partial u} L J_1(\hat{b}) J_1(\hat{b}) \times \exp\left(ik_1 \frac{v_\perp \sin\theta}{\Omega_0} + i(\hat{b}\sin\theta - \theta) \right). \tag{22}$$

Using the Bessel identity this is written

$$\sum_p = 2\pi i L \frac{\omega_p^2}{\omega^2} \int v_\perp u \, dv_\perp \, du \frac{\partial f_0}{\partial u} J_1(\hat{b} + \hat{b}_1) J_1(\hat{b}), \tag{23}$$

where $\hat{b}_1 = k_1 v_\perp/\Omega_0$. The resonant integral given by Eqs. (20) and (23) is still exact at this point. It is here that Antonsen and Manheimer make a further approximation in order to carry out their full wave treatment: they take only the first term in the expansion of $J_1(\hat{b} + \hat{b}_1)$. This procedure truncates the Fourier transformation in Eq. (21), resulting in a loss of information on the resonance. The expansion of $J_1(\hat{b})$ corresponds to the usual finite Larmor radius expansion, and this approximation is made in both Refs. 1 and 2. However, Lashmore-Davies and Dendy did not Fourier expand the resonance but evaluated the resonant integrals exactly, following the treatment of Lee *et al.*[7]

Making these expansions and assuming a Maxwellian equilibrium distribution function, Eq. (23) is approximated by

$$\sum_p \simeq -\frac{iL}{4} \frac{\omega_p^2}{\omega^2} v_T^2 \frac{k(k+k_1)}{\Omega_0^2}. \tag{24}$$

Using Eq. (24) to form the resonant current, changing the integration variables in the Fourier transform, and substituting into Maxwell's equations, one obtains the O-mode equation

$$\int_L \frac{dk}{2\pi} \left[\left(\frac{k^2c^2}{\omega^2} - 1 + \frac{\omega_p^2}{\omega^2} \right) E_z - iL \frac{\omega_p^2}{\omega^2} \int_{-\infty}^{k} dk' \frac{kk' v_T^2}{4\Omega_0^2} E_z(k') \right] e^{ikx} = 0. \tag{25}$$

This gives Eq. (9), corresponding to Eq. (30) of Ref. 1.

The approximation of the Fourier transform of the resonance term given in Eqs. (21) and (22) is equivalent to an asymptotic expansion of the resonant denominator in Eq. (20) retaining only the first two terms, since

$$\frac{1}{x + (v_\perp/\Omega_0)\sin\theta} \simeq \frac{1}{x} - \frac{v_\perp \sin\theta}{x^2\Omega_0}. \tag{26}$$

Substituting Eq. (26) into Eq. (20), the θ integration gives

$$\int d\theta \, J_1(\hat{b}) e^{i(\hat{b}\sin\theta - \theta)} \left(\frac{1}{x} - \frac{v_\perp \sin\theta}{\Omega_0} \right) \simeq \frac{1}{x} J_1^2(\hat{b}) - \frac{1}{2i} J_1(\hat{b}) J_0(\hat{b}) \frac{v_\perp}{x^2\Omega_0}, \tag{27}$$

where we have neglected a term proportional to $J_1(\hat{b})J_2(\hat{b})$. The x dependence contained in Eq. (27) may now be Fourier transformed as before to give the same result as Eq. (24). Thus, as stated, Antonsen and Manheimer's treatment of the resonance is equivalent to taking the first two terms of an asymptotic expansion of the resonant denominator. An exact treatment of this integral leads to the plasma dispersion function, which is associated with a wave–particle resonance and a smooth profile. In contrast, the treatment given by Antonsen and Manheimer results in a wave equation with a singularity at the origin. The inclusion of the second term in the asymptotic expansion of the resonance in Ref. 1 has the important consequence of satisfying the conservation of energy.[5] Without this term the full wave equation does not satisfy this condition.

IV. THE WAVE EQUATIONS AND CONSERVATION OF ENERGY

We now return to Eq. (10) for the O mode and Eqs. (16) and (17) for the X mode. Let us first consider the O mode and associate a differential equation with Eq. (10). In order to do this, we differentiate Eq. (10) with respect to k, giving

$$\frac{d^2}{dk^2} \left[\left(k^2 - \frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2} \right) \overline{\delta E}_z(k) \right] - iL \frac{\omega_p^2}{c^2} \frac{v_T^2}{4\Omega_0^2} \frac{d}{dk} \left[k^2 \overline{\delta E}_z(k) \right] - iL \frac{\omega_p^2}{c^2} \frac{v_T^2}{4\Omega_0^2} k \, \overline{\delta E}_z(k) = 0. \tag{28}$$

We have replaced L_B by $-L$ in going from Eq. (10) to Eq. (28) in conformity with the convention of Ref. 1. Equation (28) is the Fourier transform of the differential equation

$$x^2 \left(\frac{d^2}{dx^2} + \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2} \right) \delta E_z(x) + xL \frac{\omega_p^2}{c^2} \frac{v_T^2}{4\Omega_0^2} \frac{d^2}{dx^2} \delta E_z(x) - L \frac{\omega_p^2}{c^2} \frac{v_T^2}{4\Omega_0^2} \frac{d}{dx} \delta E_z(x) = 0, \tag{29}$$

which can be written

$$\frac{c^2}{\omega^2}\frac{d}{dx}\left[\left(1+\frac{L}{x}\frac{\omega_p^2}{4\Omega_0^2}\frac{v_T^2}{c^2}\right)\frac{d\delta E_z}{dx}\right] + \left(1+\frac{\omega_p^2}{\omega^2}\right)\delta E_z = 0. \tag{30}$$

The last term in Eq. (29) arises from the second term in the asymptotic expansion of the resonance given by Eq. (26). If this is neglected, Eq. (29) reduces to Budden's equation which, for comparison with Eq. (30), can be written

$$\left(\frac{c^2}{\omega^2}+\frac{L}{x}\frac{\omega_p^2}{\omega^2}\frac{v_T^2}{4\Omega_0^2}\right)\frac{d^2\delta E_z}{dx^2}+\left(1-\frac{\omega_p^2}{\omega^2}\right)\delta E_z = 0. \tag{31}$$

We note that if ϕ is a solution of Eq. (31), $d\phi/dx$ is a solution of Eq. (30). This is evidently the reason why Antonsen and Manheimer obtained the same transmission and reflection coefficients as are given by the Budden equation. The Budden equation, however, does not conserve energy.

Let us now consider the conservation properties of Eq. (30). Multiplying Eq. (30) by δE_z^* and subtracting the product of the complex conjugate equation with δE_z yields the result

$$\frac{d}{dx}\left\{\mathrm{Im}\left[\delta E_z^*\left(1+\frac{L}{x}\frac{\omega_p^2}{4\Omega_0^2}\frac{v_T^2}{c^2}\right)\frac{d\delta E_z}{dx}\right]\right\} = 0. \tag{32}$$

Hence the differential equation which results from a theory that includes the particle response to the nonuniform magnetic field does satisfy a conservation relation. The first term in Eq. (32) is the Poynting flux and the second term is the kinetic power flow of the particles. Any loss of energy from the incident wave which occurs is the result of the pole at $x=0$ which, in the usual way, is taken to give an imaginary part $\pi\delta(x)$.

Next, consider the case of the X mode at the second harmonic. This is described by Eqs. (16) and (17) taken from Ref. 1. We differentiate Eq. (17) with respect to k, giving

$$\begin{aligned}&\frac{d^2}{dk^2}\left[\left(\frac{k^2c^2}{2\omega^2}-1-\sum_{r0}\right)\overline{\delta E_+}(k)\right] \\ &\quad -\frac{c^2}{2\omega^2}\frac{d^2}{dk^2}\left[k^2\,\overline{\delta E_-}(k)\right] \\ &\quad -2iL\alpha^2\delta\frac{d}{dk}\left(\frac{k^2c^2}{\omega^2}\,\overline{\delta E_+}(k)\right) \\ &\quad -2iL\alpha^2\delta\frac{c^2k}{\omega^2}\,\overline{\delta E_+}(k) = 0.\end{aligned} \tag{33}$$

Equations (16) and (33) are the Fourier transforms of the following differential equations:

$$\frac{c^2}{2\omega^2}\frac{d^2}{dx^2}\delta E_+(x)-\left(\frac{c^2}{2\omega^2}\frac{d^2}{dx^2}+1+\sum_{r0}\right)\delta E_-(x)=0, \tag{34}$$

$$\begin{aligned}&\left(\frac{c^2}{2\omega^2}\frac{d^2}{dx^2}+1+\sum_{r0}\right)\delta E_+(x)-\frac{c^2}{2\omega^2}\frac{d^2}{dx^2}\delta E_-(x) \\ &\quad +\frac{2L\alpha^2\delta}{x}\frac{c^2}{\omega^2}\frac{d^2}{dx^2}\delta E_+(x) \\ &\quad -\frac{2L\alpha^2\delta}{x^2}\frac{c^2}{\omega^2}\frac{d}{dx}\delta E_+(x)=0.\end{aligned} \tag{35}$$

We note that the last term in Eq. (35) comes from the second term on the right-hand side of Eq. (26). Equations (34) and (35) combine to yield a fourth-order equation. Again, if ϕ is a solution of the fourth-order equation which results from neglecting the last term in Eq. (35), $d\phi/dx$ satisfies the complete fourth-order equation. As for the O mode, this means that the transmission and reflection coefficients of the X mode will be independent of whether the gyrokinetic correction is included, since k^2 for this mode tends to the same constant value on both sides of the resonance. On the other hand, the wave number of the Bernstein mode is independent of x, so that the energy flow in this wave will depend on the model used.

To analyze the conservation properties of the Antonsen and Manheimer model, we use Eqs. (34) and (35). Manipulating these equations as for the O mode, we obtain

$$\begin{aligned}&\frac{d}{dx}\left\{\mathrm{Im}\left[\delta E_+^*\left(1+4\frac{L\alpha^2}{x}\delta\right)\frac{d}{dx}\delta E_+ +\delta E_-^*\frac{d}{dx}\delta E_-\right.\right. \\ &\quad \left.\left. -\delta E_+^*\frac{d}{dx}\delta E_- -\delta E_-^*\frac{d}{dx}\delta E_+\right]\right\}=0.\end{aligned} \tag{36}$$

This is the equation for the conservation of energy for the second harmonic resonance. The term proportional to x^{-1} represents the Bernstein wave, and the other terms are the Poynting flux.

This identification of a differential equation with the Fourier transform solution of Antonsen and Manheimer makes it clear how the finite Larmor radius term is crucial for energy conservation. It also makes clear how their approximation in the kernel of the integral equation loses the finite width damping profile.

V. FULL WAVE EQUATIONS WITH DAMPING

In this section we return to the exact integral equation, and develop a method of obtaining full wave equations that retains both the energy conservation properties noted in the last section and the finite damping profile as discussed by Lashmore-Davies and Dendy.[2] For the O-mode problem the integral equation can be written

$$\begin{aligned}&\epsilon(k)E(k)-iL\frac{\omega_p^2}{\omega^2}\int_{-\infty}^{k}e^{-(k-k')^2\rho^2/4}I_1\left(\frac{k'k\rho^2}{2}\right) \\ &\quad\times e^{-k'k\rho^2/2}E(k')dk'=0,\end{aligned} \tag{37}$$

where $\epsilon(k)$ is the usual cold plasma term,

$$\epsilon(k)=1-(\omega_p^2/\omega^2)-(k^2c^2/\omega^2).$$

On inverting the Fourier transform, Eq. (37) gives

$$\int_{-\infty}^{\infty} e^{ikx}\epsilon(k)E(k)dk - iL\frac{\omega_p^2}{\omega^2}\int_{-\infty}^{\infty} e^{ikx}\,dk \times\int_{-\infty}^{k} e^{-(k'-k)^2\rho^2/4}I_1\left(\frac{k'k\rho^2}{2}\right) \times e^{-k'k\rho^2/2}E(k')dk' = 0. \tag{38}$$

We now change the order of integration in the second term, which becomes

$$-iL\frac{\omega_p^2}{\omega^2}\int_{-\infty}^{\infty} dk'\,E(k')\int_{k'}^{\infty} dk\,e^{ikx}e^{-(k-k')^2\rho^2/4} \times I_1\left(\frac{kk'\rho^2}{4}\right)e^{-kk'\rho^2/2}. \tag{39}$$

If we assume that $E(k)$ contains only long-wavelength components, with $k\rho \ll 1$, then in (39) the k' integral will only be important over a range with $|k'\rho| \ll 1$. The inner integrand, over k, will then be small, because of the factor $e^{-(k-k')\rho^2/4}$, except when $k\rho$ is less than a few times unity. Combining these two conditions, we see that in the important part of the range of the double integral, $k'k\rho^2/2 \ll 1$, so that in those parts of the integrand which depend on this variable we can make a power series expansion. We begin by retaining only the lowest-order term, putting

$$I_1(kk'\rho^2/2)e^{-kk'\rho^2/2} \approx k'k\rho^2/4. \tag{40}$$

Now, (39) becomes

$$\begin{aligned}-iL\frac{\omega_p^2}{\omega^2}&\int_{-\infty}^{\infty} dk'\,E(k')\int_{k'}^{\infty} dk\,\frac{kk'\rho^2}{4}e^{ikx}e^{-(k-k')^2\rho^2/4}\\ &= -iL\frac{\omega_p^2}{\omega^2}\int_{-\infty}^{\infty} dk'\,E(k')e^{ik'x}\\ &\quad\times\int_0^{\infty} dk\,\frac{k'(k+k')\rho^2}{4}e^{ikx}e^{-k^2\rho^2/4}.\end{aligned} \tag{41}$$

Now

$$\int_0^{\infty} e^{ikx}e^{-k^2\rho^2/4}\,dk = e^{-x^2/\rho^2}\int_{-2ix/\rho^2}^{\infty} e^{-k^2\rho^2/4}\,dk = \frac{1}{i\rho}Z\left(\frac{x}{\rho}\right), \tag{42}$$

so that Eq. (40) becomes

$$\begin{aligned}-L\frac{\omega_p^2}{\omega^2}\frac{\rho}{4}&\int_{-\infty}^{\infty} dk'\,E(k')k'e^{ik'x}\left(-i\frac{d}{dx}+k'\right)Z\left(\frac{x}{\rho}\right)\\ &= L\frac{\omega_p^2}{\omega^2}\frac{\rho}{4}\left[Z'\left(\frac{x}{\rho}\right)\frac{dE}{dx}+Z\left(\frac{x}{\rho}\right)\frac{d^2E}{dx^2}\right]\\ &= L\frac{\omega_p^2}{\omega^2}\frac{\rho}{4}\frac{d}{dx}\left[Z\left(\frac{x}{\rho}\right)\frac{dE}{dx}\right].\end{aligned} \tag{43}$$

Including the inversion of the term $\epsilon(k)E(k)$ gives the differential equation

$$\frac{c^2}{\omega^2}\frac{d}{dx}\left\{\left[1-L\frac{\omega_p^2}{c^2}\frac{\rho}{4}Z\left(\frac{x}{\rho}\right)\right]\frac{dE}{dx}\right\}+\left(1-\frac{\omega_p^2}{\omega^2}\right)E=0. \tag{44}$$

If we take the asymptotic limit of the Z function in Eq. (43), we regain Eq. (30). Also Eq. (44) has a conservation law of the form

$$\frac{d}{dx}\,\mathrm{Im}\left\{E^*\left[1-L\frac{\omega_p^2}{c^2}\frac{\rho}{4}Z\left(\frac{x}{\rho}\right)\right]\frac{dE}{dx}\right\} - L\frac{\omega_p^2}{c^2}\frac{\rho}{4}Z_i\left(\frac{x}{\rho}\right)\left|\frac{dE}{dx}\right|^2=0, \tag{45}$$

representing a balance between the divergence of power flow and the dissipation due to the fact that Z has an imaginary part Z_i.

This expansion procedure is not entirely self-consistent, since $E(k)$ is assumed to have only components with $k\rho \ll 1$, although it satisfies Eq. (43), which evidently has coefficients with much shorter wavelength Fourier components. Another way of seeing this is to note that we are assuming $k\rho \ll 1$ in the first term of (37) but later assume that in the second term the important range of k extends up to $|k\rho|$ a few times unity. If we do not make any expansion, then the inversion of the Fourier transform can still be carried out, but leads to a much more complicated integrodifferential equation. This equation, and a comparison of its solutions with those of the differential equation is currently under investigation.

If we consider higher-order terms in the expansion equation (38), then it is obvious that all terms are of the form $a_n(kk')^n\rho^{2n}$ where a_n is a constant coefficient. In place of Eq. (41) this will produce a term

$$-iL\frac{\omega_p^2}{\omega^2}\int_{-\infty}^{\infty} dk'\,E(k')\int_0^{\infty} dk\,e^{i(k+k')x} \times e^{-k^2\rho^2/4}a_n\rho^{2n}k'^n(k+k')^n. \tag{46}$$

On inverting the Fourier transforms, each factor ik gives a derivative acting on $Z(x/\rho)$ and each factor ik' gives a derivative acting on $E(x)$. Thus, Eq. (46) gives

$$\frac{L}{\rho}\frac{\omega_p^2}{\omega^2}a_n\rho^{2n}(-1)^{n-1}\sum_{j=0}^{n}\binom{n}{j}\frac{d^{n+j}}{dx^{n+j}}E(x)\frac{d^{n-j}}{dx^{n-j}}Z\left(\frac{x}{\rho}\right) = \frac{L}{\rho}\frac{\omega_p^2}{\omega^2}a_n\rho^{2n}(-1)^n\frac{d^n}{dx^n}\left[Z\left(\frac{x}{\rho}\right)\frac{d^n}{dx^n}E\right]. \tag{47}$$

The procedure given here can be used to translate resonant terms in the dielectric tensor elements into differential operators to arbitrary order in an expansion in $k\rho$ with very little effort. For consistency, higher-order nonresonant terms should also be included, but these are easily obtained and do not involve x-dependent coefficients.

For comparison with the results of Lashmore-Davies and Dendy,[2] it is of interest to make the approximation

$$I_1\left(\frac{kk'\rho^2}{2}\right)e^{-k'k\rho^2/2} \approx \frac{k'k\rho^2}{4}\left(1-\frac{k'k\rho^2}{2}\right),$$

so that Eq. (37) yields

$$-\frac{1}{8}\frac{d^2}{dx^2}\left[\rho^3L\frac{\omega_p^2}{c^2}Z\left(\frac{x}{\rho}\right)\frac{d^2E}{dx^2}\right] + \frac{d}{dx}\left\{\left[1-\frac{\omega_p^2}{c^2}L\frac{\rho}{4}Z\left(\frac{x}{\rho}\right)\right]\frac{dE}{dx}\right\} + \left(\frac{\omega^2-\omega_p^2}{c^2}\right)E=0. \tag{48}$$

If a Wentzel–Kramers–Brillouin (WKB) approximation is made in Eq. (47) and terms up to k^2 retained, we can reproduce the local dispersion relation, Eq. (77) of Ref. 2. [Note that, in this equation, $\xi_{-1}Z(\xi_{-1})\approx 1$ and $\xi_{-1}Z'(\xi_{-1})\simeq 0$.] However, the present calculation shows that for a self-consistent full wave analysis of energy flow and conservation, either the last term in Eq. (77) of Ref. 2 should be neglected or extra terms corresponding to the higher derivatives of E in Eq. (47) should be added.

VI. CONCLUSIONS

We have investigated the effect of magnetic field variation across a Larmor orbit on wave propagation near a cyclotron resonance. The reason why the full wave calculations of Antonsen and Manheimer[1] did not reproduce the finite width absorption profile at perpendicular incidence, predicted by the theory of Lashmore-Davies and Dendy,[2] has been explained in detail; so has the role of the finite Larmor radius corrections in a nonuniform magnetic field in giving the correct energy conservation properties.

We have shown how to derive full wave equations which combine the features of both theories. The placing of derivatives with respect to the field and to space-dependent coefficients comes out in the form needed to give an energy conservation relation, and the resonant pole of the Antonsen and Manheimer treatment is replaced by a plasma dispersion function. The resonant contributions to the dielectric tensor elements can be expanded to arbitrarily high order in the ratio of Larmor radius to wavelength in a very simple way. The simplicity of the method compares favorably with variational and other techniques used previously for obtaining wave equations in an inhomogeneous plasma.[8–11]

Although our detailed calculation relates to the electron cyclotron O mode at perpendicular resonance, we emphasize again that this is just for convenience in illustrating the method, and that full wave equations for any wave in the vicinity of a cyclotron resonance can be obtained in this way. The analysis is of particular relevance to ion cyclotron resonance for which the relativistic broadening is negligible in comparison with the perpendicular damping mechanism discussed in this paper. Nor is the theory restricted to perpendicular incidence. We shall not discuss the inclusion of a finite value of $k_\parallel$ in detail, but simply point out that the most important change which it introduces is to replace the first exponential factor in the integral in Eq. (37) by

$$\exp[-(k-k')^2(\rho^2/4)(1+k_\parallel^2L^2)],$$

and hence the term in Eq. (42) becomes

$$\frac{1}{i\rho(1+k_\parallel^2L^2)^{1/2}}Z\left(\frac{x}{\rho(1+k_\parallel^2L^2)^{1/2}}\right).$$

The argument of the plasma dispersion function is the same as that which occurs in the local theory of Lashmore-Davies and Dendy,[12] which includes the effect of arbitrary $k_\parallel$. With the methods described in this paper, we are now in a position to extend the results already given by Lashmore-Davies and Dendy,[2,12] and use full wave calculations to find the modification to reflection and mode conversion coefficients predicted by the gyrokinetic theory.

If $k_\parallel^2L^2$ is large enough then the problems outlined above, regarding the consistency of the small $k\rho$ expansion, do not arise and, in the appropriate parameter range a perfectly self-consistent small Larmor radius approximation can be obtained. In the large Larmor radius regime, which is important in current experiments, we must again go to an integrodifferential equation.

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