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A unified theory of gyrotron and peniotron interactions

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The thin beam model of an axis-encircling relativistic electron beam in a uniform, constant, external magnetic field in a cylindrical waveguide is used to develop a unified theory of the various interactions of such a beam with the electromagnetic field. The model includes the gyrotron interaction (longitudinal displacement of the beam) and the peniotron interaction (transverse displacement of the beam). Space-charge effects are included. The resulting self-fields influence the boundary conditions of the perturbed electromagnetic field across the electron beam. For a beam with only azimuthal equilibrium flow and for propagation perpendicular to the equilibrium external magnetic field the well-known gyrotron instabilities are recovered. In addition, it is shown that the transverse beam modes (slow peniotron) are unstable for resonant interaction with the transverse electric (TE) waveguide mode. A general dispersion relation for propagation at any angle is derived for a beam with arbitrary energy and arbitrary pitch angle. This dispersion relation includes gyrotron, cyclotron autoresonance maser (CARM), peniotron, and harmonic autoresonant peniotron (HARP) interactions and the coupling between TE and transverse magnetic (TM) waveguide modes. For weak space charge, analytic expressions for the growth rates of the Doppler-shifted cyclotron resonance interaction of TE and TM modes with the gyrotron and slow peniotron modes are obtained.

I. INTRODUCTION

A relativistic electron beam rotating in an external, equilibrium magnetic field is a highly active system which is the basis of the gyrotron. Such a beam can also support a number of other interactions which might impinge on the working conditions of a gyrotron and which are also of interest in their own right. These interactions may be divided into two broad classes. The first, is of course, the gyrotron interaction that was discovered in 1959[1-3] and the second is the peniotron interaction that was proposed as the basis for a source of electromagnetic radiation[4] in 1964. It is noteworthy that even though these two basic interactions were discovered at about the same time only the gyrotron has come to fruition as a commercial device. A summary of recent research (experimental and theoretical) is given in Granatstein and Alexeff.[5] One possible reason for this is that the gyrotron interaction is inherently unstable, i.e., it does not require a resonant circuit. This point is well made, for example, in the work of Lau.[6,7] On the other hand, the peniotron does require the presence of a resonant circuit for instability.[8-10]

Both interactions couple to the electromagnetic field, by definition. However, the gyrotron interaction is often referred to as a longitudinal mode since the most significant displacement of the electron beam is in the plane of the unperturbed beam, i.e., in the azimuthal and axial directions. The peniotron interaction, on the other hand, is referred to as a transverse mode since displacement of the beam perpendicular to its unperturbed surface is a crucial element i.e., radial displacement. In the gyrotron field the problem is often simplified by neglecting the radial displacement and boundary conditions appropriate only to a longitudinal mode allowed.[6,7] In such analyses the peniotron interaction is excluded.

An early treatment of both interactions has been given by Sprangle.[11] This analysis is closely related to the present paper in that both treat the case of a thin rotating relativistic electron beam. A notational difference is that Sprangle[11] refers to the gyrotron mode as synchronous and a peniotron mode as a cyclotron mode. A more important difference is that the effect of the beam space charge is included in the present analysis whereas it has been neglected in Ref. 11.

In this paper we have extended the axis-encircling thin beam model used by Briggs and Neil[12] to derive a general, self-consistent dispersion relation. The model is valid for arbitrary beam energy and arbitrary pitch angle. By treating fully the azimuthal, axial, *and radial* displacements, both gyrotron and peniotron interactions are simultaneously included.

As noted by Lau[6,7] and Döhler and Gallagher,[13] the thin beam model enables the beam space charge to be included. The analysis is, however, limited to the case of weak space charge such that the effect of the self-fields on the equilibrium is negligible in comparison with the uniform external magnetic field. Nevertheless, the self-fields do influence the boundary conditions of the perturbed electromagnetic field across the electron beam where we show that all components of this field are discontinuous. This result has also been obtained in Ref. 13 from an alternative approach.

A detailed discussion of the nonuniform equilibria of intense non-neutral beams has been given by Davidson *et al.*[14] In addition, a comprehensive survey of the effect of self-fields on the cyclotron maser instability is contained in the recent book by Davidson.[15]

For zero axial wave number we demonstrate that the slow peniotron mode is unstable due to resonant interaction with a transverse electric (TE) circuit wave. This is in contrast to previous authors[12,16,17] who have stated that this mode is stable in the absence of a resistive wall. For arbitrary axial wave number and weak space charge we analyze the TE and transverse magnetic (TM) Doppler-shifted cyclotron resonance interactions with the longitudinal mode. Analytic solutions for the growth rates are obtained which include the cyclotron autoresonance maser (CARM).[18,19] We also obtain analytic expressions for the growth rates for the TE and TM Doppler-shifted cyclotron resonance interactions with the transverse mode. This case includes the harmonic autoresonant peniotron (HARP).[20] For both gyrotron and peniotron interactions, the TM mode growth rates vanish when the axial beam velocity is equal to the group velocity of the circuit wave i.e., grazing incidence. This property was first noted by Sprangle[11] and later for the gyrotron interaction by Lau[6] and Abubakirov.[21]

The plan of the paper is as follows. In Sec. II the boundary conditions for the perturbed electromagnetic field at the electron beam are obtained. Section III is concerned with the special case of zero axial flow and perpendicular propagation. The general dispersion relation is obtained in Sec. IV. This includes the effect of the coupling of TE and TM modes. The Doppler-shifted cyclotron resonance interaction of TE and TM modes with the gyrotron mode is discussed in Sec. V. Section VI discusses the corresponding interactions for the peniotron modes and a summary and conclusions are given in Sec. VII.

II. BOUNDARY CONDITIONS FOR A THIN BEAM

We begin this section by defining the parameters and variables required to describe the thin beam model. This is illustrated, for example, in Refs. 6 and 7, and consists of a cylindrical waveguide of radius r_w with an axis-encircling beam of radius r_L. An external, uniform equilibrium magnetic field is directed along the axis of the waveguide. We shall follow the notation of Lau[6] and denote the relativistic cyclotron frequency by ω_0,

$$\omega_0 = |e|B_0/\gamma_0 m_0, \tag{1}$$

where m_0 is the rest mass of the electron and γ_0 is the relativistic factor

$$\gamma_0 = (1 - v_{0\theta}^2/c^2 - v_{0z}^2/c^2)^{-1/2}. \tag{2}$$

The equilibrium velocity of the electron is

$$\mathbf{v}_0 = \mathbf{i}_\theta v_{0\theta} + \mathbf{i}_z v_{0z}, \tag{3}$$

where $\mathbf{i}_\theta$ and $\mathbf{i}_z$ are unit vectors in the azimuthal and axial directions and

$$v_{0\theta} = r_L \omega_0. \tag{4}$$

It is convenient to introduce

$$\beta_\perp = v_{0\theta}/c, \tag{5}$$

$$\beta_\parallel = v_{0z}/c. \tag{6}$$

We shall also require the quantities

$$\gamma_\perp = (1 - \beta_\perp^2)^{-1/2}, \tag{7}$$

$$\gamma_\parallel = (1 - \beta_\parallel^2)^{-1/2}. \tag{8}$$

In the unperturbed (equilibrium) state we assume a beam electron to be located at r_L, θ_0, and z_0. In the presence of a perturbation the electron is located at the position

$$\mathbf{r} = r_L + \xi(\theta_0,z_0,t)\mathbf{i}_r + \eta(\theta_0,z_0,t)\mathbf{i}_\theta + \chi(\theta_0,z_0,t)\mathbf{i}_z, \tag{9}$$

where $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_z$ are cylindrical unit vectors at the unperturbed position of the beam, and ξ, η and χ represent the radial, azimuthal, and axial displacement of the beam from its unperturbed position. The advantage of these dependent variables for electron beam problems and their relationship to other dependent variables has been discussed in detail by Bobroff,[22] who refers to them as "polarization variables." Bobroff[22] has pointed out that these "polarization variables" are particularly well suited to the problem of calculating the perturbed surface charges and currents that are required in order to obtain the boundary conditions at a beam–vacuum interface. Indeed, in the Eulerian description the polarization variable must be introduced in order to define the surface charges and currents. For a full discussion of these questions the reader is referred to the paper by Bobroff.[22]

In this paper we shall employ the thin beam model which has been widely used by other authors.[6,7,11-13,17] This model is analytically tractable because the physics of the relativistic electron beam arises in the motion of a surface. The coupling of the electron beam to the electromagnetic field is calculated through the boundary conditions. This procedure avoids the more difficult problem of solving the field equations in the inhomogeneous relativistic electron beam and matching the solutions to the fields outside the beam. As emphasized by Lau[6] and Döhler and Gallagher,[13] the thin beam model allows a self-consistent treatment of the beam space charge to be carried out.

The linearized equations of motion for the electron beam, written in "polarization" variables,[22] are taken from the reference by Lau[6] and are

$$\begin{aligned}\ddot{\xi} - \omega_0^2\beta_\perp^2\gamma_0\xi - \omega_0(\gamma_0^2/\gamma_\parallel^2)\dot{\eta} - \omega_0\beta_\perp\gamma_0^2\beta_\parallel\dot{\chi} \\ = (e/\gamma_0 m_0)(E_{1r} + v_{0\theta}B_{1z} - v_{0z}B_{1\theta}),\end{aligned} \tag{10}$$

$$\ddot{\eta} + \omega_0\dot{\xi} + \gamma_\parallel^2\beta_\perp\beta_\parallel\ddot{\chi} = (e\gamma_\parallel^2/m_0\gamma_0^3)(E_{1\theta} + v_{0z}B_{1r}), \tag{11}$$

$$\ddot{\chi} + \beta_\parallel\beta_\perp\gamma_\perp^2(\ddot{\eta} + \omega_0\dot{\xi}) = (e\gamma_\perp^2/m_0\gamma_0^3)(E_{1z} - v_{0\theta}B_{1r}), \tag{12}$$

where the overdot on a polarization variable denotes $i(\omega - m\omega_0 - kv_{0z})$ and we have assumed that all perturbed quantities vary as $\exp i(\omega t - m\theta - kz)$.

Let us now obtain the boundary conditions that the perturbed electromagnetic field must satisfy across the electron beam. For this we will require the perturbed surface charge and surface currents produced on the electron beam. Integrating the θ and z components of Maxwell's equation for $\nabla\times\mathbf{H}_1$ yields

$$H_{1z+} - H_{1z-} = -J_{1\theta}^s, \tag{13}$$

$$H_{1\theta+} - H_{1\theta-} = J_{1z}^s, \tag{14}$$

where the ($\pm$) subscripts denote the relevant field component just outside and just inside the electron beam and $J_{1\theta}^s$

and J^s_{1z} are the θ and z components of the perturbed surface current. We can obtain the surface current from Bobroff[22] who gives an expression for the perturbed current in terms of the perturbed polarization variable $\mathbf{r}_1$. The current is[22]

$$\mathbf{J}_1 = \rho_0\left(\frac{\partial}{\partial t}\mathbf{r}_1 + (\mathbf{v}_0\cdot\nabla)\mathbf{r}_1 - (\mathbf{r}_1\cdot\nabla)\mathbf{v}_0\right) - \mathbf{v}_0\nabla\cdot(\rho_0\mathbf{r}_1), \tag{15}$$

where ρ_0 is the unperturbed charge density of the beam. Taking the limit in which the beam thickness $\Delta r \to 0$,

$$\lim_{\Delta r \to 0} \rho_0 \Delta r = \sigma_0, \tag{16}$$

where σ_0 is the unperturbed surface charge density of the beam. In this limit we obtain the required θ and z components of the perturbed surface current from Eq. (15),

$$J^s_{1\theta} = \sigma_0 \frac{\partial \eta}{\partial t} - \sigma_0 v_{0\theta}\frac{\partial \chi}{\partial z} + \sigma_0 v_{0z}\frac{\partial \eta}{\partial z}, \tag{17}$$

$$J^s_{1z} = \sigma_0 \dot{\chi} - \sigma_0 v_{0z}\left(\frac{\xi}{r_L} + \frac{1}{r_L}\frac{\partial \eta}{\partial \theta} + \frac{\partial \chi}{\partial z}\right). \tag{18}$$

Substituting Eqs. (17) and (18) into Eqs. (13) and (14) gives two boundary conditions. For the general case, we require two more boundary conditions. In order to obtain these we follow the procedure of Briggs and Neil[12] who used the Maxwell equation $\nabla\times\mathbf{E}_1 = -(\partial/\partial t)\mathbf{B}_1$ in integral form

$$\oint \mathbf{E}\cdot d\mathbf{l} = -i\omega\mu_0 \int \mathbf{H}\cdot d\mathbf{S}. \tag{19}$$

First consider a contour in the (r,θ) plane whose surface normal points in the z direction. For this case, Briggs and Neil[12] obtained the following boundary condition:

$$E_{1\theta+} - E_{1\theta-} = \frac{-\sigma_0}{\epsilon_0}\left(\frac{1}{r_L}\frac{\partial \xi}{\partial \theta} + i\omega\frac{v_{0\theta}}{c^2}\xi\right). \tag{20}$$

In obtaining Eq. (20), the effect of the self-field, H_{0z}, which arises through the equilibrium azimuthal current, is included.

The final boundary condition is obtained by applying Eq. (19) to a contour in the (r,z) plane whose surface normal points in the θ direction. Again following Briggs and Neil,[12] we obtain

$$E_{1z+} - E_{1z-} = -\frac{\sigma_0}{\epsilon_0}\left(\frac{\partial \xi}{\partial z} - i\omega\frac{v_{0z}}{c^2}\xi\right). \tag{21}$$

Comparing with Ref. 12 we note that Eq. (21) contains an extra term due to the inclusion of an unperturbed axial beam velocity. As for Eq. (20), the effects of the space charge of the beam have again been included in Eq. (21) through the equilibrium radial electric field E_{0r} and the self-magnetic field $H_{0\theta}$ that arises from the equilibrium axial beam current.

A dimensionless parameter which is widely used in these problems[6,7,11-13] is ω_a^2/ω_0^2, where $\omega_a^2 \equiv e\sigma_0/\epsilon_0\gamma_0 m_0 r_L$, which gives a measure of the space charge of the beam. Lau[7] has argued that for $\omega_a^2/\omega_0^2 \ll 1$ the self-fields can be neglected compared with the external, equilibrium fields. However, the self-fields do contribute to the perturbed fields through the boundary conditions, as discussed above.

The boundary conditions given in Eqs. (13), (14), (17), (18), (20), and (21) are in agreement with those derived by Döhler and Gallagher[13] from a different point of view. These authors used a more formal procedure, expressing the surface charge and surface current in terms of the Dirac δ function and its derivative to show that *all* components of the perturbed electromagnetic field are discontinuous across the unperturbed position of the beam. Döhler and Gallagher[13] also derived boundary conditions for the radial components of the perturbed electromagnetic field E_{1r} and B_{1r}. These can also be obtained from a consideration of the self-fields, the two divergence equations for $\mathbf{E}_1$ and $\mathbf{B}_1$, and the expression for the surface charge,

$$\sigma_1^s = -\sigma_0\left(\frac{\xi}{r_L} + \frac{1}{r_L}\frac{\partial \eta}{\partial \theta} + \frac{\partial \chi}{\partial z}\right), \tag{22}$$

which can be obtained from the equation for the charge density in polarization variables given by Bobroff.[22]

III. THE DISPERSION RELATION FOR $v_{0z}=0$, $k=0$

Let us discuss the special case of perpendicular propagation and zero axial beam velocity. In doing this we will recover well-known results which will serve to illustrate the route to be followed in the general case, which is algebraically complicated. We shall also point out a result for the transverse modes which to the best of the author's knowledge has not been previously noted.

For $k=0$, $v_{0z}=0$, Eqs. (10) and (11), which describe the TE modes, decouple from Eq. (12), which describes the TM modes. In this section we shall restrict attention to the TE modes. Equations (10) and (11) become

$$-(\omega - m\omega_0)^2\xi - \omega_0^2\beta_1^2\gamma_0\xi - i(\omega - m\omega_0)\omega_0^2\eta = (e/\gamma_0 m_0)(E_{1r} + v_{0\theta}B_{1z}), \tag{23}$$

$$-(\omega - m\omega_0)^2\eta + i(\omega - m\omega_0)\omega_0\xi = (e/m_0\gamma_0^3)E_{1\theta}. \tag{24}$$

We follow the method described in Neil and Heckrotte[17] and Briggs and Neil[12] to obtain the dispersion relation from Eqs. (23) and (24). This is achieved by using the boundary conditions to express the perturbed electromagnetic field in terms of the beam displacement variables ξ and η. For this purpose we use the two boundary conditions given by Eqs. (13), (17), and (20). Since $k=0$, $v_{0z}=0$ the boundary conditions are

$$E_{1\theta+} - E_{1\theta-} = -\frac{\sigma_0}{\epsilon_0}\left(-\frac{im}{r_L}\xi + i\omega\frac{v_{0\theta}}{c^2}\xi\right), \tag{25}$$

$$H_{1z+} - H_{1z-} = -i\omega\sigma_0\eta. \tag{26}$$

We choose to express $E_{1\theta}$ in terms of H_{1z}. With the aid of Maxwell's equations, we find

$$E_{1\theta} = \frac{i\omega\mu_0}{p^2}\frac{\partial H_{1z}}{\partial r}, \tag{27}$$

where $p = \omega/c$. It is convenient to introduce the wave admittances defined (for example) by Lau,[6]

$$b_{h\pm} = \mp (H_{1z\pm}/H'_{1z\pm}), \tag{28}$$

where

$$H'_{1z} = \frac{\partial}{\partial (pr)} H_{1z} \tag{29}$$

and H_{1z} is a solution of Bessel's equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_{1z}}{\partial r}\right) + p^2 H_{1z} - \frac{m^2}{r^2} H_{1z} = 0. \tag{30}$$

With the aid of Eqs. (27)–(29) we obtain

$$E_{1\theta-} = (i\omega\mu_0/p)(H_{1z-}/b_{h-}), \tag{31}$$

$$E_{1\theta+} = (-i\omega\mu_0/p)(H_{1z+}/b_{h+}). \tag{32}$$

Equation (25) can now be written as

$$\frac{\omega\mu_0}{p}\left(\frac{H_{1z+}}{b_{h+}} + \frac{H_{1z-}}{b_{h-}}\right) = \frac{\sigma_0}{\epsilon_0}\left(\frac{\omega}{c}\beta_\perp - \frac{m}{r_L}\right)\xi. \tag{33}$$

Solving Eqs. (26) and (33) for $H_{1z\pm}$ in terms of ξ and η we obtain

$$H_{1z-} = \frac{ib_{h-}\sigma_0}{(b_{h+} + b_{h-})} \times \left[\omega\eta + \frac{ib_{h+}p}{\omega\mu_0\epsilon_0}\frac{m}{r_L}\left(1 - \frac{\omega}{c}\beta_\perp\frac{r_L}{m}\right)\xi\right], \tag{34}$$

$$H_{1z+} = \frac{-ib_{h+}\sigma_0}{(b_{h+} + b_{h-})} \times \left[\omega\eta - \frac{ib_{h-}pm}{\omega\mu_0\epsilon_0 r_L}\left(1 - \frac{\omega}{c}\beta_\perp\frac{r_L}{m}\right)\xi\right]. \tag{35}$$

With the aid of Eqs. (31), (32), (34), and (35) we can express $E_{1\theta\pm}$ in terms of ξ and η. Similarly, from Maxwell's equations, we obtain the relation

$$E_{1r} = -(\omega\mu_0/p^2)(m/r)H_{1z}. \tag{36}$$

Combining Eq. (36) with Eqs. (34) and (35) allows $E_{1r\pm}$ to be expressed in terms of ξ and η. The procedure outlined above enables us to calculate the average force exerted by the perturbed electromagnetic field on the electron beam. The radial and azimuthal components of this average force are defined[12,17] by the equations

$$\langle E_{1r} + v_{0\theta}B_{1z}\rangle = \tfrac{1}{2}[E_{1r+} + E_{1r-} + v_{0\theta}(B_{1z-} + B_{1z+})], \tag{37}$$

$$\langle E_{1\theta}\rangle = \tfrac{1}{2}(E_{1\theta-} + E_{1\theta+}). \tag{38}$$

Carrying out this procedure, we obtain the following expressions for the components of the average perturbed field:

$$\langle E_{1r} + v_{0\theta}B_{1z}\rangle = \frac{\sigma_0\mu_0 c\omega[(cm/\omega r_L) - \beta_\perp]}{(b_{h+} + b_{h-})} \times \left[\left(\frac{cm}{\omega r_L} - \beta_\perp\right)b_{h-}b_{h+}\xi - \frac{i}{2}(b_{h-} - b_{h+})\eta\right], \tag{39}$$

$$\langle E_{1\theta}\rangle = \frac{-\sigma_0\mu_0 c\omega}{(b_{h+} + b_{h-})}\left[\frac{i}{2}\left(\frac{cm}{\omega r_L} - \beta_\perp\right) \times (b_{h+} - b_{h-})\xi + \eta\right]. \tag{40}$$

Substituting Eqs. (39) and (40) into the equations of motion for the beam, Eqs. (23) and (24), and taking the determinant yields the desired dispersion relation

$$\begin{aligned}(\omega - m\omega_0)^2&(\omega - m\omega_0 - \omega_0)(\omega - m\omega_0 + \omega_0)\\ &= \frac{m\omega_a^2}{(b_{h+} + b_{h-})}\left[\frac{\omega r_L}{mc}\beta_\perp^2\omega_0^2 + \frac{\omega r_L}{mc}\frac{(\omega - m\omega_0)^2}{\gamma_0^2}\right.\\ &\quad + \omega_0(\omega - m\omega_0)\left(1 - \frac{\omega\beta_\perp^2}{m\omega_0}\right)(b_{h+} - b_{h-})\\ &\quad\left. - \frac{mc}{\omega r_L}(\omega - m\omega_0)^2\left(1 - \frac{\omega\beta_\perp^2}{m\omega_0}\right)^2 b_{h-}b_{h+}\right]\\ &\quad + \frac{\omega_a^4 m^2}{4\gamma_0^2}\left(1 - \frac{\omega\beta_\perp^2}{m\omega_0}\right)^2.\end{aligned} \tag{41}$$

This is the same dispersion relation as the one derived by Briggs and Neil.[12] We have written the equation in a different form which displays the longitudinal and transverse beam modes explicitly. Since the quantity (ω_a^2/ω_0^2) is generally small compared with unity, Eq. (41) is amenable to a perturbation solution. It is straightforward to reduce this equation to the dispersion relation obtained by Lau[7] by assuming $\omega = m\omega_0 + \delta\omega$. Lau showed[7] that this equation describes the negative mass, cyclotron maser, and diocotron instabilities which are all longitudinal modes of the beam. As pointed out by Lau,[7] in the nonrelativistic limit, only the diocotron instability, which comes from the last term on the right-hand side of Eq. (41), remains. In the relativistic case, these longitudinal modes are the basis of the gyrotron.

Let us now discuss the transverse modes which are given by

$$\omega = m\omega_0 \pm \omega_0 + \delta\omega. \tag{42}$$

The upper sign is often referred to as the fast peniotron mode and the lower sign as the slow peniotron mode.[13] It has been stated[12,16,17] that the transverse modes are stable in the absence of wall losses. This is evidently the case for a nonresonant circuit. However, we shall now consider the transverse solutions when the circuit is resonant.

The condition for the circuit to be resonant is

$$b_{h+} + b_{h-} = 0. \tag{43}$$

Equation (43) is the dispersion relation for TE modes of a cylindrical waveguide. The derivation of the admittance functions $b_{h\pm}$ is given in Lau.[6] For weak space charge we may use coupled mode analysis and calculate the correction to the "cold" beam and waveguide modes due to resonant interaction. Thus, in addition to Eq. (42), we also assume

$$\omega = \omega_c + \delta\omega, \tag{44}$$

where ω_c is the cutoff frequency of the TE waveguide mode. The dispersion relation, Eq. (43), reduces to

$$J'_m(pr_w) = 0. \tag{45}$$

Thus, the cutoff frequency is given by

$$\omega_c = (c/r_w)x_{nm}, \tag{46}$$

where x_{nm} is the nth zero of J'_m.

Substituting Eq. (44) into Eq. (43), it is straightforward to obtain the result[6]

$$b_{h+} + b_{h-} \simeq \alpha\delta\omega. \tag{47}$$

The quantity α is given by

$$\alpha = \frac{c}{r_L\omega_c^2}(x_{nm}^2 - m^2)\frac{J_m^2(x_{nm})}{[J'_m(x_{nm}r_L/r_w)]^2}. \tag{48}$$

We note that since $x_{nm}^2 > m^2$, α is positive definite.

We are now in a position to obtain a perturbation solution for the slow peniotron mode interacting with a resonant circuit wave. Returning to Eq. (41) we look for a solution given by Eq. (44) assuming the resonance condition

$$m\omega_0 - \omega_0 = \omega_c. \tag{49}$$

We may therefore replace ω by $m\omega_0 - \omega_0$ and b_{h+} by $-b_{h-}$ except in the two resonant terms. The dispersion relation becomes

$$(\delta\omega)^2 \simeq -\frac{\omega_a^2}{2}\frac{\omega_c r_L}{\alpha\omega_0 c} \times\left[1 + \frac{mc}{\omega_c r_L}b_{h-}\left(1 - \frac{(m-1)}{m}\beta_\perp^2\right)\right]^2, \tag{50}$$

where[6]

$$b_{h-} = J_m(pr_L)/J'_m(pr_L). \tag{51}$$

Thus, the slow peniotron mode is unstable only when it is resonant with the TE circuit mode. This is in contrast to the gyrotron (longitudinal) interaction which is unstable even for a nonresonant circuit mode. In the nonrelativistic limit where $\beta_\perp \to 0$ the slow peniotron instability still occurs whereas the cyclotron maser or negative mass instabilities disappear.

To complete this section we obtain a solution when the fast peniotron mode is resonant with the circuit. We again look for a solution given by Eq. (44) with the resonance condition appropriate to the fast peniotron mode,

$$m\omega_0 + \omega_0 = \omega_c. \tag{52}$$

For this case, Eq. (41) reduces to

$$(\delta\omega)^2 \simeq \frac{\omega_a^2}{2}\frac{\omega_c r_L}{\alpha\omega_0 c} \times\left[1 - \frac{mc}{\omega_c r_L}\left(1 - \frac{(m+1)}{m}\beta_\perp^2\right)b_{h-}\right]^2. \tag{53}$$

Clearly, the fast peniotron mode is always stable.

IV. THE GENERAL DISPERSION RELATION

In the previous section we discussed the special case of zero axial flow and perpendicular propagation. Under these circumstances the TE and TM modes decouple. We shall now consider the general case with azimuthal and axial flow and perturbations which vary axially in addition to radially and azimuthally. Thus, we assume all perturbed quantities vary as $\exp i(\omega t - kz - m\theta)$. The equations of motion for the beam, Eqs. (10)–(12), now become

$$\begin{aligned}
&(\omega - m\omega_0 - kv_{0z})^2\xi + \omega_0^2\beta_\perp^2\gamma_0^2\xi \\
&\quad + i\omega_0(\gamma_0^2/\gamma_\parallel^2)(\omega - m\omega_0 - kv_{0z})\eta \\
&\quad + i\omega_0\beta_\perp\beta_\parallel\gamma_0^2(\omega - m\omega_0 - kv_{0z})\chi \\
&\quad = -(e/\gamma_0 m_0)\langle E_{1r} + v_{0\theta}B_{1z} - v_{0z}B_{1\theta}\rangle,
\end{aligned} \tag{54}$$

$$\begin{aligned}
&(\omega - m\omega_0 - kv_{0z})^2\eta - i\omega_0(\omega - m\omega_0 - kv_{0z})\xi \\
&\quad + \beta_\perp\beta_\parallel\gamma_\parallel^2(\omega - m\omega_0 - kv_{0z})^2\chi \\
&\quad = -(e\gamma_\parallel^2/\gamma_0^3 m_0)\langle E_{1\theta} + v_{0z}B_{1r}\rangle,
\end{aligned} \tag{55}$$

$$\begin{aligned}
&(\omega - m\omega_0 - kv_{0z})^2\chi + \beta_\perp\beta_\parallel\gamma_\perp^2(\omega - m\omega_0 - kv_{0z})^2\eta \\
&\quad - i\beta_\perp\beta_\parallel\gamma_\perp^2\omega_0(\omega - m\omega_0 - kv_{0z})\xi \\
&\quad = -(e\gamma_\perp^2/\gamma_0^3 m_0)\langle E_{1z} - v_{0\theta}B_{1r}\rangle.
\end{aligned} \tag{56}$$

Since we shall follow the same procedure described in the previous section we have immediately introduced the average electromagnetic field acting on the beam in Eqs. (54)–(56). As before, this average field will be expressed in terms of the beam displacement variables with the aid of the boundary conditions and Maxwell's equations.

Using Eqs. (13), (14), (17), (18), (20), and (21) we write the boundary conditions for the general case,

$$(H_{1z+} - H_{1z-}) = -i(\omega - kv_{0z})\sigma_0\eta - ikv_{0\theta}\sigma_0\chi, \tag{57}$$

$$H_{1\theta+} - H_{1\theta-} = i(\omega - m\omega_0)\sigma_0\chi - (\sigma_0 v_{0z}/r_L)(\xi - im\eta), \tag{58}$$

$$E_{1\theta+} - E_{1\theta-} = -\frac{i\sigma_0}{\epsilon_0}\left(\frac{\omega}{c}\beta_\perp - \frac{m}{r_L}\right)\xi, \tag{59}$$

$$E_{1z+} - E_{1z-} = (i\sigma_0/\epsilon_0)[k + (\omega/c)\beta_\parallel]\,\xi. \tag{60}$$

We use Maxwell's equations to express $E_{1\theta}$ and $H_{1\theta}$ in terms of E_{1z} and H_{1z}. Thus

$$E_{1\theta} = -\frac{k}{p^2}\frac{m}{r}E_{1z} + \frac{i\omega\mu_0}{p}H'_{1z}, \tag{61}$$

$$H_{1\theta} = -\frac{i\omega\epsilon_0}{p}E'_{1z} - \frac{k}{p^2}\frac{m}{r}H_{1z}, \tag{62}$$

where,

$$p^2 = (\omega^2/c^2) - k^2 \tag{63}$$

and H'_{1z}, E'_{1z} again denote $(\partial/\partial pr)H_{1z}$ and $(\partial/\partial pr)E_{1z}$, respectively.

We note that $E_{1\theta}$ and $H_{1\theta}$ are made up of TE and TM components. We again follow Lau[6] and introduce the wave admittances for a cylindrical waveguide for the TE and TM modes. For the TE modes we have

$$b_{h\pm} = \mp(H_{1z\pm}/H'_{1z\pm}). \tag{64}$$

The reader is referred to the reference by Lau[6] for the derivation of the admittance functions. These functions contain the information concerning the propagation of the "cold" waveguide modes. For the TM modes,

$$b_{e\pm} = \mp(E'_{1z\pm}/E_{1z\pm}). \tag{65}$$

With the aid of Eqs. (61), (62), (64), and (65), $E_{1\theta}$ and $H_{1\theta}$ can be expressed in terms of E_{1z} and H_{1z}. Equations

(57)–(60) can then be solved for $H_{1z\pm}$, $E_{1z\pm}$ as functions of ξ, η, and χ. Carrying out this procedure, we obtain

$$H_{1z\pm} = \frac{ib_{h\pm}\sigma_0}{(b_{h+}+b_{h-})}\left\{\frac{ib_h \mp m}{\omega\epsilon_0\mu_0 p r_L}\left[k\left(k+\frac{\omega}{c}\beta_\parallel\right) + p^2\left(1-\frac{\omega}{c}\beta_\perp\frac{r_L}{m}\right)\right]\xi \mp (\omega - kv_{0z})\eta \mp kv_{0\theta}\chi\right\}, \quad (66)$$

$$E_{1z\pm} = \frac{(\sigma_0 p/\omega\epsilon_0)}{(b_{e+}+b_{e-})}\left\{i\left[\frac{v_{0z}}{r_L} \pm \frac{\omega}{p} b_{e\mp}\left(k+\frac{\omega}{c}\beta_\parallel\right)\right]\xi + \frac{\omega m}{p^2 r_L}\left(\frac{\beta_\parallel\omega}{c} - k\right)\eta + \left(\omega - m\omega_0 - \frac{m\omega_0 c^2 k^2}{(\omega^2 - c^2k^2)}\right)\chi\right\}. \quad (67)$$

We are now in a position to express the average electromagnetic field at the position of the beam in terms of ξ, η, and χ. In order to do this, we must also express E_{1r} and B_{1r} in terms of E_{1z} and H_{1z}. Again, using Maxwell's equations, we obtain

$$E_{1r} = -\frac{ik}{p}E'_{1z} - \frac{\omega\mu_0}{p^2}\frac{m}{r_L}H_{1z}, \quad (68)$$

$$B_{1r} = (\omega\epsilon_0\mu_0 m/p^2 r_L)E_{1z} - (ik\mu_0/p)H'_{1z}. \quad (69)$$

We may now use Eqs. (61), (62), and (64)–(69) to express $E_{1\theta\pm}$, $H_{1\theta\pm}$, $E_{1r\pm}$, and $B_{1r\pm}$ in terms of ξ, η, and χ. Using these expressions we obtain the following results for the average electromagnetic field:

$$(e/\gamma_0 m_0)\langle E_{1r} + v_{0\theta}B_{1z} - v_{0z}B_{1\theta}\rangle = a_{11}\xi + a_{12}\eta + a_{13}\chi, \quad (70)$$

$$(e\gamma_\parallel^2/\gamma_0^3 m_0)\langle E_{1\theta} + v_{0z}B_{1r}\rangle = a_{21}\xi + a_{22}\eta + a_{23}\chi, \quad (71)$$

$$(e\gamma_\perp^2/\gamma_0^3 m_0)\langle E_{1z} - v_{0\theta}B_{1r}\rangle = a_{31}\xi + a_{32}\eta + a_{33}\chi, \quad (72)$$

where

$$a_{ij} = a_{ij}^e + a_{ij}^h. \quad (73)$$

The coefficients with a superscript e refer to the "cold" TM modes and those with superscript h to the cold TE modes. Thus, the average, perturbed electromagnetic field is determined by 18 coefficients. The values of these coefficients are given in the Appendix.

Substituting Eqs. (70)–(72) into Eqs. (54)–(56) we obtain the general dispersion relation from the determinantal equation:

$$\begin{vmatrix} \varpi^2 + \omega_0^2\beta_\perp^2\gamma_0^2 + a_{11} & i\omega_0\varpi(\gamma_0^2/\gamma_\parallel^2) + a_{12} & i\omega_0\varpi\beta_\perp\beta_\parallel\gamma_0^2 + a_{13} \\ -i\omega_0\varpi + a_{21} & \varpi^2 + a_{22} & \varpi^2\beta_\perp\beta_\parallel\gamma_\parallel^2 + a_{23} \\ -i\omega_0\varpi\beta_\parallel\beta_\perp\gamma_\perp^2 + a_{31} & \varpi^2\beta_\parallel\beta_\perp\gamma_\perp^2 + a_{32} & \varpi^2 + a_{33} \end{vmatrix} = 0, \quad (74)$$

where

$$\varpi = \omega - m\omega_0 - kv_{0z}. \quad (75)$$

We shall not write out this dispersion relation explicitly since it is rather lengthy due to the complicated form of the a_{ij}'s, none of which are zero in the general case. For $k=0$, $v_{0z}=0$ the TE and TM modes decouple as is well known and Eq. (74) reproduces the equation discussed in the previous section. In the general case the TE and TM modes are coupled. The dispersion relation given by Eq. (74), in addition to describing this coupling contains, the gyrotron, CARM, peniotron, and HARP interactions. A full evaluation of this equation will require a numerical solution. However, it is possible to make further analytic progress by expanding in the space-charge parameter ω_a^2/ω_0^2.

Since the coefficients a_{ij} are all proportional to ω_a^2 we can approximate Eq. (74) by retaining terms up to ω_a^2. This means all terms either quadratic or cubic in the a_{ij}'s are neglected.

V. THE TE AND TM DOPPLER-SHIFTED GYROTRON INTERACTIONS

We begin this section by carrying out the procedure outlined at the end of the previous section. Keeping only those terms which are linear in the coefficients (for $\omega_a^2/\omega_0^2 \ll 1$) we can approximate the general dispersion relation by the form

$$\begin{aligned}\varpi^2(\varpi^2-\omega_0^2) \simeq {} & -\varpi^2 a_{11} - \frac{\gamma_0^2}{\gamma_\parallel^2\gamma_\perp^2}(\varpi^2 + \omega_0^2\beta_\perp^2\gamma_\perp^2)a_{22} \\ & - i\omega_0\varpi a_{12} + i\omega_0\varpi\frac{\gamma_0^2}{\gamma_\parallel^2}a_{21} \\ & - \frac{\gamma_0^2}{\gamma_\parallel^2\gamma_\perp^2}(\varpi^2-\omega_0^2)a_{33} \\ & + \frac{\gamma_0^2}{\gamma_\parallel^2}\beta_\parallel\beta_\perp(\varpi^2-\omega_0^2)a_{23} \\ & + \frac{\gamma_0^2}{\gamma_\perp^2}\beta_\parallel\beta_\perp(\varpi^2-\omega_0^2)a_{32}.\end{aligned} \quad (76)$$

Equation (76) contains both gyrotron and peniotron interactions for conditions of weak space charge. Let us first consider gyrotron interactions. In this case, we can simplify Eq. (76) further by setting $\varpi = 0$ on the right-hand side of this equation and also in the term $(\varpi^2 - \omega_0^2)$ on the left-hand side, giving

$$\varpi^2 \simeq \frac{\gamma_0^2}{\gamma_\parallel^2}\beta_\perp^2 a_{22} - \frac{\gamma_0^2}{\gamma_\parallel^2\gamma_\perp^2}a_{33} + \frac{\gamma_0^2}{\gamma_\parallel^2}\beta_\parallel\beta_\perp a_{23} + \frac{\gamma_0^2}{\gamma_\perp^2}\beta_\parallel\beta_\perp a_{32}. \quad (77)$$

When the gyrotron mode is not resonant with either the TE or TM circuit wave then both TE and TM coefficients are required in Eq. (77). When the beam mode is resonant with one of these modes we may approximate the coupling term by one set of coefficients.

A. The TE Doppler-shifted gyrotron interaction

For the case of resonance between a TE mode and a gyrotron beam mode we neglect the effect of the TM fields which are then off resonance. This is the most unstable situation since the off resonance fields will produce a stabilizing effect. For small enough values of the parameter ω_a^2/ω_0^2 this procedure is justified. The required coefficients are (see the Appendix)

$$a_{22}^h = \frac{-\omega_a^2}{(b_{h+}+b_{h-})}\frac{\gamma_\parallel^2}{\gamma_0^2}\left(\frac{\omega}{c}-k\beta_\parallel\right)^2\frac{r_L}{p}, \tag{78}$$

$$a_{23}^h = \frac{-\omega_a^2}{(b_{h+}+b_{h-})}\frac{\gamma_\parallel^2}{\gamma_0^2}\left(\frac{\omega}{c}-k\beta_\parallel\right)\frac{kr_L\beta_\perp}{p}, \tag{79}$$

$$a_{32}^h = \frac{-\omega_a^2}{(b_{h+}+b_{h-})}\frac{\gamma_\perp^2}{\gamma_0^2}\frac{k\beta_\perp}{p}\left(\frac{\omega}{c}-k\beta_\parallel\right)r_L, \tag{80}$$

$$a_{33}^h = \frac{-\omega_a^2}{(b_{h+}+b_{h-})}\frac{\gamma_\perp^2}{\gamma_0^2}\frac{k^2r_L}{p}\beta_\perp^2. \tag{81}$$

Substituting Eqs. (78)–(81) into Eq. (77) we obtain

$$\overline{\omega}^2 = -\omega_a^2\beta_\perp^2 pr_L/(b_{h+}+b_{h-}). \tag{82}$$

For resonant beam and TE modes we may write

$$\overline{\omega}\simeq\delta\omega, \tag{83}$$

$$b_{h+}+b_{h-}\simeq\alpha_h\delta\omega, \tag{84}$$

where

$$\alpha_h = \frac{(c^2k^2+\omega_c^2)^{1/2}}{p^2c^2pr_L}(x_{nm}^2-m^2)\frac{J_m^2(x_{nm})}{[J'_m(x_{nm}r_L/r_w)]^2}. \tag{85}$$

The dispersion relation given by Eq. (82) can now be written

$$(\delta\omega)^3\simeq-\omega_a^2\beta_\perp^2 pr_L/\alpha_h. \tag{86}$$

This is in agreement with the dispersion relation obtained by Sprangle[11] [Eq. (9a) of Ref. 11]. It is also in agreement with Lau[6] [Eq. (96) of Ref. 6] except that we have neglected the nonresonant TM terms. The agreement with Ref. 11 is due to the neglect of the TE–TM coupling which is included in Eq. (77).

B. The TM Doppler-shifted gyrotron interaction

Let us now consider the situation where the beam mode is resonant with a TM mode. For this case, the required coefficients are (see the Appendix)

$$a_{22}^e = \frac{\omega_a^2}{(b_{e+}+b_{e-})}\frac{\gamma_\parallel^2}{\gamma_0^2}\frac{m^2}{p^3r_L}\left(\frac{\omega\beta_\parallel}{c}-k\right)^2, \tag{87}$$

$$a_{23}^e = \frac{\omega_a^2}{(b_{e+}+b_{e-})}\frac{\gamma_\parallel^2}{\gamma_0^2}\frac{m^2}{\omega p}\times\left(\frac{\omega\beta_\parallel}{c}-k\right)\left(\omega-m\omega_0-\frac{m\omega_0k^2}{p^2}\right), \tag{88}$$

$$a_{32}^e = \frac{\omega_a^2}{(b_{e+}+b_{e-})}\frac{\gamma_\perp^2}{\gamma_0^2}\left(1-\frac{m\omega_0\omega}{c^2p^2}\right)\frac{m}{p}\left(\frac{\omega\beta_\parallel}{c}-k\right), \tag{89}$$

$$a_{33}^e = \frac{\omega_a^2}{(b_{e+}+b_{e-})}\frac{\gamma_\perp^2}{\gamma_0^2}\left(1-\frac{m\omega_0\omega}{c^2p^2}\right)\times\left(1-\frac{m\omega_0}{\omega}-\frac{m\omega_0k^2}{p^2\omega}\right)pr_L. \tag{90}$$

Substituting Eqs. (87)–(90) into Eq. (77) we obtain

$$\overline{\omega}^2 = \frac{\omega_a^2}{(b_{e+}+b_{e-})}\left[\frac{\beta_\perp^2m^2}{p^3r_L}\left(\frac{\omega\beta_\parallel}{c}-k\right)^2 + 2\beta_\parallel\beta_\perp\frac{m}{p}\left(\frac{\omega\beta_\parallel}{c}-k\right)\left(1-\frac{m\omega_0\omega}{p^2c^2}\right) - \frac{pr_L}{\gamma_\parallel^2}\left(1-\frac{m\omega_0\omega}{p^2c^2}\right)^2\right]. \tag{91}$$

For resonant beam and TM modes we again use Eq. (83) for the beam mode and the analog of Eq. (84) for the TM mode, which is

$$b_{e+}+b_{e-}\simeq-\alpha_e\delta\omega, \tag{92}$$

where

$$\alpha_e = (c^2k^2+\omega_c^2)^{1/2}\frac{r_w^2}{y_{nm}c^2}\frac{r_w}{r_L}\frac{[J'_m(y_{nm})]^2}{J_m^2(y_{nm}r_L/r_w)} \tag{93}$$

and y_{nm} is the nth zero of the J_m Bessel function.

The dispersion relation for the TM Doppler-shifted gyrotron interaction becomes

$$(\delta\omega)^3\simeq-\frac{\omega_a^2}{\alpha_e}\left[\frac{\beta_\perp^2m^2}{p^3r_L}\left(\frac{\omega\beta_\parallel}{c}-k\right)^2 + 2\beta_\parallel\beta_\perp\frac{m}{p}\left(\frac{\omega\beta_\parallel}{c}-k\right)\left(1-\frac{m\omega_0\omega}{p^2c^2}\right) - \frac{pr_L}{\gamma_\parallel^2}\left(1-\frac{m\omega_0\omega}{p^2c^2}\right)^2\right]. \tag{94}$$

This dispersion relation is more general than the equation obtained by Lau[6] for Doppler-shifted cyclotron resonance. In Ref. 6, the perturbed radial force was neglected and the radial displacement was approximated. The present analysis treats both these effects exactly (within the confines of a linearized theory). Fliflet[23] has also obtained a dispersion relation for the TM Doppler-shifted cyclotron resonance but he also ignored the radial displacement. Nevertheless, Eq. (94) is in agreement with Refs. 6, 11, and 23 and with Abubakirov[21] when the condition for grazing incidence is satisfied, namely $v_{0z}=v_g$ where v_g is the group velocity of the circuit wave. At grazing incidence,

$$k=\gamma_{\parallel}(\omega_c/c)\beta_{\parallel}\equiv k_g \quad (95)$$

when it is clear that the term $(\omega\beta_{\parallel}/c-k)$ vanishes. The condition for grazing incidence is

$$1/\gamma_{\parallel}^2=m^2\omega_0^2/\omega_c^2. \quad (96)$$

With the aid of Eqs. (95) and (96) it can be demonstrated that the term $(1-m\omega_0\omega/p^2c^2)$ also vanishes at grazing incidence thus confirming the result that the TM instability disappears under these conditions.

VI. THE TE AND TM DOPPLER-SHIFTED PENIOTRON INTERACTIONS

In this section we shall analyze the Doppler-shifted peniotron interactions under conditions of weak space charge, $\omega_a^2/\omega_0^2 \ll 1$. We therefore start from Eq. (76) and specialize to the peniotron modes by utilizing the fact that $\overline{\omega}^2-\omega_0^2\simeq 0$ for these modes. This enables the last three terms on the right-hand side of Eq. (76) to be neglected. Thus for peniotron interactions the dispersion relation can be approximated by

$$\overline{\omega}^2(\overline{\omega}^2-\omega_0^2)\simeq-\overline{\omega}^2a_{11}-(\gamma_0^2/\gamma_{\parallel}^2\gamma_{\perp}^2)(\overline{\omega}^2+\omega_0^2\beta_{\perp}^2\gamma_{\perp}^2)a_{22}$$
$$-i\omega_0\overline{\omega}a_{12}+i\omega_0\overline{\omega}(\gamma_0^2/\gamma_{\parallel}^2)a_{21}. \quad (97)$$

A. The TE Doppler-shifted peniotron interaction

For the assumed conditions of weak space charge, we neglect the TE–TM coupling and consider the two cases separately. For an unstable peniotron interaction the slow peniotron mode must be resonant with a circuit mode. We consider first the case of a resonant TE mode. Under these conditions we retain only the a_{ij}^h coefficients in Eq. (97). The required coefficients are (see the Appendix)

$$a_{11}^h=\frac{\omega_a^2(m/\omega p)b_{h-}b_{h+}}{(b_{h+}+b_{h-})}\left(\frac{m}{p^2r_L}(\omega-kv_{0z})-v_{0\theta}\right)$$
$$\times\left[k\left(k+\frac{\omega}{c}\beta_{\parallel}\right)+p^2\left(1-\frac{\omega}{c}\beta_{\perp}\frac{r_L}{m}\right)\right], \quad (98)$$

$$a_{12}^h=\frac{i\omega_a^2(b_{h+}-b_{h-})}{2(b_{h+}+b_{h-})}\left[\frac{m}{p^2r_L}\left(\frac{\omega}{c}-k\beta_{\parallel}\right)-\beta_{\perp}\right]$$
$$\times\left(\frac{\omega}{c}-k\beta_{\parallel}\right)r_L, \quad (99)$$

$$a_{21}^h=\frac{-i\omega_a^2(b_{h+}-b_{h-})}{2(b_{h+}+b_{h-})}\left[k\left(k+\frac{\omega}{c}\beta_{\parallel}\right)\right.$$
$$\left.+p^2\left(1-\frac{\omega}{c}\beta_{\perp}\frac{r_L}{m}\right)\right]\frac{\gamma_{\parallel}^2}{\gamma_0^2}\frac{m}{\omega p^2}(\omega-kv_{0z}), \quad (100)$$

$$a_{22}^h=\frac{-\omega_a^2}{(b_{h+}+b_{h-})}\frac{\gamma_{\parallel}^2}{\gamma_0^2}\frac{r_L}{p}\left(\frac{\omega}{c}-k\beta_{\parallel}\right)^2. \quad (101)$$

Substituting Eqs. (98)–(101) into Eq. (97) we obtain

$$\overline{\omega}^2(\overline{\omega}^2-\omega_0^2)=-\omega_a^2\overline{\omega}^2\frac{m^2}{p^3r_L}\left[1-\frac{\omega\beta_{\perp}^2}{m\omega_0}\left(1-\frac{c^2k^2}{\omega^2}\right)-kv_{0z}\right]\left[k\left(k+\frac{\omega}{c}\beta_{\parallel}\right)+p^2\left(1-\frac{\omega}{m\omega_0}\beta_{\perp}^2\right)\right]\frac{b_{h-}b_{h+}}{(b_{h+}+b_{h-})}$$
$$+\omega_a^2\frac{\overline{\omega}\omega_0}{2}\left(\frac{\omega}{c}-k\beta_{\parallel}\right)\frac{mc}{\omega}\left(\frac{\omega^2}{(\omega^2-c^2k^2)}-\frac{2\omega\beta_{\perp}^2}{m\omega_0}+\frac{k^2}{p^2}+1\right)\frac{(b_{h+}-b_{h-})}{(b_{h+}+b_{h-})}$$
$$+\omega_a^2\left(\frac{\overline{\omega}^2}{\gamma_{\perp}^2}+\omega_0^2\beta_{\perp}^2\right)\frac{r_L}{p}\frac{[(\omega/c)-k\beta_{\parallel}]^2}{(b_{h+}+b_{h-})}. \quad (102)$$

If $k=0$ and $v_{0z}=0$, Eq. (102) reduces to Eq. (41) apart from the nonresonant term proportional to ω_a^4 which has been neglected here. Let us now obtain the solution for the slow peniotron interaction for the general case $k\neq 0$, $v_{0z}\neq 0$ by assuming

$$\overline{\omega}=-\omega_0+\delta\omega, \quad (103)$$

and assuming that the circuit mode is simultaneously resonant, i.e.,

$$b_{h+}+b_{h-}\simeq 0. \quad (104)$$

We may therefore replace b_{h+} by $-b_{h-}$ and $\overline{\omega}$ by $-\omega_0$ in nonresonant terms giving

$$(\delta\omega)^2=\frac{-m\omega_a^2}{2\alpha_h\omega_0}\left\{\frac{m}{pr_L}\left[1-\frac{\omega\beta_{\perp}^2}{m\omega_0}\left(1-\frac{c^2k^2}{\omega^2}\right)-\frac{kv_{0z}}{\omega}\right]\left[\frac{k}{p^2}\left(k+\frac{\omega}{c}\beta_{\parallel}\right)+1-\frac{\omega\beta_{\perp}^2}{m\omega_0}\right]b_{h-}^2\right.$$
$$\left.+\left(\frac{\omega}{c}-k\beta_{\parallel}\right)\left(\frac{\omega^2}{(\omega^2-c^2k^2)}-\frac{2\omega\beta_{\perp}^2}{m\omega_0}+\frac{k^2}{p^2}+1\right)\frac{c}{\omega}b_{h-}+\left(\frac{\omega}{c}-k\beta_{\parallel}\right)^2\frac{r_L}{mp}\right\}. \quad (105)$$

In Eq. (105) we have written $(b_{h+}+b_{h-})\simeq\alpha_h\delta\omega$, where

$$\alpha_h=\frac{(c^2k^2+\omega_c^2)^{1/2}}{p^2c^2pr_L}(x_{nm}^2-m^2)\frac{J_m^2(x_{nm})}{[J_m'(x_{nm}r_L/r_w)]^2}. \quad (106)$$

Now the admittance function b_{h-} is given by[6]

$$b_{h-}=J_m(pr_L)/J_m'(pr_L). \quad (107)$$

For the fundamental mode $(n=1)$, b_{h-} is necessarily positive. The right-hand side of Eq. (105) is then negative definite and the TE–slow peniotron interaction is unstable, within this approximation. For higher harmonics ($n > 1$), b_{h-} could be negative, depending on the ratio r_L/r_w. The stability of these modes would then depend on this quantity.

B. The TM Doppler-shifted peniotron interaction

In order to discuss this case we retain only the a_{ij}^e coefficients in Eq. (97) corresponding to a resonant TM mode. The required coefficients are (see the Appendix)

$$a_{11}^e = \frac{\omega_a^2[(\omega/c)\beta_\parallel - k]}{(b_{e+} + b_{e-})}\left[\frac{\beta_\parallel}{2}\frac{c}{\omega}(b_{e+} - b_{e-}) + \left(k + \frac{\omega}{c}\beta_\parallel\right)\frac{r_L}{p}b_{e+}b_{e-}\right], \tag{108}$$

$$a_{12}^e = \frac{-i\omega_a^2(b_{e+} - b_{e-})}{2(b_{e+} + b_{e-})}\frac{m}{p^2}\left(\frac{\omega}{c}\beta_\parallel - k\right)^2, \tag{109}$$

$$a_{21}^e = \frac{i\omega_a^2[(\omega/c)\beta_\parallel - k]}{(b_{e+} + b_{e-})}\frac{\gamma_\parallel^2}{\gamma_0^2}\frac{m}{2p^2}\left[\frac{2pc}{\omega}\frac{\beta_\parallel}{r_L} - \left(k + \frac{\omega}{c}\beta_\parallel\right)(b_{e+} - b_{e-})\right], \tag{110}$$

$$a_{22}^e = \frac{\omega_a^2[(\omega/c)\beta_\parallel - k]^2}{(b_{e+} + b_{e-})}\frac{\gamma_\parallel^2}{\gamma_0^2}\frac{m^2}{p^3 r_L}. \tag{111}$$

Substituting Eqs. (108)–(111) into Eq. (97) we obtain the TM–peniotron interaction for weak space charge:

$$\begin{aligned}\overline{\omega}^2(\overline{\omega}^2 - \omega_0^2) = \frac{\omega_a^2}{(b_{e+} + b_{e-})}\Bigg\{&-\frac{\beta_\parallel}{pr_L}\left(\beta_\parallel - \frac{ck}{\omega}\right)m\overline{\omega}\omega_0 \\ &- \left(\frac{\overline{\omega}^2}{\gamma_\perp^2} + \omega_0^2\beta_\perp^2\right)\frac{m^2}{p^3 r_L}\left(\frac{\omega}{c}\beta_\parallel - k\right)^2 \\ &- \overline{\omega}^2\left(\frac{\omega^2}{c^2}\beta_\parallel^2 - k^2\right)\frac{r_L}{p}b_{e+}b_{e-} \\ &- \left[\overline{\omega}^2\left(\frac{\omega}{c}\beta_\parallel - k\right)\frac{\beta_\parallel}{2}\frac{c}{\omega} \right. \\ &\left. - m\overline{\omega}\omega_0\frac{k}{p^2}\left(\frac{\omega}{c}\beta_\parallel - k\right)\right](b_{e+} - b_{e-})\Bigg\}.\end{aligned} \tag{112}$$

Looking for the slow peniotron solution we again use Eq. (103) assuming simultaneous resonance with a "cold" TM mode,

$$b_{e+} + b_{e-} \simeq 0. \tag{113}$$

Proceeding as for the TE interaction we obtain the solution

$$(\delta\omega)^2 \simeq \frac{\omega_a^2(k_g - k)}{2\alpha_e\omega_0\gamma_\parallel^2}\left[\frac{m\beta_\parallel c}{pr_L\omega} - \frac{m^2}{p^3 r_L\gamma_\parallel^2}(k_g - k) + \left(\frac{\beta_\parallel c}{2\omega} + \frac{mk}{p^2}\right)2b_{e-} + (k_g + k)\frac{r_L}{p\gamma_\parallel^2}b_{e-}^2\right], \tag{114}$$

where k_g is the axial wave number for grazing incidence and is given in Eq. (95).

In Eq. (114) we have written $(b_{e+} + b_{e-}) \simeq -\alpha_e\delta\omega$, where

$$\alpha_e = (c^2k^2 + \omega_c^2)^{1/2}\frac{r_w^2}{y_{nm}c^2}\frac{r_w}{r_L}\frac{[J_m'(y_{nm})]^2}{J_m^2(y_{nm}r_L/r_w)} \tag{115}$$

and y_{nm} is the nth zero of the J_m Bessel function.

For $k = 0, \beta_\parallel = 0$ we note that $\delta\omega = 0$. If $\beta_\parallel = 0$ and k is nonzero, the interaction will be most unstable near an extremum of $J_m(pr_L)$. This can be seen by noting the form of b_{e-}, which is[6]

$$b_{e-} = J_m'(pr_L)/J_m(pr_L). \tag{116}$$

In contrast to the TE interaction, b_{e-} can be of either sign for the fundamental mode ($n = 1$). The growth rate for the TM–slow peniotron interaction is given by Eq. (114) and is evidently much more sensitive to the radial variation of the circuit mode. We note that the TM–peniotron instability also vanishes at grazing incidence.

VII. DISCUSSION AND CONCLUSIONS

The thin beam model[6,7,11–13,16,17] has been used to give a unified treatment of the electromagnetic instabilities that may occur on a relativistic electron beam in a uniform, external magnetic field aligned along the axis of a cylindrical waveguide. A general, self-consistent dispersion relation has been derived which is valid for arbitrary beam energy and arbitrary pitch angle. The effects of the beam space charge are included, which gives rise to the coupling of TE and TM modes. The analysis is restricted to weak space charge such that the self-fields are negligible in comparison with the external magnetic field. However, we have shown that the self-fields due to the beam space charge and current enter the boundary conditions for the perturbed electromagnetic field with the result that all components of this field are discontinuous at the position of the unperturbed beam.

Radial displacement of the beam has been treated in addition to azimuthal and axial displacements. This feature enabled us to include simultaneously two general classes of interaction. The first refers to longitudinal beam modes for which the dominant displacement of the beam is within the beam. The second refers to transverse beam modes where the dominant beam displacement is in the radial direction, perpendicular to the surface of the unperturbed beam. The former interaction is the basis of the gyrotron while the latter is responsible for the peniotron device.

The special case of zero axial equilibrium flow of the beam and perpendicular propagation has been considered first. This case reduces to the well-known gyrotron model.[6,7] However, we have also shown that the slow peniotron mode is unstable due to resonant interaction with a TE waveguide (circuit) mode. This is in contrast to previous statements that the transverse modes are stable in the absence of wall losses.[12,16,17] It is significant that the gyrotron interaction is unstable even when the circuit mode is off resonant.[6]

For azimuthal and axial equilibrium flow of the electron beam and for perturbations that vary three dimensionally, we have written the general dispersion relation in a form suitable for numerical solution. For weak space charge we have obtained a simplification of the general dispersion relation by retaining only those terms up to first order in the space-charge parameter ω_a^2/ω_0^2. This simplified dispersion relation has been used to analyze Doppler-shifted cyclotron resonance of the beam with resonant TE or TM circuit modes for both gyrotron and peniotron interactions. The former case includes the CARM[18,19] and the latter the HARP.[20] Analytic solutions for the growth rates for the four interactions have been obtained. We show explicitly the vanishing of the growth rates for the TM interactions for the condition of grazing incidence. The TM–peniotron interaction is evidently more sensitive to the radial variation of the circuit electromagnetic field.

A judgement of whether the peniotron interaction could be a competitor to the gyrotron will presumably require a nonlinear treatment. However, one outstanding question which could be answered by a solution of the general dispersion relation is the effect of TE/TM coupling on the Doppler-shifted cyclotron resonance gyrotron and peniotron interactions with the circuit. The analysis presented in this paper has treated only the most unstable case by neglecting nonresonant terms. A numerical solution of the general dispersion relation will be presented in a future publication.

Finally, it is worth noting that the inclusion of transverse displacements in the thin beam model makes it suitable for the study of relativistic magnetron interactions. The addition of a periodic circuit characteristic of the magnetron would be expected to single out a particular azimuthal beam mode.

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APPENDIX: DERIVATION OF THE COEFFICIENTS FOR THE PERTURBED ELECTROMAGNETIC FIELD

In this appendix we obtain the coefficients $a_{ij} \equiv a_{ij}^e + a_{ij}^h$ which define the average, perturbed electromagnetic field at the beam. The notation is explained after Eq. (73) in the main text. In order to derive the coefficients defined in Eqs. (70)–(72) we use Eqs. (64) and (69) to write

$$\langle E_{1z} - v_{0\theta} B_{1r} \rangle = \frac{1}{2} \left[\left(1 - \frac{\omega m \omega_0}{c^2 p^2} \right) (E_{1z-} + E_{1z+}) + \frac{ikv_{0\theta}}{p} \left(\frac{B_{1z-}}{b_{h-}} - \frac{B_{1z+}}{b_{h+}} \right) \right]. \tag{A1}$$

Similarly, usings Eqs. (61), (64), and (69), we obtain

$$\langle E_{1\theta} + v_{0z} B_{1r} \rangle = \frac{1}{2} \left[\frac{m}{p^2 r_L} \left(\frac{\omega}{c} \beta_\parallel - k \right) (E_{1z-} + E_{1z+}) + \frac{i(\omega - kv_{0z})}{p} \left(\frac{B_{1z-}}{b_{h-}} - \frac{B_{1z+}}{b_{h+}} \right) \right]. \tag{A2}$$

With the aid of Eqs. (62), (65), and (68) we have

$$\langle E_{1r} + v_{0\theta} B_{1z} - v_{0z} B_{1\theta} \rangle = \frac{1}{2} \left[\frac{i}{p} \left(\frac{\omega}{c} \beta_\parallel - k \right) (b_{e-} E_{1z-} - b_{e+} E_{1z+}) - \left(\frac{m}{p^2 r_L} (\omega - kv_{0z}) - v_{0\theta} \right) (B_{1z-} + B_{1z+}) \right]. \tag{A3}$$

In order to obtain the explicit form for the a_{ij}'s, the final but rather lengthy step is to substitute Eqs. (66) and (67) into Eqs. (A1)–(A3). Starting with Eq. (A3), the result is

$$a_{11}^e = \omega_a^2 \left(\frac{\omega \beta_\parallel}{c} - k \right) \left(-\frac{1}{2} \beta_\parallel \frac{c}{\omega} (b_{e-} - b_{e+}) + \frac{[k + (\omega\beta_\parallel/c)]}{p} r_L b_{e-} b_{e+} \right) (b_{e+} + b_{e-})^{-1}, \tag{A4}$$

$$a_{11}^h = \omega_a^2 \left(\frac{m}{p^2 r_L} (\omega - kv_{0z}) - v_{0\theta} \right) \left[k \left(k + \frac{\omega\beta_\parallel}{c} \right) + p^2 \left(1 - \frac{\omega\beta_\perp}{cm} r_L \right) \right] \frac{m}{\omega p} b_{h-} b_{h+} \times (b_{h+} + b_{h-})^{-1}, \tag{A5}$$

$$a_{12}^e = \frac{i}{2} \omega_a^2 \left(\frac{\omega\beta_\parallel}{c} - k \right)^2 \frac{m}{p^2} (b_{e-} - b_{e+}) \times (b_{e+} + b_{e-})^{-1}, \tag{A6}$$

$$a_{12}^h = -\frac{i}{2} \omega_a^2 \left[\frac{m}{p^2 r_L} \left(\frac{\omega}{c} - k\beta_\parallel \right) - \beta_\perp \right] \left(\frac{\omega}{c} - k\beta_\parallel \right) \times r_L (b_{h-} - b_{h+})(b_{h+} + b_{h-})^{-1}, \tag{A7}$$

$$a_{13}^e = \frac{i}{2} \omega_a^2 \left(\frac{\omega\beta_\parallel}{c} - k \right) r_L \left(1 - \frac{m\omega_0}{\omega} - \frac{m\omega_0 c^2 k^2}{\omega(\omega^2 - c^2 k^2)} \right) \times (b_{e-} - b_{e+})(b_{e+} + b_{e-})^{-1}, \tag{A8}$$

$$a_{13}^h = -\frac{i}{2} \omega_a^2 \left[\frac{m}{p^2 r_L} \left(\frac{\omega}{c} - k\beta_\parallel \right) - \beta_\perp \right] \times k\beta_\perp (b_{h-} - b_{h+})(b_{h+} + b_{h-})^{-1}. \tag{A9}$$

Next, substituting into Eq. (A2), we obtain

$$a_{21}^e = i\omega_a^2 \frac{\gamma_\parallel^2}{\gamma_0^2} \frac{m}{p} \left(\frac{\omega\beta_\parallel}{c} - k \right) \left(\frac{c\beta_\parallel}{\omega r_L} + \frac{[k + (\omega\beta_\parallel/c)]}{2p} \times (b_{e-} - b_{e+}) \right) (b_{e+} + b_{e-})^{-1}, \tag{A10}$$

$$a_{21}^h = -\frac{i}{2} \omega_a^2 \frac{\gamma_\parallel^2}{\gamma_0^2} \frac{(\omega - kv_{0z})}{\omega p^2} m \left[k \left(k + \frac{\omega\beta_\parallel}{c} \right) + p^2 \left(1 - \frac{\omega\beta_\perp}{cm} r_L \right) \right] (b_{h+} - b_{h-}) \times (b_{h+} + b_{h-})^{-1}, \tag{A11}$$

$$a_{22}^e = \omega_a^2 \frac{\gamma_\parallel^2}{\gamma_0^2} \frac{m^2}{p^3 r_L} \left(\frac{\omega\beta_\parallel}{c} - k \right)^2 (b_{e+} + b_{e-})^{-1}, \tag{A12}$$

$$a_{22}^h = -\omega_a^2 \frac{\gamma_\parallel^2}{\gamma_0^2} \left(\frac{\omega}{c} - k\beta_\parallel \right)^2 \frac{r_L}{p} (b_{h+} + b_{h-})^{-1}, \tag{A13}$$

$$a_{23}^{e} = \omega_a^2 \frac{\gamma_\parallel^2}{\gamma_0^2} \frac{m}{\omega p} \left(\frac{\omega\beta_\parallel}{c} - k \right) \left(\omega - m\omega_0 - \frac{m\omega_0 c^2 k^2}{(\omega^2 - c^2k^2)} \right) (b_{e+} + b_{e-})^{-1}, \tag{A14}$$

$$a_{23}^{h} = -\omega_a^2 \frac{\gamma_\parallel^2}{\gamma_0^2} \left(\frac{\omega}{c} - k\beta_\parallel \right) \frac{k r_L \beta_\perp}{p} (b_{h+} + b_{h-})^{-1}. \tag{A15}$$

Finally, substituting into Eq. (A1), we have

$$a_{31}^{e} = i\omega_a^2 \frac{\gamma_\perp^2}{\gamma_0^2} \left(1 - \frac{m\omega_0\omega}{c^2p^2} \right) \left[\frac{cp\beta_\parallel}{c} + \frac{1}{2} \left(k + \frac{\omega\beta_\parallel}{c} \right) \times r_L (b_{e-} - b_{e+}) \right] (b_{e+} + b_{e-})^{-1}, \tag{A16}$$

$$a_{31}^{h} = -\frac{i}{2} \omega_a^2 \frac{\gamma_\perp^2}{\gamma_0^2} \frac{k v_{0\theta}}{\omega} \frac{m}{p^2} \left[k \left(k + \frac{\omega\beta_\parallel}{c} \right) + p^2 \left(1 - \frac{\omega\beta_\perp}{cm} r_L \right) \right] (b_{h+} - b_{h-}) \times (b_{h+} + b_{h-})^{-1}, \tag{A17}$$

$$a_{32}^{e} = \omega_a^2 \frac{\gamma_\perp^2}{\gamma_0^2} \left(1 - \frac{m\omega_0\omega}{c^2p^2} \right) \frac{m}{p} \left(\beta_\parallel \frac{\omega}{c} - k \right) \times (b_{e+} + b_{e-})^{-1}, \tag{A18}$$

$$a_{32}^{h} = -\omega_a^2 \frac{\gamma_\perp^2}{\gamma_0^2} \frac{k v_{0\theta}}{pc} \left(\frac{\omega}{c} - k\beta_\parallel \right) r_L (b_{h+} + b_{h-})^{-1}, \tag{A19}$$

$$a_{33}^{e} = \omega_a^2 \frac{\gamma_\perp^2}{\gamma_0^2} \left(1 - \frac{m\omega_0\omega}{c^2p^2} \right) \left(1 - \frac{m\omega_0}{\omega} - \frac{m\omega_0 c^2 k^2}{\omega(\omega^2 - c^2k^2)} \right) \times p r_L (b_{e+} + b_{e-})^{-1}, \tag{A20}$$

$$a_{33}^{h} = -\omega_a^2 \frac{\gamma_\perp^2}{\gamma_0^2} \frac{k^2 r_L}{p} \beta_\perp^2 (b_{h+} + b_{h-})^{-1}. \tag{A21}$$

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