AIP | Physics of Plasmas

Toroidal effects on the nonlinearly saturated m=1 island in tokamaks

K. Avinash, F. A. Haas, and A. Thyagaraja

Citation: Phys. Fluids B **4**, 1671 (1992); doi: 10.1063/1.860075
View online: http://dx.doi.org/10.1063/1.860075
View Table of Contents: http://pop.aip.org/resource/1/PFBPEI/v4/i6
Published by the American Institute of Physics.

The purpose of this Brief Communications section is to present important research results of more limited scope than regular articles appearing in Physics of Fluids B. Submission of material of a peripheral or cursory nature is strongly discouraged. Brief Communications cannot exceed three printed pages in length, including space allowed for title, figures, tables, references, and an abstract limited to about 100 words.

Toroidal effects on the nonlinearly saturated $m=1$ island in tokamaks

K. Avinash,[a)] F. A. Haas, and A. Thyagaraja
AEA Fusion, Culham Laboratory (Euratom/UKAEA Fusion Association), Abingdon, Oxfordshire, OX14 3DB, England

(Received 6 November 1990; accepted 18 February 1992)

This Brief Communication investigates the influence of toroidal effects (due to the coupling of various poloidal harmonics) on the nonlinear saturation of the $m=1$ island. Bounds are obtained relating the aspect ratio, the shear at the $q=1$ surface, and the saturated island width. Provided these bounds are satisfied, it is then found that the cylindrical $m=1$ island theory of Thyagaraja and Haas [Phys. Fluids B 3, 580 (1991)] is valid for toroidal geometry.

It is well-known theoretically[1,2] that in tokamaks with finite shear at the $q=1$ surface (q is the safety factor) and low β (the ratio of plasma to magnetic pressure), the resistive $m=1$ mode reconnects the magnetic field lines in the nonlinear phase, until the whole plasma core is encompassed. However, in experimental observations of the sawtooth phenomenon (which seems to involve the growth of $m=1$ perturbations) on JET[3] and TEXTOR,[4] there is very little evidence of this. Recently Thyagaraja and Haas[5] have shown that provided the shear at $q=1$ surface is small, the resistive kink may saturate at small amplitudes. The analysis is done in low β ordering in a cylinder, with $B_z \to \infty$ and $a/R \to 0$, keeping q fixed; here B_z defines the axial field, and a and $2\pi R$ are the radius and periodicity length of the cylinder, respectively. There is some evidence that on JET[6] and TEXTOR,[4] q profiles may be of this nature. In this case the results of Ref. 5 may be of relevance to the sawtooth phenomenon. However, as toroidal effects due to coupling of various poloidal harmonics are known to be important for the $m=1$ mode,[7-10] it is not clear whether the results of Ref. 5, which are based on a cylindrical low β approximation, are relevant (at least qualitatively) to a tokamak. In this Brief Communication we show that provided the shear at the $q=1$ surface is not too small (in a sense to be discussed), the results of Ref. 5 may be extended to a large aspect ratio, low β tokamak. In fact, we obtain general bounds on the smallness of a/R for a given small shear at the $q=1$ surface to ensure that toroidal effects are negligible in the nonlinear saturation of the $m=1$ island. These bounds may be useful in toroidal numerical codes simulating the sawtooth phenomenon.

To obtain these bounds we need to calculate the nonlinear energy integral δW to order ξ^4, where ξ is the amplitude of the fluid displacement normalized to a, and $\xi \ll 1$. In calculating these terms we use the method of Taylor[11] (see also Ref. 12), which is as follows. Consider an equilibrium distorted by a velocity $\mathbf{v}$ for a time t_0, where t_0 is fixed. We choose $|\mathbf{v}|=\xi/t_0$ where ξ is regarded as independent of time; ξ and t_0 are thus fixed parameters in the problem and, as we shall see, either of them could be regarded as the small parameter used to calculate the perturbed energy δW order by order. The iteration scheme can be described as follows. From Faraday's and Ohm's law we have

$$\frac{\partial \mathbf{B}}{\partial t}=\nabla\times\mathbf{v}\times\mathbf{B}(t). \tag{1}$$

Using $\mathbf{v}=\xi/t_0$ and defining $\delta B_1=t_0[dB(t)/dt]|_{t=0}$, we have

$$\delta\mathbf{B}_1=\nabla\times[\xi\times\mathbf{B}(t)]. \tag{2}$$

This equation can now be iterated to generate higher-order contributions to $B(t)$. To do this we expand $B(t)$ around $t=0$ as

$$\mathbf{B}(t)=\mathbf{B}(0)+t_0\frac{d\mathbf{B}(t)}{dt}\bigg|_{t=0}+\frac{t_0^2}{2!}\frac{d^2\mathbf{B}(t)}{dt^2}\bigg|_{t=0}+\frac{t_0^3}{3!}\frac{d^3\mathbf{B}(t)}{dt^3}+\frac{t_0^4}{4!}\frac{d^4\mathbf{B}(t)}{dt^4}\bigg|_{t=0}. \tag{3}$$

Next, let

$$\mathbf{B}(t)-\mathbf{B}(0)=\delta\mathbf{B}_1+\delta\mathbf{B}_2+\delta\mathbf{B}_3+\delta\mathbf{B}_4. \tag{4}$$

Then from Eqs. (2)–(4)

$$\delta\mathbf{B}_1=\nabla\times[\xi\times\mathbf{B}(t)], \tag{5}$$

$$\delta\mathbf{B}_2=(1/2!)\nabla\times\xi\times\nabla\times[\xi\times\mathbf{B}(t)], \tag{6}$$

$$\delta\mathbf{B}_3=(1/3!)\nabla\times\xi\times\nabla\times\xi\times\nabla\times[\xi\times\mathbf{B}(t)], \tag{7}$$

$$\delta\mathbf{B}_4=(1/4!)\nabla\times\xi\times\nabla\times\xi\times\nabla\times\xi\times\nabla\times[\xi\times\mathbf{B}(t)]. \tag{8}$$

Now for incompressible displacements $\nabla\cdot\xi=0$, and we have

 0899-8213/92/061671-03$04.00

$$\delta W = \frac{1}{2}\int (\delta B)^2\, d\tau + \int \delta \mathbf{B}\cdot\mathbf{B}\, d\tau. \tag{9}$$

Substituting $\delta\mathbf{B}$ from Eq. (4) and retaining terms up to order ξ^4 we have

$$\delta W = \frac{1}{2}\int (\delta B_1)^2\, d\tau + \int \delta \mathbf{B}_2\cdot\mathbf{B}\, d\tau + \frac{1}{2}\int (\delta B_2)^2\, d\tau + \int \delta\mathbf{B}_1\cdot\delta\mathbf{B}_3\, d\tau + \int \delta\mathbf{B}_4\cdot\mathbf{B}\, d\tau, \tag{10}$$

where $\delta\mathbf{B}_1$, $\delta\mathbf{B}_2$, $\delta\mathbf{B}_3$, and $\delta\mathbf{B}_4$ are given by Eqs. (5)–(8). To make contact with the linear δW calculated by Bernstein *et al.*,[13] we can easily manipulate the first two terms in Eq. (10) using the boundary conditions $\hat{n}\cdot\mathbf{B} = \hat{n}\cdot\boldsymbol{\xi} = 0$ ($\hat{n}$ is the unit vector normal to the magnetic surface), to show that

$$\delta W_L = \frac{1}{2}\int \{[\nabla\times(\boldsymbol{\xi}\times\mathbf{B})]^2 - \boldsymbol{\xi}\times(\nabla\times\mathbf{B})\cdot\nabla\times(\boldsymbol{\xi}\times\mathbf{B})\}\, d\tau. \tag{11}$$

This expression is identical to the plasma volume terms δW_L of Bernstein *et al.*[13] if we identify ξ as the linear eigenfunction. Thus we will regard ξ as the eigenfunction that is to be obtained from the linear theory.

Now we consider the specific problem discussed in Ref. 5. The q profile is monotonic with $q_0 - 1 \sim O(1)$, where q_0 is the value of q at the magnetic axis. We assume that the shear at the $q=1$ surface [i.e., $rq'(r=r_i)$] is small, where r_i is the position of the $q=1$ surface normalized to a and the primes indicate radial derivatives.

The linear radial eigenfunction ξ_τ for this q profile is the "top hat," i.e.,

$$\begin{aligned} &\xi_\tau = \text{const}, && 0 < r < r_i, \\ &\xi_\tau = 0, && r_i < r < a. \end{aligned} \tag{12}$$

Now clearly, whereas the present iteration scheme is suitable for continuous ξ_τ, this is not the case for the eigenfunction given by Eq. (12), which is discontinuous at $r=r_i$. The δW contains terms proportional to radial derivatives of ξ_τ which are infinite at $r=r_i$. For the iteration scheme to converge, the singularity around $r=r_i$ must be resolved. For this, the terms involving gradients of ξ_τ should be calculated using appropriate layer theory. In the present case we use a small resistivity $\eta(\eta\to 0)$ to resolve the singularity at $r=r_i$. This procedure has already been followed in Ref. 5, where a vanishingly small resistivity is used in the nonlinear layer around $r=r_i$ to calculate the radial derivatives of ξ_τ. We now use δW given by Eq. (10) to investigate the nonlinear saturation or marginal stability of the $m=1$ mode; more specifically, we wish to investigate the importance of toroidal corrections to the cylindrical terms obtained in Ref. 5 for the saturation of the $m=1$ island. The justification for this is as follows. The first two terms in δW are of the form $\alpha\xi^2$, and if negative, make the mode linearly unstable. In the nonlinear stage, these terms are balanced by the last two terms. The latter are of the form $\alpha\xi^4$, and lead to $\delta W=0$, which implies saturation of the mode. One may question the use of δW to investigate the stability of a resistive mode. Indeed, in the presence of finite dissipation, energy is not conserved and as a result the operator is not self-adjoint. However, in the present case resistivity is not present in "full strength." That is, resistivity is used only to resolve the singularity around $r=r_i$ and in principle it could be vanishingly small; both the present calculation and that of Ref. 5 are considered to be in this limit. In other words, we compare the energies of equilibria of two different topologies, the one without the island being used to infer marginal stability. This analysis is similar to an earlier analysis by Rebut,[14] where bifurcation of an ideal equilibrium to an equilibrium with an island of lower energy was shown to infer marginal stability. In fact, an energy principle for tearing modes has recently been proposed.[15] We now briefly discuss the properties of δW_L for this mode. Evaluating δW_L using the eigenfunction given in Eq. (12) gives

$$\delta W_L = 0, \tag{13}$$

i.e., the mode is ideally marginally stable. In Ref. 5 the resistive version of this mode is considered, i.e., the resistive kink mode with growth $\gamma\sim\eta^{1/3}$. However, δW_L has been evaluated for a large aspect ratio, low β tokamak by Bussac *et al.*,[7] who have shown that in order to be ideally unstable, β_p (β_p is roughly the ratio of pressure to poloidal magnetic field) should be greater than a critical value β_{pc}. They further showed that near linear marginal stability $\delta W_L \sim \epsilon^2\xi_\tau^2$, where $\epsilon = a/R$ is the inverse aspect ratio. Thus, in the large aspect ratio, low β version of the problem considered in Ref. 5 the modes are marginal. In this case we consider $\beta_p = \beta_{pc}$, rather than the $R\to\infty$ limit; at marginal stability $\delta W_L = 0$, while near it, $\delta W_L \sim -\epsilon^2\xi_\tau^2$.

Next we calculate the nonlinear part $\delta W_{\rm NL}$. Using Eqs. (6)–(11) we formally write the leading-order terms to be

$$\delta W_{\rm NL} \simeq [\alpha_1\xi^4 + \alpha_2(q-1)^2|_{r=r_i}(\xi^4/\delta^4)\delta + \epsilon^2\alpha_3\xi^4 + \cdots], \tag{14}$$

where α_1, α_2, and α_3 depend on equilibrium quantities and are $O(1)$; δ is the width of the nonlinear layer, i.e., the island size. The first term is the nonlinear cylindrical term from the ideal region. The second term is the nonlinear cylindrical term from the layer, while the third term is the toroidal correction from the ideal region due to coupling to various poloidal harmonics. All distances are normalized to a. The key to the form of Eq. (14) is to note from Eqs. (5) to (8) that the δB's are linear in the helical flux which is proportional to $(q-1)$, and hence $\delta W_{\rm NL}$ is quadratic in $(q-1)$. However, this factor is not made explicit in the first and third terms because, in the ideal region, $(q-1)\sim O(1)$, and hence can be included in α_1 and α_3. Now the first term has been evaluated by Avinash[12] where it is shown that, under very general conditions (irrespective of the type of q profile), the "top-hat" eigenfunction satisfies the cylindrical equilibrium to all orders and hence α_1 is identically zero. It is now straightforward to interpret the calculation of Ref. 5. In this, only the second term is retained and calculated using resistive equations in the nonlinear layer. The authors of Ref. 5 show that

$\alpha_2 \sim O(1) > 0$, that is, the second term is stabilizing provided the shear at $q=1$ is small. Clearly then, in the large aspect ratio version of this problem, the leading-order terms in $\delta W_{\rm NL}$ are the second and third terms. Expanding $(q-1)^2 \simeq (\delta q')^2$ around $r=r_i$, the condition for the toroidal term to be small is

$$q'^2 \gg \epsilon^2 \delta. \tag{15}$$

If this inequality is satisfied, then the cylindrical approximation can be expected to be valid in the nonlinear stage. Now according to Thyagaraja and Haas[5], $q' \sim \delta$; in this case, if $q' \gg \epsilon^2$ (but not so large that the saturation of the $m=1$ island is invalid), their results would be expected to be at least qualitatively correct for large aspect ratio, low β tokamaks. It should be carefully noted that since we have not determined the sign of α_3, this condition is only sufficient and may not be necessary. However, it will be useful in verifying the results of Thyagaraja and Haas[5] by toroidal computer codes that take into account the coupling of the $m=1$ mode to various poloidal harmonics. It should also be noted that these results are typical of the "top-hat" eigenfunction and the low β approximation. If the eigenfunction differs significantly from the "top hat," as in the case of ultraflat q profiles where the pressure gradient drive becomes dominant, toroidal effects may indeed be important in the linear as well as the nonlinear stages.[16–18] If we now balance the second and largest term in Eq. (14) with the linear δW_L near the marginal point, that is, where

$$\delta W_L \simeq -\epsilon^2 \xi_\tau^2,$$

then it follows, that

$$\delta W = [-\epsilon^2 \xi_\tau^2 + \alpha_2 (q-1)^2 (\xi_\tau^4/\delta^3)],$$

with $(q-1)^2 = \delta^2 q'^2$ at $r = r_i$.

The saturation of the mode determines ($\delta W=0$) the amplitude as $\xi_\tau^2 \sim \epsilon^2 \delta / q'^2 \ll 1$. This justifies dropping the contributions due to higher-order terms in Eq. (14).

Finally, we comment on the method used to derive Eqs. (5)–(8). It will be recalled that we wrote $v=\xi/t_0$, where ξ is independent of t. As we have shown, however, this choice gives the linear δW of Bernstein *et al.* Thus the parameter t_0 can be regarded as a convenient device to label the various powers of ξ_τ, so that at least the magnetic part of δW can be easily developed to any desired order.

ACKNOWLEDGMENTS

One of us (K. A.) is grateful to R. J. Hastie for interesting and helpful discussions, and to Culham Laboratory for the hospitality shown during the collaboration. The authors would also like to thank the referees for useful suggestions.

[a]Permanent address: Institute for Plasma Research, Bhat, Gandhinagar, India.

[1]B. B. Kadomtsev, Fiz. Plasmy **1**, 710 (1975) [Sov. J. Plasma Phys. **1**, 389 (1975)].

[2]R. D. Hazeltine, J. D. Meiss, and P. J. Morrison, Phys. Fluids **29**, 1633 (1986).

[3]A. W. Edwards, D. J. Campbell, W. W. Engelhardt, H.-U. Fahrbach, R. D. Gill, R. S. Granetz, S. Tsuji, B. J. D. Tubbing, A. Weller, J. Wesson, and D. Zasche, Phys. Rev. Lett. **57**, 210 (1986).

[4]H. Soltwisch, W. Stodiek, J. Manickam, and J. Schluter, in *Plasma Physics and Controlled Nuclear Fusion Research, 1986*, Proceedings of the 11th International Conference, Kyoto (IAEA, Vienna, 1987), Vol. I, p. 27.

[5]A. Thyagaraja and F. A. Haas, *Proceedings of the 17th European Conference on Controlled Fusion and Plasma Heating*, Amsterdam (EPS, Geneva, 1990), Vol. II, p. 942; Phys. Fluids B **3**, 580 (1991).

[6]R. D. Gill, A. W. Edwards, B. Keegan, E. Lazzaro, J. O'Rourke, A. Weller, and D. Zasche, *Proceedings of the 16th European Conference on Controlled Fusion and Plasma Heating*, Venice (EPS, Geneva, 1989), Vol. 13B, Part II, p. 469.

[7]M. N. Bussac, R. Pellat, D. Edery, and J. L. Soule, Phys. Rev. Lett. **35**, 1638 (1975).

[8]R. C. Grimm, R. L. Dewar, J. Manickam, S. C. Jardin, A. H. Glasser, and M. S. Chance, in *Plasma Physics and Controlled Nuclear Fusion Research, 1982*, Proceedings of the 9th International Conference, Baltimore (IAEA, Vienna, 1983), Vol. III, p. 85.

[9]J. W. Connor, S. C. Cowley, R. J. Hastie, T. C. Hender, A. Hood, and T. J. Martin, Phys. Fluids **31**, 577 (1988).

[10]A. Y. Aydemir, J. C. Wiley, and D. W. Ross, Phys. Fluids B **1**, 774 (1989).

[11]J. B. Taylor (private communication, 1987).

[12]K. Avinash, Phys. Fluids B **2**, 2373 (1990).

[13]L. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. R. Soc. London Ser. A **244**, 17 (1958).

[14]P. H. Rebut, J. Nucl. Energy C **4**, 159 (1962).

[15]M. S. Chu, M. S. Chance, J. M. Greene, and T. H. Jensen, Phys. Fluids B **1**, 62 (1989).

[16]F. L. Waelbroeck and R. D. Hazeltine, Phys. Fluids **31**, 1217 (1988).

[17]F. L. Waelbroeck, Phys. Fluids B **1**, 2372 (1989).

[18]R. J. Hastie and T. C. Hender, Nucl. Fusion **28**, 585 (1988).