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The magnetoacoustic cyclotron instability of an extended shell distribution of energetic ions

R. O. Dendy, C. N. Lashmore-Davies, and K. F. Kam
AEA Fusion, Culham Laboratory (Euratom/UKAEA Fusion Association), Abingdon, Oxfordshire, OX14 3DB, England

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The magnetoacoustic cyclotron instability is a mechanism by which waves on the perpendicular fast Alfvén-ion Bernstein branch can be excited through cyclotron resonance with an energetic ion population. It is a candidate emission mechanism for the superthermal ion cyclotron radiation, apparently associated with the products of fusion reactions, that has been observed from tokamak plasmas. In the present paper, an extended shell model is adopted for the energetic ion distribution function, $f_\alpha(\mathbf{v}) \sim n_\alpha \exp[-(v-v_0)^2/v_T^2]$. An analytical formulation of the dispersion relation is obtained, whose numerical solution yields quantitative information on the role of v_T in stabilizing wave growth at ion cyclotron harmonics. The results show that, for typical plasma parameters of interest, the degree of instability is significantly depressed, relative to its level for $v_T=0$, once $v_T/v_0 \simeq 0.1$. Gaps appear in typical multiple cyclotron harmonic excitation patterns for $0.1 \lesssim v_T/v_0 \lesssim 0.2$, and most harmonics are stable for $v_T/v_0 \gtrsim 0.25$. Thus the energetic ion shell-driven magnetoacoustic cyclotron instability typically occurs only when the shell is relatively narrow in velocity space.

I. INTRODUCTION

The magnetoacoustic cyclotron instability is of interest at present, in part because of its role as a possible emission mechanism[1] for the superthermal radiation at multiple ion cyclotron harmonics observed on the Joint European Torus (JET),[2–4] whose intensity is proportional to the measured fusion reactivity; for a recent description of JET, see, for example, Ref. 5 and references contained therein. Pioneering work on the theory of this instability was carried out some years ago by Russian, Ukrainian, and Georgian scientists; and, indeed, the magnetoacoustic cyclotron instability possesses considerable intrinsic theoretical interest, as a classic example of the physically distinctive collective relaxation phenomena which are accessible to energetic ions in magnetized plasmas.[6–10] Its essential feature, for present purposes, is the excitation by a minority energetic ion population of waves on the fast Alfvén and ion Bernstein branches, propagating perpendicular to the magnetic field with frequencies at ion cyclotron harmonics.

If the potential of ion cyclotron emission as a diagnostic of the fusion product population is to be realized, it will be necessary to understand the physics of the underlying emission mechanism (or mechanisms), and its parametric dependence. Some form of velocity space instability may be responsible; in Ref. 1, the analysis of Ref. 9 was extended to a lower-frequency regime, applicable to the JET observations, and it was then shown that the magnetoacoustic cyclotron instability appears to be a plausible emission mechanism. For example, simultaneous instability occurs for multiple cyclotron harmonics, an appropriate wave mode is excited, the energy and concentration thresholds of the instability seem encouraging, and the collective nature of the instability could lead naturally to superthermal levels of signal intensity. However, there are a number of features that are not reproduced by the magnetoacoustic cyclotron instability. These include the observed excitation of spectral peaks at cyclotron harmonics of the background plasma when they are not degenerate with energetic ion cyclotron harmonics (see, for example, Fig. 3 of Ref. 2), and the fact that the linear stability threshold provided by the positive-energy loading of the background ions is apparently sufficient to preclude excitation at the lowest ($l=1$) cyclotron harmonic of the energetic ions. The first problem presents a challenge to any theory of the excitation, which is cyclotronic in character and driven by fusion products—and at present, the overwhelming weight of experimental evidence is that ion cyclotron emission has these twin properties. It is possible that some nonlinear coupling mechanism leads to equipartition of energy between all available cyclotron harmonic modes, including nonresonant background ion harmonics and those energetic harmonics that are linearly stable, such as $l=1$. In the present state of theoretical development, this must remain a conjecture. Meanwhile, we concentrate here on elucidating the key parametric dependence of the linear magnetoacoustic cyclotron instability.

The model distribution function for the energetic ions adopted in Ref. 1 was chosen to be as simple as possible, while having clear links to the key experimental variables. It contained two free parameters, namely the concentration of the fusion products and their birth energy—in velocity space the distribution was an infinitesimally thin shell at

$v=v_0$. Instability growth rates and thresholds were calculated in terms of these parameters for a range of experimentally relevant plasma conditions.

It now seems appropriate to investigate the dependence of the magnetoacoustic cyclotron instability on further parameters that may characterize the fusion product distributions in present or future reacting plasmas. We emphasize that the number of physical effects that determine the structure of the fusion product velocity distribution is large, particularly in the edge plasma where the ion cyclotron emission observed in JET appears to originate. They may include the large-orbit effects studied by Stringer,[11] edge losses, ripple losses, center-of-mass dynamical effects, and particle diffusion. Furthermore, Sigmar[12] has pointed out how particle loss and time-dependent source terms can create hollow velocity distributions. In the present paper, we extend the model of Ref. 1 by introducing a third free parameter v_T, which governs the spread in velocity of an extended shell distribution about the birth speed v_0. Physically, such a spread can be caused for example by nonzero velocities of the centroids of pairs of fusing nuclei, due to the thermal spread of the fuel plasma. This distribution is isotropic in velocity space, so that it cannot give rise to instabilities associated with, for example, anisotropy or directional beams. Its range of applicability does not extend to experiments where the energetic particle population is dominated by injected beam ions, as, for example, was the case for the observed emission spectrum traced by the solid line in Fig. 1 of Ref. 2.

The role of velocity spread in the magnetoacoustic cyclotron instability has been briefly addressed in Refs. 8 and 9. Mathematically, the condition for instability depends on the sign of a combination of Bessel functions whose argument, in the limit of an infinitesimally thin shell at $v=v_0$, is kv_0/Ω_α, where Ω_α is the cyclotron frequency of the energetic ion species. When a velocity spread v_T is introduced, contributions with arguments of order $k(v_0+v_T)/\Omega_\alpha$ will appear, weakening the instability driving term in comparison to stabilizing effects. This will tend to suppress the instability. In the present paper, we quantify this effect, and others, in detail. First, we calculate the required dielectric tensor elements due to the energetic ion population, which do not appear to have been obtained previously. These dielectric tensor elements are then substituted into the full dispersion relation for waves on the fast Alfvén-ion Bernstein branch, in the low-frequency regime that extends down to $\omega\simeq\Omega_\alpha$, as distinct from the higher harmonic regime of Ref. 9. Numerical solution then gives a clear quantitative measure of the extent to which velocity spread affects the magnetoacoustic cyclotron instability, yielding a possible link between observations of ion cyclotron emission and the value of an additional parameter of the model velocity distribution of the energetic ions.

II. CALCULATION OF THE DIELECTRIC TENSOR ELEMENTS

Let us begin by considering how the dielectric tensor for any distribution of ions that is isotropic in velocity space can be constructed in terms of the dielectric tensor for a delta-function shell distribution. A general dielectric tensor element is of the form

$$\epsilon_{ij}=\int d^3\mathbf{v}\,\hat{D}^{ij}_{\mathbf{v}} f(\mathbf{v}), \tag{1}$$

where $\hat{D}^{ij}_{\mathbf{v}}$ is a dimensionless operator that involves partial derivatives with respect to the components of $\mathbf{v}$. Now let us restrict attention to velocity distributions that are isotropic in velocity space, and write

$$f(\mathbf{v})=f(v)\equiv\int dv_a\, f(v_a)\delta(v-v_a). \tag{2}$$

Here, the last step is an identity, involving the dummy variable v_a, which we can exploit on substituting Eq. (2) back into Eq. (1) to give

$$\begin{aligned}\epsilon_{ij}&=\int dv_a \int d^3\mathbf{v}\ \hat{D}^{ij}_{\mathbf{v}} f(v_a)\delta(v-v_a)\\ &=\int dv_a\, 4\pi v_a^2 f(v_a)\int d^3\mathbf{v}\ \hat{D}^{ij}_{\mathbf{v}}\frac{1}{4\pi v_a^2}\delta(v-v_a).\end{aligned} \tag{3}$$

Referring to Eq. (1), we see that the second integral in Eq. (3) is simply the dielectric tensor for a zero-thickness shell distribution function with velocity v_a, as considered in Refs. 1 and 9. We may accordingly write

$$\int d^3\mathbf{v}\ \hat{D}^{ij}_{\mathbf{v}}\frac{1}{4\pi v_a^2}\delta(v-v_a)=-\sum_l\frac{l^2\omega_{p\alpha}^2}{\omega(\omega-l\Omega_\alpha)}\frac{1}{z_a^2}\Pi_{ij}, \tag{4}$$

$$z_a=kv_a/\Omega_\alpha, \tag{5}$$

with the functions Π_{ij} defined in terms of z_a by Appendix A of Ref. 9, as corrected for typographical errors in the appendix of Ref. 1. Then relabeling the dummy velocity variable, we may use Eqs. (3) and (4) to write

$$\epsilon_{ij}=-\sum_l\frac{l^2\omega_{p\alpha}^2}{\omega(\omega-l\Omega_\alpha)}Q_{ij}, \tag{6}$$

$$Q_{ij}=\int dv\, 4\pi v^2 f(v)\,\frac{1}{z^2}\Pi_{ij}(z), \tag{7}$$

$$z=kv/\Omega_\alpha. \tag{8}$$

These expressions apply for arbitrary distributions $f(v)$ that are isotropic in velocity space, and from Ref. 1,

$$\Pi_{xx}=J_{2l}(2z), \tag{9}$$

$$\Pi_{xy}=-\frac{iz}{2l}[J_{2l-1}(2z)-J_{2l+1}(2z)], \tag{10}$$

$$\Pi_{yy}=\left(1-\frac{z^2}{l^2}\right)J_{2l}(2z)+\frac{z}{2l^2}\int_0^{2z}J_{2l}(x)dx. \tag{11}$$

The general dispersion relation for waves in the ion cyclotron range of frequencies, propagating perpendicular to the magnetic field in the presence of a distinct energetic delta-function shell ion population was given in Ref. 1, at Eq. (67). We are, therefore, in a position to short-circuit the considerable amount of algebra that would be involved in an *ab initio* calculation of the dispersion relation for more general isotropic energetic ion velocity distributions, as follows. Having chosen a distribution function, we evaluate the functions Q_{ij} analytically or, if need be, numerically. Next, in Eq. (67) and the preceding expressions of Ref. 1, we systematically replace $(1/z_\alpha^2)\Pi_{ij}(z_\alpha)$ by Q_{ij}. The dispersion relation is then ready for immediate evaluation.

In the present paper, we aim to pursue the analytical path as far as possible and choose an energetic ion distribution of the form

$$f_\alpha(\mathbf{v})=\frac{n_\alpha}{4\pi^{3/2}v_0^2v_T}\exp\left(-\frac{(v-v_0)^2}{v_T^2}\right), \qquad (12)$$

which describes an extended shell centred at v_0, whose spread depends on the magnitude of v_T. We confine attention to the case where v_T is sufficiently small compared to v_0 that

$$\exp(-v_0^2/v_T^2)\ll 1, \qquad (13)$$

and the normalization of Eq. (12) assumes this. As we shall see, the condition Eq. (13) is not unduly restrictive in many physical scenarios where an extended shell is an appropriate model.

Let us now proceed with the calculation of the dielectric tensor elements. From Eqs. (7) to (11), for arbitrary $f(v)$ we have

$$Q_{xx}=\frac{4\pi\Omega_\alpha^2}{k^2}\int dv\, f(v)J_{2l}\left(\frac{2kv}{\Omega_\alpha}\right), \qquad (14)$$

$$Q_{xy}=-\frac{i}{l}\frac{2\pi\Omega_\alpha}{k}\int dv\, vf(v)\left[J_{2l-1}\left(\frac{2kv}{\Omega_\alpha}\right)-J_{2l+1}\left(\frac{2kv}{\Omega_\alpha}\right)\right], \qquad (15)$$

$$Q_{yy}=\frac{4\pi\Omega_\alpha^2}{k^2}\int dv\, f(v)J_{2l}\left(\frac{2kv}{\Omega_\alpha}\right)-\frac{4\pi}{l^2}\int dv\, v^2f(v)\times J_{2l}\left(\frac{2kv}{\Omega_\alpha}\right)+\frac{2\pi\Omega_\alpha}{l^2k}\int dv\, vf(v)\int_0^{2kv/\Omega_\alpha}J_{2l}(x)dx. \qquad (16)$$

Substituting the distribution function Eq. (12) into Eq. (14), we have

$$Q_{xx}=\frac{\Omega_\alpha^2}{k^2v_0^2}\frac{1}{\pi^{1/2}}\int_0^\infty\frac{dv}{v_T}\exp\left(-\frac{(v-v_0)^2}{v_T^2}\right)J_{2l}\left(\frac{2kv}{\Omega_\alpha}\right). \qquad (17)$$

Defining dimensionless variables

$$u=(v-v_0)/v_T,\quad u_0=v_0/v_T, \qquad (18)$$

and using the identity Eq. (A2) from the Appendix, Eq. (17) can be written

$$Q_{xx}=\frac{\Omega_\alpha^2}{k^2v_0^2}\sum_{m=-\infty}^{\infty}J_{2l-m}\left(\frac{2kv_0}{\Omega_\alpha}\right)\frac{1}{\pi^{1/2}}\int_{-u_0}^\infty du\times\exp(-u^2)J_m\left(\frac{2kv_T}{\Omega_\alpha}u\right). \qquad (19)$$

Subject to the restriction given by Eq. (13) on the relative values of v_0 and v_T, we may to good approximation replace the lower limit on the integral in Eq. (19) by $-\infty$. In this case, the expression for Q_{xx} becomes analytically tractable as follows. First, from the parity of the functions in the integrand, the integral is nonzero only for $m=2n$, where n is integer. Then we may use the identity Eq. (A3) to obtain

$$Q_{xx}=\frac{\Omega_\alpha^2}{k^2v_0^2}\exp\left(-\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\sum_{n=-\infty}^{\infty}J_{2(l-n)}\times\left(\frac{2kv_0}{\Omega_\alpha}\right)I_n\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right). \qquad (20)$$

Note that in the limit of an infinitesimally thin shell, where v_T/v_0 tends to zero, Eq. (20) yields $Q_{xx}=(\Omega_\alpha/kv_0)^2\Pi_{xx}$, as required.

Turning now to Eq. (15) and substituting the distribution function Eq. (12), we may use Eqs. (18) and (A2) to write

$$Q_{xy}=-\frac{i}{2l}\frac{\Omega_\alpha}{kv_0}\frac{v_T}{v_0}\frac{1}{\pi^{1/2}}\int_{-u_0}^\infty du(u+u_0)\exp(-u^2)\times\sum_{m=-\infty}^{\infty}J_m\left(\frac{2kv_T}{\Omega_\alpha}u\right)\left[J_{2l-1-m}\left(\frac{2kv_0}{\Omega_\alpha}\right)-J_{2l+1-m}\left(\frac{2kv_0}{\Omega_\alpha}\right)\right]. \qquad (21)$$

Replacing the lower limit on the integral by $-\infty$ as before, it follows from parity arguments and the identity Eq. (A4) that in Eq. (21),

$$\sum_{m=-\infty}^{\infty}\frac{1}{\pi^{1/2}}\int_{-\infty}^\infty du\, u\exp(-u^2)J_m\left(\frac{2kv_T}{\Omega_\alpha}u\right)=\frac{kv_T}{2\Omega_\alpha}\exp\left(-\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\sum_{n=-\infty}^{\infty}\left[I_n\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)-I_{n+1}\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\right], \qquad (22)$$

where $m=2n+1$. Then, using Eqs. (A3) and (22), Eq. (21) yields

$$Q_{xy}=-\frac{i}{2l}\frac{\Omega_\alpha}{kv_0}\exp\left(-\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\sum_{n=-\infty}^{\infty}\left\{I_n\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\right.$$
$$\times\left[J_{2(l-n)-1}\left(\frac{2kv_0}{\Omega_\alpha}\right)-J_{2(l-n)+1}\left(\frac{2kv_0}{\Omega_\alpha}\right)\right]$$
$$+\frac{v_T}{v_0}\frac{kv_T}{2\Omega_\alpha}\left[I_n\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)-I_{n+1}\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\right]$$
$$\left.\times\left[J_{2(l-n-1)}\left(\frac{2kv_0}{\Omega_\alpha}\right)-J_{2(l-n)}\left(\frac{2kv_0}{\Omega_\alpha}\right)\right]\right\}. \quad (23)$$

It can be shown that Eq. (23) reduces to $(\Omega_\alpha/kv_0)^2\Pi_{xy}$ in the limit where v_T/v_0 tends to zero.

Finally, we consider Q_{yy} given by Eq. (16), for the case of the distribution function Eq. (12). Let us denote the three integrals appearing in Eq. (16) as $Q_{yy}^{(s)}$, $s=1, 2, 3$; that is,

$$Q_{yy}=Q_{yy}^{(1)}+Q_{yy}^{(2)}+Q_{yy}^{(3)}, \quad (24)$$

where by inspection

$$Q_{yy}^{(1)}=Q_{xx}. \quad (25)$$

Using the notation of Eq. (18) and the identity Eq. (A2),

$$Q_{yy}^{(2)}=-\frac{1}{l^2}\frac{v_T^2}{v_0^2}\sum_{m=-\infty}^{\infty}J_{2l-m}\left(\frac{2kv_0}{\Omega_\alpha}\right)\frac{1}{\pi^{1/2}}\int_{-u_0}^{\infty}du$$
$$\times(u+u_0)^2\exp(-u^2)J_m\left(\frac{2kv_T}{\Omega_\alpha}u\right). \quad (26)$$

Provided that u_0 is sufficiently large that we may approximate the lower limit on the integral by $-\infty$, the identities Eqs. (A3), (A4), and (A5) can be used in Eq. (26) to give

$$Q_{yy}^{(2)}=-\frac{1}{l^2}\exp\left(-\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\sum_{n=-\infty}^{\infty}\left(J_{2(l-n)}\left(\frac{2kv_0}{\Omega_\alpha}\right)\right.$$
$$\times\left\{\left(1+\frac{v_T^2}{2v_0^2}-\frac{v_T^2}{v_0^2}\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)I_n\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)+\frac{v_T^2}{2v_0^2}\frac{k^2v_T^2}{2\Omega_\alpha^2}\right.$$
$$\left.\times\left[I_{n-1}\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)+I_{n+1}\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\right]\right\}+\frac{2v_T}{v_0}\frac{kv_T}{2\Omega_\alpha}$$
$$\left.\times J_{2(l-n)-1}\left(\frac{2kv_0}{\Omega_\alpha}\right)\left[I_n\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)-I_{n+1}\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\right]\right). \quad (27)$$

The final term in Q_{yy} can be written

$$Q_{yy}^{(3)}=\frac{1}{l^2}\frac{\Omega_\alpha}{2kv_0}\frac{v_T}{v_0}\frac{1}{\pi^{1/2}}\int_{-u_0}^{\infty}du(u+u_0)\exp(-u^2)$$
$$\times\int_0^{(2kv_T/\Omega_\alpha)(u+u_0)}J_{2l}(x)dx. \quad (28)$$

As usual, we replace $-u_0$ by $-\infty$ as the lower limit on the integral, invoking Eq. (12). Then, using the chain of reasoning employed in deriving Eq. (A6), we obtain

$$Q_{yy}^{(3)}=\frac{1}{l^2}\frac{\Omega_\alpha}{2kv_0}\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}du\exp(-u^2)$$
$$\times\int_0^{2kv_T/\Omega_\alpha(u+u_0)}J_{2l}(x)dx+\frac{1}{l^2}\frac{v_T^2}{2v_0^2}$$
$$\times\exp\left(-\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\sum_{n=-\infty}^{\infty}J_{2(l-n)}\left(\frac{2kv_0}{\Omega_\alpha}\right)I_n\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right). \quad (29)$$

Combining Eqs. (25), (27), and (29) in Eq. (24), with the aid of Eq. (20) we obtain

$$Q_{yy}=\exp\left(-\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\sum_{n=-\infty}^{\infty}J_{2(l+n)}\left(\frac{2kv_0}{\Omega_\alpha}\right)I_n\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\left(\frac{\Omega_\alpha^2}{k^2v_0^2}-\frac{1}{l^2}\right)+\frac{1}{l^2}\frac{\Omega_\alpha}{2kv_0}\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}du\exp(-u^2)$$
$$\times\int_0^{2kv_T/\Omega_\alpha(u+u_0)}J_{2l}(x)dx-\frac{1}{l^2}\frac{v_T}{v_0}\frac{kv_T}{\Omega_\alpha}\exp\left(-\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\sum_{n=-\infty}^{\infty}J_{2(l-n)-1}\left(\frac{2kv_0}{\Omega_\alpha}\right)\left[I_n\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)-I_{n+1}\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\right]$$
$$-\frac{1}{l^2}\frac{v_T^2}{v_0^2}\frac{k^2v_T^2}{4\Omega_\alpha^2}\exp\left(-\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\sum_{n=-\infty}^{\infty}J_{2(l-n)}\left(\frac{2kv_0}{\Omega_\alpha}\right)\left[I_{n-1}\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)-2I_n\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)+I_{n+1}\left(\frac{k^2v_T^2}{2\Omega_\alpha^2}\right)\right]. \quad (30)$$

In the limit where v_T/v_0 tends to zero, Eq. (30) yields the value $(\Omega_\alpha/kv_0)^2\Pi_{yy}$, as required—note that Π_{yy} defined at Eq. (A6) of Ref. 9 contains a typographical error, and that the correct expression is given by Eq. (11).

We are now in a position to complete the strategy that was outlined after Eq. (11). The expressions for Q_{ij} that we have calculated can be used to replace $(\Omega_\alpha/kv_0)^2\Pi_{ij}$ in the dispersion relation Eq. (67) of Ref. 1, and in the preceding expressions in Ref. 1. Thus we have

$$\frac{(\omega^2-c_A^2k^2)}{c_A^2k^2}=\mu_1(s,z_i)\,\frac{\omega}{\omega-s\Omega_i}+\xi\mu_2(l,z_\alpha)\,\frac{\omega}{\omega-l\Omega_\alpha}+\frac{\Omega_i^2}{c_A^2k^2}$$
$$\times\left(1-\frac{\Omega_i^2}{\omega^2}\right)\left(\xi\mu_3(l,z_\alpha,s,z_i)\right.$$
$$\times\frac{\omega^2}{(\omega-s\Omega_i)(\omega-l\Omega_\alpha)}+\mu_4(s,z_i)$$
$$\left.\times\frac{\omega^2}{(\omega-s\Omega_i)^2}+\xi^2\mu_5(l,z_\alpha)\frac{\omega^2}{(\omega-l\Omega_\alpha)^2}\right), \quad (31)$$

$$\mu_1(s,z_i)=\left(1-\frac{\Omega_i^2}{\omega^2}+\frac{2\Omega_i^2}{c_A^2k^2}\right)f_s(z_i)-\frac{2\Omega_i^2}{c_A^2k^2}\left(\frac{s\omega}{\Omega_i}-z_i^2\right)g_s(z_i), \quad (32)$$

$$\mu_2(l,z_\alpha)=l^2\left[\left(1-\frac{\Omega_i^2}{\omega^2}+\frac{\Omega_i^2}{c_A^2k^2}\right)Q_{xx}+\frac{\Omega_i^2}{c_A^2k^2}Q_{yy}\right.$$
$$\left.-2i\frac{\Omega_i^2}{c_A^2k^2}\frac{\omega}{\Omega_i}Q_{xy}\right], \quad (33)$$

$$\mu_3(l,z_\alpha,s,z_i)=l^2\{f_s(z_i)(Q_{xx}+Q_{yy})-2g_s(z_i)$$
$$\times(isQ_{xy}-z_i^2Q_{xx})\}, \quad (34)$$

$$\mu_4(s,z_i)=f_s^2(z_i)-s^2g_s^2(z_i)+2z_i^2f_s(z_i)g_s(z_i), \quad (35)$$

$$\mu_5(l,z_\alpha)=l^4(Q_{xx}Q_{yy}+Q_{xy}^2). \quad (36)$$

In the above expressions,

$$z_\alpha=kv_0/\Omega_\alpha, \quad (37)$$

$$\xi=\omega_{p\alpha}^2/\omega_{pi}^2, \quad (38)$$

$$z_i^2=k^2v_{Ti}^2/2\Omega_i^2, \quad (39)$$

$$f_s(z_i)=\frac{s^2I_s(z_i^2)\exp(-z_i^2)}{z_i^2}, \quad (40)$$

$$g_s(z_i)=\exp(-z_i^2)[I_s'(z_i^2)-I_s(z_i^2)], \quad (41)$$

where subscript i refers to the thermal ion population [v_{Ti} in Eq. (39) is, of course, distinct from v_T introduced in Eq. (12)], and $l\Omega_\alpha$ and $s\Omega_i$ are, respectively, the energetic and thermal ion cyclotron harmonics that are closest to resonance with the wave frequency. For further details, we refer to Ref. 1.

Equation (31) is the dispersion relation for perpendicular propagating waves in the ion cyclotron range of frequencies, in the presence of an energetic ion population whose distribution function is given by Eq. (12), subject to the constraint Eq. (13). It includes wave branches corresponding to the fast Alfvén and ion Bernstein modes, the latter supported by both the thermal and the energetic ion populations. As discussed in Ref. 1, a reactive instability can occur at the mode resonance

$$\omega\simeq c_Ak\simeq l\Omega_\alpha. \quad (42)$$

If, in addition,

$$\omega\simeq s\Omega_i, \quad (43)$$

the positive energy loading contributed by the resonant thermal ion Bernstein wave acts to diminish, or even eliminate, the instability. Physically, this effect is the analogue, for the perpendicular propagation considered here, of cyclotron damping—which would, of course, require $k_\parallel$ to be nonzero. As we discussed in the Introduction, the spread in velocity of the energetic ion population, represented by v_T in our model, will also reduce or suppress the instability, relative to the zero-width case. In the following section, we use Eqs. (31) to (41) to quantify this effect in detail.

To conclude this section, we note that it has proved possible to obtain analytical expressions for the dielectric tensor elements for the model distribution function Eq. (12). As a result, we can study the parametric dependence of the dispersion relation in a way which would not be possible if only numerical integral representations of the dielectric tensor elements were available, as would be the case for less analytically tractable distributions.

III. SUPPRESSION OF THE MAGNETOACOUSTIC CYCLOTRON INSTABILITY

Let us first confirm that the restriction Eq. (13), which makes possible the analytical integrations in Sec. II, does not significantly constrain the range of applicability of our dispersion relation in the results which follow. Since $\exp(-9)\sim10^{-4}$, for practical purposes Eq. (13) requires

$$v_0/v_T\gtrsim3. \quad (44)$$

The parameters involving v_T that are of interest in the dispersion relation are kv_T/Ω_α and its square. From the resonance condition Eq. (42), it follows that in the regime of interest,

$$\frac{kv_T}{\Omega_\alpha}\simeq\frac{v_T}{v_0}\frac{lv_0}{c_A}. \quad (45)$$

Typically, for instability to be possible, v_0 will be in the range of one to several times c_A, and l is an integer multiple of unity. Thus, in the range of interest, whose upper limit occurs when the Bessel function with argument $2lv_0/c_A$ becomes negligible, we have

$$1<lv_0/c_A\lesssim50. \quad (46)$$

It follows from Eqs. (44) to (46) that

$$0<kv_T/\Omega_\alpha\lesssim15, \quad (47)$$

which renders accessible the full range of modified Bessel function and inverse exponential dependence on $k^2v_T^2/2\Omega_\alpha^2$ in the dielectric tensor elements derived in Sec. II.

The dispersion relation has been solved using the Culham complex zero-finder code. Some examples of the effect of different values of v_T on the magnetoacoustic cyclotron instability are shown in Figs. 1–3. In Fig. 1, the magnitude of the growth rate is plotted as a function of dimensionless wave number $z_\alpha(=kv_0/\Omega_\alpha)$, for fixed parameter values appropriate to an energetic proton (subscript α) population in a deuterium (subscript i) discharge. In the limit where v_T is zero (solid line), Fig. 1 agrees with the result ob-

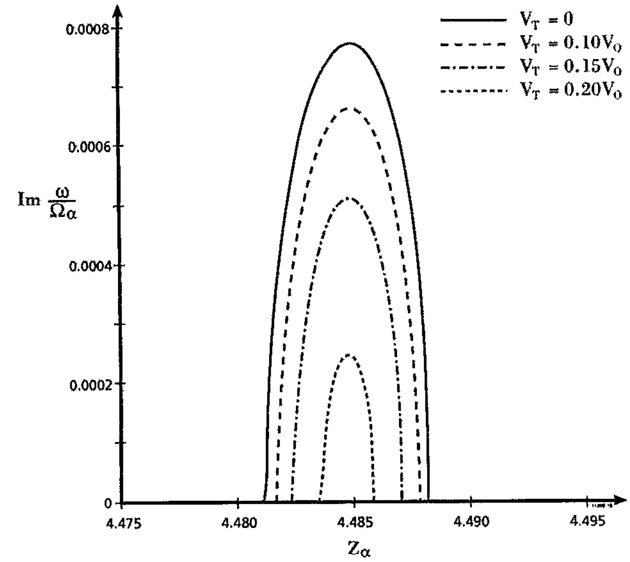

FIG. 1. Magnetoacoustic cyclotron instability of an energetic proton population (α) in a deuterium plasma (i) for $l=2$. Growth rate (normalized to Ω_α) is shown as a function of dimensionless wave number $z_\alpha=kv_0/\Omega_\alpha$ for different values of v_T. Peak instability occurs at $z_\alpha=lv_0/c_a$. Plasma parameters $B=3.1$ T, $\epsilon_i=1$ keV, $\epsilon_\alpha=3$ MeV, $n_i=2\times10^{19}m^{-3}$, $n_\alpha/n_i=10^{-5}$.

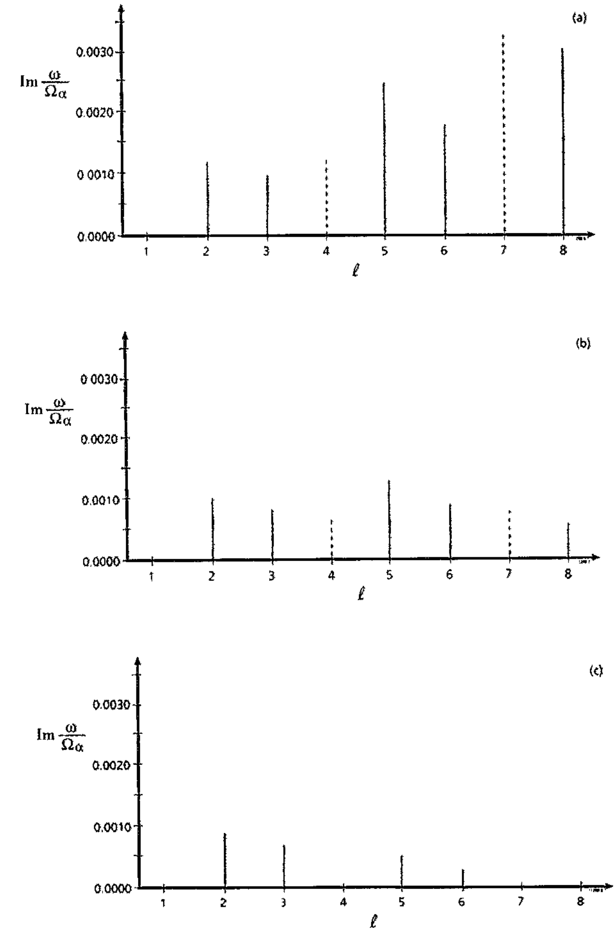

FIG. 3. Multiple cyclotron harmonic instability for $v_T=0(a)$, $0.1v_0(b)$, and $0.15v_0(c)$. Plasma parameters $B=2.1$ T, $\epsilon_i=1$ keV, $\epsilon_\alpha=3$ MeV, $n_\alpha/n_i=10^{-5}$, $n_i=10^{19}m^{-3}(1.2\times10^{19}m^{-3}$ for $l=4$ and 7).

tained for the zero-thickness shell distribution of Ref. 1, as required. The degree of instability weakens as v_T is increased, and the range of unstable wave numbers becomes narrower. For the parameter values considered, the instability is totally suppressed once v_T reaches a value $\simeq 0.22v_0$. This effect is displayed as a continuous function of v_T in Fig. 2, where the maximum growth rate (which always occurs at $z_\alpha=lv_0/c_A\equiv z_{\text{res}}$) is plotted against v_T/v_0. A distinctive feature of the magnetoacoustic cyclotron instability is the possibility of simultaneous growth at multiple ion cyclotron harmonics. This is shown in Fig. 3(a) for a case which is identical to Fig. 3 of Ref. 1 in the limit where $v_T=0$. When $v_T=0.1v_0$ [Fig. 3(b)], all the cyclotron harmonics growth rates are significantly reduced, and when v_T is increased to $0.15v_0$ [Fig. 3(c)], the cyclotron harmonics at $l=4$, 7, and 8 become stable. In parallel to this, the linear growth rates at the remaining unstable cyclotron harmonics are reduced. Eventually, all the harmonics become stable as v_T is increased above $0.2v_0$.

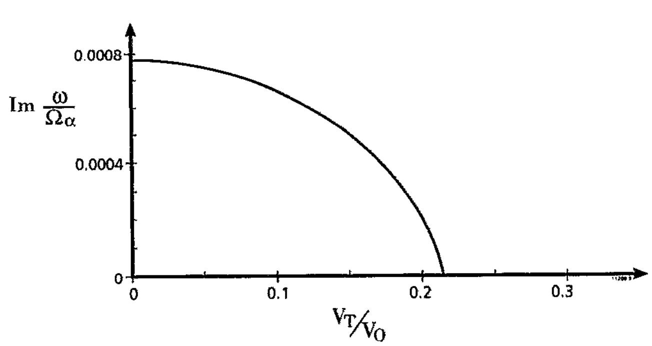

FIG. 2. Maximum instability growth rate as a function of v_T/v_0. Plasma parameters, as for Fig. 1.

Some insight into these results can be obtained by returning to the analytical expressions of Sec. II. The instability is driven by the terms Q_{ij}, which typically have three components, as follows. First, Bessel functions with argument $2kv_0/\Omega_\alpha$; these terms survive in some form in the limit where v_T is zero, and determine the wave numbers at which maximum instability occurs,

$$kv_0/\Omega_\alpha=lv_0/c_A. \tag{48}$$

When v_T is nonzero, these Bessel functions always occur multiplied by modified Bessel functions whose argument is $k^2v_T^2/2\Omega_\alpha^2$. These are the second class of component of Q_{ij}, and should be viewed in combination with the third class of component, which is an overall multiplying inverse exponential with argument $k^2v_T^2/2\Omega_\alpha^2$. For instability at given l, the summation over Bessel function indices is dominated by the term where the modified Bessel function has index zero. Thus, when v_T is finite, the terms which drive the instability in the limit, where v_T is zero, are scaled down by dominant multipliers that have a form which is—very broadly—equivalent to the plot of $\exp(-x)I_0(x)$ shown in Fig. 9.8 of Ref. 13. This is an initially sharply declining function, which falls to 0.3 at $x=2$. Depending on the magnitude of the stabilizing terms in the dispersion relation, the associated decline in the driving term will reduce or eliminate the instability as v_T is increased. For the wave number corresponding to maximum growth rate defined by Eq. (47), the argument

$$x=\frac{k^2v_T^2}{2\Omega_\alpha^2}=\frac{l^2}{2}\frac{v_T^2}{v_0^2}\frac{v_0^2}{c_A^2}=\frac{v_T^2}{2v_0^2}z_{\text{res}}^2. \tag{49}$$

It follows that the higher-l harmonics will be stabilized earliest, as v_T is increased. This is reflected in Fig. 3, where the instabilities of the higher harmonics (especially $l=7$ and 8) can be seen to reduce significantly as v_T is increased to $0.1v_0$. A further increase to $v_T=0.15v_0$ leads first to the stabilization of $l=8$, followed by $l=7$ and then $l=4$. The stabilization of $l=4$ before $l=5$ and 6 simply reflects the lower growth rate of $l=4$ compared to the higher harmonics, shown in Fig. 3(a).

IV. CONCLUSIONS

In the present paper, we have studied the magnetoacoustic cyclotron instability driven by a minority energetic ion population whose distribution function is given by Eq. (12), in the presence of a majority thermal ion species. A fusion product population which can be modeled in this way may perhaps be responsible for, in particular, the time-dependent component of superthermal ion cyclotron emission whose sawtooth oscillation is described in Ref. 3. The dependence of the instability on energetic ion density n_α and birth speed v_0 has been investigated in Ref. 1. Here, we have quantified the effect of the velocity spread, determined by the parameter v_T. This has been achieved by exploiting the isotropic structure of Eq. (12), together with a number of Bessel function integral identities, to arrive at an analytical formulation of the dispersion relation for perpendicular fast Alfvén-ion Bernstein waves, which is valid for $v_0/v_T \gtrsim 3$. We find that, for typical parameters of interest, the degree of instability is significantly reduced once v_T/v_0 exceeds ~ 0.1. Gaps appear in typical multiple cyclotron harmonic excitation patterns for $0.1 \lesssim v_T/v_0 \lesssim 0.2$. Most harmonics are stable for $v_T/v_0 \gtrsim 0.25$ in the examples chosen. The diagnostic implication of these results is that, in situations where the observed ion cyclotron emission may be driven by the perpendicular magnetoacoustic cyclotron instability of a shell-type fusion product distribution, the shell must be relatively narrow in velocity space, with $v_T/v_0 \lesssim 0.25$.

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APPENDIX: BESSEL FUNCTION INTEGRALS

The following identities, involving Bessel functions and their integrals in combination with exponential functions, are required for the calculation of the dielectric tensor elements. First,[14]

$$J_n(y+z)=\sum_{m=-\infty}^{\infty} J_m(y)J_{n-m}(z). \tag{A1}$$

Hence, in particular,

$$J_{2l}\left(\frac{2kv}{\Omega_\alpha}\right)=\sum_{m=-\infty}^{\infty} J_{2l-m}\left(\frac{2kv_0}{\Omega_\alpha}\right)J_m\left(\frac{2k(v-v_0)}{\Omega_\alpha}\right). \tag{A2}$$

Next,[15]

$$\int_0^\infty J_{2\nu}(at)\exp(-p^2t^2)dt=\frac{\pi^{1/2}}{2p}\exp\left(-\frac{a^2}{8p^2}\right)I_\nu\left(\frac{a^2}{8p^2}\right). \tag{A3}$$

It follows from Eq. (A3) that

$$\begin{aligned}\int_0^\infty J_{2\nu}(at)\exp(-p^2t^2)t\,dt&=-\frac{1}{2p^2}\int_0^\infty J_{2\nu}(at)\times\left(\frac{\partial}{\partial t}\exp(-p^2t^2)\right)dt\\&=\frac{\delta_{0\nu}}{2p^2}+\frac{a}{2p^2}\int_0^\infty J_{2\nu}'(at)\exp(-p^2t^2)dt\\&=\frac{\delta_{0\nu}}{2p^2}+\frac{a\pi^{1/2}}{8p^3}\exp\left(-\frac{a^2}{8p^2}\right)\left[I_{\nu-1/2}\left(\frac{a^2}{8p^2}\right)-I_{\nu+1/2}\left(\frac{a^2}{8p^2}\right)\right].\end{aligned} \tag{A4}$$

Using Eq. (A3), we can also derive

$$\begin{aligned}&\int_0^\infty J_{2\nu}(at)\exp(-p^2t^2)t^2\,dt\\&=-\frac{\partial}{\partial p^2}\int_0^\infty J_{2\nu}(at)\exp(-p^2t^2)dt\\&=\frac{\pi^{1/2}}{4p^3}\exp\left(-\frac{a^2}{8p^2}\right)\left\{I_\nu\left(\frac{a^2}{8p^2}\right)+\frac{a^2}{8p^2}\left[I_{\nu-1}\left(\frac{a^2}{8p^2}\right)-2I_\nu\left(\frac{a^2}{8p^2}\right)+I_{\nu+1}\left(\frac{a^2}{8p^2}\right)\right]\right\}.\end{aligned} \tag{A5}$$

Finally, we also require the identity

$$\begin{aligned}&\int_0^\infty dt\exp(-p^2t^2)t\int_0^{at}J_{2l}(x)dx\\&=-\frac{1}{2p^2}\int_0^\infty dt\left(\frac{\partial}{\partial t}\exp(-p^2t^2)\right)\int_0^{at}J_{2l}(x)dx\\&=\frac{a}{2p^2}\int_0^\infty dt\exp(-p^2t^2)J_{2l}(at)\\&=\frac{a\pi^{1/2}}{4p^3}\exp\left(-\frac{a^2}{8p^2}\right)I_l\left(\frac{a^2}{8p^2}\right),\end{aligned} \tag{A6}$$

using Eq. (A3) in the final step.

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