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A kinetic model of fast wave propagation in the vicinity of the minority ion cyclotron resonance in a toroidal magnetic field

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Nearly all kinetic treatments of fast wave minority heating of inhomogeneous plasma in the cyclotron range of frequencies assume the magnetic field varies in the direction perpendicular to the magnetic field. However, the toroidal magnetic field of a tokamak varies along a field line due to the rotational transform and causes a small number of trapped particles to turn in the region of cyclotron resonance. In order to include the effects of rotational transform and, hence, trapped particles in the kinetic plasma response, a simplified, concentric circle flux surface model of a tokamak is employed. The most important result of this work is the derivation of response functions for Maxwellian and bi-Maxwellian minority ions which generalize and extend previous replacement Z function forms obtained from a slab approximation of a tokamak (which also retains the variation of the strength of the magnetic field along a field line). The plasma response functions obtained include both passing and trapped ions, off-axis heating, and are valid for arbitrary minority ion concentrations. The response function for a bi-Maxwellian in the case of strong anisotropy substantially modifies the Maxwellian result. Anisotropy and the effects of toroidal geometry are illustrated graphically and tend to enter at higher toroidal mode numbers. For minority concentrations of the order or less than a critical value, the plasma response functions are used to obtain the standard transmission coefficient previously obtained for straight magnetic-field models. The expression for the transmission coefficient is shown to be valid for more general unperturbed distribution functions of pitch angle and speed on each flux surface provided $k_{\|} \rho \ll 1$, where $k_{\|}$ is the parallel wave number and ρ the minority gyroradius.

I. INTRODUCTION

Fast wave heating of tokamak plasmas in the ion cyclotron range of frequencies (ICRF) is well established in both the minority and second harmonic regimes.[1–4] For both these schemes an understanding of the factors which determine the (spatial) power deposition profile is highly desirable since this would lead to greater control over the heating. Three important factors which influence the resonance width are the Doppler effect, the spatial variation of the equilibrium magnetic field (including the effects of trapped particles), and the anisotropy of the distribution function. All three effects are retained in the present treatment, which for simplicity specifically considers only the fast wave minority heating case, even though the key techniques are valid more generally.

The model which has been used in most calculations of ion cyclotron absorption in a tokamak is that of an equilibrium magnetic field with straight field lines and a perpendicular gradient in strength.[5] This model has produced much useful information and includes the two essential ingredients of Doppler broadening and perpendicular magnetic-field inhomogeneity. However, a straight-field line model cannot account for effects due to the variation of the strength of the magnetic field along a field line.

In a tokamak magnetic field an ion (even in the zero Larmor radius approximation considered here) experiences a variation in the equilibrium magnetic-field strength due to the rotational transform. This feature has a profound effect on resonant ions compared with the straight magnetic-field model. For the latter case, an ion if resonant along a particular field line will remain in resonance since the field lines are straight. All other ions (i.e., with different values of $v_{\|}$) on this particular field line will always be out of resonance. This situation also applies to the finite Larmor radius broadening, where a given ion is either in resonance over the whole of its orbit or not at all.[6] For the toroidal model considered in this paper the situation is quite different. Since the field lines are curved, ions with different values of $v_{\|}$ can be in resonance on the same field line but at different spatial locations. Furthermore, these ions do not remain in resonance permanently, as in the straight-field model, but pass in and out of resonance. Thus, for the straight-field line case there are fewer resonant ions interacting strongly and indefinitely with the wave; whereas in the curved field line model there are many more resonant ions interacting for a finite time and, therefore, more weakly with the wave.

In order to include the effect of the rotational transform, a simplified concentric circular flux surface model of the toroidal magnetic field of a tokamak is employed. Using this model means that the effect of trapped ions is included in the plasma response function. In particular, the procedure employed retains the effects of the trapped particles which turn in the resonance layer, as well as the

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remaining trapped and passing particles. To date, there are very few treatments of the effect of the variation in the magnitude of the magnetic field along a field line or of trapped ions on fast wave heating at an ion cyclotron resonance. Notable exceptions are the work of Faulconer[7] and Smithe *et al.*[8] in which the model employed is partially motivated by the earlier work of Itoh *et al.*[9] Their plasma response function is found by retaining the leading order effect of the field inhomogeneity, as well as Doppler broadening, but without introducing the full complications of tokamak geometry.

Although the fraction of trapped ions is small they may be expected to be good absorbers, and in any case ion cyclotron absorption tends to make the ions non-Maxwellian by pushing them towards the trapped region. Grekov *et al.*[10] retain the effect of trapped particles on the cyclotron absorption of fast waves at the second and higher harmonics for a bi-Maxwellian, but do not give simple, explicit results for the plasma response function. Chen and Tsai[11] also consider trapped particle effects, but do not attempt to find a simple, explicit expression for the plasma response function. Instead, they evaluate the resonant energy absorption in the Maxwellian case using the full collisionless trajectories. The model employed in the work presented here differs from Ref. 10 by considering the minority heating regime. It also differs from Ref. 11 by implicitly assuming a collisional disruption of the particle motion along the field relative to the wave occurs between interactions with the radio frequency (rf), as described in more detail in Refs. 8 and 12–14. As a result of the collisions, only the most recent details of the particle's trajectory need be retained in the vicinity of the resonance layer. This simplified treatment of the trajectory retains the finite wave–particle interaction time by permitting the trapped and passing particles to pass through resonance, but dramatically simplifies the mathematical description. These simplifications ultimately make it possible to model the effects of anisotropy by evaluating the plasma response function for a bi-Maxwellian unperturbed distribution function.

The plan of the paper is as follows. In Sec. II, the model of the toroidal magnetic field with circular flux surfaces is defined. The gyrokinetic equation[15] is used to calculate the perturbed minority ion current neglecting finite Larmor radius and drift effects. In Sec. III, the resonant response function is obtained for the minority ions assuming that their equilibrium velocity distribution is Maxwellian. The response function is appropriate to trapped, as well as passing ions, is valid for arbitrary minority densities, and generalizes the results of Faulconer[7] and Smithe *et al.*[8] Section IV uses this response function for a perturbative calculation of the optical depth and transmission coefficient for the fast wave crossing the minority resonance. These particular quantities are, therefore, valid only for minority densities of the order or less than a critical value given in Sec. IV. Section V generalizes the optical depth analysis further to an arbitrary minority ion distribution function and closes by evaluating the plasma response function for an arbitrary minority concentration bi-Maxwellian unperturbed distribution function.

II. PERTURBED MINORITY CURRENT DENSITY

In the vicinity of the fundamental minority resonance the cold fluid ion response is modified by kinetic effects due to Doppler shifted cyclotron damping in the inhomogenous toroidal magnetic field. In this section, a formalism for evaluating the kinetic modifications of the perturbed minority current density in a tokamak is presented. The model considered, which neglects all finite orbit effects, is the simplest, experimentally relevant limit retaining trapped, as well as passing particle effects.

The independent spatial variables are taken to be the minor radius r, the poloidal angle β, and the toroidal angle ζ. The major radius is $R=R_0+r\cos\beta$ with R_0 the location of the magnetic axis. Concentric circular magnetic flux surfaces are employed with $|\mathbf{B}|=B\approx B_0(1-\epsilon\cos\beta)$, $\mathbf{B}\cdot\nabla\zeta\approx B/R$, and $BR\approx$ constant to the requisite order, where $\epsilon=r/R_0$, in order to illustrate the procedure without introducing flux coordinates.

The velocity of a particle $\mathbf{v}$ is written as $\mathbf{v}=\mathbf{v}_\perp+v_\parallel\,\hat{n}=v_\perp\,(\hat{r}\cos\phi+\hat{n}\times\hat{r}\sin\phi)+v_\parallel\,\hat{n}$ with $v_\parallel=\hat{n}\cdot\mathbf{v}$, $\hat{n}=\mathbf{B}/B$, $v_\perp=|\hat{n}\times\mathbf{v}|$, $\hat{r}=\nabla r/|\nabla r|$ and ϕ the gyrophase. The independent velocity space variables v, λ, and ϕ are employed with $v=|\mathbf{v}|$ and $\lambda=B_0v_\perp^2/Bv^2=2\mu B_0/v^2\approx Rv_\perp^2/R_0v^2$, where $\mu=v_\perp^2/2B$ is the magnetic moment and $v_\parallel^2=v^2(1-\lambda B/B_0)$.

Using the preceding variables and neglecting finite orbit effects, the linearized gyrokinetic equation[14,15] for the perturbed minority distribution function f simplifies to

$$\frac{\partial f}{\partial t}+\frac{v_\parallel}{R}\left(\frac{\partial f}{\partial\zeta}+\frac{1}{q}\frac{\partial f}{\partial\beta}\right)-\Omega\frac{\partial f}{\partial\phi}=-S \tag{1}$$

with $\Omega=ZeB/Mc$, $S=\mathbf{a}\cdot\nabla_vF$, $\mathbf{a}$ the acceleration, and F the unperturbed minority distribution function which is independent of gyrophase, poloidal angle and toroidal angle. For second harmonic heating the distinction between the guiding center $\mathbf{R}$ and particle $\mathbf{r}$ location must be retained in S to obtain the familiar Bessel functions. Retaining the distinction between $\mathbf{R}$ and $\mathbf{r}$ would also allow nonlocal absorption effects due to the variation of the equilibrium magnetic field across the finite Larmor radius orbits to be treated.[6]

Perturbed magnetic-field effects do not enter Eq. (1) when F is isotropic in velocity space. For more general unperturbed distribution functions the neglect of finite gyroradius corrections in Eq. (1) also allows perturbed magnetic field effects to be ignored. In particular, for anisotropic F, when $k_\perp v_\perp/\omega\ll1$ with $k_\perp$ a typical perpendicular wave number and ω the applied rf wave frequency, the acceleration $\mathbf{a}$ may be taken to be simply

$$\mathbf{a}=(Ze/M)\mathbf{e},$$

where $\mathbf{e}$ is the perturbed electric field. Then S may be written as

$$S=V\frac{\partial F}{\partial v}+\Lambda\frac{\partial F}{\partial\lambda}, \tag{2}$$

with

$$V = \mathbf{a}\cdot\nabla_v v, \quad \Lambda = \mathbf{a}\cdot\nabla_v \lambda,$$
$$\nabla_v \lambda = (2/v_\perp^2)[(B_0/B) - \lambda]\mathbf{v}_\perp , \tag{3}$$

and $\nabla_v v = \mathbf{v}/v$.

Considering a monochromatic wave of frequency ω and Fourier decomposing the toroidal angle and gyrophase dependence, any linearized function g may be written as

$$g = \mathrm{Re}\exp(-i\omega t)\sum_{l,n=-\infty}^{\infty} g_{ln}\exp(in\zeta - il\phi), \tag{4}$$

where Re denotes that the real part is to be taken. Using the preceding on Eq. (1) gives

$$\left(\frac{v_\parallel}{qR}\right)\left(\frac{\partial f_{ln}}{\partial\beta}\right) - i\left[\omega - l\Omega - \left(\frac{nv_\parallel}{R}\right)\right] f_{ln} = -S_{ln}.$$

Introducing the timelike variable τ via

$$d\tau = qRd\beta/v_\parallel \tag{5}$$

with $\beta(\tau)=\beta$ and $\beta(\tau=0)=0$ such that $\tau=0$ denotes the *last* pass through $\beta=0$, then the preceding kinetic equation becomes

$$\frac{\partial f_{ln}}{\partial\tau} - i\left[\omega - l\Omega - \left(\frac{nv_\parallel}{R}\right)\right] f_{ln} = -S_{ln}. \tag{6}$$

Integrating Eq. (6) from $\tau=-\infty$ (where $f_{ln}=0$) to τ gives

$$f_{ln} = -\int_{-\infty}^{\tau} d\tau'\, S_{ln}(\tau')\exp[i(\chi-\chi')], \tag{7}$$

where

$$\chi \equiv \int_0^\tau d\tau''[\omega - l\Omega'' - (nv_\parallel''/R'')]$$
$$= \int_0^\tau d\tau''(\omega - l\Omega'') - nq\beta \tag{8}$$

and primes denote functions to be evaluated at the primed variables τ',β' (or τ'',β'') rather than at τ,β.

The total perturbed current density $\mathbf{j}$ is

$$\mathbf{j} = \sum Ze\int d^3v\,\mathbf{v}f$$
$$= \mathrm{Re}\exp(-i\omega t)\sum_{n=-\infty}^{\infty}\mathbf{j}_n\exp(in\zeta)$$

with

$$\mathbf{j}_n = \sum Ze\sum_{l=-\infty}^{\infty}\int d^3v\,\mathbf{v}f_{ln}\exp(-il\phi),$$

where the first summation is over all species. The only kinetic modifications of the perturbed current density of interest are those affecting the minority resonance ($l=1$), namely those entering

$$\mathbf{j}_{1n} \equiv Ze\int d^3v\,\mathbf{v}f_{1n}\exp(-i\phi). \tag{9}$$

Making the usual assumption that the parallel electric field is negligible for fast waves and using

$$(2\pi)^{-1}\int_0^{2\pi} d\phi\,\mathbf{v}\exp(-i\phi) = \frac{1}{2}v_\perp(\hat{r} - i\hat{n}\times\hat{r}),$$

$$V_{1n} = Zev_\perp e_n^+/Mv \quad\text{and}\quad \Lambda_{1n} = 2B_0v_\parallel^2 V_{1n}/Bv^3,$$

with $e_n^+ \equiv (\hat{r} - i\hat{n}\times\hat{r})\cdot\mathbf{e}_n$ the left-hand polarized electric-field component, gives

$$S_{1n} = \left(\frac{Zev_\perp e_n^+}{Mv}\right)\left[\frac{\partial F}{\partial v} + \left(\frac{2B_0v_\parallel^2}{Bv^3}\right)\frac{\partial F}{\partial\lambda}\right]. \tag{10}$$

As a result, the $l=1$ portion of the perturbed minority current density becomes

$$\mathbf{j}_{1n} = -(\hat{r} - i\hat{n}\times\hat{r})\frac{(Ze)^2}{2M}\int d^3v\left(\frac{v_\perp}{v}\right)\int_{-\infty}^{\tau} d\tau'\left(\frac{\partial F}{\partial v} + \frac{2B_0v_\parallel^2(\tau')}{v^3B(\tau')}\frac{\partial F}{\partial\lambda}\right)e_n^+(\tau')v_\perp(\tau')\exp[i(\chi-\chi')]. \tag{11}$$

Essentially the same form can be obtained for a completely general flux coordinate representation.

Because of the resonance points $\omega = \Omega + nv_\parallel/R$ encountered by the minority ions during their motion, the τ' integral cannot be evaluated by integrating by parts, as in the nonresonant or cold cases. In the next section $\mathbf{j}_{1n}$ will be approximately evaluated for a Maxwellian F by a technique that permits trapped, as well as passing particles to pass through and/or reflect in the cyclotron resonance region.

III. RESONANT RESPONSE FUNCTION FOR MAXWELLIAN IONS

In order to obtain the resonant response function for a toroidally inhomogeneous plasma the expression for $\chi'-\chi$ in Eq. (11) must be simplified. Assuming that collisions[8,12–14] or orbit stochasticity[16] disrupt the ion motion between passes through the minority resonance, then only the last pass need be retained. As a result, the most straightforward way to proceed is to approximate the rapidly varying phase factor by expanding χ' about τ. This procedure is particularly appropriate to the present toroidal model, where the ions only interact strongly with the wave in the resonance region. This region is a narrow layer in which the ions spend only a small interval of time during their motion on a particular flux surface. Expanding χ' to leading order in the inhomogeneity gives the result

$$\chi' = \chi + \dot{\chi}(\tau'-\tau) + \tfrac{1}{2}\ddot{\chi}(\tau'-\tau)^2 + \tfrac{1}{6}\dddot{\chi}(\tau'-\tau)^3\cdots, \tag{12}$$

where $\dot{\beta} = v_\parallel/qR$, $\dot{\Omega} = \dot{\beta}\partial\Omega/\partial\beta$, $\dot{\chi} = \omega - \Omega - nv_\parallel/R$, $\ddot{\chi} = -\dot{\Omega} - nq\ddot{\beta}$, and $\dddot{\chi} = -\ddot{\beta}\partial\Omega/\partial\beta - (\dot{\beta})^2\partial^2\Omega/\partial\beta^2 - nq\dddot{\beta}$ with

$$\ddot{\beta} = \frac{\dot{\beta}^2}{\Omega}\frac{\partial\Omega}{\partial\beta}\left(1 - \frac{\lambda v^2R_0}{2v_\parallel^2 R}\right). \tag{13}$$

Assuming the coefficients $e_n^+(\tau')$, $v_\parallel(\tau')$, $v_\perp(\tau')$, and $B(\tau')$ are slowly varying so that they can be evaluated at τ, Eq. (11) is approximated by

$$\mathbf{j}_{1n} \approx -(\hat{r} - i\hat{n}\times\hat{r})e_n^+ \frac{(Ze)^2}{2M} \int d^3v(v_\perp^2/v) \times\left(\frac{\partial F}{\partial v} + \frac{2B_0 v_\perp^2}{Bv^3}\frac{\partial F}{\partial \lambda}\right)\int_0^\infty dt \exp[i(\chi-\chi')] \quad (14)$$

with $t=\tau-\tau'$ and

$$\chi'-\chi = -\left[\omega - \Omega - \left(\frac{nv_\parallel}{R}\right)\right]t - \left(\frac{t^2}{2qR}\right)\left[v_\parallel + \left(\frac{n}{\Omega R}\right) \times\left(v_\parallel^2 - \frac{1}{2}v_\perp^2\right)\right]\frac{\partial\Omega}{\partial\beta} + \left[\left(\frac{\partial^2\Omega}{\partial\beta^2} + \frac{1}{\Omega}\left(\frac{\partial\Omega}{\partial\beta}\right)^2\right) \times v_\parallel^2 - \frac{v_\perp^2}{2\Omega}\left(\frac{\partial\Omega}{\partial\beta}\right)^2\right]\frac{t^3}{6q^2R^2} + \cdots, \quad (15)$$

where $nv_\perp/\Omega R \ll v_\parallel/v_\perp \sim \epsilon^{1/2}$ is assumed in order to neglect $\ddot{\beta}$ corrections in $\ddot{\chi}$. In obtaining Eq. (15) the concentric flux surface model is employed to write $B\sim 1/R$, but since BR is approximately a flux function this is not expected to be a serious limitation. If the β dependence of e_n^+ is not sufficiently slow, a Fourier decomposition in β must be employed to replace n/R with $k_\parallel = (nq-m)/qR$ in Eq. (15), with m the poloidal mode number. In addition, the assumed slow variation of e_n^+ is related to the width of the resonance, a quantity that will emerge *a posteriori*. Note that the variation of $v_\parallel(\tau')$ with τ' is monotonic as the particle passes through resonance so that the approximation of Eq. (15) retains those trapped ions which turn in the resonance region.

To obtain explicit results for the plasma response (reactive and dissipative) a tractable distribution function must be assumed. Taking F to be the Maxwellian

$$F_M = (N/\pi^{3/2}v_T^3)\exp(-v^2/v_T^2), \quad (16)$$

the velocity space integrals can be evaluated by employing ϕ, $v_\perp$, and $v_\parallel$ as the variables of integration to find

$$\mathbf{j}_{1n} \approx \frac{(Ze)^2 NR}{iM|n|v_T}(\hat{r} - i\hat{n}\times\hat{r})e_n^+ Z(z,\alpha,\sigma,\xi), \quad (17)$$

where $Z(z,\alpha,\sigma,\xi)$ is the generalized plasma dispersion function

$$Z(z,\alpha,\sigma,\xi) \equiv i\int_0^\infty dx \exp\left(izx - \frac{(1-\frac{1}{2}\alpha x)^2 x^2}{4[1-i(\alpha - \frac{1}{3}\alpha^2 x - \frac{1}{3}\xi x)\sigma x^2]}\right) \Bigg/ \left[1 + \frac{1}{2}i\left(1-\frac{1}{3}\alpha x\right)\sigma\alpha x^2\right]^2 \times\left[1 - i\left(\alpha - \frac{1}{3}\alpha^2 x - \frac{1}{3}\xi x\right)\sigma x^2\right]^{1/2}, \quad (18)$$

$$x = t|n|v_T/R, \quad z \equiv \frac{(\omega-\Omega)R}{|n|v_T}, \quad \alpha \equiv R\frac{\partial\Omega}{\partial\beta}\Big/ qv_T n|n|, \quad \xi \equiv \frac{\Omega R^2 \partial^2\Omega}{q^2 n^4 v_T^2 \partial\beta^2}, \quad \text{and} \quad \sigma \equiv \frac{|n|v_T}{2\Omega R} \ll 1. \quad (19)$$

When the poloidal dependence of e_n^+ is Fourier decomposed $n \to k_\parallel R = (nq-m)/q$ in the preceding. Equation (18) is expected to be valid for $\epsilon < 1/3 - 1/10$ since the $\epsilon \ll 1$ assumption does not enter in a sensitive way. The parameter z characterizes the usual wave-particle, cyclotron resonant, interaction or Doppler broadening effect and σ arises because of Doppler modifications in the nonuniform magnetic field. The variation of the magnitude of the magnetic field along a field line is characterized by α with $\sigma\alpha$ and $\sigma\alpha^2$ the Doppler modifications. Effects due to the resonance occurring off the magnetic axis enter through ξ (which changes from positive on the low-field side to negative on the high) with $\sigma\xi$ the Doppler modification.

In the fully toroidal model considered here the rotational transform is manifestly responsible for the variation of the magnetic field along a field line via Eq. (5). To remove the rotational transform it is necessary to let $q \to \infty$ in Eqs. (18) and (19). However, the Faulconer[7] and Smithe *et al.*[8] forms are obtained without explicitly introducing a rotational transform since their replacement Z function depends on a magnetic-field scale length $L_\parallel$. By identifying their $L_\parallel$ as qR/ϵ and suppressing collisions all the descriptions become identical for an on-axis resonance and $\sigma \to 0$. By using the substitution $n/R \to k_\parallel$ the asymmetry in the $n = k_z R$ spectrum noted in Ref. 8 is recovered.

When the terms involving σ are sufficiently small ($\sigma \ll |\alpha| \ll 1$, $\sigma|\xi| \ll |\alpha|^3$ or $|\alpha| \geqslant 1$, $\sigma(\alpha^2 + |\xi|) \ll |\alpha|^{3/2}$), Eq. (18) reduces to the result of Faulconer[7] and the collisionless form of the replacement plasma response function of Smithe *et al.*,[8] namely

$$Z(z,\alpha) = i\int_0^\infty dx \exp\left[izx - \left(1 - \frac{1}{2}\alpha x\right)^2 (x^2/4)\right], \quad (20a)$$

which they obtain heuristically by employing an ingenious set of simplified trajectories that conserve both energy and magnetic moment and are motivated by the earlier work of Itoh *et al.*,[12] which noted the important role of the inhomogeneity as $n \to 0$. As $n \to 0$, $|\alpha|^{1/2} \gg 1$ and Eq. (18) simplifies to

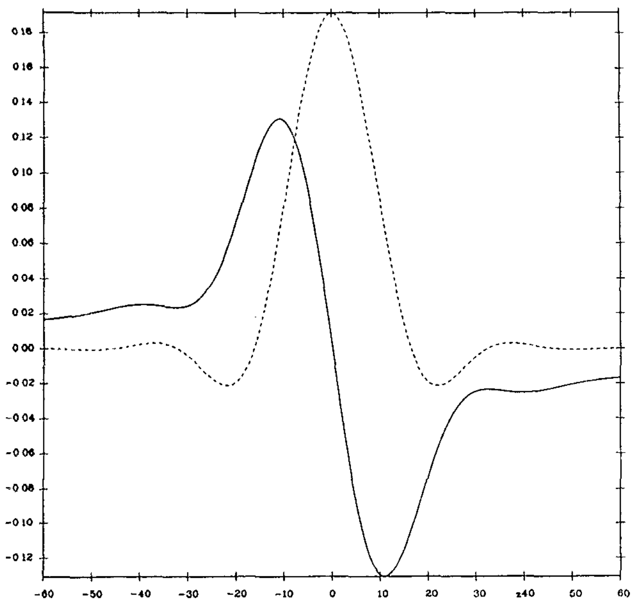


FIG. 1. The real (solid line) and imaginary (broken line) parts of $Z(z,\alpha,\sigma,\xi)$ plotted as a function of z for $\alpha=100$ and $\sigma=0=\xi$, corresponding to an on-axis minority resonance. The width of the resonance layer is given by $|z|\sim|\alpha|^{1/2}$ to be $\Delta R\sim(r\rho/q)^{1/2}$, where $\rho=v_T/\omega$.

$$Z(z,\alpha,\sigma,\xi)\,|_{n\to 0}$$
$$=\frac{2i}{|\alpha|^{1/2}}\int_0^\infty \frac{dy\exp(isy-y^4/[1+i(\kappa+\gamma)y^3])}{(1-\frac{1}{2}i\kappa y^3)^2[1+i(\kappa+\gamma)y^3]^{1/2}}, \tag{20b}$$

where $y=\frac{1}{2}|\alpha|^{1/2}x$, $s\equiv 2z/|\alpha|^{1/2}$, $\kappa\equiv 8\sigma|\alpha|^{1/2}/3$, and $\gamma\equiv 8\sigma\xi/3|\alpha|^{3/2}$ are independent of n. Since $\kappa\sim(r\rho)^{1/2}/R\ll 1$ and $\gamma\sim(\rho/r)^{1/2}\ll 1$, the new terms give only small corrections for $|\alpha|^{1/2}\gg 1$ to the results of Refs. 7 and 8, where $\rho=v_T/\omega$.

When $|\alpha|$ is sufficiently large the new terms are negligible, the inhomogeneity dominates, and the strongest wave-particle interactions are near the cold minority resonance for both signs of α, as can be seen from the $n\to 0$ form of Eq. (20b). For $|\alpha|\gg 1$, the power deposition layer broadens[8] to $|z|\sim|\alpha|^{1/2}$ from $|z|\sim 1$ and the amplitude of Z is reduced[8] by $|\alpha|^{1/2}$, as shown in Fig. 1. This case is expected to be of particular interest for small $|n|$ operation.

When the inhomogeneity is weak it is tempting to neglect the α dependence of Eq. (20) and simply replace it by the usual Z function. However, Faulconer[7] and Smithe *et al.*[8] have shown that for $\alpha>0$ there are important corrections. For $1\gg\alpha>0$ these contributions to Eqs. (18) and (20) arise from the region $x\approx 2/\alpha$ and lead to oscillatory behavior about $z=0$. Evaluating the contribution to Eq. (18) from $x\approx 2/\alpha$ by a saddle point method for $0\leqslant\alpha|z|\ll 1$, $\sigma\lesssim\alpha\ll 1$, and $\sigma|\xi|\lesssim\alpha^3$ results in

$$Z(z,\alpha,\sigma,\xi)=Z(z)+i2\pi^{1/2}\left(1-\frac{i4\sigma}{3\alpha}\right)^{-2}\left[1-\frac{i4\sigma}{3\alpha}\left(1-\frac{2\xi}{\alpha^2}\right)\right]^{-1}\times\exp\left[\frac{i2z}{\alpha}-z^2\left(1-\frac{i4\sigma}{3\alpha}\left(1-\frac{2\xi}{\alpha^2}\right)\right)\right], \tag{21}$$

where $Z(z)$ is the usual Z function (and the saddle at $x\approx 1/\alpha$ gives a negligible contribution as long as α is sufficiently small). The contribution from the saddle at $x\approx 2/\alpha$ arises because the particle motion along the twisting field line causes it to pass through an effective resonance region, where the field inhomogeneity modification cancels the Doppler shift [at the time at which the two terms linear in $v_\parallel$ cancel in Eq. (15)]. For an asymmetric spectrum this causes an up–down asymmetry in the wave propagation and absorption. As $\alpha\to 0$, collisional disruption of a particle's streaming motion[12,14] removes the oscillatory behavior in the plasma response function[8] (different collisional mechanisms are considered in each of these references).

The fine-scale oscillations introduced by $\exp(i2z/\alpha)$ for $|z|\lesssim 1$ as $\alpha\to 0$ are collisionlessly modified by the new terms when $\sigma\sim\alpha$ or $\sigma|\xi|\sim\alpha^3$; in most cases making it unnecessary to retain collisions. For still smaller $\partial\Omega/\partial\beta$ and/or larger $|n|$ such that $0<\alpha\ll\sigma$ or $\alpha^3\ll\sigma|\xi|$ the amplitude of the fine-scale oscillations become vanishingly small with only $Z(z)$ surviving in Eq. (21) as $\alpha\to 0$, as shown in the figures. Figures 2(a)–2(c) plot the real (solid line) and imaginary (broken line) parts of Eq. (18) for the minority resonance at the magnetic axis ($\partial^2\Omega/\partial\beta^2=0$) and illustrate the effect of finite $\sigma\gtrsim\alpha$ which requires $n^3\gtrsim rR/q\rho^2$. A large value of $\sigma=0.1$ is used since the $\alpha=0.1$ oscillations can be more conveniently displayed than those arising for $\alpha=0.01=\sigma$. For Figs. 3(a)–3(b), the minority resonance is shifted off axis to show $\sigma|\xi|\gtrsim\alpha^3$ and $\sigma|\xi|\gtrsim 1$ effects which occur for $n^3\gtrsim r^2/q\rho^2$ and $n^3\lesssim r/q^2\rho$, respectively. The plot for $\alpha=0.1$, $\sigma=0.01$, and $\xi=0$ is quite similar to Fig. 2(a). For Fig. 3(b), ξ is so large that α no longer plays a role and can be set to zero to obtain the same plot. Positive values of ξ result in similar plots but with more of the structure at $z>0$, rather than at $z<0$.

In the next section, the cold fluid theory of fast wave minority heating is modified by making the substitution

$$\frac{1}{\omega-\Omega}\to\frac{-RZ(z,\alpha,\sigma,\xi)}{|n|v_T} \tag{22}$$

(notice that for $|z|\gg 1$, $Z\to -1/z$). This procedure allows the optical depth and transmission coefficient for fast wave minority heating to be evaluated analytically by a perturbation expansion in the minority concentration.

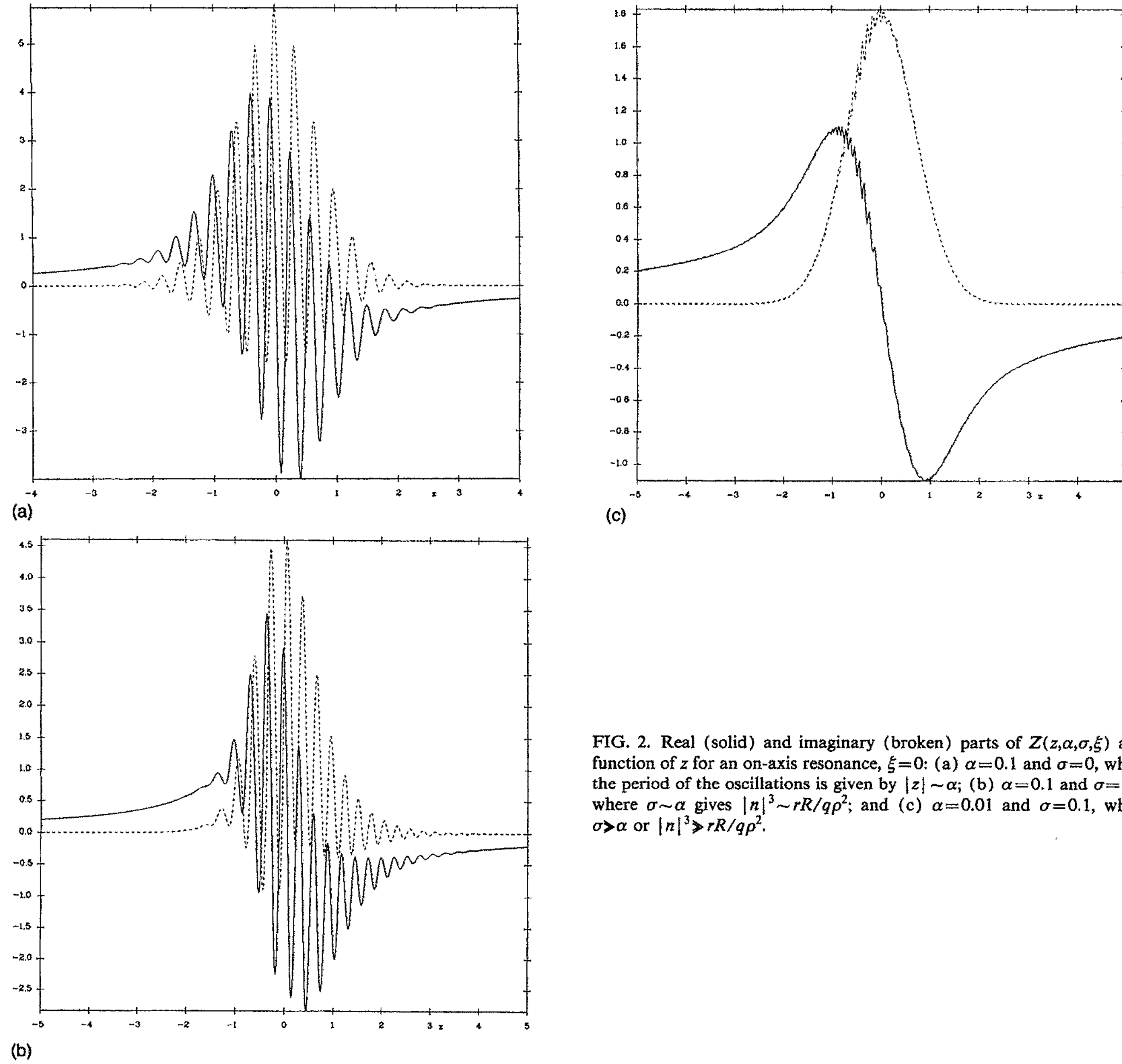


FIG. 2. Real (solid) and imaginary (broken) parts of $Z(z,\alpha,\sigma,\xi)$ as a function of z for an on-axis resonance, $\xi=0$: (a) $\alpha=0.1$ and $\sigma=0$, where the period of the oscillations is given by $|z|\sim\alpha$; (b) $\alpha=0.1$ and $\sigma=0.1$, where $\sigma\sim\alpha$ gives $|n|^3\sim rR/q\rho^2$; and (c) $\alpha=0.01$ and $\sigma=0.1$, where $\sigma\gg\alpha$ or $|n|^3\gg rR/q\rho^2$.

IV. OPTICAL DEPTH AND TRANSMISSION COEFFICIENT FOR A MAXWELLIAN *F*

The fast (or compressional Alfvén) wave in a cold fluid description of minority heating satisfies the dispersion relation[12,17,18]

$$n_\perp^2=(\epsilon_\perp+\epsilon_x-n_\parallel^2)(\epsilon_\perp-\epsilon_x-n_\parallel^2)/(\epsilon_\perp-n_\parallel^2) \quad (23a)$$

with $n_\parallel=k_\parallel c/\omega$, $n_\perp=k_\perp c/\omega$,

$$\epsilon_\perp=1-\sum_j \omega_{pj}^2\Big/(\omega^2-\Omega_j^2) \quad \text{and}$$

$$\epsilon_x=\sum_j \omega_{pj}^2\Omega_j\Big/\omega(\omega^2-\Omega_j^2).$$

For a plasma with a majority ion species ($j=i$) and a single minority species (unsubscripted), the replacement (22) in minority terms proportional to $1/(\omega-\Omega)$ yields

$$\epsilon_\perp\approx-\omega_{pi}^2\left(\frac{1}{\omega^2-\Omega_i^2}-\frac{\eta RZ(z,\alpha,\sigma,\xi)}{2\omega|n|v_T}\right)$$

and

$$\epsilon_x\approx\omega_{pi}^2\left(\frac{\omega}{\Omega_i(\omega^2-\Omega_i^2)}-\frac{\eta RZ(z,\alpha,\sigma,\xi)}{2\omega|n|v_T}\right) \quad (23b)$$

for $\omega\approx\Omega$, $\omega_{pi}^2/\omega\Omega_i\gg1$ and $\eta=Z^2M_iN/Z_i^2MN_i\ll1$, where $\omega_{pi}^2=4\pi(Z_ie)^2N_i/M_i$ and $\Omega_i=Z_ieB/M_ic$.

To evaluate the optical depth κ of the minority resonance and, therefore, the transmission coefficient

$$T=\exp(-\kappa), \quad (24)$$

$n_\perp-n_\perp^*=2i\,\mathrm{Im}(n_\perp)$ is required, where Im and Re denote the imaginary and real parts. The transmission coefficient T is a measure of how much energy the incident wave loses in crossing the interaction region. The evaluation of T

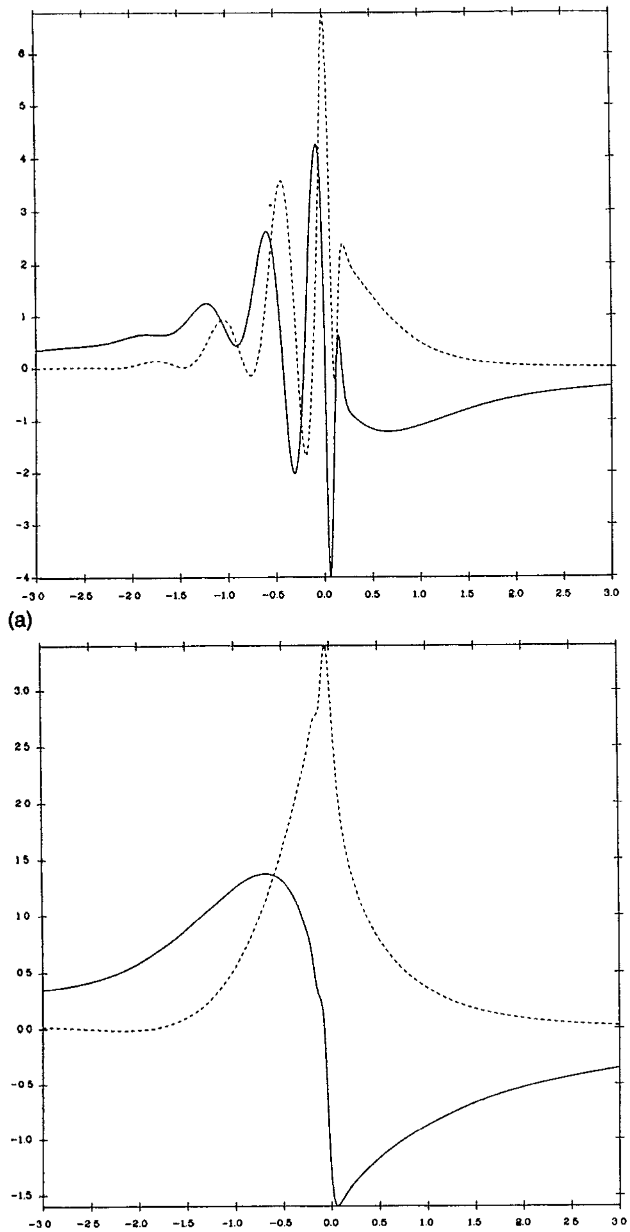


FIG. 3. Real (solid) and imaginary (broken) parts of $Z(z,\alpha,\sigma,\xi)$ as a function of z for an off-axis resonance, $\xi \neq 0$, with $\alpha=0.1$ and $\sigma=0.01$: (a) $\xi=-1$, where $\sigma|\xi| \gg \alpha^3$ or $|n|^3 \gg r^2/q\rho^2$ and (b) $\xi=-100$, where $\sigma|\xi| \sim 1$ gives $|n|^3 \sim r/q^2\rho$.

gives no information on reflection, absorption, or mode conversion; all of which must be obtained from a full wave description.

Taylor expanding $n_\perp^2$ about $\eta=0$ and denoting quantities evaluated at $\eta=0$ by an overbar gives

$$n_\perp^2 = \bar{n}_\perp^2 + \frac{\eta\omega_{pi}^2 R Z(z,\alpha,\sigma,\xi)(\bar{\epsilon}_\perp + \bar{\epsilon}_x - n_\parallel^2)^2}{2\omega|n|v_t(\bar{\epsilon}_\perp - n_\parallel^2)^2} + \cdots$$

so that

$$n_\perp - n_\perp^* = \frac{\eta\omega_{pi}^2 R(\bar{\epsilon}_\perp + \bar{\epsilon}_x - n_\parallel^2)^2[Z(z,\alpha,\sigma,\xi) - Z^*(z,\alpha,\sigma,\xi)]}{4\omega|n|v_t\bar{n}_\perp(\bar{\epsilon}_\perp - n_\parallel^2)^2} \tag{25}$$

to the requisite order, where a superscript asterisk denotes complex conjugate.

When the scale lengths of the variations of the electric field $\mathbf{e}_n$ are long compared to the width of the minority resonance the optical depth

$$\kappa = 2\left|\int dR \,\mathrm{Im}\, k_\perp\right| = \left(\frac{\omega}{c}\right)\left|\int dR(n_\perp - n_\perp^*)\right| \tag{26}$$

is only sensitive to the R dependence of $z=R(\omega-\Omega)/|n|v_t$. Using $dR \approx (|n|v_t/\omega)dz$, Eqs. (25) and (26) yield

$$\kappa \approx \frac{\eta\omega_{pi}^2 R(\bar{\epsilon}_\perp + \bar{\epsilon}_x - n_\parallel^2)^2}{4c\omega\bar{n}_\perp(\bar{\epsilon}_\perp - n_\parallel^2)^2}\bigg|_{\Omega=\omega}\left|\int_{-\infty}^{\infty} dz[Z(z,\alpha,\sigma,\xi) - Z^*(z,\alpha,\sigma,\xi)]\right|, \tag{27}$$

where all spatially varying quantities outside the z integral are slowly varying and are to be evaluated at the minority resonance as indicated. Noting that the only sensitive spatial variation in Z is via the $\exp(izx)$ and gives

$$\int_{-\infty}^{\infty} dz[Z(z,\alpha,\sigma,\xi) - Z^*(z,\alpha,\sigma,\xi)] = 4\pi i\int_0^\infty dx\,\delta(x) = 2\pi i \tag{28}$$

so that the optical depth becomes

$$\kappa \approx \frac{\pi\eta\omega_{pi}^2 R(\bar{\epsilon}_\perp + \bar{\epsilon}_x - n_\parallel^2)^2}{2c\omega\bar{n}_\perp(\bar{\epsilon}_\perp - n_\parallel^2)}\bigg|_{\Omega=\omega} \equiv \bar{\kappa}, \tag{29}$$

which for small $n_\parallel$ reduces to

$$\kappa \approx \frac{\pi\eta\omega_{pi} R(\omega-\Omega_i)^2}{2c\omega\Omega_i}\bigg|_{\Omega=\omega} \equiv \kappa_0, \tag{30}$$

since $\bar{n}_\perp \to \omega_{pi}/\Omega_i$. Equations (29) and (30) agree with the results of Francis, Bers, and Ram[19] and with Lashmore-Davies and Dendy[20] if L_B is identified as R. When $n_\parallel \propto k_\parallel = (nq-m)/qR$ is employed in Eq. (29) there is an asymmetric spectrum shift in T due to the poloidal field modification (m/qR) of n/R in essentially the same way as found in the numerical results of Smithe *et al.*[8]

The preceding evaluation provides yet another demonstration of the robustness in the optical depth and transmission coefficient when η is small since the result is independent of the details of the absorption process and, therefore, Doppler broadening (n or $k_\parallel$ in Z) and inhomogeneity effects (α and ξ in Z). This robustness is investigated further in the next section, where κ and T are evaluated for more general equilibrium distribution functions, but does not imply that the plasma response function and, hence, the power deposition profile display the same insensitivity.

Before concluding this section we observe that the formula for the optical depth given by Eq. (29) was obtained by an expansion in the minority to majority density ratio. This expansion will be valid as long as the minority resonance is indistinguishable from the hybrid resonance. However, as the minority to majority density ratio increases, these two resonances begin to separate and when they are well separated the quantity $\epsilon_\perp - n_\parallel^2$, which occurs in the denominator of Eq. (25), can vanish. In this region, the expansion clearly breaks down. One might still apply the previous analysis at the minority resonance but now important phenomena occur in the region of the hybrid resonance, namely reflection of the incident fast wave for a low-field side antenna and mode conversion to the ion Bernstein wave for high-field side incidence. In general, the larger the separation of the minority and ion–ion hybrid resonances, the weaker the damping in the vicinity of the minority resonance.

An important quantity for fast wave minority heating is the minority to majority density ratio for which the hybrid and minority resonances are degenerate. This critical ratio is also close to the condition for maximum minority absorption. As the minority to majority density ratio is increased beyond this critical value the two resonances move apart until they become quite distinct. The critical density ratio can be calculated for the present toroidal model using the argument given in Ref. 21. For small values of $n_\parallel$ (i.e., $\omega_{pi} \ll ck_\parallel$) the condition for hybrid resonance is $\text{Re}(\epsilon_\perp)=0$. Degeneracy of the minority and ion–ion hybrid resonances occurs when $\text{Re}Z(z,\alpha,\sigma,\xi)$ takes its maximum value at the position where $\text{Re}(\epsilon_\perp)=0$. Assuming $\omega=\Omega$, the equation $\text{Re}(\epsilon_\perp)=0$ can be solved for the critical density ratio by inserting the maximum value of $\text{Re}Z(z,\alpha,\sigma,\xi)$ giving

$$\frac{N_{\text{crit}}}{N_i} \simeq \frac{(2M_i|n|v_t)}{\{M\omega R[(\omega/\Omega_i)^2-1](\text{Re}\,Z)_{\max}\}}. \tag{31}$$

We note that for a straight field line model $|n|/R$ is identified as $k_\parallel$ and Eq. (31) reduces to Eq. (37) of Ref. 21, since the plasma response function becomes the plasma dispersion function in this limit. However, the toroidal calculation is more general and contains additional resonance broadening terms. Thus, referring to Fig. 1, where $\alpha=100$, we see that $(\text{Re}Z)_{\max}\simeq 0.13$ compared with unity for the plasma dispersion function. This means that the resonance is much broader for this case and, hence, the critical density ratio is much larger thus allowing strong damping for a wider range of minority densities. We also note that the damping remains even in the limit $k_\parallel=0$, as already pointed out by Smithe *et al.*[8] Of course, in the toroidal model, $k_\parallel$ cannot be zero over the whole resonance region but only at a point in the region.

V. NON-MAXWELLIAN DISTRIBUTION FUNCTIONS

The evaluation of the optical depth (or transmission coefficient) is generalizable to arbitrary isotropic and anisotropic distribution functions F that need not be even functions of parallel velocity (but are still independent of ϕ, β, and ζ). To this end a further generalization of the plasma response function, $\hat{Z}$, is defined by comparing Eqs. (11), (14), and (17) to find

$$\hat{Z} \equiv \frac{-i|n|v_t}{2NR} \int d^3v \frac{v_\perp^2}{v}\left(\frac{\partial F}{\partial v}+\frac{2B_0 v_\parallel^2}{Bv^3}\frac{\partial F}{\partial \lambda}\right) \times \int_{-\infty}^{\tau} d\tau' \exp[i(\chi-\chi')], \tag{32}$$

which reduces to Eq. (18) when Eq. (15) is employed and F is Maxwellian.

Once χ' is Taylor expanded about τ as in Eqs. (12) or (15) and it is observed that the only rapid spatial variation is in $\dot{\chi}=\omega-\Omega-nv_\parallel/R$, then proceeding as in Eq. (28) and letting $t=\tau-\tau'$ gives

$$(|n|v_t/R)\int_{-\infty}^{\infty} dz \int_{-\infty}^{\tau} d\tau' \exp[i(\chi-\chi')] \approx 2\pi \int_0^\infty dt\, \delta(t)=\pi, \tag{33}$$

where $z=[R(\omega-\Omega)-nv_\parallel]/|n|v_t$ and $dz=[1-(n\lambda v^2R_0/2\omega v_\parallel R^2)](\omega dR/|n|v_t)$. Therefore, a general expression for the optical depth is

$$\kappa=\bar{\kappa}\Upsilon \tag{34}$$

with $\bar{\kappa}$ defined in Eq. (29) and

$$\Upsilon \equiv -\left(\frac{1}{2N}\right)\int d^3v\, v_\perp \left.\frac{\partial F}{\partial v_\perp}\right|_{\Omega=\omega-v_\parallel/R} \tag{35}$$

provided $nv^2/v_\parallel \omega R \ll 1$. In carrying out the preceding steps

$$(1/v)\left[\frac{\partial F}{\partial v}+\left(\frac{2B_0v_\parallel^2}{Bv^3}\right)\frac{\partial F}{\partial \lambda}\right]=\left(\frac{1}{v_\perp}\right)\frac{\partial F}{\partial v_\perp} \tag{36a}$$

is employed. Then using $d^3v=2\pi v_\perp\, dv_\perp\, dv_\parallel$ and

$$\int_0^\infty dv_\perp\, v_\perp^2 \frac{\partial F}{\partial v_\perp}=-2\int_0^\infty dv_\perp\, v_\perp F, \tag{36b}$$

Eq. (35) becomes

$$\Upsilon=(1/N)\int d^3v\, F|_{\Omega=\omega-v_\parallel/R}=1, \tag{37}$$

where the last step follows because F must be a flux function and all other poloidal angle dependences are slow. As a result of the preceding argument, the optical depth (or transmission coefficient) is insensitive to the details of the minority distribution function, as well as Doppler and inhomogeneity effects, for minority concentrations less than the critical value given by Eq. (31). Only the quantities entering $\bar{\kappa}$ (or κ_0 if $k_\parallel=0$) matter. Therefore, under such conditions the optical depth is only sensitive to the cold fluid minority response, since replacing Z by its cold approximation gives the only contribution that survives, namely,

$$\int_{-\infty}^{\infty} dz\, Z \rightarrow \int_{-\infty}^{\infty} \frac{d\Omega}{(\omega-\Omega)} = \pi i. \tag{38}$$

The insensitivity of Υ to Doppler broadening, inhomogeneity effects, and the speed and pitch angle dependence of the unperturbed distribution function is quite striking, but does not imply that the plasma response function $\hat{Z}$ of Eq. (32) displays the same insensitivity. Inserting Eq. (36a) into Eq. (32) gives

$$\hat{Z} = \frac{-i|n|v_t}{2NR} \int d^3v\, v_\perp \frac{\partial F}{\partial v_\perp} \int_{-\infty}^{\tau} d\tau' \exp[i(\chi-\chi')]. \tag{39}$$

Under the conditions noted prior to Eq. (20a) the t^3 and nt^2 terms can be neglected in Eq. (15) to obtain

$$\chi-\chi' = \left[\omega - \Omega - \left(\frac{nv_\parallel}{R}\right)\right] t + \left(\frac{v_\parallel t^2}{2qR}\right) \frac{\partial \Omega}{\partial \beta}. \tag{40}$$

In this simplified limit, Eq. (36b) may be used to express Eq. (39), with $\chi-\chi'$ given by Eq. (40), as

$$\hat{Z} = \frac{i|n|v_t}{NR} \int d^3v F \int_0^\infty dt \exp[i(\chi-\chi')]. \tag{41}$$

From this last form it is seen that $\hat{Z}$ is only sensitive to the $v_\parallel$ dependence of F in this simplified limit, making the plasma response function $\hat{Z}$ more sensitive to Doppler and inhomogeneity effects and the details of the unperturbed distribution function than the optical depth and transmission coefficient. If the $v_\parallel$ dependence of the distribution function F (which must be a function of v, λ, and ϵ only) is Maxwellian in $v_\parallel$ then Eq. (41) reduces to Eq. (20a) for an arbitrary $v_\perp$ dependence.

In order to obtain a plasma response function more general than Eq. (20a) it is necessary to employ Eq. (39) with the full expression for $\chi-\chi'$, as given by Eq. (15). Consider, for example, the bi-Maxwellian

$$F = \eta_0 \exp(-\beta_\perp w^2 - \alpha_\parallel u^2), \tag{42}$$

where w and u are the perpendicular and parallel velocities evaluated at $\beta=0$, namely,

$$w = v(\lambda - \epsilon\lambda)^{1/2} \quad \text{and} \quad u = v(1-\lambda+\epsilon\lambda)^{1/2}, \tag{43a}$$

and $\beta_\perp$, $\alpha_\parallel$, and η_0 must be flux functions. Using $u^2 = v^2 - w^2$, $v^2 = v_\perp^2 + v_\parallel^2$, and $v_\perp^2 = B\lambda v^2/B_0$ gives

$$w^2 = (1-\epsilon)(B_0/B)v_\perp^2 \quad \text{and}$$
$$u^2 = v_\parallel^2 + [1-(1-\epsilon)(B_0/B)]v_\perp^2. \tag{43b}$$

As a result, F may be rewritten as

$$F = \eta_0 \exp(-\alpha_\perp v_\perp^2 - \alpha_\parallel v_\parallel^2) \tag{44}$$

with

$$\alpha_\perp \equiv \beta_\perp (1-\epsilon)(B_0/B) + \alpha_\parallel [1-(1-\epsilon)(B_0/B)]. \tag{45}$$

Evaluating the density $N = \int d^3vF$ by integrating over $v_\perp$ and $v_\parallel$ gives

$$\eta_0 = \pi^{-3/2} \alpha_\perp \alpha_\parallel^{1/2} N, \tag{46}$$

where $\alpha_\perp N$ must be a flux function.

Next, the plasma response function of Eq. (39) is evaluated by inserting Eqs. (15), (44), and (46) and carrying out the $v_\perp$ and $v_\parallel$ integrations, just as in the Maxwellian case. The only new feature that arises is in the $v_\perp$ integral because $\alpha_\perp \neq \alpha_\parallel$. To characterize the anisotropy let

$$\alpha_\parallel = 1/v_t^2 \quad \text{and} \quad \alpha_\perp \equiv 1/Av_t^2, \tag{47}$$

where the anisotropy factor A is approximately the ratio of the perpendicular over the parallel energy content of the bi-Maxwellian. Then, the plasma response function for the bi-Maxwellian of Eq. (44) is

$$Z_b(z,\alpha,\sigma,\xi,k) = i \int_0^\infty dx \exp\left(izx - \frac{(1-\frac{1}{2}\alpha x)^2 x^2}{4[1-i(\alpha - \frac{1}{3}\alpha^2 x - \frac{1}{3}\xi x)\sigma x^2)} \right) \Bigg/ [1+i(1-\tfrac{1}{3}\alpha x)kx^2]^2 [1-i(\alpha-\tfrac{1}{3}\alpha^2 x - \tfrac{1}{3}\xi x)\sigma x^2]^{1/2}. \tag{48}$$

In Eq. (48) the only new parameter due to the anisotropy is

$$k = \frac{A\partial\Omega/\partial\beta}{4qn\Omega}, \tag{49}$$

which carries the same sign as α and enters to enhance the role of the perpendicular portion of the distribution function and, therefore, the trapped particles, via the $k\alpha x^3$ and kx^2 terms.

Superficially, Eq. (48) is very much the same as Eq. (18); however, $\alpha\sigma/2$ has been replaced by k in one term in the denominator and $2k/\sigma\alpha = A \gtrsim 10$ for many high power minority heating experiments. For very large A the σ terms may be safely neglected in Eq. (48) to obtain the less complicated form

$$Z_b(z,\alpha,k) = i \int_0^\infty \frac{dx \exp[izx - (1-\frac{1}{2}\alpha x)^2 (x^2/4)]}{[1+i(1-\frac{1}{3}\alpha x)kx^2]^2}, \tag{50}$$

which is the bi-Maxwellian generalization of Eq. (20a). Notice that anisotropy modifies the influence of the inhomogeneity in the regions $x \lesssim 1$ and in the vicinity of $x = 2/\alpha$ by altering the fraction of particles that resonate with the wave. In particular, if $k \gtrsim \alpha^2$ or $An^3 \gtrsim rR/q\rho^2$ for

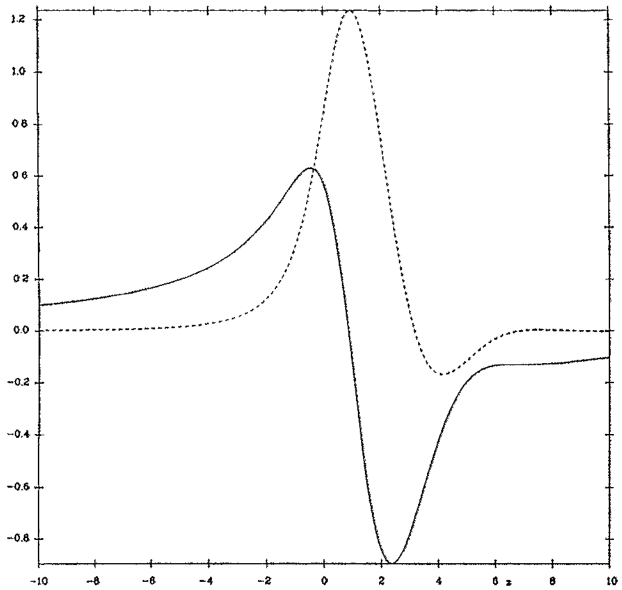


FIG. 4. The real (solid) and imaginary (broken) parts of $Z_b(z,\alpha,\sigma,\xi,k)$ plotted as a function of z for an on-axis resonance and an anisotropic bi-Maxwellian minority unperturbed distribution function for $\alpha=0.1$, $k=1$, and $\sigma=0=\xi$, where $1\gtrsim k^{1/2}\gg\alpha>0$ gives $q|n|/\epsilon\gtrsim A\gg rR/q\rho^2|n|^3$.

$0\leqslant\alpha\lessdot 1$ anisotropy results in a substantial modification of the plasma response. In addition, for $|k|\sim 1$ and $|\alpha|\sim 1$ $[A\sim q^{1/2}R/(r\rho)^{1/2}]$ or $|k|\gtrsim|\alpha|^{1/2}\gg 1$ $[A\gtrsim q^{1/2}R/(r\rho)^{1/2}$ at $|n|\lesssim(r/q\rho)^{1/2}]$ anisotropy modifications occur, but typically require very large A.

The effects of anisotropy are illustrated in Figs. 4–6. Figure 4 shows that anisotropy collisionlessly removes the fine-scale $1\gg\alpha>0$ oscillatory z structure (recall Fig. 2(a)], thereby causing Z_b to approach the homogeneous $Z(z)$ when $k\gtrsim\alpha^2$ and leaving the power deposition layer unchanged at $|z|\lesssim 1$ until k approaches unity. For larger $|k|\gtrsim 1$, strong anisotropy increases the width of the power deposition layer to $|z|\sim|k|^{1/2}$ while decreasing the amplitude of Z_b. At $\alpha\sim 1\sim k$ the oscillatory structure reappears on one side of the minority resonance for $\alpha>0$, as illustrated in Figs. 5(a)–5(c) (negative values of α have very little structure). The oscillatory structure becomes more extended for $k^{1/2}>1\sim\alpha$ and the power deposition layer broadens to $|z|\sim k^{1/2}$, as shown in Fig. 5(c). Figures 5(b)–5(c) require very large A and are shown to indicate the possible complications that strong velocity space anisotropy can introduce. For larger α and k the $|z|\sim\alpha$ oscillations are still present as illustrated in Fig. 6. Once $|k|\gtrsim|\alpha|^{1/2}\gg 1$, Z_b becomes approximately independent of the signs of α and k, as can be seen by taking the $n\to 0$ limit of Eq. (50) to obtain

$$Z_b(z,\alpha,k)\,|_{n\to 0}=\frac{2i}{|\alpha|^{1/2}}\int_0^\infty\frac{dy\exp(isy-y^4)}{[1-i(8|k|/3|\alpha|^{1/2})y^3]^2},$$

where s and y are defined, as in Eq. (20b). In such cases the width of the power deposition layer is given by $|z|\sim|\alpha k|^{1/3}$, as shown in Fig. 6.

Equation (48) includes the effects of an anisotropy in the distribution function, the rotational transform of the magnetic field, trapped and passing particles, and Doppler broadening. The substitution noted in Eq. (22) allows Eq. (48) to be used to generalize cold plasma models of fast wave heating in tokamaks to include these kinetic effects on the minority species.

VI. DISCUSSION

In the preceding sections the plasma response function is derived for fast wave minority heating from a fully toroidal description that retains the inhomogeneity of the magnetic field, including trapped particle effects, as well as the more familiar Doppler effect. Equation (18) is the general result for a Maxwellian, while Eq. (48) is a still more general expression since it is valid for a bi-Maxwellian distribution function. Both expressions retain trapped particles, including those turning in the resonance layer; however, the integrations over all velocity space make it difficult to identify specific effects associated with the trapped population. Roughly speaking, the terms σx^3 and kx^3, which come from the t^3 terms in Eq. (15), are expected to be most strongly influenced by the trapped particles. In particular, recall that Eqs. (18) and (48) differ only by the replacement of $\sigma\alpha x^2$ by $2kx^2$ in the denominator and this change arises from the $v_\perp$ integration which is somewhat more sensitive to the trapped particles. To identify clearly the role of the trapped fraction would require a far more realistic anisotropic unperturbed distribution function F having the appropriate pronounced velocity space structure due to the trapped particles that turn in the vicinity of the minority resonance. We have not yet found such an equilibrium distribution function which is sufficiently simple to enable an analytically tractable plasma response function to be obtained. Other potentially important effects that are neglected include finite Larmor radius and banana width effects and a rigorous treatment of pitch angle scattering collisional modifications.

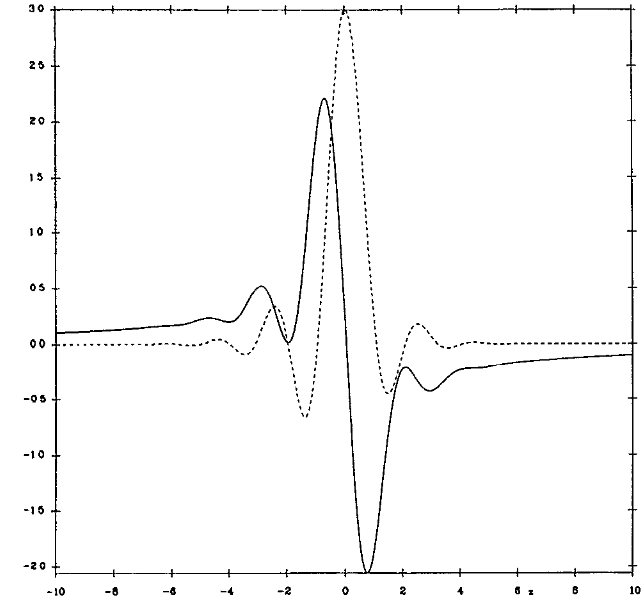
In the absence of toroidal effects the resonance or power deposition layer width is given by $|z|=|(\omega-\Omega)R/nv_t|\sim 1$. For on-axis heating $\Omega R=\omega R_0$ may be inserted to obtain the width

$$\frac{|R-R_0|}{R}\sim\frac{|n|v_T}{\omega R}\approx k_\parallel\,\rho.$$

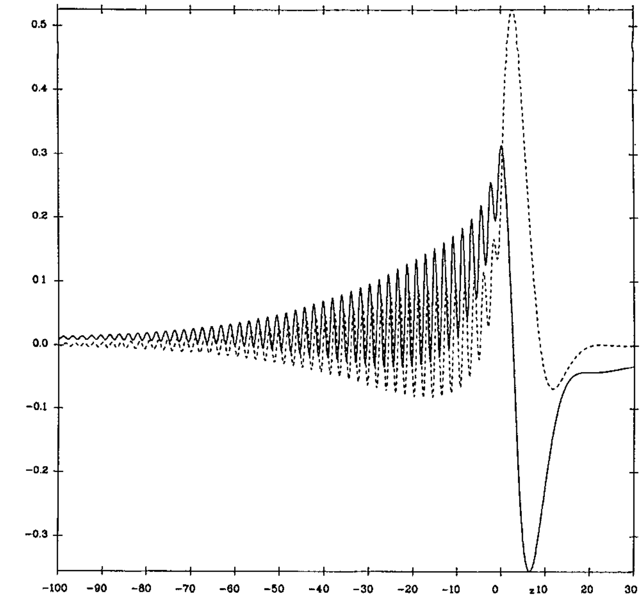
When toroidal effects are retained for the Maxwellian case additional scale lengths enter. For $0<\alpha\sim r/q\rho n^2<1$ oscillations are introduced in the plasma response[8] with a scale of $|z|\sim\alpha$ or

$$\frac{|R-R_0|}{R}\sim\frac{\epsilon}{q|n|},$$

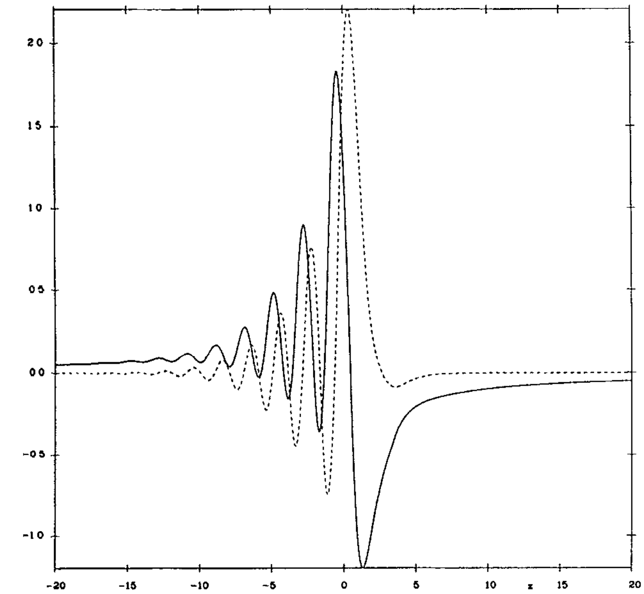
as can be seen from Eq. (21). These oscillations become less pronounced when $n^3\gtrsim rR/q\rho^2$ $(\sigma\gtrsim\alpha)$ for on-axis heating or $n^3\gtrsim r^2/q\rho^2$ $(\sigma|\xi|\gtrsim\alpha^3)$ for off-axis heating. For monopole operation or for parameters such that toroidicity dominates $(|\alpha|^{1/2}\gg 1)$, Eq. (20a) or Eq. (20b) gives the extended power deposition width from $|z/\alpha^{1/2}|\sim 1$ to be



(a)



(c)



(b)

FIG. 5. Real (solid) and imaginary (broken) parts of $Z_b(z,\alpha,\sigma,\xi,k)$ as a function of z for an on-axis resonance with $\alpha=1$ and $\sigma=0=\xi$: (a) $k=0.1$; (b) $k=1$, where the $|z|\sim\alpha$ oscillations reappear for $k\sim 1\sim\alpha$ or $A\sim R(q/\rho r)^{1/2}$; and (c) $k=10$, where the resonance layer broadens to $|z|\sim k^{1/2}$ or $\Delta R\sim\rho(A\epsilon|n|/q)^{1/2}$ and the $|z|\sim\alpha$ oscillations extend substantially further.

$$\frac{|R-R_0|}{R}\sim\left(\frac{\epsilon\rho}{qR}\right)^{1/2},$$

since σ and ξ corrections are small.

For the bi-Maxwellian case with $1\gg\alpha>0$ the power deposition width remains at $|R-R_0|\sim k_\parallel\,\rho R$ and the amplitude of the $|z|\sim\alpha$ oscillations in Z_b starts to decrease when $|k|\sim\alpha^2$ or $A\gtrsim rR/qn^3\rho^2$. The small α oscillations disappear by $k\sim 1$ $(A\gtrsim qn/\epsilon)$ when the power deposition width begins to broaden to $|z|\sim k^{1/2}$. As α is increased to $k\sim 1$ the oscillations return but only for $z<0$. For large anisotropies such that $|k|\gg 1\sim|\alpha|^{1/2}$ the $z<0$ oscillations become much more extended than the power deposition width which is given by $|z/k^{1/2}|\sim 1$ to be

$$\frac{|R-R_0|}{R}\sim\left(\frac{A\epsilon|n|}{q}\right)^{1/2}\frac{\rho}{R}.$$

When $|k|\gtrsim|\alpha|^{1/2}\gg 1$, $z<0$ oscillations are much less rapid and not as extended and the power deposition width is found from $|z|\sim|\alpha k|^{1/3}$ for this strong anisotropy case to be given by

$$\frac{|R-R_0|}{R}\sim\left(\frac{Ar^2\rho^2}{q^2R^4}\right)^{1/3}.$$

Recall that to remove the electric field from under the trajectory integral in Eq. (11) it was necessary to assume that it varied on a scale much longer than the power deposition width. The preceding estimates indicate that the deposition width varies between $|n|\rho$ and $(r\rho/q)^{1/2}$ for a Maxwellian, and over a similar range for a bi-Maxwellian unless it is highly anisotropic [in which case the width can become either $(A\epsilon|n|/q)^{1/2}\rho$ or $(Ar^2\rho^2/q^2R)^{1/3}$.] A full wave treatment is required to verify *a posteriori* that the electric field varies on a much longer scale length.

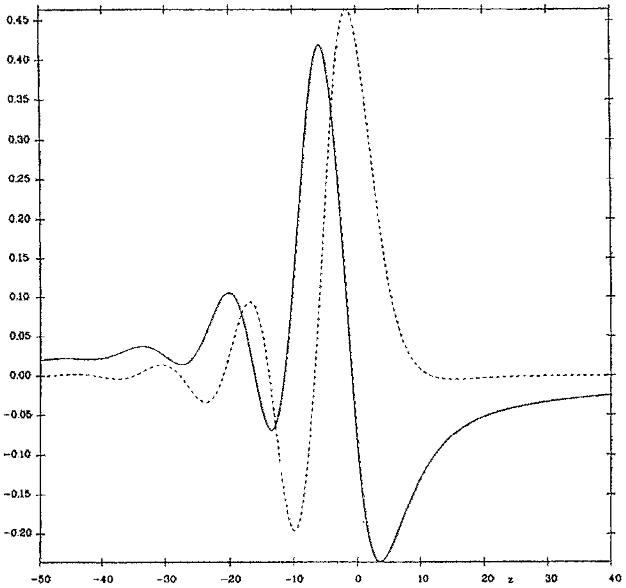


FIG. 6. Real (solid) and imaginary (broken) parts of $Z_b(z,\alpha,\sigma,\xi,k)$ as a function of z for an on-axis resonance with $\alpha=10$, $k=10$, and $\sigma=0=\xi$ for $k \gg \alpha^{1/2} \gg 1$ or $A \gg R(q/pr)^{1/2} \gg q/\epsilon|n|$. The resonance layer width is given by $|z| \sim |\alpha k|^{1/3}$ to be $\Delta R \sim (Ar^2\rho^2/q^2R)^{1/3}$.

The preceding discussion of the behavior of the plasma response function and the results of Secs. III and V indicate that the toroidicity of the magnetic field and the anisotropy of the unperturbed distribution function will complicate substantially a full wave evaluation of the transmission, reflection, and absorption/mode conversion coefficients of the fast wave in a minority heated tokamak during either monopole (small $|n|$) or dipole (large $|n|$) operation. For the Maxwellian case the inhomogeneity effect shown in Eq. (20a) and first found in Refs. 7 and 8 will be more important for low $|n|$ operation than for large $|n|$. The new inhomogeneity terms from Eq. (18) that do not appear in Eq. (20a) are more important in dipole operation, where the larger $|n|$ are more likely to make $\sigma \gtrsim \alpha$ or $\sigma|\xi| \gtrsim \alpha^3$. However, for off-axis heating with the resonance nearly tangent to a flux surface ($\alpha \to 0$), the $\sigma\xi$ term tends to be more important for monopole operation. Anisotropic effects also tend to be more likely in dipole operation since for $\alpha < 1$ only $k \gtrsim \alpha^2$ need be satisfied. However, for very strong anisotropies, $|k| \sim 1 \sim |\alpha|$ and $|k| \gtrsim |\alpha|^{1/2} \gg 1$ are more easily satisfied in monopole operation. In addition, α and k vary substantially along cylindrical surfaces of constant R. More realistic unperturbed distribution functions are not yet tractable, but are certain to introduce further complications since they cannot be characterized by two temperatures.

We emphasise that the resonant plasma response functions calculated in this paper are valid for arbitrary minority ion densities. On the other hand, the results obtained in Secs. IV and V, demonstrating the insensitivity of the optical depth and transmission coefficients to inhomogeneity and anisotropy, are only valid for minority to majority density ratios less than or of the order of the critical ratio given by Eq. (31). For larger ratios the ion–ion hybrid resonance occurs for which a full wave theory is required. The fully toroidal, kinetic evaluation of the plasma response function for the resonant minority ions is an essential step towards the construction of both one and two-dimensional full wave theories for fast wave minority heating.

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