Intermittency, dissipation, and scaling in two-dimensional magnetohydrodynamic turbulence

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Direct numerical simulations (DNS) provide a means to test phenomenological models for the scaling properties of intermittent MHD turbulence. The well-known model of She and Leveque, when generalized to MHD, is in good agreement with the DNS in three dimensions, however, it does not coincide with DNS in two dimensions (2D). This is resolved here using the results of recent DNS of driven MHD turbulence in 2D which directly determine the scaling of the rate of dissipation. Specifically, a simple modification to generalized refined similarity is proposed that captures the results of the 2D MHD simulations. This leads to a new generalization of She and Leveque in MHD that is coincident with the DNS results in 2D. A key feature of this model is that the most intensely dissipating structures, which are responsible for the intermittency, are thread-like in 2D, independent of whether the underlying phenomenology of the cascade is Kolmogorov or Iroshnikov Kraichnan. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2409528]

I. INTRODUCTION

In many astrophysical and laboratory plasmas, the magnetic and kinetic Reynolds numbers are large and the magnetohydrodynamic (MHD) approximation is valid. Such plasmas typically exhibit MHD turbulence. The question arises whether MHD turbulence, unlike hydrodynamic turbulence, should display similar cascade properties in two and three dimensions.[1] Recent numerical simulations of MHD turbulence show differences in the statistical behavior between flows in two and three dimensions; see, e.g., Refs. 2–7 and Refs. 8 and 9 for 2D and 3D results, respectively. Furthermore, numerically simulated dynamics of 3D MHD turbulence with an applied mean magnetic field $\mathbf{B}_0$ cross over to 2D as $\mathbf{B}_0$ increases.[10] The 2D equations of MHD are often used to model turbulence where the perturbed magnetic field is small compared to $\mathbf{B}_0$ and is essentially perpendicular to it, with particular relevance to solar system and astrophysical applications. The relationship between 2D and 3D MHD turbulence is thus of great physical interest, as is the underlying phenomenology in both cases.

In this paper we will investigate the scaling properties of MHD turbulence in terms of the Elsässer field variables $\mathbf{z}^{\pm} \equiv \mathbf{v} \pm \mathbf{B}(\mu_0\rho)^{-1/2}$ (Ref. 1) and consider the structure functions

$$S^p_{\pm}(l) = \langle |\delta_l z^{\pm}|^p \rangle, \tag{1}$$

where $\delta_l z^{\pm}$ represents the longitudinal increment in the Elsässer fields, and angular brackets denote a spatial ensemble average. Statistical self-similarity occurs in the inertial range over length scales $l_0 \gg l \gg \eta$, where l_0 is the macroscopic driving scale and η is the dissipation scale below which the fluid becomes dissipation-dominated. This is captured by scaling laws of the form $S^p_{\pm}(l) \sim l^{\zeta^{\pm}_p}$. An important element of MHD turbulence research is to characterize the exponents $\zeta^{\pm}_p$ and compare them to those recovered in hydrodynamics.[11–13]

Dimensional analysis of an inertial range process which simply transfers energy from one length scale to the next in a hydrodynamic flow (i.e., random eddy scrambling), yields Kolmorogov's classic phenomenology (K41),[14] that is, for differences in the velocity field, $\langle |\delta_l v|^p \rangle \sim l^{p/3}$. For $p=3$, this scaling follows directly from the Navier Stokes equations (Kolmogorov's "4/5" law, see, e.g., Ref. 15).

For MHD flows the cascade is mediated by Alfvénic phenomenology and one anticipates a scaling $\zeta^{\pm}_p = p/a$. The parameter a is anomalous in that it no longer follows from a dimensional analysis in the sense of K41. In particular, for MHD flows there is no result for the scaling of the moments analogous to the 4/5 law (although see Ref. 16). Instead, the choice of phenomenology invoked to model the energy cascade determines the parameter a. For example, the weak, isotropic, incompressible phenomenology of Iroshnikov and Kraichnan (IK),[17,18] where the dominant process is Alfvén wave collisions, gives $a=4$ or $\zeta^{\pm}_p = p/4$.

However, both direct numerical simulations (DNS) and observations show *intermittency*; that is, the ζ_p are found to vary nonlinearly with p. A phenomenological framework that expresses the intermittency in terms of physically meaningful parameters is that of She and Leveque (SL),[19] extended to MHD by Politano and Pouquet (SL-IK).[20] In addition to the parameter a, this phenomenology depends upon two parameters that describe the most intensely dissipating structures: their co-dimension, C_0; and the exponent α which specifies their scaling with length scale l. Specifically, the ensemble averaged local rate of dissipation $\langle \epsilon_l \rangle$ on length scale l scales as $\langle\epsilon_l^{(\infty)}\rangle \sim l^{-\alpha}$ as we approach the length scales of the most intensely dissipating structures.

In principle, analysis of the Elsässer fields generated by DNS can be used to determine the ζ_p, and thus both test a given phenomenology, and determine the parameters a, C_0, and α. Numerical simulations rarely achieve an extensive inertial range such that $S_\pm^p(l) \not\sim l^{\zeta_p^\pm}$. Instead, it is found that extended self-similarity (ESS) holds, that is, that ratios of the structure functions are power law in l over an extended range of scales:

$$S_\pm^p(l) \sim (S_\pm^q(l))^{\zeta_p^\pm/\zeta_q^\pm}. \tag{2}$$

This is consistent with scaling laws of the form $S_\pm^p(l) \sim l^{\zeta_p^\pm}$ if we replace l with some function $\xi(l)$; that is, $S_\pm^p(l) \sim \xi(l)^{\zeta_p^\pm}$ (generalized refined similarity, or GRS, Ref. 21). As we discuss below, GRS has been tested for the first time in the context of 2D MHD by the recent DNS of driven 2D MHD turbulence.[7] These simulations show an extended range of scaling under ESS as in Eq. (2) which however does not conform to GRS.

Recent 2D MHD simulations[2–4,7] give values of $\zeta_p^\pm/\zeta_3^\pm$ that deviate more strongly from linearity with p than those obtained in 3D, implying that they are more spatially intermittent. Also in these 2D simulations the time averaged total energy spectrum decreases more slowly with wave number k than would be expected from K41 and is closer to that of IK.[4,5,7]

The scaling found in both driven and decaying 3D MHD turbulence simulations[8,9] shows good agreement with the intermittency correction of She and Leveque. However, the SL model does not predict the ratios of exponents found in 2D.[2–4,7] Furthermore, the adoption of IK MHD phenomenology ($a=4$) to extend SL to MHD (SL-IK),[20] is found to *decrease* the level of agreement between simulation and model.[2,7]

Closer agreement is achieved in the study by Müller *et al.*[10] of 3D forced and decaying MHD turbulence simulations with an applied $\mathbf{B}_0$. Their scaling exponents, calculated from structure functions with the differencing vector perpendicular to $\mathbf{B}_0$, approach those found in 2D studies as $\mathbf{B}_0$ is increased. They suggest an intermittency model which captures this trend in the scaling exponents with $\mathbf{B}_0$. Their model relies on assumptions that are yet to be tested, specifically relating to the scaling properties of the local rate of dissipation.

In this paper we will propose a simple modification to GRS that captures the results of the 2D MHD simulations. We will then use this to generalize SL-IK. Fitting our resulting model to the simulation results then determines the parameters a, C_0 and α. This fitting procedure reveals the same C_0 and α for both $a=3$ and $a=4$, that is, for K-41 or IK phenomenology. The values of C_0 and α that we obtain are physically meaningful, providing strong support for our proposed formalism and suggesting a route to develop a full phenomenological model for 2D MHD turbulence.

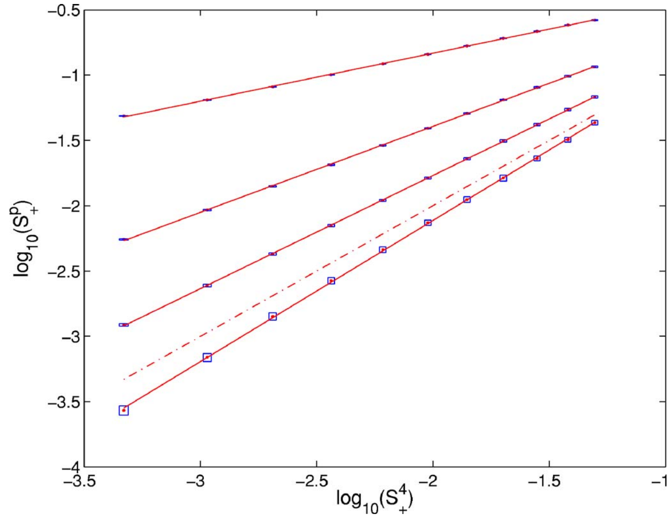

FIG. 1. (Color online) Extended self-similarity in the $\mathbf{z}^+$ Elsässer field from the 2D driven MHD turbulence simulation of Ref. 7, for $p=3$ (bottom) to $p=7$ (top). Compare Eq. (2) with $q=4$. The broken line represents the omitted plot for $p=4$ which necessarily scales perfectly.

II. GENERALIZED REFINED SIMILARITY

Numerical simulations rarely achieve an extensive inertial range, so that $S_\pm^p(l) \not\sim l^{\zeta_p^\pm}$. Instead, extended self-similarity (ESS) (Ref. 21), Eq. (2) is taken to extend into the dissipation range yielding an invariant ratio of the exponents $\zeta_p^\pm/\zeta_q^\pm$. We show ESS in the 2D MHD DNS (Ref. 7) in Fig. 1, which plots S_+^p against S_+^4 on a log-log scale.

In the DNS,[7] as well as determining the scaling of the Elsässer variables as above, we also directly determine the local rate of dissipation which we now use to explore GRS. Our starting point is to obtain an expression for the local rate of dissipation in terms of structure functions of the Elsässer variables that is consistent with ESS.

Inertial range scaling is expressed in terms of the structure functions as:[15]

$$S_\pm^p(l) \sim l^{p/a}\langle\epsilon_l^{p/a}\rangle \sim l^{\zeta_p^\pm} \tag{3}$$

and setting $p=a$ in the above gives $S_\pm^a(l) \sim \langle\epsilon_l\rangle l$. Here $\langle\epsilon_l\rangle$ is the mean energy transfer rate on length scale l, equivalent to the local rate of dissipation averaged over a volume of linear extent l, and $a=3$ or 4 reflects K41 or IK phenomenology, respectively. We now replace length scale l with a generalized length scale $\xi(l)$, such that

$$S_\pm^p(l) \sim \xi(l)^{p/a}\langle\epsilon_l^{p/a}\rangle \sim \xi(l)^{\zeta_p^\pm} \tag{4}$$

which satisfies the ESS [Eq. (2)]. Putting $p=a$ gives $\xi(l) = S_\pm^a(l)/\langle\epsilon_l\rangle$, see also Eqs. (12)–(14) of Ref. 22. We then have

$$S_\pm^p \sim \frac{\langle\epsilon_l^{p/a}\rangle}{\langle\epsilon_l\rangle^{p/a}}(S_\pm^a)^{p/a} \tag{5}$$

as an expression of generalized refined similarity (GRS).[23–26]

The simulation results of Ref. 7 have been used to test Eq. (5) directly for both $a=3$ (K41) and $a=4$ (IK) by plotting $S_\pm^p$ versus $(S_\pm^a)^{p/a}\langle\epsilon_l^{p/a}\rangle$; see Fig. 7 and Table III of Ref. 7. Such plots reveal a clear power law, but with exponents λ_p^a

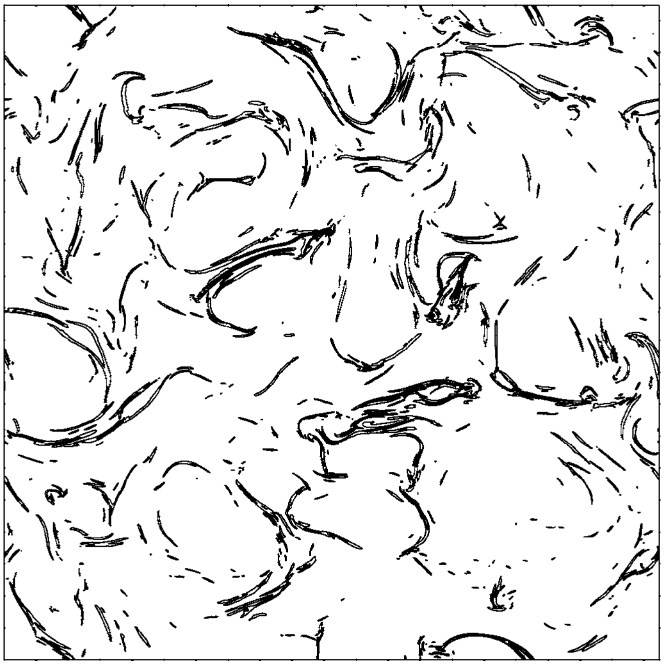

FIG. 2. Contour plot of the full tensor local rate of dissipation from the 2D driven MHD turbulence simulation of Ref. 7 at a time after steady state is achieved. The plot shows the entire simulation domain.

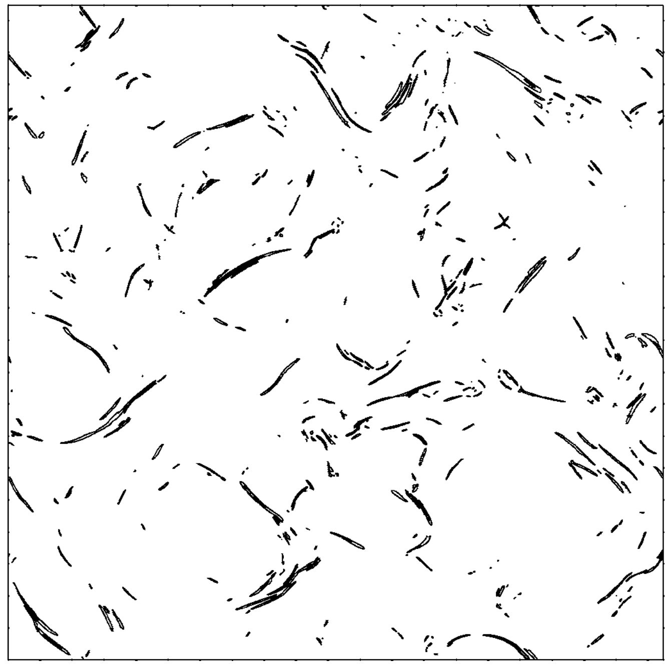

FIG. 3. Contour plot of the 1D proxy $\chi_\pm=(\partial_i z_i^\pm)^2/\rho$ from the 2D driven MHD turbulence simulation of Ref. 7 at the same time as for Fig. 2. The plot shows the entire simulation domain.

differing from 1, implying deviation from the scaling of Eq. (5). Guided by these simulations, we therefore propose a simple anomalous scaling that generalizes Eq. (5) to capture the simulation results

$$S_\pm^p \sim \left[\frac{\langle \epsilon_l^{p/a}\rangle}{\langle \epsilon_l\rangle^{p/a}}(S_\pm^a)^{p/a}\right]^{\lambda_p^a}, \tag{6}$$

where the exponents λ_p^a are given by the empirical values obtained from the simulations; see Table III of Ref. 7.

III. MODELS FOR THE INTERMITTENCY

We next generalize SL-IK to be consistent with this anomalous scaling Eq. (6). Following Ref. 22, we will first write the SL model in a form consistent with ESS.

The hypotheses of SL are expressed through a dimensionless energy dissipation rate (see Eq. 14 of Ref. 22) which is normalized to $\epsilon_l^{(\infty)}$, the dissipation rate associated with the most intermittent dissipative structures. This is assumed to have divergent scale dependence with an exponent α so that $\langle\epsilon_l^{(\infty)}\rangle \sim l^{-\alpha}$; for hydrodynamic flows, that is, K41 scaling, SL obtain $\alpha=2/3$.[19] For MHD, Politano and Pouquet[20] propose an extension of SL for the inertial range (SL-IK):

$$\zeta_p^\pm = \frac{p}{a} - \frac{\alpha p}{a} + C_0\left(1-\left(1-\frac{\alpha}{C_0}\right)^{p/a}\right), \tag{7}$$

where C_0 and α have the same meaning as above. While α is a free parameter, for MHD a value $\alpha=1/2$ has been identified[20] to be consistent with IK scaling.

Following Dubrulle,[22] we generalize Eq. (7) to hold for ESS

$$\frac{\zeta_p^\pm}{\zeta_a^\pm} = \frac{p}{a} - \frac{\alpha' p}{a} + C_0'\left(1-\left(1-\frac{\alpha'}{C_0'}\right)^{p/a}\right). \tag{8}$$

We identify $\alpha'\zeta_a^\pm$ in Eq. (8) with α in Eq. (7), and anticipate that α will fall in the range $[1/2,2/3]$. We identify $C_0'\zeta_a^\pm$ with C_0; C_0' is the co-dimension of the most intensely dissipating structures now expressed in terms of the generalized length scale $\xi(l)$. Importantly, Eq. (8) expresses the ratios of scaling exponents in terms of the generalized length scale $\xi(l)$ rather than the inertial range $S_\pm^a \sim l$. Essentially, the substitution $l\to\xi(l)$ maps the SL model, originally expressed for inertial range turbulence, onto a space appropriate for the dissipation range where ESS holds, appropriate to the simulations.

We finally extend Eq. (8) to encompass Eq. (6). Introducing our anomalous scaling Eq. (6), it follows from Eq. (2) that

$$\frac{\zeta_p^\pm}{\zeta_a^\pm} = \left[\frac{p}{a} - \frac{\alpha' p}{a} + C_0'\left(1-\left(1-\frac{\alpha'}{C_0'}\right)^{p/a}\right)\right]\lambda_p^a. \tag{9}$$

We now verify this against the high resolution 2D simulation of Ref. 7.

IV. PARAMETER ESTIMATES FROM THE SIMULATIONS

We begin by inferring the co-dimension of the most intensely dissipating structures directly from the simulation results. Figures 2 and 3 show contour plots of the local rate of dissipation from the 2D MHD simulations of Ref. 7. Contour plots are shown of the local rate of dissipation as calculated by the full tensor expression $(\partial_i B_j/\rho - \partial_j B_i/\rho)^2\eta/2 + (\partial_i v_j/\rho + \partial_j v_i/\rho)^2\mu/2$, Fig. 2, and as approximated by the 1D expression $\chi_\pm=(\partial_i z_i^\pm)^2/\rho$, Fig. 3. Both these figures show that the

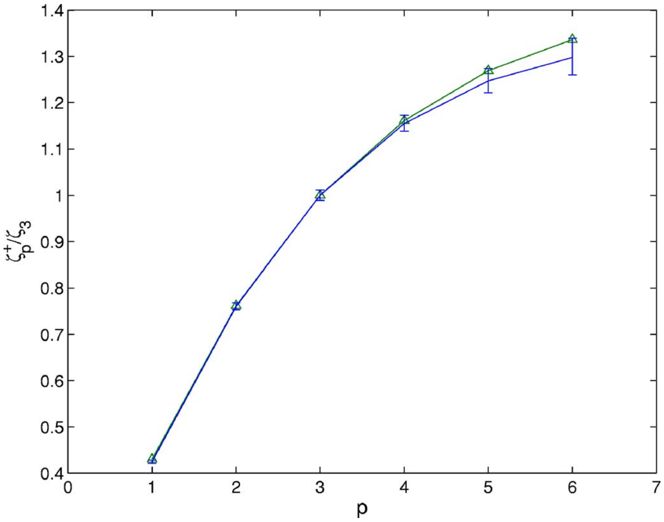

FIG. 4. (Color online) Generalized SL model Eq. (9) with $a=3$ (K41), as applied to the simulation of Ref. 7. Error bars (blue) denote the exponents obtained directly from the simulation; triangles (green), the inferred values from Eq. (9), with $\alpha=0.56\pm0.06$ and $C_0=1\pm0.1$.

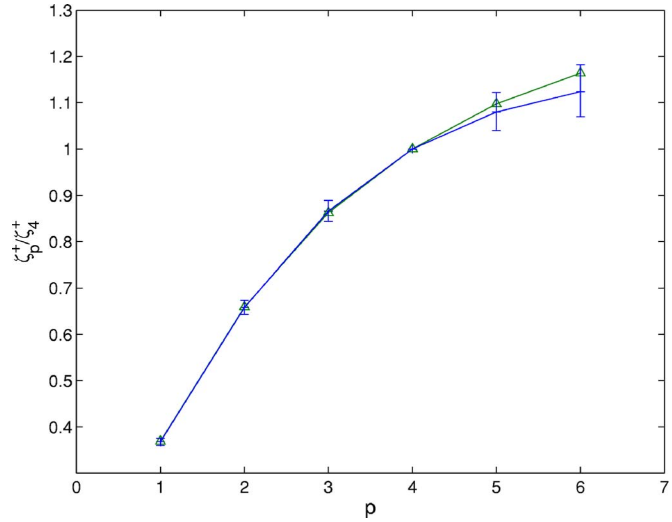

FIG. 5. (Color online) Generalized SL model Eq. (9) with $a=4$ (IK), as applied to the simulation of Ref. 7, format as in the previous figure, with $\alpha=0.56\pm0.07$ and $C_0=1\pm0.11$.

most intensely dissipating structures are mainly thread-like, suggesting that $C_0\approx1$. More structures are visible in Fig. 2 than in Fig. 3, principally because the contrast in intensity between structures is less for the full tensor dissipation, so that the same contour level used in both figures picks out more features in Fig. 2. Given this difference in relative intensities, some difference in scaling properties might be expected. Although the full tensor treatment describes the local rate of dissipation more accurately, we will consider scaling exponents obtained from the 1D proxy since it enables comparison with other experimental and observational studies, e.g., Ref. 27 for MHD and Ref. 28 for hydrodynamic turbulence.

We will perform two investigations, for $a=3$ (K41 phenomenology) and $a=4$ (IK phenomenology). We first estimate values of the $\zeta_a^{\pm}$ which relate the parameters C_0' and α' to C_0 and α. This is done using the value of the spectral index obtained from: the time averaged total energy spectrum $E(k)$ (see Fig. 6 of Ref. 7), the relation $E(k)\sim k^{-(\zeta_2+1)}$,[15] and measured values of $\zeta_a^{\pm}/\zeta_2^{\pm}$ using ESS [see Eq. (2)]. Comparatively large errors are present in these estimated values, mainly due to the relatively short scaling range of the energy spectrum. We next set $C_0=1$ in line with the topology of the most intensely dissipating structures seen in Fig. 3 above, giving $C_0'=1/\zeta_a^{\pm}$.

To obtain α' we fit Eq. (9) to the ratios of the exponents $\zeta_p^{\pm}/\zeta_a^{\pm}$ obtained directly from the simulations. We fit preferentially to the lowest four p since we have the greatest statistical confidence in these. Figures 4 and 5 then show the values of $\zeta_p^{\pm}/\zeta_a^{\pm}$, for $a=3$ and 4, respectively, obtained directly from the simulations (bars) overlaid with our model Eq. (9) (triangles). There is good agreement between these curves for both K41 and IK phenomenologies. In both cases we choose $C_0=1$. The uncertainty in C_0 can be inferred from the errors on the exponents $\zeta_p^{\pm}/\zeta_a^{\pm}$ obtained directly from the simulations, giving $C_0=1\pm0.1$ for $a=3$, and $C_0=1\pm0.11$ for $a=4$. The best fit value of α' implies a value of $\alpha=0.56\pm0.06$ for $a=3$, and $\alpha=0.56\pm0.07$ for $a=4$. The values of the parameters C_0 and α are thus indistinguishable for the two phenomenologies, once one extends GRS with the anomalous scaling λ_p^a in Eq. (6).

V. CONCLUSIONS

In this paper, we make a significant step toward a successful phenomenological model for 2D MHD turbulence, in that we have shown that such a model could be of the form Eqs. (6) and (9). By determining the λ_p^a via DNS we show that such a model would not need to be built specifically upon the scaling of either Kolmogorov or Iroshnikov Kraichnan and would be based on thread-like structures for the dissipation ($C_0=1$) with exponents $\alpha\simeq1/2$. This points toward a full phenomenological model for 2D MHD turbulence that would then predict the λ_p^a, given that the dissipation is characterized by $C_0=1$ and $\alpha\simeq1/2$.

We have proposed a simple modification to GRS that for the first time captures the results of 2D MHD simulations of driven turbulence in the context of the intermittency correction of She and Leveque. If our generalization Eq. (9) is used, no distinction needs to be drawn between the K41 ($a=3$) and IK ($a=4$) approaches, since both yield the same behavior. Specifically, both yield the same exponent α for the divergent scale dependence for the most intensely dissipating structures $\langle\epsilon_l^{(\infty)}\rangle\sim l^{-\alpha}$, and the same co-dimension C_0. The value for the co-dimension $C_0=1$ indicates that intermittency is governed by the formation of thread-like structures that dissipate most intensely. These correspond to sheet-like structures in 3D. This creates a point of contact with intermittency in 3D MHD turbulence, which Biskamp and Müller[8] found to be governed by the formation of sheet-like structures. Importantly, the inferred exponent α, which is related to energy transfer time, is found to lie between the IK value of 1/2 and the K41 value of 2/3 as anticipated. This mirrors the behavior of the total energy spectrum in the 2D MHD simulations, which also suggests an exponent lying between that predicted by K41 and IK.[7]

The model parameters C_0 and α obtained for the K41 or IK phenomenologies are indistinguishable within the errors. Our model then provides a unifying framework for exploring two- and three-dimensional MHD turbulence without the need to insist upon K41 or IK phenomenology *a priori*. Instead one simply uses the data (here, the DNS results) to compute the anomalous exponents λ_p^a. The remaining challenge is to develop a physical understanding of MHD turbulence that predicts these λ_p^a. We conjecture that the λ_p^a depend on a in such a manner that there is, in fact, only one intermittency correction (function of p) needed to characterize 2D MHD turbulence. This function would depend only on the properties of the most intensely dissipating structures, the C_0 and α, irrespective of a. The fact that we find that both the C_0 and α are identical (within errors) for K41 and IK suggests that there is a single dissipation scaling and topology in operation.

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