A destabilizing effect of rotation shear on magnetohydrodynamic ballooning modes

J. W. Connor, R. J. Hastie, and A. J. Webster
EURATOM/UKAEA Fusion Association, Culham Science Centre, Abingdon, Oxfordshire, OX14 3DB, United Kingdom

(Received 9 January 2007; accepted 2 March 2007; published online 30 April 2007)

The destabilization of ideal magnetohydrodynamic ballooning modes at finite rotation shear is demonstrated for the model s-α equilibrium by exploiting low magnetic shear, s, to simplify the two-dimensional stability problem to a one-dimensional eigenvalue problem. This simpler calculation captures the same features as exhibited by a full two-dimensional treatment, namely that stable values in the s-α stability diagram become unstable above a critical rotation shear. The first and second stability boundaries at low s are calculated as functions of rotation shear.
[DOI: 10.1063/1.2718909]

The stability of plasma to ideal magnetohydrodynamic (MHD) ballooning modes provides a limit to the maximum pressure that can be maintained. For the most dangerous, high-n modes (where n is the toroidal mode number in an axisymmetric plasma), this limit can be calculated for a stationary plasma using the ballooning transformation; this reduces the problem to a one-dimensional ordinary differential equation for the plasma perturbation.[1,2] However, the introduction of a sheared plasma flow destroys the translational symmetry required for the application of the ballooning transform, and leads to a two-dimensional eigenvalue problem.[3,4] Equivalently, one can seek a time-dependent solution in terms of Floquet solutions.[5,6] We parametrize the toroidal rotation shear as $\Omega_q=d\Omega/dq$, where $\Omega=\Omega_T/\omega_A$ is the toroidal plasma rotation frequency, Ω_T, normalized to the Alfvén frequency, $\omega_A=RB_T/(m_in_i)^{1/2}$, with m_i and n_i the plasma ion mass and density, respectively, R the plasma major radius, and B_T the toroidal magnetic field; Ω_q is a flux surface function and thus can be considered as a function of the safety factor, q. As Ω_q approaches zero, it can be shown analytically that the ballooning mode growth rate is reduced from its value in a stationary plasma (essentially averaging its value over the so-called ballooning angle parameter, θ_0—or time in the time-dependent formulation—instead of selecting the most unstable value of θ_0). This has been confirmed[7] by considering the low rotation shear limit of the solutions of the two-dimensional problem appropriate to finite shear, specifically for the model s-α equilibrium [$s=(r/q)dq/dr$ is the magnetic shear and $\alpha=-(2Rq^2/B_T^2)dp/dr$ is the normalized pressure gradient parameter[2]]. With increasing rotation shear, the growth rate continues to decrease (indeed the mode can become stable) but, surprisingly, above a critical value, the growth rate begins to increase, before finally stabilizing as $\Omega_q\to 1$ (see Fig. 1, taken from Ref. 4 where the high-n ballooning limit corresponds to $\Lambda^2=\infty$, since $\Lambda^2=nq/a$). To understand this numerical result, we have considered the limit of low s and α to obtain a much simpler problem (involving the solution of a simple second-order differential eigenvalue equation), which displays these very same features. In the limit of very low s, with $s\propto\alpha^2$, we demonstrate the two-dimensional results converge to those from the simpler treatment.

Reference 4 derives a set of coupled differential equations for the amplitudes, $u_m(x)$, of a poloidal Fourier harmonic decomposition of the plasma perturbation, where the two-dimensional perturbation of toroidal mode number n is written as $\varphi(x,\theta,\xi)=\Sigma_m e^{in\zeta-im\theta}u_m(x)$,

$$
\begin{aligned}
&s^2\frac{d}{dx}[(x+m)^2+\Gamma^2]\frac{du_m}{dx}-[(x+m)^2+\Gamma^2]u_m+\alpha\\
&\times\Bigg\{-\frac{s}{2}\frac{d}{dx}([(x+m)^2+\Gamma^2][u_{m+1}-u_{m-1}])-\frac{s}{2}[(x+m)^2\\
&+\Gamma^2+1]\frac{d}{dx}[u_{m+1}-u_{m-1}]-s(x+m)\frac{d}{dx}[u_{m+1}+u_{m-1}]\\
&+\frac{1}{2}[u_{m+1}+u_{m-1}]\Bigg\}+\frac{\alpha^2}{2}\Bigg\{[(x+m)^2+\Gamma^2+1]\Bigg[u_m\\
&-\frac{1}{2}(u_{m+2}+u_{m-2})\Bigg]-(x+m)[u_{m+2}-u_{m-2}]\Bigg\}=0 \qquad (1)
\end{aligned}
$$

Here $x=(r-r_0)/nqs$ is a radial coordinate measured from a reference surface, r_0, in units of $1/nqs$, so that it increases by unity in moving to the next resonant surface, and $\Gamma=\gamma+i\Omega_qx$ represents the Doppler shift arising from the sheared plasma rotation, where all frequencies are normalized to the Alfvén frequency, ω_A.

We consider an ordering scheme for low s and α, but finite Ω_q,

$$\gamma\sim s\sim\alpha^2\ll 1,\quad \Omega_q\sim 0(1) \qquad (2)$$

This generalizes to finite flow shear a treatment of the stationary s-α ballooning stability problem at low s.[8] It is convenient to introduce a scaled distance measured from the mth resonant surface $m=nq(x_m):x+m=st$, with $t\sim 0(1)$; then $x+m\pm j=st\pm j$ and $\Gamma=s(\hat{\gamma}+i\Omega_qt)\equiv s\hat{\Gamma}(t)$, where $\hat{\gamma}\sim 0(1)$.

If we consider a "central" Fourier harmonic u_m, this will couple to adjacent ones, generating $u_{m\pm j}\sim 0(\alpha^j)$, so we truncate at $j=2$, defining $u_m\equiv u, u_{m\pm j}\equiv u_{\pm j}$; it is convenient to introduce the notation $u_j\pm u_{-j}\equiv u_j^{\pm}$.

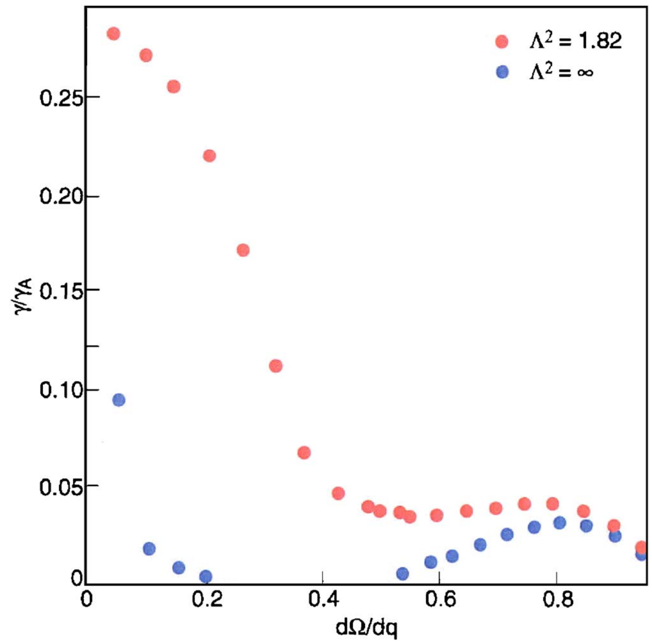

FIG. 1. (Color online) The destabilizing effect of rotation shear, Ω_q, on ideal MHD ballooning modes from Ref. 4; here $s=1$, $\alpha=1$. The high-n limit corresponds to the case $\Lambda^2=\infty$ (i.e., the lower curve with blue dots) since $\Lambda^2=nq/a$, whereas the upper curve, for $\Lambda^2=1.82$, retains finite-n effects.

Taking account of the above orderings, we find that the equation for u takes the form

$$s^2\left[\frac{d}{dt}(t^2+\hat{\Gamma}^2)\frac{d}{dt}-(t^2+\hat{\Gamma}^2)\right]u+\alpha\left[-\frac{1}{2}\frac{du_1^-}{dt}-st\frac{du_1^+}{dt}+\frac{u_1^+}{2}\right]$$
$$-\frac{\alpha^2}{2}u+\frac{\alpha^2}{4}u_2^+=0. \tag{3}$$

This equation for u is exact for the localized solutions with $t\sim 0(1)$ that we seek. However, to close the system of equations, we must truncate those for the $u_j^\pm$ at $0(\alpha^2)$. Thus

$$Lu_1^+ + 2s\left[\frac{d}{dt}\left(\frac{tdu_1^-}{dt}-tu_1^-\right)\right]+\alpha\left[u+2s\frac{d}{dt}(tu)-\frac{5}{2}\frac{d}{dt}u_2^-+u_2^+\right]$$
$$-\alpha^2u_1^+=0, \tag{4a}$$

$$Lu_1^- + 2s\left[\frac{d}{dt}\left(\frac{tdu_1^+}{dt}\right)-tu_1^+\right]+\alpha\left[\frac{du}{dt}-\frac{5}{2}\frac{d}{dt}u_2^+ +\frac{u_2^-}{2}\right]-\alpha^2u_1^-$$
$$=0, \tag{4b}$$

where we have introduced the operator $L\equiv d^2/dt^2-1$. Finally, for $u_2^\pm$ we obtain

$$Lu_2^+ + \frac{5}{8}\alpha\frac{d}{dt}u_1^- + \frac{1}{8}\alpha u_1^+ + \frac{1}{8}\alpha^2 u=0, \tag{5a}$$

$$Lu_2^- + \frac{5}{8}\alpha\frac{d}{dt}u_1^+ + \frac{1}{8}\alpha u_1^-=0. \tag{5b}$$

Equation (3) is our fundamental equation, into which we substitute Eqs. (4) and (5) for the $u_j^\pm$. At lowest order, $0(\alpha)$, we find Eq. (3) implies

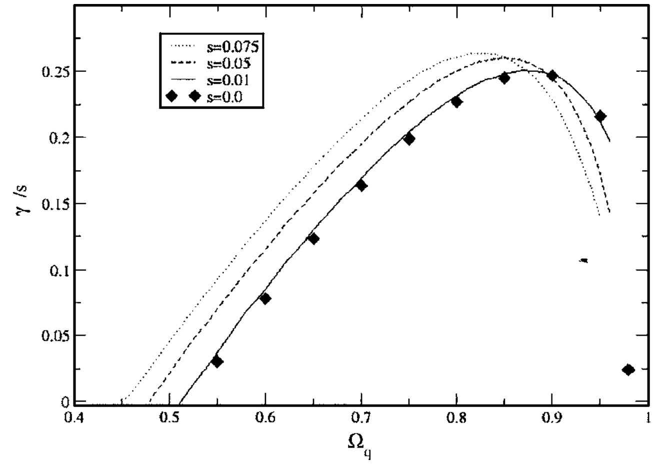

FIG. 2. The convergence of the numerical results from Ref. 4 to the solution of the eigenvalue problem of Eq. (10), shown by full diamonds, as $s\to 0$ at fixed $\sigma=0.53$. There is a critical value of $\Omega_q\simeq 5.2$ above which there is instability.

$$U\equiv -\frac{1}{2}\frac{du_1^-}{dt}+\frac{u_1^+}{2}-\frac{\alpha u}{2}=0 \tag{6}$$

for the leading-order contribution to $u_1^\pm$. This is consistent with the leading-order parts of $u_1^\pm$ from Eqs. (4),

$$Lu_1^+ + \alpha u=0, \quad Lu_1^- + \alpha\frac{du}{dt}=0 \tag{7}$$

since Eq. (7) implies $LU=0$, with the solution satisfying vanishing boundary conditions on $u^\pm$ at large $|t|$ being given by $U=0$. Thus to obtain Eq. (3) correct to $0(\alpha^4)$ we must obtain the $0(\alpha^2)$ corrections to $u_1^\pm$ from Eq. (4).

It is convenient to apply the operator L^2 to Eq. (3) to eliminate $u_j^\pm$ in terms of u, using Eqs. (4) and (5). After some algebra, we obtain

$$L^2\left[\frac{d}{dt}(t^2+\hat{\Gamma}^2)\frac{d}{dt}-(t^2+\hat{\Gamma}^2)\right]u=2\sigma\left(\frac{3}{16}\sigma-1\right)u \tag{8}$$

when $\sigma=\alpha^2/s$. Recalling the definition of $\hat{\Gamma}$ and introducing a shift in the origin of t, Eq. (8) can be written as

$$\hat{L}^2\left\{\frac{d}{d\tau}\left[\tau^2+\frac{\hat{\gamma}^2}{(1-\Omega_q^2)^2}\right]\frac{d}{d\tau}u-\left[\tau^2+\frac{\hat{\gamma}^2}{(1-\Omega_q^2)^2}\right]\mathrm{u}\right\}$$
$$-\frac{2\sigma}{1-\Omega_q^2}\left(\frac{3}{16}\sigma-1\right)u=0; \quad \hat{L}=\frac{d^2}{d\tau^2}-1, \tag{9}$$

where $\tau=[t+i\hat{\gamma}\Omega_q/(1-\Omega_q^2)]$. Finally, it is convenient to introduce $\hat{u}(k)$, the Fourier transform of u, to obtain

$$\frac{d}{dk}(1+k^2)\frac{d}{dk}\hat{u}-\frac{\hat{\gamma}^2}{(1-\Omega_q^2)^2}(1+k^2)\hat{u}$$
$$+\frac{1}{(1+k^2)^2}\frac{2\sigma}{(1-\Omega_q^2)}\left(1-\frac{3\sigma}{16}\right)\hat{u}=0. \tag{10}$$

This is an eigenvalue equation for $\hat{\gamma}(\sigma,\Omega_q)$, with the effect of the $s-\alpha$ equilibrium being encapsulated in the single parameter $\sigma=\alpha^2/s$.

In Fig. 2, we compare solutions of the full Eqs. (1) from

Ref. 4 as $s \sim \alpha^2 \to 0$, but holding σ constant. (The value $\sigma = 0.53$ is less than the marginal stability point for the first stability boundary at $\sigma = 0.84$.[8]) One sees that at sufficiently small s, the results converge to the universal curve obtained from Eq. (10). On multiplying Eq. (10) by $(1 - \Omega_q^2)$ it is clear that $\hat{\gamma}^2 \to 0$ as $\Omega_q \to 1$. Furthermore, increasing Ω_q reduces the stabilizing field line bending term (the second derivative term) more rapidly than the ballooning drive proportional to σ, so that a stable point in the $s - \alpha$ diagram can be destabilized above a finite critical value, $\Omega_q^{\text{crit}}(\sigma)$, as is evident from Fig. 1. Clearly, the growth rate reaches a maximum for $\Omega_q = \Omega_q^{\max}$, where $\Omega_q^{\text{crit}} < \Omega_q^{\max} < 1$. Although these results are only valid for $s \sim \alpha^2 \ll 1$, similar trends are seen in the full solution of Ref. 4 for finite s and α.

It is of interest to calculate the function $\Omega_q^{\text{crit}}(\sigma)$ or, equivalently, $\sigma^{\text{crit}}(\Omega_q)$ from the marginally stable ($\hat{\gamma} = 0$) form of Eq. (10). This differential equation has an eigenvalue

$$\lambda \equiv \frac{2\sigma}{(1 - \Omega_q^2)}\left(1 - \frac{3\sigma}{16}\right). \tag{11}$$

With the substitution $k = \tan y$, Eq. (10) can be transformed to a Mathieu equation,

$$\frac{d^2\hat{u}}{dy^2} + \frac{\lambda}{2}(1 + \cos 2y)\hat{u} = 0. \tag{12}$$

A variational treatment with the simple trial solution $\hat{u} = \cos y$ yields the eigenvalue $\lambda = 4/3$; inclusion of a $\cos 3y$ term in the trial function gives good agreement with the numerical solution of Eq. (12), which yields a more accurate value, $\lambda = 1.316$. We can then obtain the marginal values of σ as

$$\sigma = \frac{8}{3}\left\{1 \pm \left[1 - \frac{3}{8}(1 - \Omega_q^2)\lambda\right]^{1/2}\right\}. \tag{13}$$

The two solutions for σ are plotted in Fig. 3, the lower (higher) values corresponding to the parabolic boundaries of the first (second) stability regions of the s-α diagram. Clearly, the unstable region of the s-α diagram broadens as Ω_q increases toward 1, where the first stability boundary approaches $\sigma = 0$ while the second stability boundary approaches $\sigma = 16/3 \simeq 5.3$. Figure 3 shows clearly how initially stable values of σ migrate into the unstable region as Ω_q increases. [We note that using the variational value $\lambda = 4/3$ for $\Omega_q = 0$, we recover the results of Ref. 8, $\sigma = 8/3(1 \pm 1/\sqrt{2})$.]

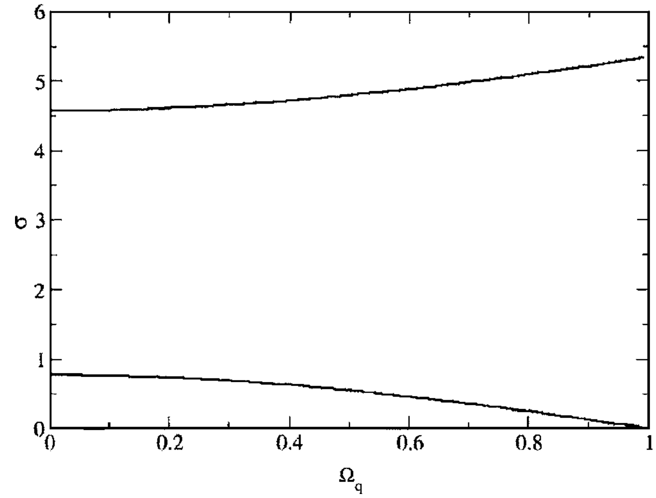

FIG. 3. The effect of rotation shear, Ω_q, on the first and second stability boundaries as characterized by the parameter σ at low s. The lower (upper) curves, calculated from Eq. (13), correspond to the first (second) stability boundaries of the s-α diagram.

Although these effects result from finite values for $\Omega_q \sim (r/sqv_A)(dv_T/dr)$, where v_A and v_T are the Alfvén and toroidal plasma velocities, respectively, one should note that at low s and with steep gradients in toroidal velocity, (dv_T/dr), as is typical near internal transport barriers, this does not require Alfvénic plasma flows.[9]

This work was funded by the UK Engineering and Physical Sciences Research Council and by the European Communities under the contract of Association between EURATOM and UKAEA. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

[1] J. W. Connor, R. J. Hastie, and J. B. Taylor, Proc. R. Soc. London, Ser. A **365**, 1 (1979).
[2] J. W. Connor, R. J. Hastie, and J. B. Taylor, Phys. Rev. Lett. **40**, 396 (1978).
[3] F. L. Waelbroeck and L. Chen, Phys. Fluids B **3**, 601 (1991).
[4] A. J. Webster, H. R. Wilson, and A. M. M. Scaife, Phys. Plasmas **11**, 2135 (2004).
[5] W. A. Cooper, Plasma Phys. Controlled Fusion **30**, 1805 (1988).
[6] R. L. Miller, F. L. Waelbroeck, A. B. Hassam *et al.*, Phys. Plasmas **2**, 3676 (1995).
[7] J. W. Connor, R. J. Hastie, and J. B. Taylor, Plasma Phys. Controlled Fusion **46**, B1 (2004).
[8] D. Lortz and J. Nührenberg, Phys. Lett. **68A**, 49 (1978).
[9] A. Webster and H. R. Wilson, Phys. Rev. Lett. **92**, 165004 (2004).