Two-stream instability, wave energy, and the energy principle

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A generalized Poynting theorem for a system of uniform electron beams is obtained. Two examples of the two-stream instability with beams of equal density are used to discuss the relation between negative wave energy and negative potential energy, which arises in the energy principle of ideal magnetohydrodynamics. In the first example, $v_{10} > v_{20}$, while in the second example, $v_{20} = -v_{10}$, where $v_{10,20}$ are the equilibrium beam velocities. Both cases can be interpreted in terms of the energy density arising from the generalized Poynting theorem. The first instability is due to the coupling of negative and positive energy waves at a frequency $k(v_{10}+v_{20})/2$. The second instability is due to the coupling of the same two perturbations, but at zero frequency. In this case, there is no oscillatory (wave) energy, but the beam electrons still make a negative contribution to the total energy. [DOI: 10.1063/1.2768016]

I. INTRODUCTION

Two-stream instability is an example of a general class of instabilities that can occur in a conservative system. Such instabilities have been referred to as reactive.[1]

Two-stream instabilities and the related beam-plasma instabilities are, of course, very well known. However, they continue to be of interest in a variety of situations. For example, Startsev and Davidson[2] recently gave a part analytical, part numerical analysis of a two-stream instability for a longitudinally compressing charged particle beam. In another recent study, a Vlasov-Poisson simulation of electron beam interaction was also described by Silin, Sydora, and Sauer.[3]

Reactive instabilities can be understood through the use of conservation theorems,[4–6] which lead to the concept of negative energy waves. A few years after the introduction of this concept, the subject of ideal magnetohydrodynamic (MHD) stability theory was advanced by the discovery of the energy principle.[7] This principle requires the potential energy to be negative. A reactive instability occurs when two linear waves couple (or coalesce) at a critical frequency and where one of the modes has negative wave energy. On the other hand, for a stationary equilibrium, an ideal MHD instability occurs when two modes couple (coalesce) at zero frequency. For this case, there is no oscillatory (wave) energy, but the potential energy, as defined by the energy principle, must be negative. Although both negative wave energy and negative potential energy arise from conservation theorems, it is not clear what, if any, is the connection between the two.

The extension of the energy principle to nonstationary equilibria is relevant to contemporary problems, such as magnetic fusion, since equilibrium flows are widespread and of considerable interest. This extension is also relevant to other fields, for example solar physics, space physics, and astrophysics. Davidson[8] addressed this question for nondissipative flows in incompressible conservative systems that include ideal MHD.

In this paper, two-stream instability is revisited in order to illustrate a number of basic points concerning reactive instabilities. In addition, a link is noted between the concepts of a negative energy wave and negative potential energy, used in the analysis of ideal MHD instabilities. In conservative systems possessing free energy, the existence of negative energy waves is demanded by the appropriate conservation theorem.[5] Similarly, the ideal MHD energy principle for stationary equilibrium can be obtained from a generalized Poynting theorem.[9]

The reactive instabilities arising from the interaction between cold electron beams, although much simpler than ideal MHD instabilities of magnetically confined plasmas, allow some insight into the link between the two. For two beams, with equilibrium velocities v_{10} and v_{20} ($v_{10} > v_{20}$), two-stream instability occurs at a frequency $k(v_{10}+v_{20})/2$. The instability arises when the negative energy slow space-charge wave on the faster beam couples (coalesces) with the positive energy fast space-charge wave on the slower beam. For the special case in which the beams have equal and opposite velocities, $v_{10}+v_{20}=0$, the instability occurs at zero frequency, as in the case of ideal MHD.

In Sec. II, the generalized Poynting theorem for electrostatic fluctuations of a system of cold electron beams is obtained. This allows the identification to be made of the wave energy density for the unperturbed beam modes. It is demonstrated in Sec. III how the expression for the small signal energy is able to account for instability for both finite values of the frequency and for the special case in which Re $\omega=0$. In both cases, solutions of the dispersion relation are obtained that demonstrate explicitly the coalescence (or coupling) of the relevant beam modes. A summary and conclusions are given in Sec. IV.

II. THE GENERALIZED POYNTING THEOREM

Consider a system of j-electron beams, each of uniform density n_{j0} and velocity $v_{j0}=v_{j0}\mathbf{z}$, where $\mathbf{z}$ is the unit vector along the z axis. There is no equilibrium magnetic field. The analysis is restricted to one-dimensional, electrostatic perturbations to the uniform equilibrium. The linearized equations for the beams are

$$\left(\frac{\partial}{\partial t}+v_{jo}\frac{\partial}{\partial z}\right)v_{j1z}=-\frac{e}{m_e}E_{1z}, \tag{1}$$

$$\frac{\partial n_{j1z}}{\partial t}+\frac{\partial}{\partial z}(n_{j0}v_{j1z}+n_{j1}v_{j0})=0, \tag{2}$$

$$J_{1z}+\varepsilon_0\frac{\partial E_{1z}}{\partial t}=0, \tag{3}$$

$$\text{where } J_{1z}=-e\sum_j(n_{jo}v_{j1z}+n_{j1}v_{jo}). \tag{4}$$

Multiply Eq. (1) by $m_e n_{jo} v_{j1z}^*$ and the complex conjugate equation by $m_e n_{jo} v_{j1z}$ and add, where the * denotes the complex conjugate. Now, multiply Eq. (1) by $m_e n_{j1}^* v_{jo}$ and the complex conjugate equation by $m_e n_{j1} v_{jo}$ and again add. Combining the resulting pair of equations gives

$$\begin{aligned}&\frac{\partial}{\partial t}(n_{jo}m_e|v_{j1z}|^2)+\frac{\partial}{\partial z}(n_{jo}m_e v_{jo}|v_{j1z}|^2)+m_e v_{jo}^2 n_{j1}^*\frac{\partial v_{j1z}}{\partial z}\\&\quad+m_e v_{jo}^2 n_{j1}\frac{\partial v_{j1z}^*}{\partial z}+m_e v_{jo} n_{j1}^*\frac{\partial v_{j1z}}{\partial t}+m_e v_{jo} n_{j1}\frac{\partial v_{j1z}^*}{\partial t}\\&\quad=-e(n_{jo}v_{j1z}^*+n_{j1}^*v_{jo})E_{1z}-e(n_{jo}v_{j1z}+n_{j1}v_{jo})E_{1z}^*.\end{aligned} \tag{5}$$

Similarly, multiply Eq. (2) by $m_e v_{jo} v_{j1z}^*$ and its complex conjugate by $m_e v_{jo} v_{j1z}$ and add, giving

$$\begin{aligned}&m_e v_{jo}\left(v_{j1z}^*\frac{\partial n_{j1}}{\partial t}+v_{j1z}\frac{\partial n_{j1}^*}{\partial t}\right)+m_e v_{jo}\frac{\partial}{\partial z}(n_{jo}v_{j1z}v_{j1z}^*)\\&\quad+m_e v_{jo}v_{j1z}^*\frac{\partial}{\partial z}(v_{jo}n_{j1})+m_e v_{jo}v_{j1z}\frac{\partial}{\partial z}(n_{j1}^*v_{jo})=0.\end{aligned} \tag{6}$$

Now use the relation

$$\frac{\partial}{\partial z}(v_{j1z}^*n_{j1})=v_{j1z}^*\frac{\partial n_{j1}}{\partial z}+n_{j1}\frac{\partial v_{j1z}^*}{\partial z} \tag{7}$$

and its complex conjugate, to obtain

$$\begin{aligned}v_{j1z}^*\frac{\partial n_{j1}}{\partial z}+v_{j1z}\frac{\partial n_{j1}^*}{\partial z}&=\frac{\partial}{\partial z}(v_{j1z}^*n_{j1}+v_{j1z}n_{j1}^*)-n_{j1}^*\frac{\partial v_{j1z}}{\partial z}\\&\quad-n_{j1}\frac{\partial v_{j1z}^*}{\partial z}.\end{aligned} \tag{8}$$

Combining Eqs. (6) and (8),

$$\begin{aligned}&m_e v_{jo}\left(v_{j1z}^*\frac{\partial n_{j1}}{\partial z}+v_{j1z}\frac{\partial n_{j1}^*}{\partial t}\right)+m_e v_{jo}\frac{\partial}{\partial z}(n_{jo}v_{j1z}v_{j1z}^*)\\&\quad+m_e v_{j0}^2\frac{\partial}{\partial z}(v_{j1z}^*n_{j1}+v_{j1z}n_{j1}^*)-m_e v_{jo}^2 n_{j1}\frac{\partial v_{j1z}^*}{\partial z}\\&\quad-m_e v_{jo}^2 n_{j1}^*\frac{\partial v_{j1z}}{\partial z}=0.\end{aligned} \tag{9}$$

Now add Eqs. (5) and (9) giving

$$\begin{aligned}&\frac{\partial}{\partial t}[n_{jo}m_e|v_{j1z}|^2+m_e v_{jo}(n_{j1}^*v_{j1z}+n_{j1}v_{j1z}^*)]\\&\quad+\frac{\partial}{\partial z}[2m_e n_{jo}v_{jo}|v_{j1z}|^2+m_e v_{jo}^2(v_{j1z}^*n_{j1}+v_{j1z}n_{j1}^*)]\\&\quad=-e(n_{jo}v_{j1z}^*+n_{j1}^*v_{jo})E_{1z}-e(n_{jo}v_{j1z}+n_{j1}v_{jo})E_{1z}^*.\end{aligned} \tag{10}$$

Summing Eq. (10) over all the j species of electron beams and taking account of Eq. (4), Eq. (10) can be written

$$\begin{aligned}&\frac{\partial}{\partial t}\sum_j\left\{\frac{1}{2}n_{jo}m_e|v_{j1z}|^2+\frac{1}{2}m_e v_{jo}(v_{j1z}n_{j1}^*+v_{j1}^*n_{j1})\right\}\\&\quad+\frac{\partial}{\partial z}\left\{\sum_j\left[m_e n_{jo}|v_{j1z}|^2+\frac{1}{2}m_e v_{jo}(v_{j1z}^*n_{j1}\right.\right.\\&\quad\left.\left.+v_{j1z}n_{j1}^*)\right]v_{jo}\right\}=\frac{1}{2}(J_{1z}^*E_{1z}+J_{1z}E_{1z}^*).\end{aligned} \tag{11}$$

Multiplying Eq. (3) by E_{1z}^* and its complex conjugate equation by E_{1z} yields

$$\frac{1}{2}(J_{1z}E_{1z}^*+J_{1z}^*E_{1z})=-\frac{1}{2}\varepsilon_o\frac{\partial}{\partial t}|E_{1z}|^2. \tag{12}$$

The generalized Poynting theorem for one-dimensional perturbations to an equilibrium system consisting of j-electron beams follows from Eqs. (11) and (12),

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\sum_j\left[\frac{1}{2}n_{jo}m_e|v_{j1z}|^2+\frac{1}{2}m_e v_{jo}(n_{j1}^*v_{j1z}+n_{j1}v_{j1z}^*)\right]\right.\\&\quad\left.+\frac{1}{2}\varepsilon_o|E_{1z}|^2\right\}+\frac{\partial}{\partial z}\left\{\sum_j\left[m_e n_{jo}|v_{j1z}|^2\right.\right.\\&\quad\left.\left.+\frac{1}{2}m_e v_{jo}(v_{j1z}^*n_{j1}+v_{j1z}n_{j1}^*)\right]v_{jo}\right\}=0.\end{aligned} \tag{13}$$

The equation expresses the conservation of small signal energy for perturbations to the above system of electron beams. The first term is the time rate of change of the energy density and the second term is the energy flow. For electrostatic fluctuations and cold beams, the energy flow is due to the equilibrium beam velocities. Since the equilibrium is uniform, averaging over a period of oscillation (or a wavelength) gives

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\frac{1}{2}\varepsilon_o|E_{1z}|^2+\sum_j\left[\frac{1}{2}n_{jo}m_e|v_{j1z}|^2+\frac{1}{2}m_e v_{jo}(n_{j1}^*v_{j1z}\right.\right.\\&\quad\left.\left.+n_{j1}v_{j1z}^*)\right]\right\}=0.\end{aligned} \tag{14}$$

Denoting the expression inside the curly brackets in Eq. (14) by ξ, the small signal energy density, the equation becomes

$$\frac{\partial \xi}{\partial t}=0. \tag{15}$$

The constancy of ξ implies that instability can only occur when the system permits some of the contributions to ξ to be negative.[5]

III. SMALL SIGNAL ENERGY AND INSTABILITY

The physical content of Eq. (14) is illustrated in this section with two specific examples of two-stream instability. Assuming that all perturbed (small signal) quantities vary as $\exp i(kz-\omega t)$, Eqs. (1) and (2) give

$$v_{j1z}=\frac{iq_j}{m_j}\frac{E_{1z}}{(\omega-kv_{jo})}, \tag{16}$$

$$n_{ji}=\frac{ikn_{jo}q_jE_{1z}}{m_j(\omega-kv_{jo})^2}. \tag{17}$$

With the aid of Eqs. (16) and (17), ξ can be written as

$$\xi=\frac{1}{4}\varepsilon_o|E_{1z}|^2\left\{1+\sum_j\left[\frac{\omega_{pj}^2}{(\omega-kv_{jo})^2}+\frac{2kv_{jo}\omega_{pj}^2}{(\omega-kv_{jo})^3}\right]\right\}. \tag{18}$$

The first example of two-stream instability to be considered is when both streams have the same density and stream 1 moves faster than stream 2,

Case a) $\omega_{p1}=\omega_{p2}=\omega_p;\quad v_{10}>v_{20}$.

For simplicity, the subscript "0" on the equilibrium beam velocities is dropped so that $v_1>v_2$. The dispersion relation for this case can be found in most plasma physics text books, see, e.g., Stix,[10]

$$1-\frac{\omega_p^2}{(\omega-kv_1)^2}-\frac{\omega_p^2}{(\omega-kv_2)^2}=0. \tag{19}$$

The unstable solutions for this equation are well known and are given, for example, in Ref. 10, where the solution for maximum growth is given as

$$\omega=\frac{k(v_1+v_2)}{2}\pm\frac{i\omega_p}{2}. \tag{20}$$

In Ref. 10, it was pointed out that the stable solutions of Eq. (19) reduce to the fast and slow space-charge waves on the two beams, which propagate independently for large k. However, these waves also play a key role in the unstable range of wave numbers, as will now be demonstrated.

The result given in Eq. (20) can be obtained by recognizing that the space-charge waves on each beam will perturb each other. The dispersion relation given by Eq. (19) can be written as

$$(\omega-kv_1-\omega_p)(\omega-kv_1+\omega_p)\times(\omega-kv_2-\omega_p)(\omega-kv_2+\omega_p)=\omega_p^4. \tag{21}$$

This form is physically revealing, since it suggests that the instability can be described in terms of the coupling of the fast and slow space-charge waves carried by the two beams. The fast space-charge waves are given by $\omega=kv_{1,2}+\omega_p$ and the slow space-charge waves by[11] $\omega=kv_{1,2}-\omega_p$. By substituting the fast space-charge wave frequencies into Eq. (18), it can be seen that the energy density of the fast waves will always be positive since $\omega-kv_{1,2}>0$. On the other hand, substituting the slow space-charge wave frequencies into Eq. (18) shows that the slow waves carry negative energy, since $\omega-kv_{1,2}<0$.

Since $v_1>v_2$, instability can be expected when the slow wave on beam 1 couples to the fast wave on beam 2. The coupling condition is

$$kv_1-\omega_p=kv_2+\omega_p. \tag{22}$$

In order to demonstrate this, assume a solution of Eq. (21) of the form

$$\omega=kv_1-\omega_p+\delta\omega. \tag{23}$$

Substituting Eq. (23) into Eq. (21), making use of the coupling condition, Eq. (22), and neglecting $\delta\omega$ in nonresonant terms, Eq. (21) reduces to

$$(\delta\omega)^2\approx-\frac{\omega_p^2}{4}. \tag{24}$$

Hence, Eqs. (22)–(24) yield

$$\omega=\frac{k(v_1+v_2)}{2}\pm\frac{i\omega_p}{2} \tag{25}$$

in agreement with the exact solution given by Eq. (20). The wave number at the threshold for instability can be obtained from the condition that $\xi=0$. Using the coupling condition, Eq. (22), the frequency $\omega=kv_1-\omega_p=kv_2+\omega_p$ is substituted into Eq. (18) to give

$$\xi=\frac{1}{4}\varepsilon_0|E_{1z}|^2\left[3-\frac{2k(v_1-v_2)}{\omega_p}\right]. \tag{26}$$

Hence, the wave number at the threshold for instability resulting from the quadratic approximation is $k=3\omega_p/2(v_1-v_2)$. For comparison, the exact value[10] is $k=2\sqrt{2}\omega_p/(v_1-v_2)$. The wave number corresponding to the maximum growth rate is $\sqrt{3}\omega_p/(v_1-v_2)$. The coupling condition, Eq. (22), which is expected to correspond to maximum growth, gives the value $k=2\omega_p/(v_1-v_2)$, which is in fair agreement.

Case b) $\omega_{p1}=\omega_{p2}=\omega_p;\quad v_1=v,\ v_2=-v$.

This is a rather special case in which the two beams again have equal densities but equal and opposite velocities. The dispersion relation for this case is

$$1-\frac{\omega_p^2}{(\omega-kv)^2}-\frac{\omega_p^2}{(\omega+kv)^2}=0. \tag{27}$$

The solution can be obtained from the exact result given by Eq. (20) by putting $v_1=v$, $v_2=-v$, and is

$$\omega=\pm i\frac{\omega_p}{2}. \tag{28}$$

Note that for this special case, Re $\omega=0$, which is analogous to ideal MHD instability. This result can again be obtained in terms of the coupling of the space-charge waves. It is clear that for this case there can only be coalescence of roots, and therefore instability, at zero frequency. The coupling condition corresponding to Eq. (22) is now

$$kv - \omega_p = -kv + \omega_p. \tag{29}$$

Assuming

$$\omega = kv - \omega_p + \delta\omega \tag{30}$$

and making use of Eq. (29), Eq. (21) yields the result given in Eq. (28).

Although there is no oscillatory energy for this case, since Re $\omega=0$, the energy expression given by Eq. (18) still holds and must take the value $\xi=0$, at the threshold for instability. Substituting $v_1=v$, $v_2=-v$ in Eq. (18), it can be seen that there are now two negative contributions to ξ,

$$\xi = \frac{1}{4}\varepsilon_o|E_{1z}|^2\left\{1 + \frac{\omega_p^2}{(\omega-kv)^2} + \frac{2kv\omega_p^2}{(\omega-kv)^3} + \frac{\omega_p^2}{(\omega+kv)^2} - \frac{2kv\omega_p^2}{(\omega+kv)^3}\right\}. \tag{31}$$

Imposing the threshold condition $\omega=0$, Eq. (31) yields

$$\xi_{\text{th}} = \frac{1}{4}\varepsilon_o|E_{1z}|^2\left(1 - \frac{2\omega_p^2}{k^2v^2}\right). \tag{32}$$

This result gives $k=\sqrt{2}\omega_p/v$ as the wave number at the instability threshold, which is in agreement with the value obtained from the solution of the dispersion relation, given by Eq. (27). The wave number corresponding to maximum growth is given by the exact solution of the dispersion relation[10] and is $k=\sqrt{3}\omega_p/2v$. For comparison, the resonance condition [Eq. (29)] gives ω_p/v, in reasonable agreement. It should be emphasized that the negative contributions to ξ in Eq. (31) still relate to the kinetic energy of the beam electrons. However, at zero frequency these contributions are equivalent to $\delta W<0$ in the case of ideal MHD instability.

IV. SUMMARY AND CONCLUSIONS

The generalized Poynting theorem for a system of cold, uniform electron beams with no equilibrium magnetic field has been derived. This provides the basis for a discussion of reactive instabilities, which result from the coalescence of two roots of the linear dispersion relation. The well known two-stream instability suggests a possible connection between negative small signal energy and negative potential energy ($\delta W<0$) characteristic of ideal MHD instability, where Re $\omega=0$, and for which there is no oscillatory energy.

A reactive instability occurs when two wave modes of the linear system couple at a critical frequency, where one of the modes carries positive energy and the other negative energy. Ideal MHD instabilities are a special case of reactive instability, and occur when two linear modes couple at zero frequency for a stationary equilibrium. The ideal MHD energy principle is restricted to stationary plasmas. Since equilibrium flows are common to many situations, it would be useful to have a corresponding result to the energy principle for nonstationary plasmas.

Two-stream instability has been discussed for two cases. In the first, two beams of equal density have drift speeds v_1 and v_2 with $v_1>v_2$. Instability is shown to occur when the negative energy slow space-charge wave carried by the faster beam couples with the positive energy fast space-charge wave carried by the slower beam. The requirement that the frequencies of the two modes should be equal gives the coupling (coalescence) condition, which allows a quadratic approximation to the dispersion relation, yielding the exact values of the frequency and maximum growth rate. A value for threshold wave number for the instability is obtained from the condition that the total energy is zero at the critical (coupling) frequency. This is compared with the exact result.

In the second case, the two beams of equal density have equal and opposite drift speeds $v_1=v$ and $v_2=-v$. Coalescence can only occur in this case at zero frequency, analogous to an ideal MHD instability where there is no oscillatory energy but where $\delta W<0$. For the second example, it is again shown how the coupling of two beam modes reduces the dispersion relation to a quadratic approximation, which nevertheless yields the exact growth rate. The expression for the nonoscillatory energy density yields the threshold wave number from the condition that the total energy is zero at threshold. Again, the wave number for maximum growth is compared with the wave number obtained from the coupling condition.

It is worth noting an interesting distinction between the two cases. For the finite frequency instability ($v_1>v_2$), there is only one negative contribution to the total energy, which comes, of course, from the negative energy wave on the faster beam. On the other hand, for the case with equal and opposite flows ($v_1+v_2=0$), both beams give negative contributions to the total energy.

Since the generalized Poynting theorem leads to a consistent interpretation of both oscillatory and nonoscillatory reactive instabilities for a simple two-beam example, it is suggested that the corresponding generalized Poynting theorem for ideal MHD for a nonstationary plasma would yield useful information through the identification of negative energy waves. For nonstationary equilibria, ideal MHD instabilities will occur at finite frequencies. It should be emphasized that the analysis given in this paper applies to the case in which the beam velocity is much greater than the thermal spread of the beam.

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[1] A. Hasegawa, Phys. Rev. **169**, 204 (1968).
[2] E. A. Startsev and R. C. Davidson, Phys. Plasmas **13**, 062108 (2006).
[3] I. Silin, R. Sydora, and K. Sauer, Phys. Plasmas **14**, 012106 (2007).
[4] L. J. Chu, *Proceedings of Electron Devices Conference IRE-PGED* (Institute of Radio Engineers, University of New Hampshire, Durham, 1951).
[5] P. A. Sturrock, Ann. Phys. (N.Y.) **4**, 306 (1958).

[6] R. L. Dewar, Phys. Fluids **13**, 2710 (1970).
[7] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy. Soc. A **223**, 17 (1958).
[8] P. A. Davidson, J. Fluid Mech. **402**, 329 (2000).
[9] W. M. Manheimer and C. N. Lashmore-Davies, *MHD and Microinstabilities in Confined Plasma* (Adam Hilger, Bristol, 1989), pp. 49–52.
[10] T. H. Stix, *Waves in Plasmas* (American Institute of Physics, New York, 1992), pp. 154–155.
[11] W. H. Louisell, *Coupled Mode and Parametric Electronics* (Wiley, New York, 1960), pp. 39–47.