Saturation of tearing modes in reversed field pinches with locally linear force-free magnetic fields and its application to quasi-single-helicity states

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A simple formula for predicting the width of a saturated island, formed as a consequence of tearing perturbation of linear force-free fields in cylindrical geometry, is derived. The formula makes it possible to calculate the saturated island width in terms of the values of parameters characterizing the initial force-free equilibrium and can be applied to equilibria of interest for reversed field pinches. In particular it is applied, in this paper, to force-free equilibria with piecewise constant radial profile of the pinch parameter, which have been recently suggested to be relevant for the formation of quasi-single-helicity states. The main result is that the island width becomes larger as a parameter, that quantifies the departure from a relaxed Taylor state, increases. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2913263]

I. INTRODUCTION

Reversed field pinches (RFPs) are toroidal devices for magnetic plasma confinement in which the intensities of the poloidal and toroidal components of the magnetic field are of the same order. In addition to the importance due to the role that such devices play in the fusion context, where they provide an example of a confinement scheme alternative to tokamak, RFPs attract interest also because they represent a paradigm for self-organization in laboratory plasmas.[1] Indeed, typical RFP discharges are characterized by an initial, strongly turbulent phase followed by the reaching of a more quiescent state, which is largely independent of the details of the initial configuration. A classical theory that is able to properly describe a number of qualitative, and in some cases quantitative, aspects of this self-organization process is the magnetic relaxation theory proposed by Taylor.[2,3] According to Taylor's theory, plasma in a RFP is a turbulent medium which, after the initial violently unstable phase, tends, due to dissipative processes, to relax to a state of minimum magnetic energy under the constraints of conservation of the total magnetic toroidal flux and of the total magnetic helicity. Mathematically, the equation derived from this constrained minimization process is given by

$$\nabla \times \mathbf{B} = \mu \mathbf{B}, \tag{1}$$

where $\mathbf{B}$ is the magnetic field and μ is a constant.[4] Therefore, according to Taylor, plasmas in RFPs tend to reach a state of vanishing Lorentz force, i.e., a so-called force-free state. The hypothesis that only the total helicity is preserved during the relaxation process (whereas the helicities of the volumes associated with the internal magnetic surfaces can be altered by dissipative processes) implies that the resulting configuration is a linear force-free field, i.e., the function μ is constant. Moreover, in the cylindrically symmetric version of this model, only two quantities in the initial configuration, namely the toroidal flux and the global helicity, determine the values of the parameters of the final state, i.e., the values of μ and of the amplitude of the magnetic field. Note that, if the conservation of the helicities of the internal magnetic surfaces had been imposed, then the resulting μ would have become, in general, a flux function.

In the cylindrical approximation of a toroidal domain, Eq. (1) admits the following cylindrically symmetric solutions:

$$B_r = 0, \quad B_\theta = B_0 J_1(\mu r), \quad B_z = B_0 J_0(\mu r), \tag{2}$$

expressed in terms of Bessel functions and where B_0 is an arbitrary constant. The above solutions represent the so-called Bessel function model (BFM). In spite of its simplicity, which makes it also amenable to analytical treatments, the BFM is able to explain some important features observed in RFPs. In particular the BFM accounts for the reversal of the toroidal field near the plasma edge and describes to a good extent current density profiles in the plasma core region. Corrections to the BFM yielding current density profiles closer to the experimental situation have been considered, for instance, in Ref. 5. However, in spite of the success of Taylor's theory, it is however important to mention that in recent years a considerable effort has been devoted in trying to explain RFP dynamics from a perspective which is different from the one adopted by Taylor. In particular, on the basis of numerical, analytical, and experimental results[6–9] it has been argued that RFP plasmas need not be intrinsically turbulent but could reach a helically symmetric, laminar equilibrium state[10,11] characterized also by better confinement properties.

Ideal stability of the cylindrical BFM has been studied by Voslamber and Callebaut,[12] whereas the linear stability

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analysis with respect to tearing modes has been carried out by Gibson and Whiteman[13] (ideal and resistive instabilities in the presence of pressure gradients, on the other hand, have been investigated in Refs. 14 and 15). In particular, the analysis by Gibson and Whiteman shows that if μ, normalized with respect to the cylinder radius exceeds 3.11, then the BFM equilibrium is linearly unstable with respect to tearing modes with poloidal wave number $m=1$. Thus, for sufficiently large values of μ, linear theory predicts that tearing perturbations of the BFM equilibria can lead to magnetic reconnection processes at the resonant surfaces and to the formation and growth of magnetic islands. Numerical simulations showing this process have been described in Ref. 16. On the other hand, it has been recently shown[17,18] that even a small departure from a μ profile, constant over the whole domain, can significantly modify the stability properties of the force-free equilibrium. In particular, in these works, it was proposed that RFP plasmas, after reaching a force-free state with constant $\mu=\mu_T<3.11$ according to Taylor's theory, could tend to drift away from it, because of central Ohmic heating, which causes the current density profile to peak in the center and thus to induce radial gradients in μ. The model for such force-free, nonconstant μ equilibria, adopted in these works, is the stepped-μ equilibrium, in which μ is defined by

$$\mu(r)=\begin{cases}\mu_0 & \text{if } 0 \leqslant r \leqslant r_{\text{step}},\\ \mu_1 & \text{if } r_{\text{step}} < r \leqslant 1,\end{cases} \tag{3}$$

where r_{step} is the radius where the step is located and μ_0 and μ_1 are two constants with $\mu_0>\mu_1$ for the sake of compatibility with the experimental results which show a profile of $\mu(r)$ decreasing with r. Although very simple, this model shows that the presence of a variation in the μ profile can significantly alter the linear stability properties of the equilibrium, compared to the constant-μ case. In particular it showed that, whereas for a constant-μ case the mode that becomes unstable at the critical value $\mu=3.11$ resonates in the region of reversed B_z, i.e., near the edge, the presence of a step in μ could first destabilize the innermost resonant mode. Moreover, for a μ profile such as Eq. (3), the threshold value of μ at the resonant surface, at which the instability appears, is lower than the value of 3.11 corresponding to the BFM. This result suggested that the presence of gradients in μ might be related to the destabilization of the dominant mode characterizing the so-called QSH states observed in a number of RFP experiments (see, e.g., Refs. 9 and 19–24, and references therein). Indeed, QSH states are regimes of reduced turbulence characterized by a magnetic spectrum dominated by the innermost resonant mode. Such regimes attracted particular interest not only because they provide an example of magnetic self-organization but also because they correspond to a condition of improved particle confinement.[25]

The main purpose of the present work is to complement the linear stability analysis presented in Ref. 18 by considering what happens in the nonlinear stage, after the growth of the magnetic island has reached a saturation phase. More precisely we aim at providing a relation that allows prediction of the magnitude of the island of the helically symmetric saturated state in terms of the equilibrium parameters. In particular, in connection to the above mentioned mechanism proposed for the formation of QSH states from stepped-μ equilibria, we aim at deriving a relation that allows, for given μ_T, r_{step} and toroidal magnetic flux, to see how the saturated island width in the QSH state varies with the "departure" (measured by $\Delta\mu=\mu_0-\mu_1$) from an initial Taylor state. Technically this is made possible after realizing that a perturbative technique, which has been widely applied to a number of cases in the tokamak context,[26–31] can be transferred, with appropriate modifications, to the RFP case, where the assumption of strong toroidal field is no longer valid. Besides the stepped-μ case, this technique, and consequently also the main result of this paper, can also be applied to more general force-free equilibria, provided that μ is constant over a sufficiently large region comprising the resonant surface. On the other hand, the saturation relation that we intend to derive, can of course also be applied to the classical BFM.

We also note that the analysis related to the stepped-μ equilibria can also be potentially of interest for tokamaks, given that, in a series of recent works,[32–34] such equilibria were considered in relation to the issue of internal transport barriers.

Finally we note that, in a recent paper,[35] Arcis *et al.* independently derived a relation which, in a certain limit, is similar to Eq. (59) presented in this article. However, we would like to mention that two different problems have been addressed in these two works. Whereas Arcis *et al.* consider the tearing perturbation of a nonreversed Ohmic equilibrium with a laminar equilibrium velocity field and spatially varying μ, we consider as an initial state a reversed quasiequilibrium corresponding, at least locally, to a Taylor state, in the sense that we require μ be constant around the resonant surface. We assume that this initial magnetic configuration is sustained by a turbulent velocity field which keeps the equilibrium quasistationary.

The paper is organized as follows: In Sec. II the geometry of the problem and the basic model equations are introduced. Section III is devoted to the description of the class of force-free equilibria considered in our analysis. In Sec. IV we describe the derivation of the saturation relation, whereas in Sec. V we apply the result to the stepped-μ equilibria and to the BFM. Section VI is devoted to the conclusions, while the Appendix details concerning the matching procedure are provided.

II. MODEL EQUATIONS

We approximate the toroidal chamber of a RFP with a cylinder described by cylindrical coordinates (r,θ,z), where r is the distance from the cylinder axis, θ is the azimuthal angle, and z is the distance along the cylinder axis. Henceforth, we consider lengths as normalized with respect to the radius of the cylinder. For our purposes, the RFP plasma dynamics can be adequately described by means of the resistive magnetohydrodynamics (MHD) equations consisting of the equation of motion,

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]=\mathbf{j}\times\mathbf{B}-\nabla p, \tag{4}$$

and of Ohm's law

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{j}. \tag{5}$$

In the above equations, ρ indicates the mass density, that we assume to be constant, $\mathbf{v}$ is the plasma velocity, $\mathbf{B}$ is the magnetic field, p is the plasma pressure, $\mathbf{j}=\nabla\times\mathbf{B}$ is the current density, $\mathbf{E}$ is the electric field, and η indicates the plasma resistivity.

The subsequent analysis will concern a magnetic field that in the saturated state is helically symmetric. More precisely, such field will be function only of r and of the helical coordinate $u=m\theta+kz$, with m poloidal wave number and $k=-n/R$, where n is the toroidal wave number and R is the RFP major radius. It is then convenient to introduce the following representation valid for helically symmetric divergence-free vector fields:

$$\mathbf{B}(r,u)=\nabla\chi(r,u)\times\mathbf{h}+g(r,u)\mathbf{h}, \tag{6}$$

where χ is the helical magnetic flux function. In Eq. (6) $\mathbf{h}=[r/(m^2+k^2r^2)]\nabla r\times\nabla u$ so that g denotes the amplitude of the field along the helical direction $\mathbf{h}$.

III. EQUILIBRIA

According to the classical theory developed by Taylor,[2] the mean magnetic field $\mathbf{B}$ in RFPs tends to reach a force-free equilibrium state (1), with constant μ, as a consequence of a turbulent relaxation that leaves the initial values of total toroidal magnetic flux and of total helicity unchanged. Our analysis refers to the cylindrically symmetric version of this important class of equilibria. In addition, the assumptions we require allow us to extend the results to a wider class of force-free equilibria such as, for instance, the stepped-μ equilibria, which could be relevant for the formation of quasi-single-helicity states. In particular, we require μ to be constant not necessarily over the whole cylindrical domain as for Taylor equilibria, but only on an annulus $[(r,\theta):r_1\leq r\leq r_2, 0\leq\theta<2\pi]$. In particular, the resonant surface r_s, around which we intend to expand the outer solution, has to lie between r_1 and r_2. Considering Taylor equilibria as a paradigmatic case, it is worth recalling that cylindrically symmetric solutions of Eq. (1), already anticipated in Sec. I, read

$$B_r=0,\quad B_\theta=B_0J_1(\mu r),\quad B_z=B_0J_0(\mu r). \tag{7}$$

In terms of the representation introduced in Eq. (6), these solutions read

$$\chi(r)=\frac{B_0}{\mu}[mJ_0(\mu r)-krJ_1(\mu r)],\quad g(r)=\mu\chi(r). \tag{8}$$

The cases of interest for RFPs in particular are those with $\mu>2.404$, for which the toroidal field B_z reverses within the chamber.

The force-free condition implies that at equilibrium, in the force balance equation (4), the contributions from the inertial term and from the pressure gradient are negligible compared to the electromagnetic term. On the other hand, it is known that reversed, strictly cylindrically symmetric, steady force-free states cannot satisfy a classical Ohm's law[8] such as Eq. (5). Nevertheless, small amplitude perturbations that break cylindrical symmetry could develop into a turbulent scenario that restores a relaxed force-free state. In this sense, a sort of quasisteady Ohmic equilibrium could be established on a sufficiently long time scale.[1] We express the presence of these turbulent fluctuations by means of an effective electric field $\mathbf{E}_{\text{eff}}=\mathbf{E}_0+\mathbf{E}_{\text{turb}}$, which would be able to sustain the equilibrium reversed current density parallel to the mean cylindrically symmetric equilibrium magnetic field so that $\mathbf{E}_{\text{eff}}=\eta\mathbf{j}_{\text{eq}}$ at the (quasi) equilibrium. The effective electric field is given by the sum of the field $\mathbf{E}_0$, due to the externally applied voltage, and of the "turbulent" electric field $\mathbf{E}_{\text{turb}}$ generated by the turbulent fluctuations.

The description of the equilibria that we consider in our analysis can be formalized with the help of the representation (6). Let us consider first generic helically symmetric magnetic fields and later focus on the cylindrically symmetric equilibria of interest. In the helically symmetric case, the projection along $\mathbf{h}$ of $\nabla\times\mathbf{B}=\mu\mathbf{B}$, with $\mu=\mu(r,u)$ and $\mathbf{B}=\mathbf{B}(r,u)$, in terms of the fields χ and g, yields

$$(g,\chi)=0. \tag{9}$$

In Eq. (9), (g,χ) is equal to $\partial_r g\partial_u\chi-\partial_u g\partial_r\chi$. Equation (9) thus implies $g=g(\chi)$, i.e., g is a flux function. Making use of this property, the projection of $\nabla\times\mathbf{B}=\mu\mathbf{B}$ along ∇r (or equivalently along ∇u) yields

$$L_h\chi-\left(\iota-\frac{dg}{d\chi}\right)g=0, \tag{10}$$

where L_h is an operator defined as

$$L_h=\frac{\partial^2}{\partial r^2}+\frac{1}{f}\frac{df}{dr}\frac{\partial}{\partial r}+\frac{1}{rf}\frac{\partial^2}{\partial u^2} \tag{11}$$

with $f(r)=r/(m^2+k^2r^2)$. The factor ι is defined by

$$\iota=-\frac{2mk}{m^2+k^2r^2}. \tag{12}$$

It is important to point out that for helically symmetric force-free states also $\mu=\mu(\chi)$ turns out to be a flux function and is connected to g by the relation $\mu(\chi)=dg/d\chi$. Using Ampere's law and the force-free condition (10), one also finds that the current density is related to g and χ by the relation

$$\mathbf{j}=\nabla g\times\mathbf{h}+\frac{1}{2}\frac{dg^2}{d\chi}\mathbf{h}. \tag{13}$$

Notice that the force-free equation (10) alone is not sufficient to determine both χ and g but it can be solved with respect to χ once the functional dependence of g on χ is prescribed. Now we consider the equilibrium state, where we assume that the mean magnetic field is cylindrically symmetric, i.e., a solution of Eq. (10) depending only on r. This can be represented by two functions $\chi_{\text{eq}}(r)$ and $g_{\text{eq}}[\chi_{\text{eq}}(r)]$, with the former solution of Eq. (10) and the latter a prescribed function of $\chi_{\text{eq}}(r)$. In particular, as anticipated above, for our class of equilibria the choice of g_{eq} is constrained by imposing $\mu = dg_{\rm eq}/d\chi_{\rm eq}$ constant over a region enclosing the resonant surface. We are reminded that, with regard to the equilibrium flow, we assume the existence of turbulent fluctuations that sustain the equilibrium current density. The presence of such turbulent flow is represented in the effective electric field so that at equilibrium Ohm's law reads

$$\mathbf{E}_{\rm eff} = \eta \mathbf{j}_{\rm eq}. \tag{14}$$

The current density $\mathbf{j}_{\rm eq}$ is derived from $\chi_{\rm eq}$ and $g_{\rm eq}$ with the help of Eq. (13), whereas the resistivity is assumed to be locally constant.

IV. MATCHING PROCEDURE

We consider the evolution, governed by the resistive MHD equations (4) and (5), of a helically symmetric perturbation of the equilibrium states described above. If the equilibrium is unstable, a magnetic island will form at the resonant surface. Assuming that the nonlinear time evolution of the system terminates with the reaching of a steady state, such a new equilibrium will be characterized by the presence of a magnetic island of width w, which is defined by $w^2 = 16\tilde{\chi}(r_s)/\chi''_{\rm eq}(r_s)$, where $\tilde{\chi}(r_s)$ indicates the amplitude of the perturbation at the resonant surface. We can then obtain information on the size of the saturated island by making use of the technique adopted in Ref. 31 for the "small island case." This technique allows one to obtain a relation between w and parameters characterizing the initial equilibrium state.

The adopted technique is based on a matching procedure according to which the cylindrical domain is separated in two regions: a narrow annular inner region, of thickness $\delta \sim w$, centered around the resonant surface, and an outer region complementary to the inner region. In the two regions, different approximations are made in order to obtain perturbative solutions in the small parameter w. Asymptotic expansions of such solutions are then matched in an overlap domain yielding the relation between w and the equilibrium parameters.

A. Outer region

In order to describe the nonlinear evolution of a magnetic island, nonlinear terms are essential only in a narrow layer of width $\delta \sim w$, centered around the resonant surface, that we define to be the inner region. In the outer region the linear approximation is adequate. Moreover, given that the Lundquist number for the plasmas of interest is very large, far from the resonant surface the resistive term in Ohm's law (5) is negligible. Therefore, in the outer region we assume the ideal MHD approximation be valid. In this region we consider a magnetic field, given by the sum of the equilibrium part and of a helically symmetric perturbation. Such a field can then be written in terms of

$$\chi(r,u,t) = \chi_{\rm eq}(r) + \sum_{N=-\infty}^{+\infty} \tilde{\chi}_N(r) e^{\gamma t + iNu},$$
$$g(r,u,t) = g_{\rm eq}(r) + \sum_{N=-\infty}^{+\infty} \tilde{g}_N(r) e^{\gamma t + iNu}. \tag{15}$$

As stated above we assume negligible equilibrium flow and pressure gradient in the momentum equation. Moreover, given that RFP plasmas are low-β plasmas (with β indicating here the ratio between plasma and magnetic pressure), we assume that perturbations in the pressure gradients can be neglected.

On the basis of the linear stability results of Refs. 13 and 18, the most unstable perturbations, for the equilibria of interest in this analysis, are those with $m = N = 1$ and it will be these that we will consider in the analysis of the outer region. For a given magnetic equilibrium state linearization of Eq. (10) implies that a generic eigenfunction $\tilde{\chi}(r)$ must satisfy

$$\frac{d^2\tilde{\chi}}{dr^2} + \frac{1}{f}\frac{df}{dr}\frac{d\tilde{\chi}}{dr} + \left(\mu^2 - \frac{1}{rf} + \frac{g_{\rm eq}}{\frac{d\chi_{\rm eq}}{dr}}\frac{d\mu}{dr} - \iota\mu \right)\tilde{\chi} = 0, \tag{16}$$

and that $\tilde{g}(r)$ follows from

$$\tilde{g} = \mu(r)\tilde{\chi}. \tag{17}$$

Perturbations for the electric and velocity field are assumed to be of the same form of those given in Eq. (15).

With regard to the choice of the equilibrium magnetic field, we will be restricted to the case of equilibria in which μ is constant on a sufficiently large region centered around the resonant radius so that it would be legitimate to perform expansions of the outer solution about the resonant surface considering μ as a constant. Such class of equilibria includes of course Taylor's equilibria (7) for which μ is constant over the whole domain. Notice that the term proportional to $d\mu/dr$ in Eq. (16) vanishes wherever μ is constant. In particular, for the class of equilibria under consideration, this term gets suppressed at the resonant surfaces, r_s, for which $d\chi_{\rm eq}/dr(r_s) = 0$ so that in this case the equation reduces to the one derived in Ref. 12. Note that if μ had not been constant at the resonant surface, then Eq. (16) would have had a singular term.

We consider then the expansion of Eq. (16) about r_s (for $\mu = \text{const}$),

$$\frac{d^2\tilde{\chi}}{dx^2} + \left[\frac{1}{f}\frac{df}{dx}(0) + \frac{d}{dx}\left(\frac{1}{f}\frac{df}{dx}\right)(0)x \right]\frac{d\tilde{\chi}}{dx} + \left\{ \mu^2 - \frac{1}{r_s f(0)} - \iota(0)\mu - \frac{d}{dx}\left[\frac{1}{(x+r_s)f} + \iota\mu \right](0)x \right\}\tilde{\chi} = 0, \tag{18}$$

where the variable $x = r - r_s$ has been used. In the vicinity of $x = 0$ the outer solution for $\tilde{\chi}$ could be expanded as

$$\tilde{\chi}_{\text{out}}^{\pm}(x)=\tilde{\chi}_0+\tilde{\chi}_0\frac{A\pm\Delta'}{2}x+a_2\frac{x^2}{2}+\cdots, \tag{19}$$

where the sign + and − refer to the solution for $x>0$ and $x<0$, respectively, and

$$\tilde{\chi}_0=\lim_{x\to 0}\tilde{\chi}_{\text{out}}, \tag{20}$$

$$A=\frac{1}{\tilde{\chi}_0}\left(\lim_{x\to 0^+}\frac{d\tilde{\chi}_{\text{out}}}{dx}+\lim_{x\to 0^-}\frac{d\tilde{\chi}_{\text{out}}}{dx}\right), \tag{21}$$

$$\Delta'=\frac{1}{\tilde{\chi}_0}\left(\lim_{x\to 0^+}\frac{d\tilde{\chi}_{\text{out}}}{dx}-\lim_{x\to 0^-}\frac{d\tilde{\chi}_{\text{out}}}{dx}\right). \tag{22}$$

The form (19) adopted to write the expansion of the outer solution makes explicit the dependence of such solution on the parameter Δ', which determines the stability of the equilibrium.[36] It will be shown that the matching procedure will require Δ' to be $\mathcal{O}(w)$. The expression for the constant coefficient a_2 in Eq. (19) can be obtained by inserting the expansion (19) into Eq. (18).

It is convenient to introduce at this point the matching function[31]

$$M(x)=\frac{1}{\tilde{\chi}_0}\left[\frac{d\tilde{\chi}}{dx}(x)-\frac{d\tilde{\chi}}{dx}(-x)\right]. \tag{23}$$

The relation between w and Δ' is derived by matching the asymptotic expansion of $M(x)$ in the outer and inner region. By making use of the outer solution (19), we can write the expression for the matching function in the outer region. Since the matching is performed between inner and outer expansions, both written in terms of the inner variable, $X=x/w$, it is convenient to write the outer matching function in terms of this rescaled variable. The result is

$$M_{\text{out}}(X)=\Delta'-2wX\left[\mu^2-\frac{1}{r_s f(0)}-\iota(0)\mu+A\frac{1}{f}\frac{df}{dx}(0)\right]+O(w^2). \tag{24}$$

B. Inner region

In the inner region, we rescale dependent and independent variables in the following way:

$$X=\frac{x}{w},\quad \Omega(X,u)=\frac{\chi_{\text{eq}}(0)-\chi(X,u)}{w^2},\quad G(\Omega)=\frac{g(\Omega)}{w}. \tag{25}$$

In this region, which encloses the resonant surface where the reconnection process takes place, nonlinear and resistive terms cannot be neglected. However, thanks to the assumption of having a narrow inner region, local Taylor expansion about the resonant surface is legitimate. In particular, in the inner region, we can expand the coefficients of the momentum equation about $x=0$. This reads

$$-\frac{\partial^2\Omega}{\partial X^2}-w\left\{\frac{1}{f(0)}\frac{df}{dx}(0)+w\frac{d}{dx}\left[\frac{1}{f(x)}\frac{df}{dx}\right](0)X\right\}\frac{\partial\Omega}{\partial X}-w^2\frac{1}{r_s f(0)}\frac{\partial^2\Omega}{\partial u^2}=w\left[\iota(0)+w\frac{d\iota}{dx}(0)X+w^2\frac{d^2\iota}{dx^2}(0)\frac{X^2}{2}\right]G+\frac{1}{2}\frac{dG^2}{d\Omega}. \tag{26}$$

This equation is obtained by assuming that at the lowest order the saturated state is a force-free state, then expanding Eq. (10) and rescaling the variables.

Next, we consider the projection along $\mathbf{B}$ of Ohm's law (5),

$$\mathbf{E}\cdot\mathbf{B}=\eta\mathbf{j}\cdot\mathbf{B}. \tag{27}$$

Along the $\mathbf{B}$ direction, the electric field $\mathbf{E}=\mathbf{E}_{\text{eff}}-\nabla\phi$ in the saturated state is assumed to be given by the superposition of the equilibrium effective electric field $\mathbf{E}_{\text{eff}}=\eta\mathbf{j}_{\text{eq}}$ with a perturbative part $-\nabla\phi$ where ϕ is the corresponding electrostatic potential. The validity of this assumption is based on the conjecture that, along the field lines of $\mathbf{B}$, the turbulent dynamo term $\mathbf{E}_{\text{turb}}$ is not affected by the presence of the saturated magnetic island. This conjecture could be taken as reasonable considering that our treatment refers only to small islands, whose presence should not drastically alter the properties of the random fluctuations that sustain the reversed equilibrium current on the time scale of interest. At this point we would also like to mention a possible alternative to this assumption. Indeed one could argue that the turbulent mechanism required to sustain a reversed current might operate mainly at the plasma edge, namely, where reversal occurs. The reversal region is indeed densely populated with resonant surfaces and the turbulence could be locally generated, for instance by g-modes driven by small pressure gradients in the presence of unfavorable field curvature, such as that of RFPs. On the other hand, around the resonant surface of the dominant mode of a QSH state, which is located in the plasma core, one could imagine that the contribution of the term $\mathbf{E}_{\text{turb}}$ be modest and that locally a mostly laminar equilibrium flow exist. In this case the treatment of the inner region would follow the one carried out by Arcis *et al.*[35] and would lead to the presence of an additional term in the final saturation relation. However, as will be shown in Sec. V, for typical values of interest for QSH states, such an additional term would represent a small correction. In the following, therefore, we follow the conjecture of having a turbulent dynamo term in the inner region and comments on the different results obtained by following the two approaches will be given in Sec. V.

Making use of the representation of $\mathbf{j}$ and $\mathbf{B}$ in terms of the fields Ω and G, one can rewrite Eq. (27) as

$$\eta\mathbf{j}_{\text{eq}}\cdot\mathbf{B}-\mathbf{B}\cdot\nabla\phi=-w\eta\frac{dG}{d\Omega}(|\nabla\Omega|^2+G^2)h^2. \tag{28}$$

The term $\mathbf{B}\cdot\nabla\phi$ in Eq. (28) can be annihilated under the action of the flux surface average operator. The flux surface average of a generic function $\mathcal{F}(r,u)$ with respect to a flux surface Ω is defined as

$$\langle \mathcal{F} \rangle_\Omega = \left(\oint \left| \frac{\partial \Omega}{\partial X} \right|^{-1} \mathcal{F} du \right) \Big/ \left(\oint \left| \frac{\partial \Omega}{\partial X} \right|^{-1} du \right), \tag{29}$$

where the integration is taken over $(0,2\pi)$ for open flux surfaces and between values of u, where $\partial\Omega/\partial X$ vanishes for closed magnetic surfaces enclosed within the island. Applying this operator to Eq. (28) and making use of the representation $\mathbf{j}_{\rm eq}=\mu(\nabla\chi_{\rm eq}\times\mathbf{h}+g_{\rm eq}\mathbf{h})$ one obtains

$$\begin{aligned}
&\frac{\mu}{G}\left\langle \chi_{\rm eq}' \frac{\partial\Omega}{\partial X} h^2 \right\rangle_\Omega - \mu\langle g_{\rm eq}h^2\rangle_\Omega \\
&\quad = \frac{1}{G}\frac{dG}{d\Omega}\left\langle \left(\frac{\partial\Omega}{\partial X}\right)^2 h^2 \right\rangle_\Omega \\
&\quad\quad + \frac{w^2}{G}\frac{dG}{d\Omega}\left\langle \left(\frac{\partial\Omega}{\partial u}\right)^2 \left(\frac{m^2}{w^2X^2+r_s^2}+k^2\right) h^2 \right\rangle_\Omega \\
&\quad\quad + \frac{1}{2}\frac{dG^2}{d\Omega}\langle h^2\rangle_\Omega,
\end{aligned} \tag{30}$$

where we also made use of the fact that flux functions such as $G=G(\Omega)$ are invariant under the action of the flux surface average. Given that we are considering the inner region, we can perform a Taylor expansion of $\chi_{\rm eq}''$, $g_{\rm eq}$, h^2, and $m^2/(w^2X^2+r_s^2)+k^2$, which are given functions of X, about the resonant radius, similar to what was done for the momentum equation (26). We obtain

$$\begin{aligned}
&w\frac{\mu}{G}\left\langle \chi_{\rm eq}''(0) X \frac{\partial\Omega}{\partial X} h_0 \right\rangle_\Omega \\
&- \mu\left\langle \left(g_{\rm eq}(0)+w^2 g_{\rm eq}''(0)\frac{X^2}{2}\right)(h_0+wh_1X+w^2h_2X^2) \right\rangle_\Omega \\
&= \frac{1}{G}\frac{dG}{d\Omega}\left\langle \left(\frac{\partial\Omega}{\partial X}\right)^2 h_0 \right\rangle_\Omega \\
&\quad + \frac{w^2}{G}\frac{dG}{d\Omega}\left\langle \left(\frac{\partial\Omega}{\partial u}\right)^2 \left(\frac{m^2}{r_s^2}+k^2\right) h_0 \right\rangle_\Omega \\
&\quad + \frac{1}{2}\frac{dG^2}{d\Omega}\langle (h_0+wh_1X+w^2h_2X^2)\rangle_\Omega,
\end{aligned} \tag{31}$$

where h_0, h_1, and h_2 are coefficients in the expansion of h^2. Then, Eqs. (26) and (31) form a two-equation system to be solved in the inner region with respect to the unknown variables Ω and G. We recall that the solutions must also satisfy the condition (9) which, in terms of the inner variables, reads

$$(G,\Omega)=0. \tag{32}$$

We proceed by solving the system of equations perturbatively. Motivated by the form of the expansion of the outer solutions in terms of the inner variables, we choose the following expansions for Ω and G in the inner region, adopting w as an expansion parameter,

$$\begin{aligned}
\Omega(X,u) &= \Omega_0 + w\Omega_1 + w^2\Omega_2 + \mathcal{O}(w^3), \\
G(X,u) &= \frac{G_{-1}}{w} + G_0 + wG_1 + w^2G_2 + \mathcal{O}(w^3).
\end{aligned} \tag{33}$$

From Eq. (32) at the lowest order, we obtain

$$\frac{1}{w}(G_{-1},\Omega_0)=0, \tag{34}$$

which implies $G_{-1}=G_{-1}(\Omega_0)$. We can make use of this result and consider Eq. (31) at its lowest order. This yields

$$\frac{1}{w^2}G_{-1}(\Omega_0)\frac{dG_{-1}(\Omega_0)}{d\Omega_0}h_0=0, \tag{35}$$

which implies

$$G_{-1}=c_{-1} \tag{36}$$

with constant c_{-1}. The arbitrariness in this constant can be removed by matching the lowest order term, i.e., the term of order $1/w$, of the expansion of the inner solution for G as $X\to\infty$, with the corresponding term of the outer solution as $X\to 0$. The expansions about $X=0$ of the outer solutions for χ and g, written in terms of the inner variables, are

$$\begin{aligned}
\Omega_{\rm out} &= -\chi_{\rm eq}''(0)\frac{X^2}{2} - \frac{\chi_{\rm eq}''(0)}{16}\cos u \\
&\quad - w\left[\chi_{\rm eq}'''(0)\frac{X^3}{6} + \chi_{\rm eq}''(0)\frac{A\pm\Delta'}{2}\frac{X}{16}\cos u\right] + \mathcal{O}(w^2), \\
G_{\rm out} &= \frac{g_{\rm eq}(0)}{w} + w\left[g_{\rm eq}''(0)\frac{X^2}{2} + \mu\tilde{\chi}_0\cos u\right] + \mathcal{O}(w^2) \\
&= \mu\frac{\chi_{\rm eq}(0)}{w} + w\left[\mu\chi_{\rm eq}''(0)\frac{X^2}{2} + \mu\frac{\chi_{\rm eq}''(0)}{16}\cos u\right] + \mathcal{O}(w^2).
\end{aligned} \tag{37}$$

By considering the term of order $1/w$ in $G_{\rm out}$ one then obtains

$$G_{-1}=g_{\rm eq}(0)=\mu\chi_{\rm eq}(0). \tag{38}$$

Notice that, in order to derive Eq. (37), we used the relation $\tilde{\chi}_0=w^2\chi_{\rm eq}''(0)/16$, which comes from conservation of magnetic flux on an island separatrix, and considered only the fundamental harmonic in the perturbation $\Sigma_{N=-\infty}^{+\infty}\tilde{\chi}_N(r)\exp(iNu)$.

From considering Eq. (32) at the next order, and taking into account that G_{-1} is a constant, one obtains

$$(G_0,\Omega_0)=0, \tag{39}$$

therefore $G_0=G_0(\Omega_0)$. From Eq. (31) at order $1/w$ one obtains

$$G_{-1}\frac{dG_0}{d\Omega_0}=0, \tag{40}$$

so that G_0 is a constant c_0. The matching with $G_{\rm out}$ tells us that, in particular,

$$G_0 = 0. \tag{41}$$

Iterating the procedure at the next order we see that Eq. (32), combined with the results obtained at lower orders, gives

$$(G_1, \Omega_0) = 0, \tag{42}$$

therefore $G_1 = G_1(\Omega_0)$. Evaluating Eq. (31) at $\mathcal{O}(1)$ gives

$$G_{-1}\frac{dG_1}{d\Omega_0} = -\mu g_{\mathrm{eq}}(0). \tag{43}$$

This relation, together with Eq. (38), yields

$$G_1 = -\mu\Omega_0 + c_1, \tag{44}$$

with c_1 as an arbitrary constant. The matching with the outer solution requires $c_1=0$ so that

$$G_1 = -\mu\Omega_0. \tag{45}$$

The knowledge of G_1, G_0, and G_1 can now be used to evaluate Eq. (26) at $\mathcal{O}(1)$. This gives

$$-\frac{\partial^2\Omega_0}{\partial X^2} = \mu[\iota(0) - g_{\mathrm{eq}}(0)]. \tag{46}$$

Upon integrating Eq. (46) and making use of the local equilibrium relation $\chi_{\mathrm{eq}}(0)'' = \mu[\iota(0) - g_{\mathrm{eq}}(0)]$, one obtains

$$\Omega_0 = -\chi''_{\mathrm{eq}}(0)\frac{X^2}{2} + A(u)X + B(u) \tag{47}$$

with $A(u)$ and $B(u)$ as arbitrary functions. The matching with the corresponding term in Ω_{out} yields

$$\Omega_0 = -\chi''_{\mathrm{eq}}(0)\frac{X^2}{2} - \frac{\chi''_{\mathrm{eq}}(0)\cos u}{16}. \tag{48}$$

The next iteration, starting again from Eq. (32), yields

$$-(\mu\Omega_0, \Omega_1) + (G_2, \Omega_0) = 0, \tag{49}$$

from which one obtains that $G_2 = \mathcal{H}(\Omega_0) - \mu\Omega_1$, with $\mathcal{H}(\Omega_0)$ some function of Ω_0. Ohm's law (31) at order w then gives

$$G_{-1}\frac{dG_2}{d\Omega} = 0. \tag{50}$$

Using this result we can consider Eq. (26) at order w, which reads

$$\begin{aligned}\frac{\partial^2\Omega_1}{\partial X^2} &= -\frac{1}{f(0)}\frac{df}{dx}(0)\frac{\partial\Omega_0}{\partial X} - \frac{d\iota}{dx}(0)G_{-1}X \\ &= \left[\frac{1}{f(0)}\frac{df}{dx}(0)\chi''_{\mathrm{eq}}(0) - \frac{d\iota}{dx}(0)g_{\mathrm{eq}}(0)\right]X.\end{aligned} \tag{51}$$

This equation can be easily integrated to give

$$\begin{aligned}\Omega_1 &= \left[\frac{1}{f(0)}\frac{df}{dx}(0)\chi''_{\mathrm{eq}}(0) - \frac{d\iota}{dx}(0)g_{\mathrm{eq}}(0)\right]\frac{X^3}{6} \\ &\quad + C(u)X + D(u)\end{aligned} \tag{52}$$

with $C(u)$ and $D(u)$ as arbitrary functions. Upon matching Eq. (52) with the corresponding terms, i.e., those of order $\mathcal{O}(w)$, in the expanded outer solution (37) one obtains immediately $D(u)=0$. Differentiating the left-hand side of Eq. (10) and expanding its coefficients about the resonant surface, one obtains a relation showing that the coefficients of X^3 in Eqs. (52) and (37) also get automatically matched. On the other hand, determining $C(u)$, requires anticipating the result that Δ' is $\mathcal{O}(w)$. This yields $C(u) = (A/16)\chi''_{\mathrm{eq}}(0)\cos u$. Similarly to the previously investigated tokamak case[31] such dependence can be obtained with the help of the matching function, as it will be shown in Sec. IV C and in the Appendix.

To summarize, the solutions we found for Ω and G at the different orders, are given by

$$\begin{aligned}\Omega_0 &= -\chi''_{\mathrm{eq}}(0)\frac{X^2}{2} - \frac{\chi''_{\mathrm{eq}}(0)\cos u}{16}, \\ \Omega_1 &= \left[\frac{1}{f(0)}\frac{df}{dx}(0)\chi''_{\mathrm{eq}}(0) - \frac{d\iota}{dx}(0)g_{\mathrm{eq}}(0)\right]\frac{X^3}{6} \\ &\quad + \frac{A}{16}\chi''_{\mathrm{eq}}(0)X\cos u,\end{aligned} \tag{53}$$

$$G_{-1} = \mu\chi_{\mathrm{eq}}(0), \quad G_0 = 0, \quad G_1 = \mu\chi''_{\mathrm{eq}}(0)\frac{X^2}{2} + \mu\frac{\chi''_{\mathrm{eq}}(0)\cos u}{16}.$$

Finally we provide the expression for the quantity $G(dG/d\Omega)$ at order w^2, which is needed for constructing the matching function in the inner region at the required order. The condition $[G(dG/d\Omega), \Omega]=0$, combined with the lower order results, tells us that $G_{-1}(dG_3/d\Omega)$ must be a function of Ω_0. Ohm's law (31) at order w^2 then gives

$$G_{-1}\frac{dG_3}{d\Omega} + G_1\frac{dG_1}{d\Omega} = -\frac{\mu}{2}g''_{\mathrm{eq}}(0)\langle X^2\rangle_{\Omega_0}, \tag{54}$$

which provides the required quantity.

C. Saturation relation

The form of the matching function in the outer region has been given in Eq. (24). In the inner region it is convenient to express it in the following integral form:

$$M_{\mathrm{in}}(x) = \frac{1}{\pi\tilde{\chi}_0}\int_{-x}^{x} dx' \int_{-\pi}^{\pi} du \frac{\partial^2\chi}{\partial x'^2}\cos u. \tag{55}$$

Notice that if one considers $\chi(x,u) = \chi_{\mathrm{eq}}(x) + \Sigma_{N=-\infty}^{+\infty}\tilde{\chi}_N \exp(iNu)$, then $M_{\mathrm{in}}(x)$ indeed corresponds to Eq. (23). Expressing Eq. (55) in terms of the inner variables and making use of the inner equation (26) we can write

$$\begin{aligned}M_{\mathrm{in}}(X) = -\frac{16}{w\pi\chi''_{\mathrm{eq}}(0)}\int_{-X}^{X} dX' \int_{-\pi}^{\pi} du\Bigg[&-w\frac{1}{f(0)}\frac{df}{dx}(0)\frac{\partial\Omega}{\partial X'} \\ &- w^2\frac{d}{dx}\left(\frac{1}{f}\frac{df}{dx}\right)(0)X'\frac{\partial\Omega}{\partial X'} - w^2\frac{1}{r_s f(0)}\frac{\partial^2\Omega}{\partial u^2} - w\iota(0)G \\ &- w\frac{d\iota}{dx}(0)X'G - w^2\frac{d^2\iota}{dx^2}(0)\frac{X'^2}{2}G - \frac{1}{2}\frac{dG^2}{d\Omega}\Bigg].\end{aligned} \tag{56}$$

One can then insert into Eq. (56), the perturbative solutions for Ω and G given in Eq. (53). The terms multiplying the same powers of w can then be collected and matched onto the corresponding terms of the asymptotic expansion of the outer matching function (24). Whereas the contributions to $M_{\mathrm{in}}(X)$ to order $\mathcal{O}(w^{-1})$ and $\mathcal{O}(1)$ vanish identically, the contribution of order $\mathcal{O}(w)$ (see Appendix for details) reads

$$M_{1\mathrm{in}}(X) = 2wX\left[\iota(0)\mu + \frac{1}{r_s f(0)} - \mu^2 + \frac{16}{\pi\chi''_{\mathrm{eq}}(0)}\frac{1}{f}\frac{df}{dx}(0)\int_{-\pi}^{\pi} du C(u)\cos u\right] + w\frac{\mu^2}{\pi}\int_{-X}^{X} dX'\int_{-\pi}^{\pi} du\langle\cos u\rangle_{\Omega_0}\cos u. \quad (57)$$

The matching with M_{out} then yields

$$\Delta' = w\frac{\mu^2}{\pi}\int_{-X}^{X} dX'\int_{-\pi}^{\pi} du\langle\cos u\rangle_{\Omega_0}\cos u. \quad (58)$$

The evaluation of the integral in Eq. (58) requires the calculation of the flux surface average referred to Ω_0. Details related to this calculation are given in Ref. 31. The final result reads

$$w = \frac{\Delta'}{0.41\mu^2}. \quad (59)$$

The quantity Δ' turns out then to be of the order w. Equation (59) expresses the sought relation providing the dependence of the width of the saturated island w on the stability parameter Δ', which refers to the initial equilibrium. A formally similar relation was found for the tokamak case in Ref. 26. The relation is remarkably simple and shows that the saturated island size of a perturbed cylindrically symmetric locally Taylor state, to the lowest order, depends on the initial equilibrium only through the values of Δ' and of the pinch parameter μ. We recall in particular that, due to the vanishing of the helical component of the equilibrium current density at the resonant radius, the island width does not depend on the global asymmetry parameter A.

The relation derived in Ref. 35 by Arcis *et al.* differs from Eq. (59) due to the presence of terms proportional to a gradient of $\mu(r)$ at the resonant surface and also due to an additional term, proportional to the global asymmetry parameter A, which originates from a different choice of the equilibrium configuration. Indeed, as anticipated in Sec. IV B, Arcis *et al.* in their work, consider a laminar equilibrium velocity field, in the presence of a nonreversed magnetic field, whereas we start from a reversed quasiequilibrium configuration and assume that a background effective electric field exists, in order to sustain the reversed configuration. It is also worth to notice that the saturation relation (59) is the same one that one would obtain with our choice of the equilibrium state but imposing incompressibility of the flow. As pointed out by Arcis *et al.* the incompressibility condition is incompatible with the existence of an Ohmic force-free equilibrium with a helical symmetry[8] therefore in principle imposing the velocity field to be divergence-free is not correct in this context. However it turns out that, with our choice for the equilibrium, the modifications due to compressibility would appear only at orders higher than the ones we considered.

In the next section we will discuss the application of the formula (59) to the stepped-μ equilibria and also consider one example of application to the BFM with $\mu > 3.11$.

V. APPLICATIONS OF THE SATURATION RELATION

As anticipated above, the formula (59) can, in principle, be applied to cylindrically symmetric force-free equilibria provided that μ is constant around the resonant surface. Equilibria satisfying this condition include, for instance, those in which μ is a step function of the radius. As mentioned in Sec. I such equilibria might be relevant, for instance, for explaining the formation of cyclic QSH states observed in RFPs.[37] Indeed, equilibria of such type can be linearly tearing unstable with respect to the dominant mode observed during QSH states while being stable with respect to the other modes. For the stepped-μ equilibria under consideration, $\mu = \mu(r)$ is defined by

$$\mu = \begin{cases} \mu_0 & \text{if } 0 \leq r \leq r_{\mathrm{step}}, \\ \mu_1 & \text{if } r_{\mathrm{step}} < r \leq 1, \end{cases} \quad (60)$$

with μ_0, μ_1 constants and r_{step}, such that $0 < r_{\mathrm{step}} < 1$, indicating the radius where the step is located. Force-free equations are solved separately for $0 \leq r \leq r_{\mathrm{step}}$ and $r_{\mathrm{step}} < r \leq 1$ and the resulting fields are matched by imposing continuity at r_{step}. Provided that the distance $|r_{\mathrm{step}} - r_s|$ is sufficiently large so that we can consider Eq. (16) with constant μ, our local analysis for the island width still holds and the relation (59) remains valid. A remarkable property of such force-free states is that for them it is possible to obtain an analytical expression for the parameter Δ'. In the case where $0 < r_s < r_{\mathrm{step}}$ (which is the most relevant one for dominant modes in QSH states) this expression reads

$$r_s\Delta'(r_s;\mu_0,\mu_1,r_{\mathrm{step}}) = -\frac{2}{\pi}\lambda_2\frac{[\mu_0^2 - k^2(r_s)][1 + k^2(r_s)r_s^2]}{G_{0J}(G_{0J} - \lambda_2 G_{0Y})}, \quad (61)$$

where

$$G_{0J} = k\sqrt{\mu_0^2 - k^2}r_s J_0(\sqrt{\mu_0^2 - k^2}r_s) + (\mu_0 - k)J_1(\sqrt{\mu_0^2 - k^2}r_s),$$

$$G_{0Y} = k\sqrt{\mu_0^2 - k^2}r_s Y_0(\sqrt{\mu_0^2 - k^2}r_s) + (\mu_0 - k)Y_1(\sqrt{\mu_0^2 - k^2}r_s),$$

and λ_2 is a constant which is fixed by a boundary condition at r_{step}. The expression (61) is derived in Ref. 18 and allows us to calculate the value of Δ' as a function of the parameters m, μ_0, μ_1, r_s, r_{step}. Combining this formula with Eq. (59), one can obtain a fully analytical expression for the island width, w, of the form

$$w(m,\mu_0,\mu_1,r_{\mathrm{step}},r_s) = \frac{\Delta'(m,\mu_1,\mu_0,r_{\mathrm{step}},r_s)}{0.41\mu_0^2}. \quad (62)$$

The relation (62) thus makes it possible to obtain the size of the saturated island directly in terms of the parameters characterizing the equilibrium state.

As mentioned in Ref. 18 and anticipated in Sec. I, the formation of such stepped-μ equilibria in RFPs might model small departures from an initial Taylor state characterized by $\mu = \mu_T$. Such a departure is imagined as a time evolution through a sequence of stepped-μ equilibria characterized by

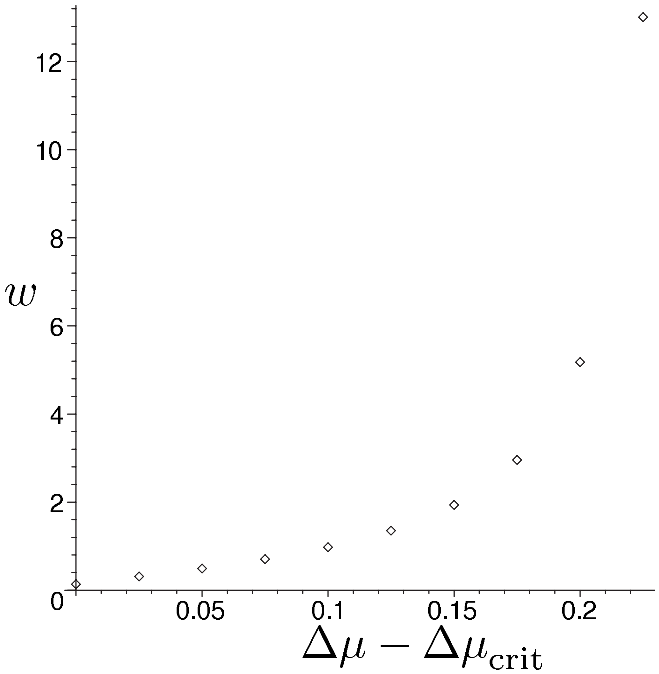

FIG. 1. Plot showing the width w of the saturated island, generated by an $(m=1,n=7)$ tearing mode, for different values of the parameter $\Delta\mu-\Delta\mu_{crit}$. Each value of $\Delta\mu-\Delta\mu_{crit}$ refers to a member of a family of force-free equilibria with a stepped-μ profile, characterized by a step placed at $r_{step}=0.296$ in a RFP with aspect ratio $R=4.34$. Such family originates by an initial Taylor equilibrium with $\mu_T=2.93$.

increasing values of $\Delta\mu=\mu_0-\mu_1$, under the constraints of toroidal magnetic flux conservation, toroidal current conservation and assuming r_{step} as fixed. In this context the parameter $\Delta\mu$ can then be loosely interpreted as a timelike variable. If one considers a starting Taylor equilibrium with $\mu_T<3.11$, which corresponds to a condition of stability to $m=1$ tearing modes,[13] the evolution through a sequence of stepped-μ equilibria with increasing $\Delta\mu$ obeys the following pattern:[17] the system departs from the initial Taylor state with $\Delta\mu=0$ and, as long as $\Delta\mu$ is below a critical value $\Delta\mu_{crit}$, the equilibrium remains stable. However, the value of Δ' increases as the value of $\Delta\mu$ gets larger, though. When the height of the step exceeds the threshold value $\Delta\mu_{crit}$ (which corresponds to $\Delta'=0$) the equilibrium becomes unstable to one tearing mode with $m=1$ at the innermost resonant surface. As $\Delta\mu-\Delta\mu_{crit}$ becomes larger and larger the value of Δ' for this mode increases, first approximately linearly, and subsequently in a much more rapid way until a threshold for ideal instability, which corresponds to infinite Δ', is possibly encountered. Such dependence of Δ' on the parameter $\Delta\mu$ of course reflects also in the evolution of the island width through the sequence of equilibria.

In order to consider a concrete case, the example shown in Fig. 1 refers in particular to the formation of QSH states in the RFX device[9] where $(m=1,n=7)$ is the dominant mode and the aspect ratio is given by $R=4.34$. Figure 1 shows the values of the island width for equilibria with different values of the parameter $\Delta\mu-\Delta\mu_{crit}$. It can be seen that the island width in the saturated state grows almost linearly for small values of $\Delta\mu-\Delta\mu_{crit}$ but then tends to explode as the step gets larger. In order to show that the dependence of w on the height of the step can become nonlinear the plot is shown for a rather wide range of values for $\Delta\mu-\Delta\mu_{crit}$. However, it is necessary to remark that the relation (62) is valid on a region where μ is constant. This means that the prediction given by Eq. (62) fails when the half island width exceeds the distance $|r_{step}-r_s|$.

As anticipated in Sec. IV B, if one had chosen a laminar velocity field at equilibrium, then, following Arcis *et al.*, an additional term, proportional to the global asymmetry parameter A, defined in Eq. (21), would have appeared in the saturation relation. The latter would have then read

$$w(m,\mu_0,\mu_1,r_{step},r_s)=\frac{\Delta'(m,\mu_1,\mu_0,r_{step},r_s)}{0.41[\mu_0^2+\mu_0(nr_s/mR)(A/2)]}. \qquad (63)$$

However, for the sake of applying this relation to QSH states, we observe that, if we refer, as typical values, to the parameters of Fig. 1 and consider a step $\Delta\mu=0.1$ (corresponding to $\Delta'=0.12$ and $A=2.34$), then the relation (63) would predict an island width which would be only 15% less than that predicted by Eq. (62). Therefore, we believe that, with regard to the application of this theory to QSH states, the presence of an equilibrium flow would lead to a small correction to the result we derive assuming the existence of a turbulent dynamo term.

The formula (62) can of course also cover the case of BFM equilibria when $\mu_0=\mu_1=\mu$, in which case the expression for Δ' reduces to the one obtained by Gibson and Whiteman.[13] As an example, let us consider a RFP with aspect ratio $R=3.87$ in a BFM state characterized by $\mu=3.114$. For such a state the mode $(m=1,n=-5)$ is tearing unstable and resonates at $r_s=0.969$. The corresponding value of Δ' is equal to 0.104. The formula (62) predicts then that the saturated island will have a width equal to $w=0.026$.

VI. CONCLUSIONS

In this paper the nonlinear saturation of a tearing mode perturbing force-free equilibria relevant for RFPs has been investigated. The analysis applies to equilibria for which the parameter μ is constant over a sufficiently large region including the resonant radius. The problem of nonlinear saturation of tearing modes is tackled by making use of a perturbative technique previously adopted for the tokamak case. In fact, interestingly, it was possible to show that such technique can be applied also without the requirement of a strong guide field, provided that pressure gradients are negligible. The saturation relation obtained by means of this technique consists of a simple relation according to which the saturated island width is proportional to the instability parameter Δ' and inversely proportional to the square of the value of μ at the resonant surface. In the context of a recent scenario proposed for explaining the occurrence of cyclic QSH, the saturation relation was applied to stepped-μ equilibria. In this case the island width has been shown to grow approximately linearly for small values of the step height $\Delta\mu-\Delta\mu_{crit}$. As the value of $\Delta\mu-\Delta\mu_{crit}$ increases the response of the island width to the increase in the step height becomes nonlinear showing a tendency toward an explosive growth for a finite value of $\Delta\mu-\Delta\mu_{crit}$. In practice, however, the theory holds only as long as the island separatrix does not reach r_{step}. The interaction of the island with the region of strong current density variation might be possibly related to the mechanism that causes the abrupt decay of the dominant mode during cyclic QSH states. Modelling this phase of the island dynamics will be subject of future investigations. Finally, the derived saturation relation is applied with one example to the BFM in the case of a realistic aspect ratio.

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APPENDIX: INNER MATCHING FUNCTION

The purpose of this appendix is to provide details about the derivation of the expression of $M_{in}(X)$ up to terms of the order $\mathcal{O}(w^2)$.

The integral form for $M_{in}(X)$, as given by Eq. (56), reads

$$M_{in}(X) = -\frac{16}{w\pi\chi''_{eq}(0)}\int_{-X}^{X} dX' \int_{-\pi}^{\pi} du\left[-w\frac{1}{f(0)}\frac{df}{dx}(0)\frac{\partial\Omega}{\partial X'} - w^2\frac{d}{dx}\left(\frac{1}{f}\frac{df}{dx}\right)(0)X'\frac{\partial\Omega}{\partial X'} - w^2\frac{1}{r_s f(0)}\frac{\partial^2\Omega}{\partial u^2} - w\iota(0)G - w\frac{d\iota}{dx}(0)X'G - w^2\frac{d^2\iota}{dx^2}(0)\frac{X'^2}{2}G - \frac{1}{2}\frac{dG^2}{d\Omega}\right]. \quad (A1)$$

After inserting the expressions for the inner solutions for Ω and G given in Eq. (53), one obtains that the lowest order contribution is of order $\mathcal{O}(w^{-1})$ and reads

$$M_{-1in}(X) = -\frac{16}{\pi w\chi''_{eq}(0)}\int_{-X}^{X} dX' \int_{-\pi}^{\pi} du \times\left[-\iota(0)G_{-1} - \frac{1}{2}\frac{dG_{-1}G_1}{d\Omega}\right]\cos u = -\frac{16}{\pi w\chi''_{eq}(0)}\int_{-X}^{X} dX' \int_{-\pi}^{\pi} du \times[-\iota(0)\mu\chi''_{eq}(0) + \mu g_{eq}(0)]\cos u = 0. \quad (A2)$$

At the next order one has

$$M_{0in}(X) = -\frac{16}{\pi\chi''_{eq}(0)}\int_{-X}^{X} dX' \int_{-\pi}^{\pi} du \times\left[-\frac{d\iota}{dx}(0)X'G_{-1} - \frac{1}{f}\frac{df}{dx}(0)\frac{\partial\Omega_0}{\partial X'}\right]\cos u = -\frac{16}{\pi\chi''_{eq}(0)}\int_{-X}^{X} dX' \int_{-\pi}^{\pi} du \times\left[-\frac{d\iota}{dx}(0)X'G_{-1} - \frac{1}{f}\frac{df}{dx}(0)\frac{\partial\Omega_0}{\partial X'}\right]\cos u = -\frac{16}{\pi\chi''_{eq}(0)}\int_{-X}^{X} dX' \int_{-\pi}^{\pi} du \times\left[-\frac{d\iota}{dx}(0)g_{eq}(0)X' + \frac{1}{f}\frac{df}{dx}(0)\chi''_{eq}(0)X'\right]\cos u = 0. \quad (A3)$$

The lowest order finite contribution to $M_{in}(X)$ comes then from terms of order $\mathcal{O}(w)$ which yield

$$M_{1in}(X) = -\frac{16w}{\pi\chi''_{eq}(0)}\int_{-X}^{X} dX' \int_{-\pi}^{\pi} du\left[-\frac{1}{2}\frac{d^2\iota}{dx^2}(0)X'^2G_{-1} - \iota(0)G_1 + \mu\frac{g''_{eq}(0)}{2}\langle X'^2\rangle_{\Omega_0} - \frac{d}{dX'}\left(\frac{1}{f}\frac{df}{dx}\right)(0)X'\frac{\partial\Omega_0}{\partial X'} - \frac{1}{f}\frac{df}{dx}(0)\frac{\partial\Omega_1}{\partial X'} - \frac{1}{r_s+f(0)}\frac{\partial^2\Omega_0}{\partial u^2}\right]\cos u = 2wX\left[\iota(0)\mu + \frac{1}{r_s f(0)}\right] - \frac{8w\mu^2}{\pi}\int_{-X}^{X} dX' \int_{-\pi}^{\pi} du\left\langle -\frac{2}{\chi''_{eq}(0)}\Omega_0 - \frac{\cos u}{8}\right\rangle_{\Omega_0}\cos u + \frac{16}{\pi\chi''_{eq}(0)}\frac{1}{f}\frac{df}{dx}(0)2wX\int_{-\pi}^{\pi} du C(u)\cos u = 2wX\left[\iota(0)\mu + \frac{1}{r_s f(0)} - \mu^2 + \frac{16}{\pi\chi''_{eq}(0)}\frac{1}{f}\frac{df}{dx}(0)\int_{-\pi}^{\pi} du C(u)\cos u\right] + w\frac{\mu^2}{\pi}\int_{-X}^{X} dX' \int_{-\pi}^{\pi} du\langle\cos u\rangle_{\Omega_0}\cos u, \quad (A4)$$

and this corresponds to the expression given in Eq. (57).

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