Techniques for studying the separatrix of tokamak plasmas

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This paper describes (physically and mathematically) how the plasma-vacuum boundary of a tokamak plasma equilibrium can be perturbed to form a separatrix with an X-point, while having an otherwise negligible affect on the plasma equilibrium. A deliberate consequence of the technique is that the radial and poloidal extent of the perturbed region may be arbitrarily localized. This has useful theoretical and physical consequences, namely (1) it is possible to take any plasma equilibrium and modify the outermost flux surface to form a separatrix with one or more additional x-points in a rigorous way, (2) subsequent studies will be able to separate the effects of shaping from those due to topological changes associated with a separatrix, for example, a circular cross-section plasma may be modified to form a separatrix that is circular everywhere except for an arbitrarily localized region that is perturbed to form an x-point, (3) because the perturbation is arbitrarily localized, there is the possibility for modifying the stability of the edge, without affecting the bulk plasma properties (or stability). Therefore the ideas presented here provide analytical and conceptual tools to study how a separatrix can affect plasma stability, and a potential experimental technique to study the stability of the plasma edge. The paper also investigates how the magnetic shear and the Mercier coefficient behave as a separatrix is approached, showing that for a nonzero toroidal current at the x-point, the Mercier coefficient always asymptotes to zero. [DOI: 10.1063/1.3046070]

I. INTRODUCTION

ITER[1] and proposed tokamak power plants are designed with a divertor onto which the outermost plasma is directed. This is accomplished by additional current carrying coils outside the plasma that shape the outermost flux surface at the plasma-vacuum boundary into a separatrix, with one or more x-points in the poloidal cross section at which the poloidal magnetic field is zero. It is widely believed that the topological change from nested flux surfaces to flux surfaces with a separatrix and one or more x-points at the edge will affect the plasma's stability.[2–7] It is also known that shaping of the plasma affects stability, making it unclear whether the strong plasma shaping usually associated with forming a separatrix is the dominant stabilizing mechanism, or if topological effects such as the safety factor q becoming infinite at the separatrix are. Additionally, stability studies of strongly shaped plasmas generally require numerical calculations, something that can obscure the physical principles that determine the observed results.

Previous authors[2,3,8] have proposed model tokamak equilibria with x-points. However these are not easy to study because they have strongly shaped plasmas, unlike the more extensively studied circular cross-section equilibria. Here a method that allows the formation of highly localized x-points is described, along with analytic solutions that describe the resulting equilibrium. Within the model the x-point region can be arbitrarily localized, something that will allow future studies to distinguish between the affects from a change in magnetic topology and the affects from shaping the plasma. The x-point's extreme localization allows the method and solutions presented here to be applied to equilibria with a circular cross section, and hence to determine how a change in plasma topology affects stability in these widely studied systems. The possibility of modifying the separatrix shape (potentially to include an additional x-point), with an otherwise negligible perturbation to the plasma equilibrium, offers a potential technique for novel plasma studies, such as using ballooning modes to reduce the edge pressure gradient. Finally we consider how the safety factor q and the Mercier coefficient D_M behave near a separatrix.

II. THE MODEL

The equilibrium is formed by taking an equilibrium without an x-point (e.g., a circular cross-section equilibrium[9]), and then adding two external coils carrying equal but opposite toroidal currents. Positioning the coils close to the plasma's edge will cause an x-point to form, but the presence of the second current carrying coil (with equal but opposite current), ensures that the perturbation to the magnetic field can be very localized. The key points are:

(1) Perturbations ψ_1 to the original magnetic flux ψ_0 are small (with $\psi_1 \ll \psi_0$), but the gradients have $\nabla\psi_1 \sim \nabla\psi_0$ near the x-point.
(2) The arbitrarily strong localization of the perturbation to the plasma, allows the region to be made small enough that both ψ_0 and the major radius R are approximately constant. So for the outer flux surfaces that are affected by the x-point, $\psi_0(r) \simeq \psi_e$ its equilibrium value at the plasma-vacuum surface, and is effectively constant.

To describe this mathematically, we start with a conven-

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tional equilibrium satisfying the Grad–Shafranov equation and some specified boundary conditions,[9]

$$\nabla \cdot \left\{ \frac{\nabla[\psi_0(\vec{x})]}{R^2} \right\} = - F[\psi_0(\vec{x}),R], \tag{1}$$

where[9] $F[\psi_0(\vec{x}),R]=p'[\psi_0(\vec{x})]+I[\psi_0(\vec{x})]I'[\psi_0(\vec{x})]/R^2$, with p the plasma pressure, $I=RB_\phi$, R the major radius, B_ϕ the toroidal magnetic field, and both p and I are functions of ψ with primes denoting derivatives with respect to ψ. Two external coils carrying currents $\pm j_1$ are then added at $\vec{x}_1$ and $\vec{x}_2$, so that the Grad–Shafranov equation is modified to

$$\nabla \cdot \left[\frac{\nabla \psi(\vec{x})}{R^2} \right] = - F[\psi(\vec{x}),R] + I_1 \delta(\vec{x} - \vec{x}_1) - I_1 \delta(\vec{x} - \vec{x}_2), \tag{2}$$

where $I_1=j_1/R$. To solve this we first write $\psi(\vec{x})=\psi_0(\vec{x})+\psi_1(\vec{x})$, where $\psi_0(\vec{x})$ is the solution to the original Grad–Shafranov equation, Eq. (1), that would be present prior to the addition of the extra coils used to form an x-point. The nonlinearity in the equations expresses itself through ψ_1, with the flux functions $p(\psi)$ and $I(\psi)$ being modified through the change in $\psi_0(\vec{x})$ to $\psi(\vec{x})=\psi_0(\vec{x})+\psi_1(\vec{x})$. Usually ψ_1 would be difficult to calculate, but because the x-point region may be arbitrarily localized, the perturbation $\psi_1(\vec{x})$ to the magnetic flux $\psi_0(\vec{x})$, can be made sufficiently small that

$$F[\psi(\vec{x}),R] \simeq F[\psi_0(\vec{x}),R] + \psi_1(\vec{x}) \left. \frac{\partial F}{\partial \psi} \right|_{\psi=\psi_0(\vec{x})} \tag{3}$$

giving

$$\nabla \cdot \left[\frac{\nabla \psi_0(\vec{x})}{R^2} \right] + \nabla \cdot \left[\frac{\nabla \psi_1(\vec{x})}{R^2} \right] = - F[\psi_0(\vec{x}),R] - \psi_1(\vec{x}) \times \left. \frac{\partial F}{\partial \psi} \right|_{\psi_0(\vec{x})} + I_1 \delta(\vec{x} - \vec{x}_1) - I_1 \delta(\vec{x} - \vec{x}_2). \tag{4}$$

With the aid of Eq. (2) this simplifies to

$$\nabla \cdot \left[\frac{\nabla \psi_1(\vec{x})}{R^2} \right] = - \psi_1(\vec{x}) \left. \frac{\partial F}{\partial \psi} \right|_{\psi_0(\vec{x})} + I_1 \delta(\vec{x} - \vec{x}_1) - I_1 \delta(\vec{x} - \vec{x}_2). \tag{5}$$

Again, because the x-point may be made arbitrarily localized, we can study a sufficiently narrow edge region that $\psi_0(\vec{x}) \simeq \psi_e$, a constant. Note that whereas ψ_0 will be approximately constant in this edge region, ψ_1 will change rapidly near the x-point (so as to form the x-point). Hence the localization of the perturbed region allows us to take ψ_0 as effectively constant near the plasma edge, but we must solve for ψ_1.

To solve these equations we make a large aspect ratio approximation. This allows us to take $\partial F(\psi_e,R)/\partial\psi \simeq \partial F(\psi_e)/\partial\psi$ which is constant, and to neglect terms in $\nabla\psi_1 . \nabla R/R^3$. Writing $-R^2 \partial F(\psi_e)/\partial\psi = k^2$, then gives

$$\nabla^2 \psi_1 + k^2 \psi_1 = R^2 I_1 [\delta(\vec{x} - \vec{x}_1) - \delta(\vec{x} - \vec{x}_2)]. \tag{6}$$

Note that because $-F(\psi)=-p'-II'/R^2=j_\phi(\psi)$ with $j_\phi=\nabla\phi.\vec{J}$ the toroidal current, then $F'(\psi_0)=-dj_\phi/d\psi \geqslant 0$ for a typical current profile with $dj_\phi/d\psi \leqslant 0$ at the plasma's edge. Hence $-R^2F'(\psi_0)=k^2$ and at the plasma's edge we expect $k^2 \geqslant 0$. When calculating k we must remember to include a factor of μ_0 that has for convenience so far been ignored. For JET plasmas, $R \simeq \langle R \rangle \simeq 3$ and a typical L-mode current profile gives $k \sim 1\ \mathrm{m}^{-1}$ and also allows the linear approximation to remain good for a few tens of percent of the minor radius. For H-mode plasmas in the JET the usual plasma modelling produces a current profile with a "bump" near the plasma's edge.[10] This will limit the linear approximation's validity to much smaller values of δ/r_0 than for the L-mode case, and the steeper current gradient near the edge leads to $k \sim 4.5\ \mathrm{m}^{-1}$. For the "dipole" configuration of two coils that is described above, the constant k will often cancel and not appear in the resulting expressions for the magnetic field and flux.

In a circular cross-section coordinate system with poloidal angle θ, Eq. (6) has solutions given in terms of Bessel functions[11] J_m and the Hankel function $H_0^{(1)}$, with $\psi_1=\psi_P+\psi_H$, and

$$\psi_H(\vec{x}) = \sum_{m=0}^{\infty} [a_m \cos(m\theta) + b_m \sin(m\theta)] J_m(kr),$$
$$\psi_P(\vec{x}) = R^2 I_1 [H_0^{(1)}(k|\vec{x} - \vec{x}_1|) - H_0^{(1)}(k|\vec{x} - \vec{x}_2|)] \tag{7}$$

with $\vec{x}=[r\cos(\theta), r\sin(\theta)]$. The a_m and b_m are constants arising from the solution of the homogeneous equation $\nabla^2\psi_H+k^2\psi_H=0$, and are chosen such that the plasma-vacuum boundary conditions are satisfied.

III. THE LOCALIZATION OF THE PERTURBATION

To illustrate the localization of the perturbed region and the accuracy of the approximation $\psi_1 \ll \psi_0$, we consider a cylindrical geometry with $\psi_0=\psi_0(r)$ and B_{p0} the equilibrium poloidal field at the outermost flux surface with radius $r=r_0$ (prior to the addition of the external field coils). For coils at $\vec{x}_1$ and $\vec{x}_2$ near the x-point, with small $|\vec{x}-\vec{x}_1|$ and $|\vec{x}-\vec{x}_2|$, we can approximate $H_0^{(1)}(k|\vec{x}-\vec{x}_i|)$ as $H_0^{(1)}(k|\vec{x}-\vec{x}_i|) \simeq -\ln(k|\vec{x}-\vec{x}_i|)$, and ψ_1 as $\psi_1 \simeq R^2 I_1 \ln(|\vec{x}-\vec{x}_2|/|\vec{x}-\vec{x}_1|)$. First, we consider the currents needed in the additional external coils.

For an x-point at $\vec{x}=(0,-r_0)$ and coils at $\vec{x}_1=(0,-r_0-\delta)$ and $\vec{x}_2=(0,-r_0-\delta-\epsilon)$, then the zero poloidal field at the x-point requires that $\vec{B}_{p0} = \nabla\phi \wedge \nabla\psi_1|_{x=(0,-r_0)}$, giving $I_1=(\delta(\delta+\epsilon)/\epsilon)(B_{p0}/R)$ and $\psi_1 \simeq (\delta R B_{p0})((\delta+\epsilon)/\epsilon) \ln(|\vec{x}-\vec{x}_2|/|\vec{x}-\vec{x}_1|)$. Note that we have neglected ψ_H for simplicity; this will be justified later for localized perturbations. Also notice that if the coil separation ϵ is increased, then the currents in the coils $j_1=RI_1$ are reduced, asymptoting to $j_1=\delta B_{p0}$ for $\epsilon \gg \delta$. Next we consider the approximation of $\psi_1 \ll \psi_0$.

At the x-point ψ_1 will usually have its maximum value in the plasma. The Appendix shows that for an x-point produced with two coils, the value of ψ_1 at the x-point is $\psi_1=(\delta R B_{p0})$ with an asymptotic behavior at distances of order ρ from the coils of $\psi_1 \sim (\delta R B_{p0})(\delta/\rho)$. Therefore provided δ is sufficiently small, then the perturbation ψ_1 to the magnetic flux may be made arbitrarily small, and the region it affects may be arbitrarily localized.

For the example of a cylindrical geometry we may estimate the poloidal extent of the region that is influenced by the coils, first writing $x=r\cos(\theta)$ and $y=r\sin(\theta)$, with the x-point located at $(r,\theta)=(r_0,-\pi/2)$. Then, if for example, $r\sim r_0$ and $|\theta+\pi/2|\gg\sqrt{\delta/r_0}$, then a simple estimate for the poloidal distance from the x-point of $\rho\sim r_0|\theta+\pi/2|\gg\sqrt{\delta r_0}$ gives $\psi_1\sim(\delta RB_{p0})\sqrt{\delta/r_0}$ and $B_1\sim B_{p0}(\delta/r_0)$. Therefore $|\theta+\pi/2|\gg\sqrt{\delta/r_0}$ and $\delta/r_0\ll 1$ ensures that away from the x-point ψ_1 and B_1 have a negligible effect on the equilibrium. The region in which $B_1\sim B_{p0}$ has $|\theta+\pi/2|\lesssim\pm\sqrt{\delta/r_0}$, and becomes arbitrarily small as δ/r_0 is reduced. The spatial localization of perturbations by different coil combinations are discussed further in Sec. VII and the Appendix.

IV. VACUUM SOLUTIONS

The magnetic field in the vacuum must satisfy

$$\vec{J}\wedge\vec{B}_V=\nabla p=0. \tag{8}$$

In the vacuum $\vec{J}=j_1[\delta(\vec{x}-\vec{x}_1)-\delta(\vec{x}-\vec{x}_2)]\vec{e}_\phi$ and at large aspect ratio $\vec{J}=\nabla\wedge\vec{B}_V$ has the solution

$$\vec{B}_V=\nabla\phi\wedge\nabla\psi_X+\nabla f, \tag{9}$$

where ψ_X satisfies

$$\frac{\nabla^2\psi_X}{R^2}=I_1[\delta(\vec{x}-\vec{x}_1)-\delta(\vec{x}-\vec{x}_2)] \tag{10}$$

and the large aspect ratio approximation of neglecting terms in $\nabla\psi_X.\nabla R/R^3$ has again been made. Equation (10) has the solution

$$\psi_X=R^2I_1[\ln(|\vec{x}-\vec{x}_1|^2)-\ln(|\vec{x}-\vec{x}_2|^2)] \tag{11}$$

for coils positioned at $\vec{x}_1$ and $\vec{x}_2$.

To ensure $\nabla.\vec{B}_V=0$ we need

$$\nabla^2 f=0 \tag{12}$$

that at large aspect ratio has solutions given by

$$f=\sum_{m=0}^{\infty}c_m r^{-m}\sin(m\theta)+d_m r^{-m}\cos(m\theta)+f_\theta\theta+f_\phi\phi \tag{13}$$

with f_θ and ϕ_ϕ constants.

V. PLASMA-VACUUM BOUNDARY CONDITIONS

In the absence of a surface/skin current (this will be considered in a future article), Ampere's law at the plasma-vacuum boundary requires equality between the vacuum and the plasma fields,[12] with $\vec{B}|_{\text{edge}}=\vec{B}_V|_{\text{edge}}$.

At the plasma-vacuum boundary, ψ is constant, but the radial position is a function of the poloidal angle θ. First we must solve $\psi(r,\theta)=\psi|_{x\text{-point}}$ for the radial position of the edge, as a function of θ. This will give the radial position as a function of the (currently) unknown constants $\vec{a}=(a_0,a_1,\ldots)$ and $\vec{b}=(b_0,b_1,\ldots)$, that arise in the solution for ψ. For example, writing the radial position of the edge as $r_e(\theta)=r_0+dr(\theta)$, we can make use of the limit $dr/r_0\ll 1$ to simplify the calculations, and expand $dr(\theta)$ as a Fourier series. The sine and cosine components of the resulting equations will give two coupled (matrix) equations, that can be solved for the sine and cosine Fourier components of dr, that will depend upon $\vec{a}$ and $\vec{b}$.

Matching $\vec{B}|_{\text{edge}}=\vec{B}_V|_{\text{edge}}$ will give two equations, for the components in the $\nabla\theta$ and ∇r directions. Substituting for $dr(\vec{a},\vec{b},\theta)$, and projecting out the cosine and sine components of these equations, then gives four (matrix) equations in terms of the four unknowns $\vec{a}=(a_0,a_1,\ldots)$, $\vec{b}=(b_0,b_1,\ldots)$, $\vec{c}=(c_0,c_1,\ldots)$, and $\vec{d}=(d_0,d_1,\ldots)$, which may be solved to give $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$. Although this procedure does in principle allow the calculation of the free parameters $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$, in practice it will be awkward due to the large numbers of Fourier modes needed to expand a highly localized x-point; and often it will not be necessary, for reasons outlined next.

First consider a less accurate approximation to $F(\psi)$ than that given in Eq. (3), instead neglecting the term linear in ψ_1 and simply taking $F(\psi)\simeq F(\psi_0)$. This is reasonable because it is possible to have $\psi_1\ll\psi_0$ for all of the edge region, despite $\nabla\psi_1\sim\nabla\psi_0$ near the x-point. This lowest order approximation neglects *all* the nonlinear effects that the change in ψ has on $F(\psi)$, i.e., it treats the plasma as being unaffected by the change in the magnetic field. [Previously the first order correction to account for the changes in ψ were included through the term linear in ψ_1, $F'(\psi_0)\psi_1$.] When we neglect the term $F'(\psi_0)\psi_1$, and consider a large aspect ratio limit, we have

$$\nabla^2\psi=R^2I_1[\delta(\vec{x}-\vec{x}_1)-\delta(\vec{x}-\vec{x}_2)]+R^2F(\psi_0). \tag{14}$$

This has the solution $\psi=\psi_0+\psi_X+\psi_H$, with ψ_X given by Eq. (11) and ψ_H the homogeneous solution in Eq. (7). Hence

$$\vec{B}=\vec{B}_0+\nabla\phi\wedge\nabla\psi_X+\nabla\phi\wedge\nabla\psi_H, \tag{15}$$

where $\vec{B}_0$ is the magnetic field prior to the addition of the external wires.

For zero skin currents the vacuum field must equal the plasma's field at the plasma-vacuum surface, and satisfy the additional boundary condition of a zero normal component at any surrounding conducting wall (that is often approximated to be at infinity). The original vacuum field may be written $\vec{B}_{V0}=\nabla f_0$, and satisfies the boundary conditions and the homogeneous equation $\nabla^2 f_0=0$. When the plasma surface is displaced from its original position, then the homogeneous solution f_0 is modified to $f=f_0+f_1$, with f_0 being modified so that the boundary conditions continue to be satisfied. Therefore the vacuum solutions are

$$\vec{B}_V=\nabla f_0+\nabla f_1+\nabla\phi\wedge\nabla\psi_X. \tag{16}$$

Using Eqs. (15) and (16), and the plasma-vacuum boundary conditions of $\vec{B}=\vec{B}_V$, then requires

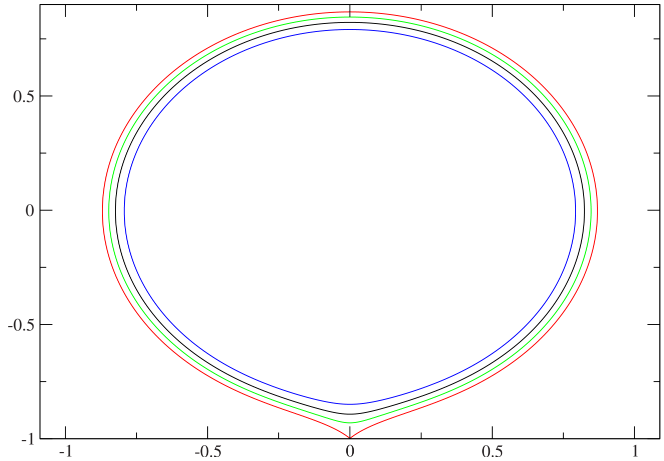

FIG. 1. (Color online) A plot of flux surface contours for a cylindrical equilibrium that is perturbed with a dipole coil configuration to form a separatrix.

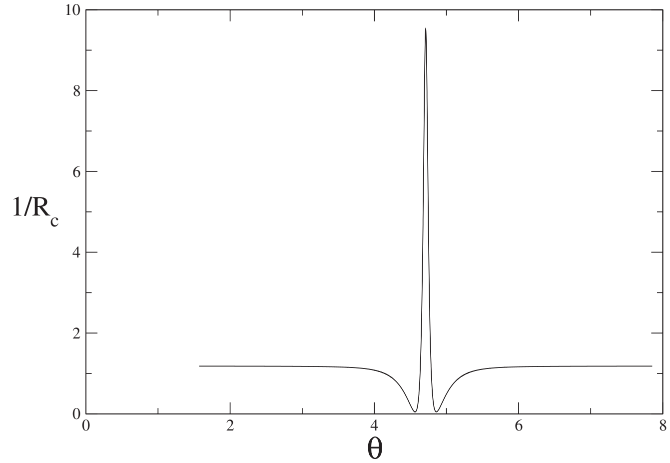

FIG. 3. The local curvature of the flux surfaces in the poloidal cross section, as a function of θ in cylindrical coordinates. R_c is the radius of curvature, and the x-point is at $\theta=3\pi/2$.

$$
\begin{aligned}
&[\vec{B}_0+\nabla\phi\wedge\nabla\psi_X+\nabla\phi\wedge\nabla\psi_H]_{\text{edge}}\\
&\quad=[\nabla f_0+\nabla f_1+\nabla\phi\wedge\nabla\psi_X]_{\text{edge}}
\end{aligned}
\tag{17}
$$

that using $\vec{B}_0=\vec{B}_{0V}=\nabla f_0$, simplifies to

$$[\nabla\phi\wedge\nabla\psi_H]_{\text{edge}}=[\nabla f_1]_{\text{edge}}. \tag{18}$$

As mentioned above ∇f_1 (and hence also $\nabla\psi_H$), are determined from the above Eq. (18) and the boundary condition on the vacuum field of $\vec{n}.\vec{B}_V=0$ at a conducting wall (or $\vec{B}_V\to 0$ in the absence of a conducting wall). Comparison with a cylindrical calculation with a plasma surface perturbed from a radial position $r=r_0$ to $r=r_0+\xi$, with a conducting wall at $r=r_b$, indicates that the field at the plasma surface will be perturbed by of order ξ/r_b for a plasma surface displacement of order ξ from its original position and a conducting wall at a distance of order r_b. For an arbitrary cross section in a large aspect ratio approximation it can be formally demonstrated using the method of conformal transformations,[13] that the same order of magnitude estimation continues to hold. Therefore although in general we need to solve for f_1 and ψ_H using the procedure outlined earlier, if the plasma displacement is small compared with the distance to a conducting wall, or for a conducting wall at "infinity," then we may neglect the perturbation to the field from ∇f_1 and take $\nabla f_1=\nabla\phi\wedge\nabla\psi_H\simeq 0$. This would be the case for $\xi/r_b\sim\delta/r_0\ll 1$ for example, for which we will have

$$
\begin{aligned}
&\vec{B}=\vec{B}_0+\nabla\phi\wedge\nabla\psi_X,\\
&\vec{B}_V=\vec{B}_{V0}+\nabla\phi\wedge\nabla\psi_X.
\end{aligned}
\tag{19}
$$

Next consider the case where terms in $F'(\psi_0)\psi_1$ are included, for which

$$
\begin{aligned}
&\vec{B}=\vec{B}_0+\nabla\phi\wedge\nabla\psi_P+\nabla\phi\wedge\nabla\psi_H,\\
&\vec{B}_V=\nabla f_0+\nabla f_1+\nabla\phi\wedge\nabla\psi_X.
\end{aligned}
\tag{20}
$$

In the region where the strong gradients in ψ_X and ψ_1 provide a non-negligible contribution to the magnetic field, then $\psi_X=-IR^2(\ln\rho_1-\ln\rho_2)=-IR^2\ln(\rho_1/\rho_2)$ and $\psi_P=IR^2[H_0^{(1)}(k\rho_1)-H_0^{(1)}(k\rho_2)]\simeq IR^2[-\ln k\rho_1-(-\ln k\rho_2)]=-IR^2\ln(\rho_1/\rho_2)$, where $\rho_1=|\vec{x}-\vec{x}_1|$ and $\rho_2=|\vec{x}-\vec{x}_2|$. Hence to a good approximation the terms in $\nabla\psi_X$ and $\nabla\psi_P$ cancel, and again provided $\xi/r_b\sim\delta/r_0\ll 1$, then we can neglect ∇f_1 and $\nabla\psi_H$ and the boundary conditions will be satisfied without the need for the evaluation of the free parameters $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ that in principle appear in the solution.

To illustrate a typical perturbed solution, we use a dipole to perturb a cylindrical equilibrium with an equilibrium flux $\psi_0(r)$, with $\delta=0.1$, and the poloidal field B_{p0} normalized by its value at the opposite side of the flux surface to the x-point. Figure 1 plots contours of constant ψ, and Figs. 2–4 plot the safety factor q, the local curvature, and the poloidal magnetic field near the plasma's edge. When plotting the safety factor q we have used the fact that q can be calculated from $q=(1/2\pi)\oint(I/R^2)(dl/B_p)$, with dl an element of arc

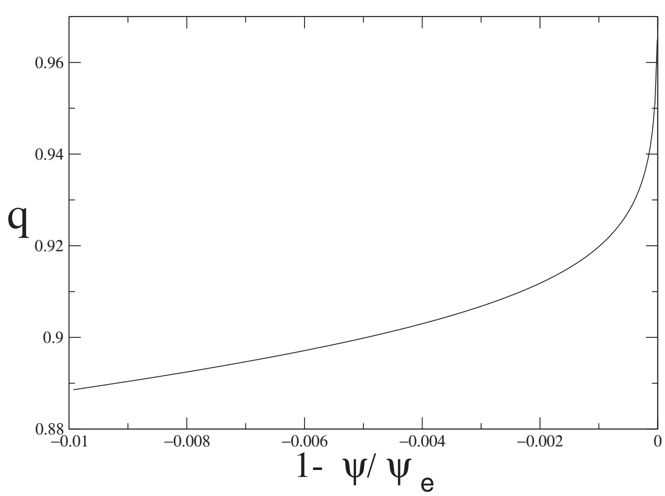

FIG. 2. The (normalized) safety factor q diverges strongly as ψ approaches ψ_e, its value at the plasma's edge.

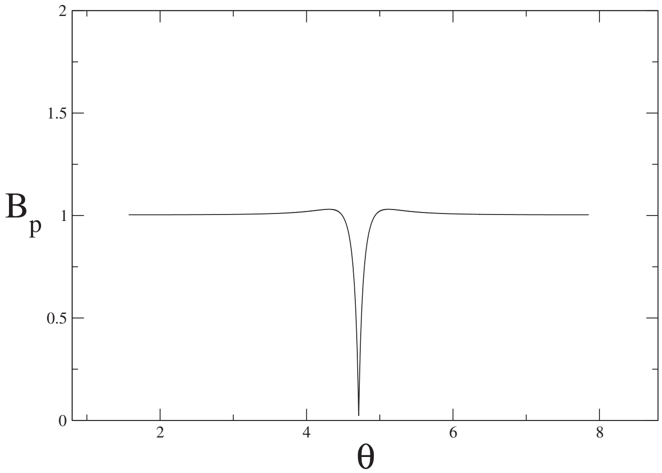

FIG. 4. The (normalized) poloidal magnetic field as a function of θ in cylindrical coordinates. The x-point is at $\theta=3\pi/2$.

length in the poloidal cross section, and normalized q such that the q we plot is $q=(1/2\pi)\oint(B_{p0}/B_p)dl$, with B_{p0} as described above.

VI. MERCIER STABILITY

Here we briefly examine the behavior of the safety factor q and the Mercier coefficient D_M near the separatrix. The safety factor $q=(1/2\pi)\oint \nu d\chi$, where χ is the poloidal coordinate in the orthogonal χ, ψ, ϕ system, and $\nu=IJ_\chi/R^2$ is the local field line pitch written in terms of the Jacobian J_χ. As mentioned above, the safety factor may alternately be written in terms of an integral along the surface in a poloidal cross section, with element of arc length $dl=(J_\chi B_p)d\chi$, and

$$q=\frac{1}{2\pi}\oint \frac{I}{R^2}\frac{dl}{B_p}. \tag{21}$$

The magnetic shear is proportional to $q'=(1/2\pi)\oint(\partial\nu/\partial\psi)d\chi$. In χ, ψ, ϕ coordinates the Grad–Shafranov equation is

$$\frac{\partial\nu}{\partial\psi}=\frac{\nu}{B_p^2}\left[-\frac{\partial}{\partial\psi}(p+B^2)+\frac{R^2B^2}{I}\frac{\partial}{\partial\psi}\left(\frac{I}{R^2}\right)\right] \tag{22}$$

hence using $\nabla\phi.\vec{J}=-p'-II'/R^2$ with $\vec{J}$ the plasma current and ϕ the toroidal angle, we have

$$q'=\frac{1}{2\pi}\oint\frac{\nu}{B_p^2}\left[\nabla\phi\,.\,\vec{J}-\frac{\partial B_p^2}{\partial\psi}\right]+\frac{1}{2\pi}\oint\nu\frac{R^2}{I}\frac{\partial}{\partial\psi}\left(\frac{I}{R^2}\right). \tag{23}$$

Provided $\nabla\phi.\vec{J}\neq 0$ at the x-point, then near the separatrix the divergence in ν/B_p^2 at the x-point will dominate the integrals, allowing the second integral to be neglected. Furthermore, because the integral is dominated by the divergence at the x-point, we may approximate $\nabla\phi.\vec{J}$ by its value there. Therefore because $\partial B_p^2/\partial\psi$ is negative, we have $q'\gtrsim\nabla\phi.\vec{J}\oint(\nu/B_p^2)d\chi\sim(\langle B_p\rangle/R^2)\oint(\nu/B_p^2)d\chi$. In practice, because q is easy to calculate, q' is most easily evaluated numerically using finite differences. However, we will find it useful to have an indication of how q' behaves near the separatrix later in this section.

The Mercier coefficient is given by[14] $D_M=-Q/P$ with

$$\begin{aligned}Q&=\frac{p'}{2\pi}\oint\frac{\partial J_\chi}{\partial\psi}d\chi-\frac{(p')^2}{2\pi}\oint\frac{J_\chi}{B_p^2}d\chi\\&\quad+Ip'\oint\frac{J_\chi}{R^2B_p^2}d\chi\left[\oint\frac{J_\chi B^2}{R^2B_p^2}d\chi\right]^{-1}\\&\quad\times\left[-q'+\frac{Ip'}{2\pi}\oint\frac{J_\chi}{R^2B_p^2}\right],\\P&=2\pi(q')^2\left[\oint\frac{J_\chi B^2}{R^2B_p^2}d\chi\right]^{-1}\end{aligned} \tag{24}$$

and determines stability to interchange modes,[1] with $D_M<1/4$ indicating stability and $D_M>1/4$ indicating instability. Using $\nu=IJ_\chi/R^2$ to replace J_χ with ν, and expressing Q with a common factor of $[\oint\nu B^2/B_p^2d\chi]^{-1}$, gives

$$\begin{aligned}Q&=\frac{p'}{2\pi I\oint\frac{\nu B^2}{B_p^2}d\chi}\left\{\left(\oint R^2\frac{\partial\nu}{\partial\psi}d\chi\right)\left(\oint\frac{\nu B^2}{B_p^2}d\chi\right)\right.\\&\quad+I\left(\oint\nu\frac{\partial(R^2/I)}{\partial\psi}d\chi\right)\left(\oint\frac{\nu B^2}{B_p^2}d\chi\right)-p'\left(\oint\frac{\nu R^2}{B_p^2}d\chi\right)\\&\quad\times\left(\oint\frac{\nu B^2}{B_p^2}d\chi\right)+I^2p'\left(\oint\frac{\nu}{B_p^2}d\chi\right)\left(\oint\frac{\nu}{B_p^2}d\chi\right)\\&\quad\left.-I^2\left(\oint\frac{\partial\nu}{\partial\psi}d\chi\right)\left(\oint\frac{\nu}{B_p^2}d\chi\right)\right\}.\end{aligned} \tag{25}$$

Writing $B^2/B_p^2=1+I^2/R^2B_p^2$, using Eq. (24) for P, and grouping terms, gives

$$\begin{aligned}D_M&=\frac{p'}{(2\pi)^2I^2q'^2}\left\{\left(\oint\nu d\chi\right)\left(\oint R^2\frac{\partial\nu}{\partial\psi}d\chi\right)+I\left(\oint\nu d\chi\right)\right.\\&\quad\times\left(\oint\nu\frac{\partial(R^2/I)}{\partial\psi}d\chi\right)-p'\left(\oint\nu d\chi\right)\left(\oint\frac{\nu R^2}{B_p^2}d\chi\right)\\&\quad+I\left(\oint\nu\frac{\partial(R^2/I)}{\partial\psi}d\chi\right)\left(\oint\frac{\nu I^2}{R^2B_p^2}d\chi\right)\\&\quad+I^2\left[\left(\oint R^2\frac{\partial\nu}{\partial\psi}d\chi\right)\left(\oint\frac{\nu}{R^2B_p^2}d\chi\right)-\left(\oint\frac{\partial\nu}{\partial\psi}d\chi\right)\right.\\&\quad\left.\times\left(\oint\frac{\nu}{B_p^2}d\chi\right)\right]+p'I^2\left[\left(\oint\frac{\nu}{B_p^2}d\chi\right)^2-\left(\oint\frac{\nu R^2}{B_p^2}d\chi\right)\right.\\&\quad\left.\left.\times\left(\oint\frac{\nu}{R^2B_p^2}d\chi\right)\right]\right\}.\end{aligned} \tag{26}$$

Alternately using the Grad–Shafranov equation, Eq. (22), this can be written as

$$D_M = \frac{p'}{(2\pi)^2 I^2 q'^2}\left\{ q \oint \nu \frac{R^2}{B_p^2}\left[-\frac{\partial}{\partial\psi}(2p+B^2)\right] d\chi + I\left\{\left(\oint \nu d\chi\right)\left[\oint \frac{\nu R^2}{B_p^2}\frac{\partial(I/R^2)}{\partial\psi}d\chi\right] - \left(\oint \nu R^4 \frac{\partial(I/R^2)}{\partial\psi}d\chi\right)\left(\oint \frac{\nu}{R^2 B_p^2}d\chi\right)\right\} + I^2\left[\left(\oint R^2\frac{\partial\nu}{\partial\psi}d\chi\right)\left(\oint \frac{\nu}{R^2B_p^2}d\chi\right) - \left(\oint \frac{\partial\nu}{\partial\psi}d\chi\right) \times \left(\oint \frac{\nu}{B_p^2}d\chi\right)\right] + p'I^2\left[\left(\oint \frac{\nu}{B_p^2}d\chi\right)^2 - \left(\oint \frac{\nu R^2}{B_p^2}d\chi\right) \times \left(\oint \frac{\nu}{R^2B_p^2}d\chi\right)\right]\right\}. \tag{27}$$

The exact behavior of D_M near a separatrix will depend on the particular geometry and equilibrium considered, however some general features of the above equations are worth pointing out. If we consider an arbitrarily large aspect ratio, so that we can treat the major radius as constant, then the final 3 bracketed terms in Eq. (27) and the final 2 bracketed terms in Eq. (26) will be zero. Similarly, provided that at the x-point $\nabla\phi.\vec{J}=-p'-II'/R^2\neq 0$, then near the separatrix the integrals are dominated by the divergence in ν/B_p^2 that acts increasingly like a delta function, so that in the region where ν/B_p^2 diverges most strongly we have $R\simeq R_X$ a constant, and hence obtain the same result as for large aspect ratio with Eq. (26) becoming

$$\begin{aligned} D_M &= \frac{p'}{I^2 2\pi q'^2}\left\{2\pi q R_X^2 q' - qp'R_X^2\left(\oint \frac{\nu}{B_p^2}d\chi\right) - II'q\left(\oint \frac{\nu}{B_p^2}d\chi\right) - \frac{I'}{I}q^2\right\} \\ &= \frac{p'R_X^2}{I^2}\left(\frac{q}{q'}\right)\left\{1 + \frac{1}{q'}\left(\oint \frac{\nu}{B_p^2}d\chi\right)\left(-p' - \frac{II'}{R_X^2}\right) - \left(\frac{q}{q'}\right)\left(\frac{I'}{IR_X^2}\right)\right\} \end{aligned} \tag{28}$$

with $q=(1/2\pi)\oint \nu d\chi$ and $q'=(1/2\pi)\oint(\partial\nu/\partial\psi)d\chi$. Previously we observed that $q'\gtrsim(\langle B_p\rangle/R^2)\oint(\nu/B_p^2)d\chi$, therefore we find that for $\nabla\phi.\vec{J}\neq 0$ at the X-point, then $D_M\sim(q/q')\to 0$ as we approach a separatrix, clearly indicating ideal MHD interchange modes are stable near the separatrix.

It should be emphasized that it is only near the separatrix where q' diverges to infinity, that the stabilization will be found. Further from the separatrix where the divergence in ν/B_p^2 is weaker and D_M is closer to 1/4, there will be a noticeable dependence on the poloidal location of the x-point, through $R\simeq R_X$ when ν/B_p^2 is a maximum. A specific case of these general observations is found in Bishop,[3] who numerically studied a particular equilibrium with an x-point. In that study,[3] in the limit of the boundary approximating a separatrix with nonzero edge current ($k=1$ and $\Lambda\neq 0$, respectively, in the notation of Ref. 3), there are no longer any values of pressure gradient for which the mode is unstable (Figs. 8 and 10 of Ref. 3). Similarly further from the separatrix ($k\neq 1$), it is found that[3] the stability boundaries depend on the poloidal location of the x-point.

VII. DISCUSSION

The plasma perturbation. In the lowest order approximation $F(\psi)\simeq F(\psi_0)$, with the result that the plasma is negligibly affected by the external fields used to form the x-point. So the lowest order approximation neglects the (nonlinear) effect of the plasma on the resulting magnetic field. The next order of approximation (described in Sec. II), included the leading order effect of the plasma on the magnetic field through the term $F'(\psi_0)\psi_1$ in Eq. (3). The consequence of including this term, is to alter the *field* at long distances from

$$B_p^X \sim \nabla[\ln(\rho)-\ln(\rho+\delta)] \sim \frac{\delta}{\rho} \tag{29}$$

to

$$B_p^1 \sim \nabla\{H_0^{(1)}(k\rho) - H_0^{(1)}[k(\rho+\delta)]\} \sim (k\delta)\,\nabla\left(\frac{e^{ik\rho}}{\sqrt{k\rho}}\right). \tag{30}$$

Hence the leading order effect of the plasma is to support a wavelike perturbation, whose amplitude falls more slowly than it would in a vacuum. Note that in the above Eq. (30) there is no dependence on the radial position in the plasma. This is because for simplicity the analysis considers a sufficiently narrow edge region for which ψ_0 is approximately constant, and consequently $F'(\psi_0)$ in Eq. (3) is also approximately constant.

Interestingly, for an equilibrium current gradient that *increases* towards the plasma edge, the Hankel function appearing in the particular solution is replaced by a zeroth order modified Bessel function. In that case the magnetic field $B_p^1\sim(k\delta)\nabla(e^{-k\rho}/\sqrt{k\rho})$ and the leading order effect of the plasma is then a diamagnetic one that localizes the perturbation more strongly than in a vacuum. Therefore we have the interesting observation that where it is possible to apply a delta function type of perturbation to the current in a region with positive current gradient (where transport barriers are often observed), the perturbation will be much more strongly localized than in regions with the usual negative current gradient. The consequences of this observation will be explored in future work.

Other coil configurations. Similar results to those described above may be obtained by considering a single wire with a sufficiently small current that is placed sufficiently close to the plasma's edge. Then we simply have

$$\psi_P = \frac{IR^2}{2\pi}H_0^{(1)}(k|\vec{x}-\vec{x}_1|) \tag{31}$$

and

$$\psi_X = -\frac{IR^2}{2\pi}\ln(k|\vec{x}-\vec{x}_1|). \tag{32}$$

However, although good approximate solutions may be obtained, the perturbed region is no longer strongly localized. Perhaps of more interest is the question of whether reasonably localized plasma perturbations can be realized with realistic coil configurations, for which the required coil separation and distance from the plasma are practical from an engineering perspective. As noted in Sec. III the coil separation ϵ can be increased, allowing the current[16] j_1 in the coils to be reduced, with $j_1 = RI_1 = (\delta B_{p0})(\delta+\epsilon)/\epsilon$ that compares with the *total* current j_0 in the plasma that has $j_0 \sim r_0 B_{p0}$. The other question is how close must the coils be to the plasma? The answer to this depends on how localized the perturbation is required to be, and how many coils we are prepared to use. For the case of a "dipole" with two coils, as discussed in the Appendix the perturbation ψ_1 at the x-point has $\psi_1 \lesssim \delta R B_{p0}$ with $\psi_1 \sim (\delta R B_{p0})(\delta/\rho)$ for an approximate distance ρ from the coils. Similarly, the perturbation to the field has $B_{p1} \sim B_{p0}(\delta/\rho)^2$. If a "quadrupole" of four coils were used, then the Appendix finds $\psi_1 \lesssim [(\delta+\epsilon)RB_{p0}](\delta+\epsilon)^2/\epsilon^2$ at the x-point, but now $\psi_1 \sim [(\delta+\epsilon)B_{p0}](\delta+\epsilon)^2/\rho^2$ and $B_{p1} \sim B_{p0}(\delta/\rho)^3$ for distances of order ρ from the coils. For example, if $\delta/\rho = 1/4$, then a dipole perturbation to the magnetic flux would be of order 1/4 of its value at the x-point, and a quadrupole perturbation would be of order 1/16 of its value. Therefore a dipole coil configuration should produce a reasonably localised perturbation, but if it were sufficiently desirable to have a more localized perturbation, it would seem possible to produce one by using four (or in principle even more), coils.

It might be worth noting that the extent of localization of the perturbation used to form an x-point might change the stability properties of the equilibrium. As the x-point becomes increasingly localized, its effect will act more like a "hard-wall" as opposed to a gently varying field/potential.

Plasma control. The advantage of a perturbation by a dipole is that it can provide a strong perturbation at the edge, but has a relatively weak and possibly negligible (direct) effect on the core equilibrium. For example, a constant perturbation could be made to strongly shape the plasma in a localized region on the outboard side. According to the results of Bishop,[2,3] the extent to which such a perturbation approximates an x-point will determine the (Ballooning mode limited) pressure gradient at the plasma edge. A reduced edge pressure gradient is likely to modify the plasma confinement and the properties of ELMs. Alternately a perturbation might be applied transiently in the hope of modifying the edge stability sufficiently to trigger ELMs; however the consequences of a time-dependent perturbation are poorly understood at present. There are many issues connected with the feasibility of such coils that require further examination—the idea of using axisymmetric coils to modify the properties of the plasma edge, and as a possible consequence the divertor power loading, is simply raised as an interesting possibility here.

VIII. SUMMARY

This paper shows how two external coils with equal but opposite currents, may be used to give an (arbitrarily localized) perturbation to a plasma equilibrium. By adjusting the currents in the coils and their proximity to each other and to the plasma, the perturbation can be used to form a separatrix and x-point in an otherwise circular cross-section plasma, for example. This provides a useful thought experiment for theoretical reasoning, and a potentially useful technique for investigating the effect of a separatrix on numerical stability calculations and possibly also for future analytical work. It is hoped that the work here will make it easier to incorporate the effects of an x-point into future studies of plasma stability, this was the original purpose of this paper, there have already been some numerical studies along these lines.[15]

Analytic calculations have been given that describe the resulting equilibrium. The solutions become increasingly accurate as the distance δ from the plasma to the nearest coil is made small compared with a measure of the minor radius r_0. The solutions include the modified field in the plasma, and the vacuum field that matches this at the plasma-vacuum boundary. These solutions are

$$\begin{aligned} \vec{B} &= \vec{B}_0 + \nabla\phi \wedge \nabla\psi_P + \nabla\phi \wedge \nabla\psi_H, \\ \vec{B}_V &= \vec{B}_{V0} + \nabla f_1 + \nabla\phi \wedge \nabla\psi_X \end{aligned} \tag{33}$$

with ψ_P given by Eq. (7) and ∇f_1 determined by the boundary conditions for the vacuum field, with ψ_H determined from ∇f_1 through $\nabla\phi \wedge \nabla\psi_H = \nabla f_1$ at the plasma-vacuum surface. Provided that the ratio between the distance from the coils to the plasma δ and the typical distance to any surrounding wall r_b has $\delta/r_b \ll 1$, then the solutions are particularly simple because both $f_1 \simeq 0$ and $\psi_H \simeq 0$ and can be neglected. The solutions include the nonlinear effect of the plasma on the magnetic flux ψ, by a linear approximation for small perturbations to the magnetic flux (that is valid for the localized perturbations considered here). By comparing these solutions with the equivalent solutions in a vacuum it has been observed that whereas in a negative current-gradient the plasma perturbation is wavelike and decays more slowly than in a vacuum, for a positive current gradient (as are often associated with transport barriers), the plasma "screens" the perturbed magnetic field that then decays more rapidly than in a vacuum.

Mercier stability is considered, and it is shown that if $\nabla\phi.\vec{J} \neq 0$ at the x-point, the Mercier coefficient D_M tends to zero as a separatrix is approached, indicating that the plasma is Mercier stable at the separatrix. Plots have illustrated the behavior of q, B_p, and the local curvature near a separatrix.

Finally some possibilities for plasma control have been raised, noting that the radially localized perturbations considered here have the desirable property of being able to strongly perturb the edge while only weakly perturbing the core plasma.

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APPENDIX: DIPOLE AND QUADRUPOLE FIELDS

Here we briefly review the properties of a magnetic field produced by a "dipole" and "quadrupole" configuration of wires. For the purpose of illustration and estimation we will take a vacuum approximation, noting that a more accurate account will need to incorporate the diamagnetic effects from the plasma, as is done more fully in the main text.

In a vacuum the magnetic flux function associated with a straight wire located at $\vec{x}_1$ carrying current RI_1, is found from the solution of Laplace's equation $\nabla^2\psi_1=0$, that has a particular solution,

$$\psi_1 = R^2 I_1 \ln(|\vec{x}-\vec{x}_1|). \tag{A1}$$

More generally for a collection of wires at positions $\{\vec{x}_i\}$ each carrying current of magnitude RI_1, we have

$$\psi_1 = R^2 I_1 \sum_i \sigma_i \ln(|\vec{x}-\vec{x}_i|) \tag{A2}$$

with $\sigma=\pm 1$ depending on the direction of the current in the ith coil. The associated magnetic field is

$$\nabla\phi\wedge\nabla\psi_1 = RI_1 \sum_i \sigma_i \frac{\vec{e}_\phi \wedge (\vec{x}-\vec{x}_i)}{|\vec{x}-\vec{x}_i|^2}. \tag{A3}$$

We first consider a "dipole" of wires with currents $+RI_1$ at $(x,y)=(0,\alpha)$ and $-RI_1$ at $(x,y)=(0,-\alpha)$. The field from these wires is given by $\nabla\phi\wedge\nabla\psi_1$ with

$$\nabla\psi_1 = R^2 I_1 \left\{ \frac{(x,y+\alpha)}{x^2+(y+\alpha)^2} - \frac{(x,y-\alpha)}{x^2+(y-\alpha)^2} \right\} \tag{A4}$$

that for an x-point at $(x,y)=(0,\beta)$ has $\nabla\psi_1 = R^2 I_1 (0,-2\alpha)/(\beta^2-\alpha^2)$. The field from $\nabla\psi_1$ must be equal to the equilibrium field B_{p0} at $(0,\beta)$ so as to form an x-point (with zero poloidal field). This requires $\vec{B}_{p0}=\nabla\phi\wedge\nabla\psi_1$, leading to $I_1=(1/2)((\beta^2-\alpha^2)/\alpha)(B_{p0}/R)$. We consider the localization of the perturbed flux and field by expanding ψ_1 and $\nabla\psi_1$ for small α relative to $\rho\sim x\sim y\sim\sqrt{x^2+y^2}$, to obtain

$$\nabla\psi_1 \simeq R^2 I_1 \left\{ (x,y)\left[-\frac{4y\alpha}{(x^2+y^2)^2}\right] + (0,\alpha)\frac{2}{(x^2+y^2)} \right\} + O(\alpha^2). \tag{A5}$$

After substituting for I_1, this has an asymptotic behavior of

$$\nabla\psi_1 \sim (RB_{p0})\left(\frac{\beta^2}{\rho^2}\right). \tag{A6}$$

Similarly,

$$\psi_1 \simeq \frac{R^2 I_1}{2}\left[\frac{4y\alpha}{x^2+y^2}\right] \tag{A7}$$

has an asymptotic behavior of

$$\psi \sim (\beta R B_{p0})\left(\frac{\beta}{\rho}\right). \tag{A8}$$

These results may be related to those in the main text by replacing β and α with $\beta=\delta+\epsilon/2$ and $\alpha=\epsilon/2$.

Next we consider a "quadrupole" of wires with positive currents at $(x,y)=(0,\pm\alpha)$ and negative current at $(x,y)=(\pm\alpha,0)$. For this configuration we have

$$\nabla\psi_1 = R^2 I_1 \left\{ \frac{(x,y+\alpha)}{x^2+(y+\alpha)^2} + \frac{(x,y-\alpha)}{x^2+(y-\alpha)^2} - \frac{(x+\alpha,y)}{(x+\alpha)^2+y^2} - \frac{(x-\alpha,y)}{(x-\alpha)^2+y^2} \right\} \tag{A9}$$

that for an x-point at $(x,y)=(0,\beta)$ has $\nabla\psi_1 = (RI_1 4\alpha^2\beta/(\beta^4-\alpha^4))(0,1)$, and therefore requires $I_1=(B_{p0}/R)(1/4)(\beta^4-\alpha^4)/\alpha^2\beta$. The asymptotic behavior of the field has

$$\nabla\psi_1 \simeq R^2 I_1 \left\{ 8(x,y)\alpha^2 \frac{y^4-x^4}{(x^2+y^2)^4} + 4(x,-y)\frac{\alpha^2}{(x^2+y^2)^2} \right\} \tag{A10}$$

and hence substituting for I_1 gives

$$\nabla\psi_1 \sim (RB_{p0})\left(\frac{\beta^3}{\rho^3}\right). \tag{A11}$$

Similarly,

$$\psi_1 \sim [(\beta+\alpha)RB_{p0}]\frac{(\beta+\alpha)^2}{\rho^2}. \tag{A12}$$

As above, $\beta=\delta+\epsilon/2$ and $\alpha=\epsilon/2$ give a quadrupole of wires with wire-separation ϵ and whose nearest wire to the plasma is a distance δ from the x-point.

[1] J. Wesson, *Tokamaks*, 3rd ed. (Clarendon, Oxford, 2004), p. 711.
[2] C. M. Bishop, P. Kirby, J. W. Connor, R. J. Hastie, and J. B. Taylor, Nucl. Fusion **24**, 1579 (1984).
[3] C. M. Bishop, Nucl. Fusion **26**, 1063 (1986).
[4] N. Mattor and R. H. Cohen, Phys. Plasmas **2**, 4042 (1995).
[5] J. R. Myra, D. A. D'Ippolito, and J. P. Goedloed, Phys. Plasmas **4**, 1330 (1997).
[6] J. R. Myra and D. A. D'Ippolito, Phys. Plasmas **5**, 659 (1998).
[7] G. T. A. Huysmans, Plasma Phys. Controlled Fusion **47**, 2107 (2005).
[8] N. Mattor, Phys. Plasmas **2**, 594 (1995).
[9] J. P. Freidberg, *Ideal Magneto-Hydro-Dynamics* (Plenum, New York, 1987).

[10]S. Saarelma, private communication (2008).
[11]G. Arfken, *Mathematical Methods for Physicists* (Academic, San Diego, 1985), p. 610.
[12]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
[13]A. J. Webster and C. G. Gimblett, “Magnetohydrodynamic stability of a toroidal plasma’s separatrix,” Phys. Rev. Lett. (in press).
[14]J. W. Connor, R. J. Hastie, H. R. Wilson, and R. L. Miller, Phys. Plasmas **5**, 2687 (1998).
[15]S. Saarelma, C. G. Gimblett, H. Meyer, A. Kirk, A. J. Webster, H. Wilson, and the MAST Team, *Proceedings of the 3rd IAEA Technical Meeting on Theory of Plasma Instabilities (York)* (IAEA, Vienna, 2007).
[16]Note that j_1 is the total current in each coil, as opposed to a current per unit area.