Magnetohydrodynamic stability at a separatrix. I. Toroidal peeling modes and the energy principle

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A potentially serious impediment to the production of energy by nuclear fusion in large tokamaks, such as ITER [R. Aymar, V. A. Chuyanov, M. Huguet, Y. Shimomura, ITER Joint Central Team, and ITER Home Teams, Nucl. Fusion **41**, 1301 (2001)] and DEMO [D. Maisonner, I. Cook, S. Pierre, B. Lorenzo, D. Luigi, G. Luciano, N. Prachai, and P. Aldo, Fusion Eng. Des. **81**, 1123 (2006)], is the potential for rapid deposition of energy onto plasma facing components by edge localized modes (ELMs). The trigger for ELMs is believed to be the ideal magnetohydrodynamic peeling-ballooning instability, but recent numerical calculations have suggested that a plasma equilibrium with an *X*-point—as is found in all ITER-like tokamaks, is stable to the peeling mode. This contrasts with analytical calculations [G. Laval, R. Pellat, and J. S. Soule, Phys. Fluids **17**, 835 (1974)] that found the peeling mode to be unstable in cylindrical plasmas with arbitrary cross-sectional shape. Here, we re-examine the assumptions made in cylindrical geometry calculations and generalize the calculation to an arbitrary tokamak geometry at marginal stability. The resulting equations solely describe the peeling mode and are not complicated by coupling to the ballooning mode, for example. We find that stability is determined by the value of a single parameter Δ' that is the poloidal average of the normalized jump in the radial derivative of the perturbed magnetic field's normal component. We also find that near a separatrix it is possible for the energy principle's δW to be negative (that is usually taken to indicate that the mode is unstable, as in the cylindrical theory), but the growth rate to be arbitrarily small. [DOI: 10.1063/1.3194270]

I. INTRODUCTION

Thermonuclear fusion requires plasmas with a pressure of at least an atmosphere and temperatures in excess of 100×10^6 K. These conditions can be achieved in tokamaks such as Joint European Torus (JET),[1] but the plasmas are subject to a number of instabilities, the consequences of which range from benign to structurally damaging. By understanding the instabilities that can occur, they can be avoided or mitigated. A class of instabilities that are only partly understood are edge localized modes (ELMs).[2] ELMs can lead to a rapid deposition of energy onto plasma facing components, and this is a potentially serious issue for proposed large tokamak devices such as ITER.[3]

Our present understanding of ELMs is based on the linear ideal magnetohydrodynamic peeling-ballooning instability,[4,5] which is thought to trigger ELMs that subsequently evolve nonlinearly. The studies upon which this understanding were based considered tokamak equilibria with a smoothly shaped magnetic flux-surface at the plasma-vacuum boundary. In contrast, modern tokamak plasmas have a cross section in which the outermost flux surface is redirected onto divertor plates, forming a separatrix with a sharp "*X*-point" where the magnetic topology changes from closed (confined plasma) to open field lines along which plasma can flow to the divertor plates.[1]

The first numerical evidence for a stabilizing effect from the separatrix was found in Ref. 6. More recently, numerical studies of the peeling-ballooning instability in these *X*-point plasmas have found that as the plasma's outermost flux surface is made increasingly close to that of a separatrix with an *X*-point, the peeling mode becomes stabilized.[7] Crucially the stabilization appeared to happen before the plasma formed a separatrix with an *X*-point, and shaping alone appeared to be sufficient to stabilize the mode. This seems to be contrary to theoretical work by Laval *et al.*[8] that finds the peeling mode unstable in cylindrical plasmas with an arbitrarily shaped cross section. In addition the ELITE code[9] has recently been used to examine peeling mode stability as the outermost flux surface approaches the separatrix. It was found that although the growth rate reduced in size as the boundary more closely approximated a separatrix, it did so increasingly slowly and its asymptotic behavior was uncertain.[10] To help understand and reconcile these results, here in the first part of this two-part paper we re-examine the assumptions made in the derivation of the peeling mode stability criterion for a cylinder, and generalize the calculation so that it applies to a toroidal tokamak plasma.

This paper generalizes previous analytic calculations in a number of ways. First it applies to axisymmetric toroidal geometries, as opposed to the cylindrical geometry in which the peeling mode has been extensively studied.[8,11,12] It allows for equilibrium poloidal currents at the plasma edge, in addition to the toroidal current that is solely included in previous analytic studies. The skin currents that are induced by a plasma perturbation are related to the difference between the magnetic field in the plasma and the vacuum, and it is found

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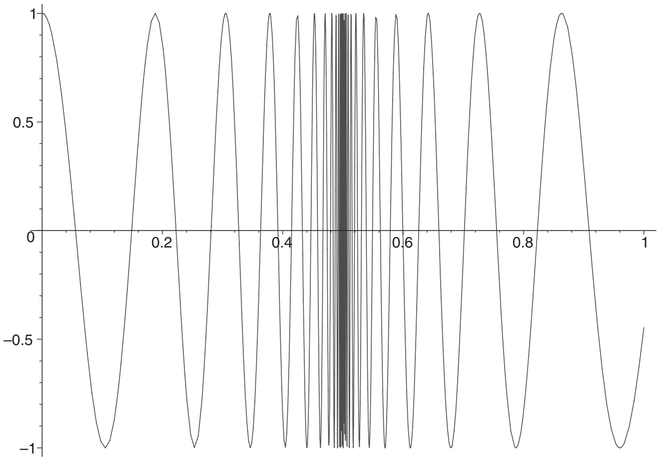

FIG. 1. A plot of the trial function $e^{im\theta}$ used by Laval *et al.* (Ref. 8) with $m \sim nq$. The second part of this paper shows that in a physically reasonable model, angle θ and the length l along a flux surface may be parametrized by α, with $\theta(\alpha) \simeq 1/q \int_{-\pi}^{\alpha} d\alpha/(\alpha^2/2+\epsilon^2)$ and $l(\alpha) \propto \int_{-\pi}^{\alpha}(\alpha^2/2+\epsilon^2)^{1/4} d\alpha$. The figure plots $e^{inq\theta(\alpha)}$ (vertical axis) and $l(\alpha)/l(\pi)$ (horizontal axis), for $\epsilon = 0.001$ and $n = 20$.

that at marginal stability a plasma perturbation induces a skin current that is parallel and proportional to the equilibrium edge current and proportional to the amplitude of the radial plasma displacement. The complicated-looking plasma-vacuum boundary condition that is usually found in association with the energy principle may be expressed as a simple relationship between the normal components of the plasma and vacuum magnetic fields. This is used to relate the generalized equations for peeling mode stability at marginal stability to the energy principle, allowing us to define the peeling mode in terms of the energy principle's[13] δW. With this energy principle for the peeling mode we can consider the trial function used by Laval *et al.*,[8] finding that a single parameter Δ' determines the sign and magnitude of δW. Finally the instability's growth rate is considered. A full and detailed derivation of the calculation outlined here is given in Ref. 14, and a summary of both parts is given in Ref. 15.

II. BACKGROUND

Laval *et al.*[8] considered a large aspect ratio ordering that neglects toroidal effects, and also neglects any equilibrium poloidal current at the plasma edge. The work suggests that the peeling mode will be unstable for a nonzero edge current, regardless of the plasma cross section. Later we will reconsider peeling mode stability for arbitrary cross section tokamak plasmas, but first we consider some properties of the trial function considered in Ref. 8.

Note that for an element of length dl along a flux surface in the poloidal plane, $dl/B_p = J_\chi B_p d\chi/B_p = \nu R^2 d\chi/I$ with J_χ the Jacobian, B_p the poloidal field, $I = RB_\phi$ with B_ϕ the toroidal field and R the major radius, $\nu = IJ_\chi/R^2$ is the local field-line pitch, and ψ, χ, and ϕ an orthogonal toroidal coordinate system (for example, see Freidberg[13] for details). Thus the poloidal angle used in Laval *et al.*, $2\pi\int_0^l (dl/B_p)/\oint(dl/B_p) = 2\pi\int^\chi \nu d\chi'/\oint \nu d\chi' = \theta$ is the same as the usual straight field line angle[16] and the "safety factor"[13] $q = (1/2\pi)\oint \nu d\chi = (1/2\pi)\oint (I/R^2)(dl/B_p)$. The perturbation they consider has a plasma displacement $\xi \sim e^{im\theta}$, so if we plot ξ versus the length along a flux surface in the poloidal plane, then near the separatrix in an X-point equilibrium ξ will oscillate arbitrarily rapidly as we approach the X-point (see Fig. 1). The most unstable modes have $m \simeq nq$, so in the figure we plot a mode with $m \simeq nq$, for which $e^{im\theta} \simeq e^{in\int^\chi \nu d\chi}$. Alternately, when $m \ll nq$, the mode is approximately constant everywhere except near the X-point. The rapid oscillation of ξ near the X-point makes it questionable whether it is physically acceptable, and other terms beyond ideal magnetohydrodynamics (MHD) need to be considered, but it certainly means that the closer we approach the separatrix the greater the number of Fourier modes required (since $m \simeq nq$), and the smaller a computer code's mesh spacing would need to be to represent the mode. Therefore as we approach the separatrix it will be increasingly difficult for a numerical calculation to represent the mode.

III. CYLINDRICAL PLASMAS

Here we outline the derivation of the marginal stability condition for a large aspect ratio (cylindrical) equilibrium with the tokamak ordering.[1,13] Starting from the usual force balance equation $\vec{J} \wedge \vec{B} = \nabla p$ we take the curl of both sides and expand to give

$$0 = \vec{B} \cdot \nabla \vec{J} - \vec{J} \cdot \nabla \vec{B}. \tag{1}$$

Linearizing the equation then gives for the equilibrium quantities

$$0 = \vec{B}_0 \cdot \nabla \vec{J}_0 - \vec{J}_0 \cdot \nabla \vec{B}_0 \tag{2}$$

and for the perturbed quantities

$$0 = \vec{B}_0 \cdot \nabla \vec{J}_1 + \vec{B}_1 \cdot \nabla \vec{J}_0 - \vec{J}_0 \cdot \nabla \vec{B}_1 - \vec{J}_1 \cdot \nabla \vec{B}_0 + \xi \cdot \nabla(\vec{B}_0 \cdot \nabla \vec{J}_0 - \vec{J}_0 \cdot \nabla \vec{B}_0). \tag{3}$$

Because Eq. (2) holds everywhere the last term in Eq. (3), that arises from the displacement of the plasma surface by ξ is zero. In the large aspect ratio approximation Eq. (3) further simplifies to

$$0 = \vec{B}_0 \cdot \nabla \vec{J}_1 + \vec{B}_1 \cdot \nabla \vec{J}_0 \tag{4}$$

and we may write $\vec{B}_1 = \vec{e}_z \wedge \nabla \tilde{\psi}$, for which

$$\vec{J}_1 = \nabla \wedge (\vec{e}_z \wedge \nabla \tilde{\psi}) = \vec{e}_z \nabla^2 \tilde{\psi} - \vec{e}_z \cdot \nabla\nabla \tilde{\psi}, \tag{5}$$

and so the $\vec{e}_z$ component of Eq. (4) gives

$$0 = \vec{B}_0 \cdot \nabla(\nabla^2 \tilde{\psi}) + \vec{B}_1 \cdot \vec{e}_r \frac{dJ_{0z}}{dr}. \tag{6}$$

Cylindrical symmetry implies that we need only consider a single mode, $\tilde{\psi} = \tilde{\psi}_m(r) e^{ikz+im\theta}$ where r, θ, and z are the usual cylindrical coordinates, with m and k the poloidal and toroidal mode numbers, respectively. Hence Eq. (6) now gives

$$0=\left(ikB_z+im\frac{B_p}{r}\right)\left\{\left(-k^2-\frac{m^2}{r^2}\right)\tilde{\psi}_m+\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\tilde{\psi}_m}{\partial r}\right\}$$
$$+B_{1r}\frac{dJ_{0z}}{dr}. \tag{7}$$

$\nabla\cdot\vec{B}=0$ requires the normal component of $\vec{B}$ to be continuous across the plasma-vacuum surface, and then $\vec{B}_1=\vec{e}_z\wedge\nabla\tilde{\psi}$ implies that $\tilde{\psi}_m$ is continuous across the surface (the perturbed field from $d\tilde{\psi}_m/dr$ may be discontinuous however). Also $J_{0z}=J_a$ just inside the plasma, but $J_{0z}=0$ in the vacuum outside the plasma, so if we integrate over an infinitesimal distance in a direction normal to the perturbed surface from plasma to vacuum (see Sec. IV A), we derive

$$0=\left(\frac{m-nq}{m}\right)\frac{\left[\left|r\frac{d\tilde{\psi}_m}{dr}\right|\right]}{\tilde{\psi}_m}+\frac{rJ_a}{B_p}, \tag{8}$$

as the condition for marginal stability. Here we have written $k=-n/R$ and $q=rB_z/RB_p$, and $[|f|]$ indicates the *difference* between f evaluated in the vacuum just outside of the plasma and f evaluated just inside the plasma.

IV. TOKAMAK PLASMAS

A. Generalizing the derivation

Here we re-examine the assumptions made in going from Eqs. (7) and (8). A careful and detailed derivation of the results in this section is given in Ref. 14. To obtain Eq. (8), we allowed the perturbed fields to be discontinuous, but required the equilibrium fields to be continuous across the plasma-vacuum boundary. Because a discontinuous magnetic field requires a skin current,[17] we will allow perturbed skin currents, but the continuous equilibrium fields imply zero equilibrium skin currents. Therefore we assume that: (a) there are no equilibrium skin currents, but $\vec{J}_0$ can be discontinuous at the plasma-vacuum interface, and (b) perturbations to the magnetic field induce surface skin currents. The first of these assumptions means that with the exception of $\vec{J}_0$, the equilibrium quantities will be continuous across the plasma-vacuum boundary. Therefore if we integrate along the unit normal to the plasma surface, from a distance ϵ just inside the surface to a distance ϵ just outside the plasma, then with the exception of $\vec{J}_0$, the integration of the equilibrium quantities will be of order ϵ and may be neglected. However for radial derivatives of $\vec{J}_0$ we have

$$\int_{l-\epsilon}^{l+\epsilon} f(l')\frac{d\vec{J}_0}{dl'}dl'=f(l)[|\vec{J}_0|] \tag{9}$$

for a path parametrized by l, and for an arbitrary continuous function $f(l)$. The second assumption (b), that the perturbation will produce a skin current $\vec{\sigma}$ at the plasma surface (i.e., that the perturbed magnetic field can be discontinuous at the perturbed plasma surface), means that for a perturbed current $\vec{J}_1$

$$\int_{l-\epsilon}^{l+\epsilon} f(l')\vec{J}_1(l')dl'=f(l)\frac{\vec{\sigma}}{RB_p}\neq 0. \tag{10}$$

This is our definition of the skin current $\vec{\sigma}$ that acts like a delta function in the perturbed current $\vec{J}_1$ at the plasma surface. As shown in Ref. 14, the integration $\int_{l-\epsilon}^{l+\epsilon}RB_p dl$ is equivalent at leading order in the plasma's displacement $\vec{\xi}$ to the integration $\int_{\psi_-}^{\psi_+}d\psi$, where ψ_- and ψ_+ are the equilibrium values of ψ just inside and just outside the equilibrium plasma surface. Hence the factor of $1/RB_p$ in Eq. (10) means that $\vec{\sigma}=\int_{\psi_-}^{\psi_+}\vec{J}_1 d\psi$.

Bearing the above remarks in mind, we now integrate Eq. (3) across the plasma surface, distinguishing perturbed quantities by a subscript of 1 and the equilibrium quantities by a subscript of 0, to get

$$0=\vec{B}_0\cdot\nabla\int_{\psi_-}^{\psi_+}\vec{J}_1 d\psi+\int_{\psi_-}^{\psi_+}(\vec{B}_1\cdot\nabla\psi)\frac{\partial\vec{J}_0}{\partial\psi}d\psi$$
$$-\left(\int_{\psi_-}^{\psi_+}\vec{J}_1 d\psi\right)\cdot\nabla\vec{B}_0+O(\xi^2). \tag{11}$$

Note that because the terms in Eq. (3) are of order ξ, the $O(\xi)$ corrections that arise when integrating along the normal to the surface produce terms of order ξ^2 and are neglected. Therefore at leading order we have

$$0=\vec{B}_0\cdot\nabla\vec{\sigma}_1+B_1^{\psi}[|\vec{J}_0|]-\vec{\sigma}_1\cdot\nabla\vec{B}_0, \tag{12}$$

where $B_1^{\psi}=\vec{B}_1\cdot\nabla\psi$. This is the generalized force balance condition for peeling mode marginal stability, valid for an arbitrary cross-section tokamak plasma. The last term is a new term that is not present in a circular cross-section cylindrical geometry. The procedure of integrating across the plasma-vacuum boundary is clearer if we project out the components before integrating across the surface. We have done this as a check, but it is algebraically cumbersome, and obscures the physical arguments presented here.

B. Ampere's law at the surface

Here we derive the relationship between discontinuities in the perturbed magnetic field and skin currents. At the plasma-vacuum interface, $\nabla\cdot\vec{B}=0$ and Ampere's law require[17]

$$[|\vec{n}\cdot\vec{B}|]=0, \tag{13}$$

$$\vec{n}\wedge(\vec{B}_V-\vec{B})=\frac{\vec{\sigma}}{RB_p}, \tag{14}$$

with $\vec{\sigma}=\int_{\psi_-}^{\psi_+}\vec{J}_1 d\psi$ as before, $\vec{n}$ denotes the unit normal to the surface and $\vec{B}_V$ the magnetic field in the vacuum. Because we assume zero equilibrium skin currents,

$$\frac{\vec{\sigma}_0}{RB_p}=\vec{n}_0\wedge(\vec{B}_0^V-\vec{B}_0)=0, \tag{15}$$

and therefore because Eq. (13) requires $\vec{n}_0\times(\vec{B}_0^V-\vec{B}_0)=0$, then $(\vec{B}_0^V-\vec{B}_0)$ is not parallel to $\vec{n}_0$ and the only way to satisfy Eq. (15) is if $\vec{B}_0^V=\vec{B}_0$.

Considering the lowest order perturbation, we have

$$\frac{\vec{\sigma}}{RB_p} = \vec{n}_1 \wedge (\vec{B}_0^V - \vec{B}_0) + \vec{n}_0 \wedge (\vec{B}_1^V - \vec{B}_1) + \xi . \nabla[\vec{n}_0 \wedge (\vec{B}_0^V - \vec{B}_0)], \quad (16)$$

which using $\vec{B}_0^V = \vec{B}_0$ and $\nabla\psi = RB_p\vec{n}_0$ gives

$$\vec{\sigma} = \nabla\psi \wedge (\vec{B}_1^V - \vec{B}_1) \quad (17)$$

and hence

$$\nabla\psi \cdot \vec{\sigma} = 0. \quad (18)$$

In the ψ, χ, and ϕ coordinate system it is possible to directly evaluate $\vec{\sigma} = \int_{\psi_-}^{\psi_+} \nabla \wedge \vec{B}_1 d\psi$ giving

$$\vec{\sigma} = R^2 \nabla \phi [|\vec{B}_p \cdot \vec{B}_1|] - R^2 \vec{B}_p [|\nabla\phi \cdot \vec{B}_1|]. \quad (19)$$

C. Skin currents at marginal stability

Using $\nabla\psi \cdot \vec{\sigma} = 0$, while projecting out components of Eq. (12), gives

$$0 = \vec{B}_0 \cdot \nabla(\vec{\sigma} \cdot \vec{B}_p) + \vec{B}_1 \cdot \nabla\psi [|\vec{B}_p \cdot \vec{J}_0|] - \vec{\sigma} \cdot \nabla B_p^2,$$
$$0 = \vec{B}_0 \cdot \nabla(\vec{\sigma} \cdot \nabla\phi) + \vec{B}_1 \cdot \nabla\psi [|\nabla\phi \cdot \vec{J}_0|] + 2I \frac{\vec{\sigma} \cdot \nabla R}{R^3}. \quad (20)$$

Because $2I\sigma \cdot \nabla R / R^3 = -\sigma \cdot \nabla(I/R^2) = -(\sigma \cdot \vec{B}_p / B_p^2)\vec{B}_p \cdot \nabla(I/R^2)$, we can rewrite Eq. (20) as

$$0 = \vec{B}_0 \cdot \nabla\left(\frac{\vec{\sigma} \cdot \vec{B}_p}{B_p^2}\right) + \vec{B}_1 \cdot \nabla\psi \frac{[|\vec{B}_p \cdot \vec{J}_0|]}{B_p^2},$$
$$0 = \vec{B}_0 \cdot \nabla(\vec{\sigma} \cdot \nabla\phi) + \vec{B}_1 \cdot \nabla\psi [|\nabla\phi \cdot \vec{J}_0|] - \left(\frac{\sigma \cdot \vec{B}_p}{B_p^2}\right)\vec{B}_p \cdot \nabla\left(\frac{I}{R^2}\right). \quad (21)$$

For an axisymmetric equilibrium, we can write[13] $\vec{B}_0 = I(\psi)\nabla\phi + \nabla\phi \wedge \nabla\psi$, with $I(\psi) = RB_\phi$, for which the equilibrium current $\vec{J}_0 = \nabla \wedge \vec{B}_0$ has $(\vec{B}_p \cdot \vec{J}_0)/(B_p^2) = -I'$ and $\nabla\phi \cdot \vec{J}_0 = -p' - II'/R^2$, and hence

$$\left[\left|\frac{\vec{B}_p \cdot \vec{J}_0}{B_p^2}\right|\right] = I_a',$$
$$[|\nabla\phi \cdot \vec{J}_0|] = p_a' + \frac{I_a I_a'}{R^2}. \quad (22)$$

Using $\vec{B}_1 \cdot \nabla\psi = \vec{B}_0 \cdot \nabla\xi_\psi$ and remembering that $I = I(\psi)$, Eqs. (21) now become

$$0 = \vec{B}_0 \cdot \nabla\left(\frac{\vec{\sigma} \cdot \vec{B}_p}{B_p^2} + I_a'\xi_\psi\right),$$
$$0 = \vec{B}_0 \cdot \nabla\left(\vec{\sigma} \cdot \nabla\phi + \xi_\psi\left(p_a' + \frac{I_a I_a'}{R^2}\right)\right) - \vec{B}_0 \cdot \nabla\left(\frac{I}{R^2}\right) \times \left(\frac{\vec{\sigma} \cdot \vec{B}_p}{B_p^2} + I_a'\xi_\psi\right). \quad (23)$$

Hence we require that

$$\frac{\vec{\sigma} \cdot \vec{B}_p}{B_p^2} = -I_a'\xi_\psi + f\left(\phi - \int^\chi \nu d\chi'\right),$$
$$\vec{\sigma} \cdot \nabla\phi = -\left(p_a' + \frac{I_a I_a'}{R^2}\right)\xi_\psi + \frac{I}{R^2} f\left(\phi - \int^\chi \nu d\chi'\right), \quad (24)$$

where f is some function of $\phi - \int^\chi \nu d\chi'$, so that $\vec{B}_0 \cdot \nabla f = 0$. Therefore using $(\vec{B}_p \cdot \vec{J}_0)/(B_p^2) = -I'$ and $\nabla\phi \cdot \vec{J}_0 = -p' - II'/R^2$, we may write Eq. (24) in terms of the equilibrium current $\vec{J}_0$, giving the skin currents at marginal stability as

$$\vec{\sigma} = \vec{J}_0\xi_\psi + \vec{B}_0 f\left(\phi - \int^\chi \nu d\chi'\right). \quad (25)$$

Because we require that $\vec{\sigma} = 0$ for $\xi_\psi = 0$ and because $\vec{B}_0 \cdot \nabla\xi_\psi \neq 0$, Eq. (24) is solved by

$$\vec{\sigma} = \vec{J}_0\xi_\psi, \quad (26)$$

i.e., the skin current due to a perturbation at marginal stability is equal to the product of the equilibrium current at the edge and the radial displacement of the plasma.

In deriving this result we have not used any expansion in terms of straight field line coordinates, for example, and we have not neglected any poloidal dependencies of the equilibrium quantities. We have implicitly assumed the existence of the ψ, χ, and ϕ coordinate system, but it is possible to re-express the coordinates in terms of arc length along a flux surface, and the consequent relations expressed in Eq. (19), for example, remain valid (except at the point of zero size that is the X-point). Hence the results appear valid at the separatrix.

D. Marginal stability

To relate this to the previous cylindrical condition for marginal stability, Eq. (8), we consider $\nabla \cdot \vec{B}$, which may be written

$$\nabla \cdot \vec{B} = \frac{1}{J_\chi}\left[\frac{\partial}{\partial\psi}(J_\chi \nabla\psi \cdot \vec{B}) + \frac{\partial}{\partial\chi}(J_\chi \nabla\chi \cdot \vec{B}) + \frac{\partial}{\partial\phi}(J_\chi \nabla\phi \cdot \vec{B})\right], \quad (27)$$

with J_χ the Jacobian. Then we evaluate $[|\nabla \cdot \vec{B}|]$, the difference between $\nabla \cdot \vec{B}$ evaluated in the vacuum just outside the plasma and $\nabla \cdot \vec{B}$ evaluated just inside the plasma. Noting that $|\nabla\chi| = 1/J_\chi B_p$, and hence $J_\chi \nabla\chi \cdot \vec{B}_1 = (1/B_p)(\nabla\chi/|\nabla\chi|) \cdot \vec{B}_1 = (\vec{B}_p \cdot \vec{B}_1 / B_p^2)$, then $[|\nabla \cdot \vec{B}_1 = 0|]$ requires that

$$0 = \left[\left|\frac{1}{J_\chi}\frac{\partial}{\partial\psi}J_\chi\nabla\psi\cdot\vec{B}_1\right|\right] + \frac{1}{J_\chi}\frac{\partial}{\partial\chi}\left[\left|\frac{\vec{B}_p\cdot\vec{B}_1}{B_p^2}\right|\right] + \frac{\partial}{\partial\phi}[|\nabla\phi\cdot\vec{B}_1|]. \quad (28)$$

Equation (19) gives

$$-\frac{\vec{\sigma}\cdot\vec{B}_p}{R^2B_p^2} = [|\nabla\phi\cdot\vec{B}_1|], \qquad \vec{\sigma}\cdot\nabla\phi = [|\vec{B}_p\cdot\vec{B}_1|]. \quad (29)$$

So we have

$$0 = \left[\left|\frac{1}{J_\chi}\frac{\partial}{\partial\psi}J_\chi\nabla\psi\cdot\vec{B}_1\right|\right] + \frac{1}{J_\chi}\frac{\partial}{\partial\chi}\left(\frac{\vec{\sigma}\cdot\nabla\phi}{B_p^2}\right) + \frac{\partial}{\partial\phi}\left(-\frac{\vec{\sigma}\cdot\vec{B}_p}{R^2B_p^2}\right). \quad (30)$$

To simplify this we consider a limit of high toroidal mode number n and write $(1/J_\chi)(\partial/\partial\chi)=-(I/R^2)(\partial/\partial\phi)+\vec{B}_0\cdot\nabla$. Then we note that $\vec{B}_0\cdot\nabla$ is of order one to prevent a large stabilizing contribution from field-line bending, but $(\partial/\partial\phi)$ is of order n. We also have[13,18] $(\partial/\partial\psi)\nabla\psi\cdot\vec{B}_1\sim n$. Because the system is axisymmetric, we need only consider a single Fourier mode in the toroidal angle and take $\vec{\sigma}\sim e^{-in\phi}$. Then we have

$$0 = \left[\left|\frac{1}{J_\chi}\frac{\partial}{\partial\psi}J_\chi\nabla\psi\cdot\vec{B}_1\right|\right] + \frac{in}{R^2B_p^2}(I\vec{\sigma}\cdot\nabla\phi + \vec{\sigma}\cdot\vec{B}_p) + O(1). \quad (31)$$

$\nabla\cdot\vec{B}=0$ requires that $[|\nabla\psi\cdot\vec{B}_1|]=0$, so there will be zero contribution from the terms involving $\partial J_\chi/\partial\psi$. Rearranging the resulting equation leaves

$$R^2B_p^2\left[\left|\frac{i}{n}\frac{\partial}{\partial\psi}\nabla\psi\cdot\vec{B}_1\right|\right] = \vec{B}_0\cdot\vec{\sigma} + O\left(\frac{1}{n}\right). \quad (32)$$

Equation (32) is our generalized criterion for marginal stability to the peeling mode at high-n, with $\vec{\sigma}$ given by Eq. (26).

E. Cylindrical limit

In cylindrical geometry Eq. (26) becomes $\vec{\sigma}=RB_p\xi_r\vec{J}_0$ and Eq. (32) becomes

$$\frac{RB_p}{rB_0}\left[\left|\frac{i}{n}\frac{db_r}{dr}\right|\right] = \frac{\vec{B}_0\cdot\vec{J}_0}{B_0}\frac{\xi_r}{r}. \quad (33)$$

Noting that $b_r=\vec{B}\cdot\nabla\xi_r$, taking $\xi\sim e^{im\theta-in\phi}$, and using $(1/nq)=(1/m)+(m-nq)/nq$ with $m\simeq nq$, give

$$0 = \frac{rJ_\parallel}{B_p} + \frac{m-nq}{m}\frac{\left[\left|r\frac{db_r}{dr}\right|\right]}{b_r}, \quad (34)$$

where for a cylinder $q=rB_0/R_0B_p$. Finally the plasma vacuum boundary conditions,[13]

$$\vec{n}_0\cdot\vec{B}_1^V = \vec{B}_0\cdot\nabla(\vec{n}_0\cdot\vec{\xi}) - (\vec{n}_0\cdot\vec{\xi})\vec{n}_0\cdot(\vec{n}_0\cdot\nabla\vec{B}_0), \quad (35)$$

in cylindrical geometry require that $b_r^V=\vec{B}_0\cdot\nabla\xi_r=b_r$. So we regain the usual condition for marginal stability to the peeling mode in cylindrical geometry of

$$\Delta_a'\Delta_a + J_\parallel = 0, \quad (36)$$

with $\Delta_a'=[|(r/b_r)(db_r/dr)|]$ and $\Delta_a=(1-nq/m)$.

V. THE ENERGY PRINCIPLE

Now we consider how the equations for marginal stability of the peeling mode relate to the energy principle. We consider the high mode number formulation given in Ref. 12 for $\delta W=\delta W_F+\delta W_S+\delta W_V$, where δW_F, δW_S, and δW_V are the so-called plasma, surface, and vacuum contributions to the energy principle's δW.[13] Looking first at δW_S, we have

$$\delta W_S = \pi\oint d\chi\frac{\xi^*}{n}J_\chi Bk_\parallel\left[\frac{R^2B_p^2}{J_\chi B^2}\frac{1}{n}\frac{\partial}{\partial\psi}(J_\chi Bk_\parallel\xi_\psi) - \frac{\vec{B}\cdot\vec{J}}{B^2}\xi_\psi\right], \quad (37)$$

with $J_\chi Bk_\parallel=iJ_\chi\vec{B}\cdot\nabla$. (Note that Connor *et al.*[12] took $\xi\sim e^{+in\phi}$, however, here we have $\xi\sim e^{-in\phi}$, which is reflected by $n\to-n$ in the expression for δW.) We replace $J_\chi Bk_\parallel$ with $iJ_\chi\vec{B}\cdot\nabla$, integrate by parts, and use $\vec{B}\cdot\nabla\xi_\psi^*=\nabla\psi\cdot\vec{B}_1^*$ to give

$$\delta W_S = \pi\oint J_\chi d\chi\left(\frac{i}{n}\nabla\psi\cdot\vec{B}_1^*\right)\times\left[-\frac{R^2B_p^2}{B^2}\frac{i}{n}\frac{1}{J_\chi}\frac{\partial}{\partial\psi}(J_\chi\nabla\psi\cdot\vec{B}_1) + \frac{\vec{B}\cdot\vec{J}}{B^2}\xi_\psi\right]. \quad (38)$$

To obtain the vacuum solution for $\vec{B}_1^V=\nabla V$, we need to solve $\nabla^2V=0$, which requires

$$0 = \frac{1}{J_\chi}\left\{\frac{\partial}{\partial\psi}(J_\chi\nabla\psi\cdot\nabla V) + \frac{\partial}{\partial\chi}(J_\chi\nabla\chi\cdot\nabla V) + \frac{\partial}{\partial\phi}(J_\chi\nabla\phi\cdot\nabla V)\right\}. \quad (39)$$

Using $(1/J_\chi)(\partial/\partial\chi)=-(I/R^2)(\partial/\partial\phi)+\vec{B}\cdot\nabla$, $|\nabla\chi|^2=1/J_\chi^2B_p^2$, and $V\sim e^{-in\phi}$, we can rewrite this as

$$0 = \frac{1}{J_\chi}\frac{\partial}{\partial\psi}(J_\chi\nabla\psi\cdot\nabla V) + \left(\vec{B}\cdot\nabla + in\frac{I}{R^2}\right)\frac{1}{B_p^2}\times\left(\vec{B}\cdot\nabla + in\frac{I}{R^2}\right)V - \frac{n^2}{R^2}. \quad (40)$$

Then using $\vec{B}\cdot\nabla\sim 1$, and taking the high-n limit, gives

$$0 = \frac{1}{J_\chi}\frac{\partial}{\partial\psi}(J_\chi\nabla\psi\cdot\nabla V) - n^2\frac{I^2}{R^4B_p^2}V - n^2\frac{1}{R^2}V, \quad (41)$$

which using $\vec{B}_1^V=\nabla V$ rearranges to give the result that for $n\gg 1$,

$$V = \frac{R^2 B_p^2}{B^2} \frac{1}{n^2} \frac{1}{J_\chi} \frac{\partial}{\partial \psi} (J_\chi \nabla \psi \cdot \vec{B}_1^V). \tag{42}$$

The vacuum contribution to δW is

$$\delta W_V = \frac{1}{2} \int \vec{dr} |\vec{B}_1^V|^2. \tag{43}$$

Using $\nabla^2 V = 0$, $\vec{B}_1 = \nabla V$, and Gauss' theorem, this may be written as an integral over the plasma surface, with

$$\delta W_V = -\frac{1}{2} \int V \vec{dS} \cdot \nabla V^*. \tag{44}$$

Using $\vec{dS} = (\nabla\psi / RB_p)(J_\chi B_p d\chi)(R d\phi)$ and integrating with respect to ϕ gives

$$\delta W_V = -\pi \oint J_\chi d\chi V \nabla \psi \cdot \nabla V^*. \tag{45}$$

The plasma-vacuum boundary conditions of Eq. (35) are identical to requiring that[14,18] $\nabla\psi \cdot \vec{B}_1 = \nabla\psi \cdot \vec{B}_1^V$. Using this and Eq. (42) for V at high-n, we get

$$\delta W_V = -\pi \oint J_\chi d\chi (\nabla \psi \cdot \vec{B}_1) \times \left[\frac{R^2 B_p^2}{B^2} \frac{1}{n^2} \frac{1}{J_\chi} \frac{\partial}{\partial \psi} (J_\chi \nabla \psi \cdot \vec{B}_1^V) \right]. \tag{46}$$

Equations (38) and (46) together give

$$\delta W_S + \delta W_V = \pi \oint J_\chi d\chi \left(\frac{i}{n}\right) (\nabla \psi \cdot \vec{B}_1^*) \times \left[\frac{R^2 B_p^2}{B^2} \frac{i}{n} \frac{1}{J_\chi} \frac{\partial}{\partial \psi} (J_\chi \nabla \psi \cdot \vec{B}_1^V) - \frac{R^2 B_p^2}{B^2} \frac{i}{n} \frac{1}{J_\chi} \frac{\partial}{\partial \psi} (J_\chi \nabla \psi \cdot \vec{B}_1) + \frac{\vec{B} \cdot \vec{J}}{B^2} \xi_\psi \right]. \tag{47}$$

Because $\nabla\psi \cdot \vec{B}_1^V = \nabla\psi \cdot \vec{B}_1$, the terms involving $\partial J_\chi / \partial\psi$ will cancel. Then using the notation $[|f|]$ to denote the difference between f evaluated just outside and just inside the plasma, we have

$$\delta W_S + \delta W_V = \pi \oint J_\chi d\chi \left(\frac{i}{n}\right) \frac{(\nabla \psi \cdot \vec{B}_1^*)}{B^2} \times \left\{ R^2 B_p^2 \left[\left| \frac{i}{n} \frac{\partial}{\partial \psi} (\nabla \psi \cdot \vec{B}_1) \right| \right] + \vec{B} \cdot \vec{J} \xi_\psi \right\}. \tag{48}$$

The term in { } is exactly Eq. (32) with $\vec{\sigma} = \vec{J}_0 \xi_\psi$ as given by Eq. (26). Therefore marginal stability of the peeling mode corresponds (at high-n) to taking the plasma's contribution to δW, $\delta W_F \equiv 0$, and then solving $\delta W_S + \delta W_V = 0$. This suggests we should define the high-n peeling mode as a mode that allows us to neglect δW_F compared to $\delta W_S + \delta W_V$ (by for example being sufficiently localized), and whose subsequent stability is determined by δW_S and δW_V.

VI. *X*-POINT PLASMAS

Now we consider the stability of peeling modes to the trial function considered by Laval *et al.*[8] that consists of a single Fourier mode with $\xi = \xi_m(\psi) e^{im\theta - in\phi}$, where $\theta = (1/q) \int^\chi \nu d\chi'$ is the usual straight field-line poloidal coordinate. Because $\nabla\psi \cdot \vec{B}_1^V = \nabla\psi \cdot \vec{B}_1$, we can rewrite $[|\partial / \partial\psi (\nabla\psi \cdot \vec{B}_1)|]$ as $[|(\partial / \partial\psi (\nabla\psi \cdot \vec{B}_1)) / (\nabla\psi \cdot \vec{B}_1)|] \nabla\psi \cdot \vec{B}_1$. Then using $\nabla\psi \cdot \vec{B}_1 = \vec{B} \cdot \nabla \xi_\psi$ and substituting the trial function into Eq. (48), we get

$$\delta W = -2\pi^2 \frac{|\xi_m|^2}{R_0} \Delta (\Delta \hat{\Delta}' + \hat{J}), \tag{49}$$

where $\Delta \equiv (m - nq)/(nq)$,

$$\hat{J} \equiv \frac{1}{2\pi} \oint dl \frac{I R_0}{R^2 B_p} \frac{\vec{J} \cdot \vec{B}}{B^2} \tag{50}$$

and

$$\hat{\Delta}' \equiv \left[\left| \frac{1}{2\pi} \oint dl R_0 B_p \frac{I^2}{R^2 B^2} \frac{\frac{\partial}{\partial \psi} (\nabla \psi \cdot \vec{B}_1)}{\nabla \psi \cdot \vec{B}_1} \right| \right], \tag{51}$$

with $dl = J_\chi B_p d\chi$ an element of arc length in the poloidal cross section and R_0 is a typical measure of the major radius such as its average, for example, that is included so that δW, $\hat{J}$, and Δ' have their usual dimensions and form. Note that because ξ_m is a Fourier component of $\vec{\xi} \cdot \nabla\psi \sim \xi (RB_p)$, the dimension of $|\xi_m|^2 / R_0$ is energy. Equation (49) may easily be minimized for Δ (or equivalently, minimized with respect to choice of toroidal mode number), finding $\Delta = -\hat{J} / (2\hat{\Delta}')$, for which

$$\delta W = \left(\frac{\pi}{2}\right)^2 \frac{|\xi_m|^2}{R_0} \left(\frac{2\hat{J}^2}{\hat{\Delta}'}\right). \tag{52}$$

For convenience in the following we will write Δ' in place of $\hat{\Delta}'$.

If Δ is chosen to maximize the growth rate, then a different value will be found, but with a very similar order of magnitude. Also, there is no reason why there should not be a more unstable mode than the trial function we have considered. However, our primary interest is stability to the trial function that was found to be unstable by Laval *et al.*,[8] whose stability is determined by Eq. (52) with a value dependent on Δ'. The calculation of Δ' when there is a separatrix with an *X*-point is the subject of the second part to this paper. For a circular cross section plasma, an estimate for Δ' is found by approximating the perturbation to the magnetic field near the edge of the plasma as being the same as for a vacuum, then solving Laplace's equation both inside and outside the plasma, and matching the solutions at the plasma-vacuum boundary. This gives $\Delta' = -2m \simeq -2nq$.

If $\vec{J}\cdot\nabla\phi=0$, then $\vec{B}\cdot\vec{J}=-I'B_p^2$, and $\hat{J}$ becomes

$$\hat{J}=\frac{1}{2\pi}\oint dl\frac{IR_0}{R^2B_p}\frac{(-I')B_p^2}{B^2}\sim 1. \quad (53)$$

So provided $\Delta'\sim nq$, we will have $\delta W\to 0$ as $q\to\infty$. If $\vec{J}\cdot\nabla\phi\neq 0$, then $\vec{B}\cdot\vec{J}=-Ip'-B^2I'$ and

$$\hat{J}\simeq\left(R_0\frac{\vec{J}\cdot\vec{B}}{B^2}\right)\oint\frac{I}{R^2}\frac{dl}{B_p}=R_0\left(\frac{\vec{J}\cdot\vec{B}}{B^2}\right)q. \quad (54)$$

For which if $\Delta'\sim -nq$, then $\hat{J}^2/\Delta'$ would be of order $-q/n$ and $\delta W<0$, suggesting that the mode would be unstable.

The sign of δW is usually taken to indicate whether a mode is unstable or not; however, the growth rate determines how unstable the mode is (i.e., how rapidly it develops). For example if our trial function $\xi_m(\psi)e^{im\theta}=\xi_\psi=\nabla\psi\cdot\vec{\xi}$ had been $\xi_m(\psi)e^{im\theta}=\nabla\psi\cdot\vec{\xi}/RB_p$ so that ξ_m had dimensions of length as opposed to dimensions of length times RB_p, then we would no longer have $\hat{J}\sim q$, despite our model only depending on the poloidal structure of the mode. The dependence of δW on the normalization of the plasma perturbation does not affect the calculation of the growth rate, however, for which the consequences of the normalization of the plasma perturbation will cancel. The growth rate is discussed later.

Are there any reasons why a computer code might fail to find an unstable mode? One possibility is that the need for $m\sim nq$ will require very high poloidal mode numbers as $q\to\infty$, and this could potentially prevent a numerical code from seeing the instability. Also important is the need to consider the most unstable mode. Minimizing δW with respect to the toroidal mode number gave $\Delta=-\hat{J}/2\Delta'$. If $\Delta'\simeq -2nq$ and $\hat{J}\simeq qR_0(\vec{J}\cdot\vec{B})/B^2$ (for $\vec{J}\cdot\nabla\phi\neq 0$), this would require $\Delta\simeq -\hat{J}/(2\Delta')=(R_0\vec{J}\cdot\vec{B}/B^2)(1/4n)$, a value independent of q and the poloidal mode number. A final possibility is that Δ' might diverge more rapidly than $\hat{J}$ as we approach the separatrix. This is considered in the second part to this paper by analytically calculating Δ' for a separatrix with an X-point, with the analytical calculation avoiding the numerical problems usually associated with an X-point. A less obvious reason why an unstable mode might not be found in computer calculations is that despite $\delta W<0$ indicating that it is energetically favorable for the mode to be unstable, the growth rate can still be vanishingly small. This possibility is explored next.

VII. THE GROWTH RATE

So far we have considered δW because its sign is usually presumed to be sufficient to indicate whether a mode is stable or not. The growth rate γ for a mode with $\xi\sim e^{\gamma t}$ determines how unstable a mode and is obtained from[13] $\gamma^2=-\delta W/\frac{1}{2}\int\rho_0|\xi|^2\vec{dr}$, where $\frac{1}{2}\int\rho_0|\xi|^2\vec{dr}$ is the kinetic energy term. Here we will estimate $\int\rho_0|\vec{\xi}|^2\vec{dr}$, first writing

$$\vec{\xi}=\xi_\psi\frac{\nabla\psi}{R^2B_p^2}+\xi_B\frac{\vec{B}}{B^2}+\xi_\perp\frac{\vec{B}\wedge\nabla\psi}{R^2B_p^2B^2} \quad (55)$$

a form in which we can use the results from a high-n ordering[18,19] that gives

$$\xi_\perp=\frac{i}{n}\nabla\psi\cdot\nabla\xi_\psi=\frac{i}{n}R^2B_p^2\frac{\partial\xi_\psi}{\partial\psi} \quad (56)$$

and

$$\vec{B}\cdot\nabla\xi_B-\xi_B\frac{\vec{B}\cdot\nabla B^2}{B^2}=\xi_\psi\frac{\partial}{\partial\psi}(2p+B^2)+\frac{I\xi_\perp}{R^2B_p^2B^2}\vec{B}\cdot\nabla B^2. \quad (57)$$

Because the trial function that we consider consists of a single Fourier mode, Eq. (57) may be solved for ξ_B, with

$$\xi_B=\frac{\xi_\psi\frac{\partial}{\partial\psi}(2p+B^2)+\frac{I\vec{B}\cdot\nabla B^2}{B^2}\left(\frac{i}{n}\frac{\partial\xi_\psi}{\partial\psi}\right)}{\frac{I}{qR^2}(im-inq)-\frac{\vec{B}\cdot\nabla B^2}{B^2}}. \quad (58)$$

Using Eqs. (55), (56), and (58) we can estimate the largest terms in $|\xi|^2$, finding[14]

$$|\vec{\xi}|^2\sim\frac{|\xi_\psi|^2}{R^2B_p^2}+\frac{R^2B_p^2}{B^2}\frac{1}{n^2}\left|\frac{\partial\xi_\psi}{\partial\psi}\right|^2. \quad (59)$$

Although we have not considered the radial structure of the mode in this paper, we will assume that near the plasma's edge ξ_m can be approximated by a power law, with

$$\xi_m=\xi_0\frac{(\psi_a-\psi_s)^p}{(\psi_a-\psi)^p}, \quad (60)$$

where $\psi=\psi_a$ at the separatrix and $\psi=\psi_s$ at the plasma surface. This is consistent with studies that consider the mode's radial structure.[11,12] We also take

$$q\simeq -q_0\ln\left(\frac{\psi_a-\psi}{\psi_a}\right) \quad (61)$$

for some constant $q_0\sim 1$, as found here[20] and elsewhere.[21] For the trial function $\xi_\psi=\xi_m(\psi)e^{im\theta}$ with $\theta=(1/q)\int^\chi\nu d\chi'$, and ξ_m given by Eq. (60), then[14] $\partial\xi_\psi/\partial\psi\propto q'$. Integrating Eq. (59) allows us to estimate the kinetic energy term, finding at leading order[14]

$$\int\rho_0|\xi|^2\vec{dr}\sim\frac{\rho_0\oint dl}{B^2}(\psi_sq')|\xi_\psi|^2|_{\psi=\psi_s}, \quad (62)$$

with $\psi_s\sim\oint R^2B_pdl/\oint dl$.

Now we can estimate the growth rate γ that for $\nabla\phi\cdot\vec{J}\neq 0$ and $\Delta'\simeq -2nq$ (Δ' is calculated in the second part to this paper), then gives

$$\gamma^2=\frac{-\delta W}{\int\vec{dr}\rho_0|\xi|^2}\sim\gamma_A^2\left(\frac{R_0\vec{J}\cdot\vec{B}}{B^2}\right)^2\left(\frac{q}{\psi_sq'}\right), \quad (63)$$

with $\gamma_A\equiv B^2/(\rho_0R\oint dl)$. Because $q'\to\infty$ more rapidly than q as we approach the separatrix, then for an outermost flux surface that is made increasingly close to that of a separatrix with $\psi_s \to \psi_a$, we have that $\gamma^2 \to 0$ and

$$\ln\left(\frac{\gamma}{\gamma_A}\right) = -\frac{1}{2}\ln\left(\frac{\psi_s q'}{q}\right), \quad (64)$$

with γ_A the Alfven frequency, indicating that the growth rate $\gamma \to 0$ as $q' \to \infty$. This argument is repeated by a different route, with the same conclusions,[22] in the Appendix. The behavior of γ/γ_A has subsequently been found to be in agreement with results from ELITE,[15] and is similar to the behavior of the Mercier coefficient[13] D_M near a separatrix.[23]

VIII. SUMMARY

This paper re-explores the stability of the peeling mode for toroidal tokamak geometry. It starts from a simple approach to peeling mode stability at marginal stability in cylindrical geometry, then generalizes this to toroidal tokamak equilibrium. In the process of doing so we find a number of interesting results, namely,

(1) At marginal stability, a plasma perturbation induces a skin current that is parallel and proportional to the equilibrium current at the edge, and proportional to the radial plasma displacement.
(2) The equilibrium conditions (force balance) for the peeling mode at marginal stability and high toroidal mode number n are identical to requiring $\delta W_S + \delta W_V = 0$, where δW_S and δW_V are the surface and vacuum contributions to the energy principle's $\delta W = \delta W_F + \delta W_S + \delta W_V$, with δW_F the plasma's contribution to the energy principle (Freidberg (Ref. 13)]. This suggests the peeling mode be defined as a mode for which $\delta W_F \ll \delta W_S + \delta W_V$. For the trial function used by Laval *et al.*[8] that consisted of a single Fourier mode in straight field line coordinates, we find that the most unstable choice of Δ gives $\delta W = (\pi/2)^2(|\xi_m|^2/R_0)(2\hat{J}^2/\Delta')$. To evaluate δW for this model, it is necessary to know Δ' for a plasma cross section with a separatrix and X-point at the plasma-vacuum boundary. The evaluation of Δ' without the approximations required by a numerical calculation, is the subject of the second part to this paper (Ref. 20).
(3) Finally we considered the growth rate and found that even with $\delta W < 0$, the growth rate can be vanishingly small. This is because the kinetic energy term was found to diverge like q' as the outermost flux surface becomes increasingly close to a separatrix. When this divergence is sufficiently rapid (as would be the case for $\Delta' \simeq -2nq$), then $\ln(\gamma/\gamma_A)$ asymptotes to $\ln(\gamma/\gamma_A) = -1/2\,\ln(\psi_s q'/q)$, a result that has been found to agree with those from the ELITE code (Ref. 15).

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APPENDIX: ALTERNATIVE DERIVATION FOR THE GROWTH RATE

Here we give an alternative derivation for the growth rate.

(1) We will continue to use the high-n ordering of Connor *et al.*,[12] for which $\nabla \cdot \vec{\xi} = 0$ requires

$$\xi_\perp = \frac{i}{n} R^2 B_p^2 \frac{\partial}{\partial \psi} \xi_\psi. \quad (A1)$$

(2) We will use the arguments from Sec. VII that lead us to expect that $\partial \xi_\psi / \partial \psi \sim (mq'/q)|\xi_m| \sim nq'|\xi_m|$, for the trial function of Laval *et al.*(Ref. 8) with $\xi_\psi = \xi_m(\psi) e^{im\theta}$.
(3) We will, as in Sec. VII, continue to assume that near the separatrix,

$$q \simeq -q_0 \ln\left(\frac{\psi_a - \psi}{\psi_a}\right) \quad (A2)$$

for some constant $q_0 \sim 1$, with ψ_a the value of ψ at the separatrix. We will also continue to assume that near the separatrix we can approximate $|\xi_m(\psi)|$ as a power law, with

$$\xi_m = \xi_0 \frac{(\psi_a - \psi_s)^p}{(\psi_a - \psi)^p}, \quad (A3)$$

where $\psi_s < \psi_a$ is the value of ψ at the plasma-vacuum surface.

With the assumptions of 1, 2, and 3, we require[14]

$$\xi_\perp \sim i R^2 B_p^2 \frac{q_0}{\psi_a - \psi} \xi_0 \frac{(\psi_a - \psi_s)^p}{(\psi_a - \psi)^p}. \quad (A4)$$

However, because we are considering a linear stability analysis, we must always have $\xi_\perp \ll 1$, even as $\psi \to \psi_s$. This requires $\xi_0 = \hat{\xi}_0(\psi_a - \psi_s)/\psi_a$ with $\hat{\xi}_0 \ll 1$ a constant, so that as $\psi \to \psi_a$ we continue to have $\xi_\perp \sim \hat{\xi}_0 \ll 1$. Therefore as a consequence

$$\xi_\psi \sim \hat{\xi}_0 \frac{(\psi_a - \psi_s)}{\psi_a} \frac{(\psi_a - \psi_s)^p}{(\psi_a - \psi)} \to \hat{\xi}_0 \frac{(\psi_a - \psi_s)}{\psi_a} \quad \text{as} \quad \psi \to \psi_a. \quad (A5)$$

Hence in the limit where the plasma surface tends to a separatrix, with $\psi_s \to \psi_a$, we have $\xi_\psi \sim \hat{\xi}_0(\psi_a - \psi_s)/\psi_a \to 0$. Therefore δW, for which ξ_m is evaluated at $\psi = \psi_s$, has

$$\delta W \sim -\frac{|\hat{\xi}_0|^2}{R_0}\left(\frac{R_0 \vec{J} \cdot \vec{B}}{B^2}\right)\left(\frac{q}{n}\right)\left(\frac{\psi_a - \psi_s}{\psi_a}\right)^2 \to 0 \quad \text{as} \quad \psi_s \to \psi_a. \quad (A6)$$

Next we consider the growth rate. From Sec. VII, we expect

$$\int \vec{dr}|\xi|^2 \sim \int \vec{dr}\frac{|\xi_\perp|^2}{R^2B_p^2B^2} \tag{A7}$$

that with $\xi_\perp$ given by Eq. (A4) gives[14]

$$\int \vec{dr}\frac{|\xi_\perp|^2}{R^2B_p^2B^2} \sim \frac{|\hat{\xi}_0|^2}{p}\frac{\oint dl}{\langle B^2\rangle}\left(\frac{\psi_a-\psi_s}{\psi_a}\right), \tag{A8}$$

where $\langle\,\rangle$ denotes a poloidal average, and we use $\psi_a \sim \langle B_p\rangle R^2$. Hence combining Eqs. (A6)–(A8) we find

$$\gamma^2 = \frac{\delta W}{\int \vec{dr}\rho_0|\xi|^2} \sim \gamma_A^2\left(\frac{R_0\vec{J}\cdot\vec{B}}{B^2}\right)\left(\frac{\psi_a-\psi_s}{\psi_a}\right)\frac{pq}{n}, \tag{A9}$$

with $\gamma_A^2 = \langle B^2\rangle/\rho_0 R_0 \oint dl$. Therefore as the outermost plasma surface more closely approximates a separatrix with $\psi_s \to \psi_a$, we have that $\gamma/\gamma_A \to 0$. Note that because we have taken $q \sim q_0 \ln[(\psi_a-\psi)/\psi_a]$, then $q' \sim q_0/(\psi_a-\psi)$, and hence $(\gamma/\gamma_A)^2 \propto q/q'$ as found in Sec. VII.

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[19] Before continuing further we make some observations on the high-n ordering that for $\nabla\cdot\vec{\xi}=0$ usually leads to $\xi_\perp=(i/n)R^2B_p^2(\partial\xi_\psi/\partial\psi)$. This analysis is used in the derivation of δW used in Sec. V onwards, and to derive the equations solved by ELITE (Ref. 9). The ordering implicitly assumes that $\partial\xi_\psi/\partial\psi \gg (\xi_\psi/J_\chi)\partial J_\chi/\partial\psi$, which is the case in the plasma core where $(1/J_\chi)\partial J_\chi/\partial\psi \sim 1$ because $\partial\xi_\psi/\partial\psi \sim n \gg 1$. Whereas a sufficiently large n can always be found to ensure $(1/n)(\xi_\psi/J_\chi)\partial J_\chi/\partial\psi \ll 1$, and terms of this type will often be negligible order one contributions anyhow, future calculations would be improved by including them. For example $\xi_\perp$ would then become $\xi_\perp=(i/n)(R^2B_p^2/J_\chi)\partial(J_\chi\xi_\psi)/\partial\psi$. For the present we will continue to use the ordering employed by Connor *et al.* (Ref. 12) that is also used to derive the equations solved by ELITE.
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[22] Observe that we could have (incorrectly) argued that whereas $\gamma \propto \sqrt{q/q'} \to 0$ at the separatrix, the plasma motion $v_\perp = \gamma\xi_\perp e^{\gamma t} \propto \gamma q'\xi_\psi \propto \sqrt{q'}\xi_\psi$ does not. This argument is incorrect, however, because the linearized stability analysis requires that $\xi_\perp \ll 1$ always, and hence that $\xi_\psi \sim 1/q'|_{\psi_s}$ where ψ_s denotes the plasma surface. Then we have $\gamma\xi_\perp \propto 1/\sqrt{q'} \to 0$. The normalization of ξ_ψ is discussed in detail in the Appendix.
[23] A. J. Webster, Phys. Plasmas **16**, 012501 (2009).