Conservative regularization of ideal hydrodynamics and magnetohydrodynamics

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Inviscid, incompressible hydrodynamics and incompressible ideal magnetohydrodynamics (MHD) share many properties such as time-reversal invariance of equations, conservation laws, and certain topological features. In three dimensions, these systems may lead to singular solutions (involving vortex and current sheets). While dissipative (viscoresistive) effects can regularize the equations leading to bounded solutions to the initial-boundary value (Cauchy) problem which presumably exist uniquely, the time-reversal symmetry and associated conservation properties are certainly destroyed. The present work is analogous to (and suggested by) the Korteweg–de Vries regularization of the one-dimensional, nonlinear kinematic wave equation. Thus the regularizations applied to the original equations of hydrodynamics and ideal MHD retain conservation properties and the symmetries of the original equations. Integral invariants which generalize those known for the original systems are shown to imply bounded enstrophy. The regularization developed can also be applied to the corresponding dissipative models (such as the Navier–Stokes equations and the viscoresistive MHD equations) and may imply interesting regularity properties for the solutions of the latter as well. The models developed thus have intrinsic mathematical interest as well as possible applications to large-scale numerical simulations in systems where dissipative effects are extremely small or even absent. © *2010 American Institute of Physics*. [doi:10.1063/1.3339913]

I. INTRODUCTION

Dissipationless (sometimes called "ideal") systems arise in many fields of continuum mechanics and kinetics. Standard continuum examples are the well-known Euler equations of an ideal, frictionless, incompressible fluid and the magnetohydrodynamics ("MHD") equations of motion. In kinetics one has the full n-particle Liouville equation for the distribution function in phase space and the "collisionless" Boltzmann equations (also called the Vlasov equation). Such equations have many important properties such as well-defined conservation laws and generally describe time-reversible dynamical evolution. However, in most cases where nonlinearities are involved in an essential way, the equations may not have solutions valid for all time, or even a sufficiently long time-scale of physical interest. There is an extensive literature concerning such questions of existence, uniqueness, and regularity of solutions to appropriate initial-boundary-value (Cauchy) problems. For well-motivated, accessible introductions the following standard works[1–4] which give extensive references to related recent papers should be consulted. The specific problem relating to the existence of finite classical solutions to the Cauchy problem in three dimensions of the Euler equations is still open (see, for example, the following Refs. 5–7). Thus it is conjectured that there could be a "finite-time blow up" of vorticity in the solutions. This can occur simply because the Euler equations fail to describe the correct physics at sufficiently short length-scales and there is always a "direct cascade" of enstrophy in three dimensions (suggested by Richardson and Taylor, as discussed in Ref. 2) from the long wavelength energy containing eddies to very short wavelength ones (or equivalently sufficiently high frequencies and wave numbers).

In many cases, the resolution of the problem of preventing such "ultraviolet catastrophes" is straightforward: one takes account of "viscous" or other dissipative processes operative at very short distances and adds terms to equations of motion with higher derivatives of the dependent variables, multiplied by a small parameter (usually the reciprocal of some "effective Reynolds number"). This is often sufficient to "regularize" the solutions to the initial value (Cauchy) problem of the original system, that is, guarantee existence and uniqueness (though not necessarily the Lyapunov stability) of the solutions. However, the extended system is no longer time-reversible and has significantly different physical (and mathematical) properties to the original one, although globally valid solutions to the Cauchy problem may well exist uniquely and could even be infinitely differentiable with respect to all independent variables.

A classic example of such a "dissipative" regularization of an initially dissipationless model equation occurs with the so-called "kinematic wave" equation (KWE) in one dimension (cf. Ref. 8). Thus consider, on the interval, $(-\infty,\infty)$ the equation satisfied by the function $u(x,t)$:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{1}$$

It is well-known that for certain types of initial data, the solution becomes multivalued in space or otherwise ceases to be a physically valid solution to the equation. Burgers in Ref. 9 introduced a simple dissipative modification,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \quad (2)$$

where ν is a small (positive) parameter with the dimensions of a diffusivity. This nonlinear "advection-diffusion" equation can be transformed into the linear heat diffusion equation of Fourier, and solved exactly for all values of $t>0$ using the Cole–Hopf transformation (cf. Ref. 8). It describes an array of shocks with the thickness determined by the parameter ν which "damps" excitations at very high wave numbers responsible for the failure of Eq. (1) to have solutions for sufficiently long times. The diffusive term added by Burgers to regularize the KWE rapidly dissipates the energy of very short wavelength fluctuations which is lost to the system as heat due to the resultant entropy production. In view of these facts, the solutions of the Burgers equation exist as C_∞ functions for all future times, a property not shared by the original KWE, Eq. (1).

A different type of regularization is known for Eq. (1). It was discovered by Korteweg and de Vries (KdV)[10] that dispersive nonlinear water waves satisfy a different generalization of this equation involving the third spatial derivative of u. Thus, one writes down the KdV equation, which also arises in many other contexts (discussed in detail in Refs. 1, 8, and 11):

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \quad (3)$$

where α,β are real constants. After suitable rescalings, this equation is often stated in its standard form corresponding to the choices, $\alpha=1$ and $\beta=1$ (sometimes, $\alpha=6$ is also employed). For our purposes, to exhibit explicitly the connection with the KWE [Eq. (1)], the choices, $\alpha=1$ and $\beta=-k$, are made, where k is a "regularizing parameter" multiplying the (linear) dispersive term (u_{xxx}) of the equation.

Unlike the Burgers equation, the KdV equation describes conservative, nonlinear dispersive wave processes. It has many remarkable properties: an infinite set of conservation laws, Hamiltonian structure, recurrent motions along with exact integrability using the inverse scattering transform and Gelfand–Levitan equations (cf. Refs. 1, 8, 12, and 13). We have here an excellent instance of a "conservative regularization" of the original, nondissipative, nonlinear KWE Eq. (1). Such a regularization permits one to study solutions for arbitrarily long times without losing the principal qualitative features of the original model, namely, its conservation laws and time-reversibility. Regularizations of this nondissipative or reactive type do not involve any entropy production in the system and may be appropriate for physical systems in which it is known that dissipative effects are very weak (or even totally absent) in comparison with conservative nonlinearities.

A second classic instance of a regularized nonlinear dispersive wave system is represented by the "cubic" nonlinear Schrödinger equation,

$$i\frac{\partial \Psi}{\partial t} = -\frac{\partial^2 \Psi}{\partial x^2} + V_0|\Psi|^2\Psi. \quad (4)$$

This equation also admits exact solutions and has been much studied (see Refs. 8, 14, 12, and 13 for some of its properties).

Another much less well-understood example (e.g., considered in Refs. 15–18) is the extension of the ideal MHD Ohm's law to include electron inertia. It is known from these works that ideal MHD singularities (for example, those associated with current and vortex sheets) can be resolved using resistivity and viscosity (an instance of dissipative regularization) and also, alternatively, by electron inertia and the Hall effect. The two latter are time-reversible effects, unlike resistivity or viscosity which lead to entropy/heat production. It is not yet clear whether these effects alone will be able to prevent the enstrophy (integral over the solution domain of the squared vorticity) in the system from increasing unboundedly without the inclusion of some additional conservative regularization.

A famous instance of dissipative regularization of Eulerian (inviscid) hydrodynamics is the introduction of viscosity and the generalization to Navier–Stokes equations (cf. Refs. 2–4). This is exactly analogous to the Burgers regularization but preceded it historically. However, in this particular case, in three dimensions, it is still an open problem (attached with a Clay prize to the solver) to prove with mathematical rigor that the solution to the Cauchy (initial/boundary value) problem for the Navier–Stokes equations exists uniquely for all $t>0$. Thus, it is not yet proved that the Navier–Stokes/viscous regularization of the inviscid Euler equations in three dimensions does in fact lead to unique solutions with bounded enstrophy (it has been known since the work of Leray and Ladyzhenskaya[19,20] that it does in two dimensions). This question may not be of great physical importance since it is well-known that if the viscosity is small enough (i.e., at high enough Reynolds numbers) virtually all solutions to the Navier–Stokes equations (assuming they exist!) become linearly (and occasionally nonlinearly as in Ref. 21) unstable, and the flow is turbulent. The resultant dynamics involves a very large number of effective degrees of freedom and is also chaotic (i.e., Lyapunov unstable at any instant, as discussed by Frisch in Ref. 2). In a classic paper, Taylor and Green[22] raised the interesting question of the growth of enstrophy in the Navier–Stokes system when the kinematic viscosity $\nu\to 0$. Their studies have been replicated with modern computers (cf. Refs. 2 and 7). As stated earlier, the problem of rigorously establishing the boundedness of entropy has only been solved by introducing some "extra" regularizations such as "hyperviscosity" (*op. cit.* Refs. 2 and 20). As far as is known to the author, no such result is available for "conservative" regularizations considered in this work.

The question addressed in this paper is a mathematical one: can ideal hydrodynamics and MHD be regularized while retaining time-reversibility and conservation properties? Such a regularization represents a "conservative/time-reversible" extension of the models, analogous to the KdV modification of the KWE. The motivation for this is twofold:

intrinsically, it would be mathematically useful to have such a regularized extension in order to investigate the possible classes of singular solutions of Eulerian hydrodynamics. Second, it would be of considerable benefit in numerical solutions to have regularized equations provided one is interested only in describing features above a certain "microscale" and do not need to worry about "microstructures." For example, in hydrodynamics, no one is usually interested in what happens at the molecular level (at length-scales corresponding to the collisional mean-free-path of the molecules composing the fluid), with rare exceptions such as superfluid dynamics. A solution to this problem is presented for two important special cases of (a) inviscid Eulerian hydrodynamics and (b) ideal, incompressible MHD. The guiding principles in the notion of a regular extension are the following: the extended system should share as many of the formal symmetry properties of the original one as possible. It should involve only local interactions with at most quadratic nonlinearities and should employ as few extra spatial derivatives as possible. This means that the new, reactive terms introduced must be "small" in some sense, and just sufficient to avoid the singularities of the original system. Finally, the initial value specification should not involve more variables. This implies that the original Cauchy data should suffice for the extended system, although the higher order spatial boundary conditions may require additional specifications. This is in line with the fact that Navier–Stokes equations need more boundary (but not initial) specifications than the Euler equations.

II. CONSERVATIVE REGULARIZATION OF IDEAL INCOMPRESSIBLE HYDRODYNAMICS

We consider an ideal, isentropic fluid in which the speed of sound is sufficiently large compared to the flow velocity. This allows us to consider incompressible flows. The density and pressure variations then satisfy the equations, $d\rho/dt=0$; $p/p_0=(\rho/\rho_0)^\gamma$. From the integrated equation of continuity, we infer that the total mass of the fluid ($M=\int \rho dV$) is an invariant. The standard, incompressible, inviscid, Eulerian hydrodynamic equations of motion may then be written in the form:

$$\nabla\cdot\mathbf{v}=0, \tag{5}$$

$$\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\frac{\gamma}{\gamma-1}\nabla\left(\frac{p}{\rho}\right). \tag{6}$$

This model also applies to the common case when the equilibrium density and pressure are uniform everywhere and the fluctuating pressure is of order $(1/2)\rho\mathbf{v}\cdot\mathbf{v}$. We assume, for simplicity, that the flow takes place in a fixed, simply connected, three-space dimensional, bounded domain and always consider the initial value problem with impermeable boundaries. Extensions to other cases including conservative external forces (such as gravity) and unbounded flows are of course possible and relatively straight forward.

A well-known transformation enables the momentum equation to be written in the form:

$$\frac{\partial\mathbf{v}}{\partial t}+\mathbf{W}\times\mathbf{v}=-\nabla\sigma, \tag{7}$$

$$\mathbf{W}=\nabla\times\mathbf{v}, \tag{8}$$

where the "specific enthalpy" (also called stagnation pressure in the following) σ is defined by

$$\sigma=\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}. \tag{9}$$

Thus, in this theory, the vorticity satisfies:

$$\frac{\partial\mathbf{W}}{\partial t}+\nabla\times(\mathbf{W}\times\mathbf{v})=0. \tag{10}$$

The last equation expresses the Lagrange–Cauchy–Kelvin (circulation) theorem (see, for example, Refs. 4 and 3) on the "freezing-in" of vorticity in the flow and also the phenomenon of "vortex stretching" which can cause the enstrophy $\Omega=(1/2)\int\mathbf{W}\cdot\mathbf{W}dV$ to be unbounded. This is a standard example of a direct cascade of enstrophy from long to short wavelengths, similar to the ultraviolet catastrophe in the classical theory of radiation. It should be noted (as is well-known) that in two spatial dimensions, the vorticity equation Eq. (10) admits an infinity of integral invariants and thus the Euler equations need no further "regularization" except possibly to avoid finite discontinuities in vorticity (i.e., vortex sheets). Returning to the three-dimensional, generic case, the example of the KdV equation leads us to propose that the equation of motion [Eq. (7)] be modified as follows, while retaining the continuity equation [Eq. (5)] and the definition of the scalar field σ [Eq. (9)]:

$$\frac{\partial\mathbf{v}}{\partial t}+\mathbf{W}\times\mathbf{v}=-\nabla\sigma-\lambda^2\mathbf{W}\times(\nabla\times\mathbf{W}), \tag{11}$$

where the constant λ has the dimensions of length. Note that, whereas the unregularized Euler equation involved only first order spatial derivatives, the extended Eq. (11) contains second spatial derivatives of velocity. In this respect, this equation resembles the Navier–Stokes equation. However, it is time-reversible and as will be demonstrated shortly, it possesses a positive-definite integral invariant, whereas the Navier–Stokes equations are dissipative.

We introduce the vector $\mathbf{T}$ (which represents a purely reactive stress, unlike the dissipative viscous stress of Navier–Stokes):

$$\mathbf{T}=\mathbf{W}\times(\nabla\times\mathbf{W})=\nabla\left(\frac{\mathbf{W}\cdot\mathbf{W}}{2}\right)-\nabla\cdot(\mathbf{W}\mathbf{W}). \tag{12}$$

Note that, by definition, $\mathbf{T}\cdot\mathbf{W}=\mathbf{T}\cdot(\nabla\times\mathbf{W})=0$. Furthermore, if the flow is irrotational, the model reduces to standard potential flow theory.

Bearing in mind the continuity equation, upon taking the divergence of the modified equation of motion [Eq. (11)], we see that the function σ ("stagnation pressure") must satisfy (as a consistency condition) the Poisson equation,

$$\nabla^2\sigma = -\nabla \cdot [(\mathbf{W}\times\mathbf{v}) + \lambda^2\mathbf{T}]. \tag{13}$$

We next investigate the behavior of the vorticity. The vorticity in this model evolves in time according to

$$\frac{\partial \mathbf{W}}{\partial t} + \nabla\times(\mathbf{W}\times\mathbf{v}) = -\lambda^2[\nabla\times\mathbf{T}]. \tag{14}$$

If the vorticity vanished everywhere at $t=0$, the equation assures us that it will remain zero for all time (Lagrange's theorem of Eulerian hydrodynamics). In this event, we simply recover the results of irrotational hydrodynamics. Of course, this result assumes that solutions to the Cauchy problem exist for $\lambda \neq 0$.

The flow "specific kinetic energy" defined as $(\mathcal{E}=\int_V(\mathbf{v}\cdot\mathbf{v}/2)dV)$ in the domain is governed by,

$$\frac{d\mathcal{E}}{dt} = -\lambda^2\int_V \mathbf{v}\cdot[\mathbf{W}\times(\nabla\times\mathbf{W})]dV.$$

We also find that the flow enstrophy $(\Omega=\int(\mathbf{W}\cdot\mathbf{W}/2)dV)$ satisfies,

$$\frac{d\Omega}{dt} = -\int_V \nabla\times\mathbf{W}\cdot[\mathbf{W}\times\mathbf{v}]dV = \int_V \mathbf{v}\cdot[\mathbf{W}\times(\nabla\times\mathbf{W})]dV.$$

We therefore have the remarkable invariant ("swirl energy") $\mathcal{E}^*=\mathcal{E}+\lambda^2\Omega$:

$$\frac{d\mathcal{E}^*}{dt} = \frac{d}{dt}\left(\int_V[\mathbf{v}\cdot\mathbf{v}+\lambda^2\mathbf{W}\cdot\mathbf{W}]dV\right) = 0. \tag{15}$$

This is associated with a local conservation law. To see this we make use of the following relations.

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right) &= -\nabla\cdot(\mathbf{v}\sigma) - \lambda^2\mathbf{v}\cdot\mathbf{T}\frac{\partial}{\partial t}\left(\frac{\mathbf{W}\cdot\mathbf{W}}{2}\right)\\ &= -\mathbf{W}\cdot\nabla\times(\mathbf{W}\times\mathbf{v}) - \lambda^2\mathbf{W}\cdot[\nabla\times\mathbf{T}]\\ &= -\nabla\cdot[(\mathbf{W}\times\mathbf{v})\times\mathbf{W}] - (\mathbf{W}\times\mathbf{v})\cdot(\nabla\times\mathbf{W})\\ &\quad -\lambda^2\nabla\cdot[\mathbf{T}\times\mathbf{W}]\\ &= -\nabla\cdot[(\mathbf{W}\times\mathbf{v})\times\mathbf{W}+\lambda^2(\mathbf{T}\times\mathbf{W}] + \mathbf{v}\cdot\mathbf{T}.\end{aligned}$$

Multiplying the second equation by λ^2 and adding, we get,

$$\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\cdot\mathbf{v}}{2} + \lambda^2\frac{\mathbf{W}\cdot\mathbf{W}}{2}\right] + \nabla\cdot\mathbf{F} = 0, \tag{16}$$

where the flux $\mathbf{F}$ is given by,

$$\mathbf{F} = \sigma\mathbf{v} + \lambda^2[(\mathbf{W}\times\mathbf{v})\times\mathbf{W}] + \lambda^4[\mathbf{T}\times\mathbf{W}]. \tag{17}$$

Next we consider the evolution of the **flow helicity** $H=\int_V\mathbf{v}\cdot\mathbf{W}dV$. We find in this case,

$$\mathbf{v}\cdot\frac{\partial\mathbf{W}}{\partial t} = -\nabla\cdot[(\mathbf{W}\times\mathbf{v})\times\mathbf{v}] - \lambda^2\nabla\cdot[\mathbf{T}\times\mathbf{v}],$$

$$\mathbf{W}\cdot\frac{\partial\mathbf{v}}{\partial t} = -\nabla\cdot[\sigma\mathbf{W}], \tag{18}$$

$$\frac{\partial}{\partial t}(\mathbf{v}\cdot\mathbf{W}) + \nabla\cdot\mathbf{G} = 0,$$

$$\mathbf{G} = \sigma\mathbf{W} + [(\mathbf{W}\times\mathbf{v})\times\mathbf{v}] + \lambda^2[\mathbf{T}\times\mathbf{v}]. \tag{19}$$

Assuming that the boundary terms vanish by a suitable choice of conditions, we see that the helicity is indeed a constant of the motion.

The equations, Eqs. (5) and (11) share some key properties with the Euler equations. Under the transformations, $t\to -t$, $\mathbf{v}\to-\mathbf{v}$, it is seen that $\sigma\to\sigma$, $\mathbf{W}\to-\mathbf{W}$, and both equations are invariant. This is "time-reversal" symmetry. Changing $\mathbf{x}\to-\mathbf{x}$, $\mathbf{v}\to-\mathbf{v}$; $\mathbf{W}\to\mathbf{W}$, we again find the equations remain invariant. This is "parity" transformation. It is also evident that the equations transform exactly as the Euler equations do under Galilean transformations.

We introduce a new vector, $\mathbf{v}_*=\mathbf{v}+\lambda^2\nabla\times\mathbf{W}=\mathbf{v}-\lambda^2\nabla^2\mathbf{v}$. It is evidently solenoidal. We may now write Eqs. (11)–(13) in the forms:

$$\frac{\partial\mathbf{v}}{\partial t} + \mathbf{W}\times\mathbf{v}_* = -\nabla\sigma, \tag{20}$$

$$\nabla^2\sigma = -\nabla\cdot(\mathbf{W}\times\mathbf{v}_*), \tag{21}$$

$$\frac{\partial\mathbf{W}}{\partial t} + \nabla\times(\mathbf{W}\times\mathbf{v}_*) = 0. \tag{22}$$

We infer that,

$$\frac{\partial\mathbf{v}_*}{\partial t} + \mathbf{W}\times\mathbf{v}_* = -\nabla\sigma - \lambda^2\nabla\times[\nabla\times(\mathbf{W}\times\mathbf{v}_*)]. \tag{23}$$

From the above, we obtain,

$$\mathbf{v}_* \cdot \frac{\partial \mathbf{v}}{\partial t} = -\nabla \cdot (\sigma \mathbf{v}_*),$$

$$\begin{aligned}\mathbf{v} \cdot \frac{\partial \mathbf{v}_*}{\partial t} &= -\mathbf{v} \cdot (\mathbf{W} \times \mathbf{v}_*) - \nabla \cdot (\mathbf{v}\sigma) - \lambda^2 \mathbf{v} \cdot \nabla \times [\nabla \times (\mathbf{W} \times \mathbf{v}_*)] \\ &= -\lambda^2 \mathbf{v} \cdot (\mathbf{W} \times \nabla \times \mathbf{W}) - \nabla \cdot (\mathbf{v}\sigma) - \lambda^2 \mathbf{v} \cdot \nabla \times [\nabla \times (\mathbf{W} \times \mathbf{v}_*)] \\ &= \lambda^2 \nabla \times \mathbf{W} \cdot (\mathbf{W} \times \mathbf{v}) - \lambda^2 \mathbf{W} \cdot [\nabla \times (\mathbf{W} \times \mathbf{v}_*)] - \lambda \nabla \cdot \{[\nabla \times (\mathbf{W} \times \mathbf{v}_*)] \times \mathbf{v}\} - \nabla \cdot (\mathbf{v}\sigma) \\ &= -\lambda^2 \nabla \cdot [(\mathbf{W} \times \mathbf{v}_*) \times \mathbf{W}] - \lambda^2 \nabla \cdot \{[\nabla \times (\mathbf{W} \times \mathbf{v}_*) \times \mathbf{v}]\} - \nabla \cdot (\mathbf{v}\sigma).\end{aligned} \tag{24}$$

Adding, we obtain,

$$\frac{\partial}{\partial t}(\mathbf{v} \cdot \mathbf{v}_*) = -\nabla \cdot \mathbf{K}, \tag{25}$$

$$\mathbf{K} = \sigma \mathbf{v} + \sigma \mathbf{v}_* + \lambda^2 [(\mathbf{W} \times \mathbf{v}_*) \times \mathbf{W}] + \lambda^2 \{[\nabla \times (\mathbf{W} \times \mathbf{v}_*)] \times \mathbf{v}\}. \tag{26}$$

We observe that

$$\begin{aligned}\mathbf{v} \cdot \mathbf{v}_* &= \mathbf{v} \cdot \mathbf{v} + \lambda^2 \mathbf{v} \cdot (\nabla \times \mathbf{W}) \\ &= \mathbf{v} \cdot \mathbf{v} + \lambda^2 \mathbf{W} \cdot \mathbf{W} + \lambda^2 \nabla \cdot (\mathbf{W} \times \mathbf{v}) \\ &= \mathbf{v} \cdot \mathbf{v} + \lambda^2 \mathbf{W} \cdot \mathbf{W} - \lambda^2 \nabla^2 \sigma - \lambda^2 \nabla \cdot \mathbf{T}.\end{aligned} \tag{27}$$

This shows that $\int_V \mathbf{v} \cdot \mathbf{v}_* dV = \int_V [\mathbf{v} \cdot \mathbf{v} + \lambda^2 \mathbf{W} \cdot \mathbf{W}] dV = \mathcal{E}^*$. Furthermore, Eq. (22) implies that the vorticity of the fluid is **frozen** in to $\mathbf{v}_*$ rather than the flow velocity, $\mathbf{v}$ itself (Lagrange–Kelvin–Helmholtz theorem).

It is clear from the integral invariants that if $\lambda = l$, where l is a "fundamental length" much smaller than any length-scale we wish to model, the regularized equations imply bounded enstrophy. From a physical point of view, if, for example, λ is chosen to be of the order of the fluid mean-free-path (usually far smaller than most length-scales of interest in hydrodynamics) it is clear that the above regularization will not change the physics except in the inner structures of vortex/boundary layers. It can be conjectured (but not proven definitively) that this regularization is unique when the highest order spatial derivatives of $\mathbf{v}$ in Eq. (11) are restricted to two and the nonlinearities to be quadratic. It is also not known if this system can be put in a Hamiltonian form using suitable transformations. In analogy with the KdV equation, this is what one would expect, although it is unlikely that there exist an infinity of distinct constants of the motion. The system can be expected to exhibit "conservative chaos" similar to those seen in generic, nonlinear Hamiltonian systems with more than three degrees of freedom. It would be instructive to investigate this system and characterize its long-term behavior, especially as regards the spectral (and spatio-temporal) distributions of kinetic energy and the enstrophy. Since the swirl energy $\mathcal{E}^*$ is a positive-definite invariant, the dynamical equations of motion can be represented in terms of a group of unitary transformations operating on a suitable Hilbert space. This could facilitate numerical modeling of the system (such as those carried out by Boratav *et al.*[23]) by appropriately truncated spectral expansions of the velocity and pressure. Furthermore, statistical mechanics of vortex motions considered by Chorin[24] would be more self-consistent when made in the context of a bounded enstrophy, guaranteed by the conserved swirl energy.

It is expected that the solutions for long times would "phase mix" but exhibit recurrence, depending upon the initial data (see, for example, Refs. 12, 13, and 25). A statistical study of the spectra of energy, enstrophy, and the correlation functions of such a system could be of interest if the large-scale structure and statistics can be shown to be essentially independent of the "cut-off" length-scale, l, chosen to be smaller than length-scales of physical interest. The "Taylor–Green" problem could be readily studied using the regularized system. It is expected to throw light on the vortical structures which develop and saturate in the resultant flows.

III. REGULARIZATION OF INCOMPRESSIBLE IDEAL MHD

As in the preceding section, we consider a single-fluid, ideal isentropic magnetized plasma. The pressure and mass density are assumed to be related by the adiabatic equation of state and the motions considered will be assumed to be of such low Mach numbers that the incompressibility assumptions are satisfied. It should be noted in this context that the ideal MHD energy principle states that incompressible perturbations of an equilibrium are the most unstable (since compressional energy of the plasma is always stabilizing). The ideal equations in their standard, unregularized form are:

$$\nabla \cdot \mathbf{v} = 0, \tag{28}$$

$$\rho_m \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right] = -\nabla p + \mathbf{j} \times \mathbf{B}, \tag{29}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \tag{30}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{31}$$

Here ρ_m is the mass density. It satisfies the equations, $d\rho_m/dt = 0$; $p \propto \rho_m^\gamma$. As in the preceding section, we may rewrite the momentum balance equation and obtain the vorticity equation:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{W} \times \mathbf{v} = -\nabla\sigma + \frac{1}{\rho_m}\mathbf{j} \times \mathbf{B}, \quad (32)$$

$$\frac{\partial \mathbf{W}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{W}) + \frac{1}{\rho_m} \nabla \times [\mathbf{j} \times \mathbf{B}]. \quad (33)$$

To regularize this system, we use extended equations involving three arbitrary parameters, λ, ν, κ and including a maximum of two spatial derivatives of each quantity being advanced in time. The maximal nonlinearity allowed is quadratic. These take the following general forms:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{W} \times \mathbf{v} = -\nabla\sigma + \frac{1}{\rho_m}\mathbf{j} \times \mathbf{B} + \lambda^2(\nabla \times \mathbf{W}) \times \mathbf{W}, \quad (34)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \lambda^2 \nabla \times [(\nabla \times \mathbf{W}) \times \mathbf{B}] + \nu \nabla \times [\mathbf{W} \times \mathbf{j}] + \kappa \nabla \times [\mathbf{j} \times \mathbf{B}], \quad (35)$$

$$\frac{\partial \mathbf{W}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{W}) + \frac{1}{\rho_m} \nabla \times [\mathbf{j} \times \mathbf{B}] + \lambda^2 \nabla \times [(\nabla \times \mathbf{W}) \times \mathbf{W}], \quad (36)$$

where λ is a constant with the dimensions of length; the constants ν, κ can have either sign. In SI units, $\nu/\mu_0, \kappa/\mu_0$ have nontrivial dimensions $[l^2],[l^2/Bt]$, respectively. It is also evident that κ has the dimensions of $1/en$, where e is the electric charge and n is number density, which effectively makes it a Hall coefficient.

We can demonstrate that Eqs. (34)–(36) have the symmetries of the original system [Eqs. (28)–(31)]: for example, under time-reversal, $t \to -t$; $\mathbf{x} \to \mathbf{x}$, we know that, $\mathbf{v} \to -\mathbf{v}$, $\mathbf{W} \to -\mathbf{W}$, $\mathbf{j} \to -\mathbf{j}$, $\mathbf{B} \to -\mathbf{B}$, and $p \to p$. If λ, ν, κ are invariant, the equations are clearly seen to be invariant. Similarly, under parity transformation: $t \to t$; $\mathbf{x} \to -\mathbf{x}$; $\nabla \to -\nabla$, $\mathbf{v} \to -\mathbf{v}$, $\mathbf{W} \to \mathbf{W}$, $\mathbf{j} \to -\mathbf{j}$, $\mathbf{B} \to \mathbf{B}$, and $p \to p$. Again, we see that the equations remain invariant. Invariance under the Euclidean and Galilean transformations is also assured.

Next, we derive an integral invariant which shows that the enstrophy and the magnetic field energy, as well as the flow kinetic energy are bounded. In this, we have assumed that the relative density fluctuations, $\tilde{\rho}/\rho_m$ are small like the Mach number, so that the time-dependent part of density is negligible compared with all other time-varying perturbations. If this assumption is not made, we would have to include the fluctuations of internal energy in the total energy budget and will essentially lead to compressible plasma energetics. The regularization of compressible gases and plasmas is outside the terms of reference of this work.

Thus retaining only the equilibrium (but possibly spatially stratified density ρ_m), the equations and appropriate boundary data imply the following integral relations assuming all "surface" contributions vanish:

$$\frac{1}{2}\frac{d}{dt}\int \mathbf{v}\,.\,\mathbf{v}dV = \frac{1}{\rho_m}\int \mathbf{v}\,.\,(\mathbf{j} \times \mathbf{B})dV + \lambda^2\int \mathbf{v}\,.\,[(\nabla \times \mathbf{W}) \times \mathbf{W}]dV \frac{1}{2}\frac{d}{dt}\int \left(\frac{\mathbf{B}\,.\,\mathbf{B}}{\mu_0\rho_m}\right)dV$$
$$= -\frac{1}{\rho_m}\int \mathbf{v}\,.\,(\mathbf{j} \times \mathbf{B})dV - \frac{\lambda^2}{\rho_m}\int \mathbf{j}\,.\,[\mathbf{B} \times (\nabla \times \mathbf{W})]dV \frac{1}{2}\frac{d}{dt}\int \mathbf{W}\,.\,\mathbf{W}dV$$
$$= -\int \mathbf{v}\,.\,[(\nabla \times \mathbf{W}) \times \mathbf{W}]dV + \frac{1}{\rho_m}\int \mathbf{j}\,.\,[\mathbf{B} \times (\nabla \times \mathbf{W})]dV.$$

Hence, setting $H^* = (1/2)\int[\mathbf{v}.\mathbf{v} + (\mathbf{B}.\mathbf{B}/\mu_0\rho_m) + \lambda^2\mathbf{W}.\mathbf{W}]dV$, we obtain the positive-definite integral invariant (it seems appropriate to term this, "specific hydromagnetic swirl energy"),

$$\frac{dH^*}{dt} = 0. \quad (37)$$

However, it is not difficult to show that the regularized system does allow "slip" between flows and fields. It is also easy to see that

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} - \lambda^2(\nabla \times \mathbf{W}) \times \mathbf{B} - \nu\mathbf{W} \times \mathbf{j} - \kappa\mathbf{j} \times \mathbf{B}$$
$$= -\mathbf{v}_* \times \mathbf{B} - \kappa\mathbf{j} \times \left[\mathbf{B} - \left(\frac{\nu}{\kappa}\right)\mathbf{W}\right]. \quad (38)$$

Thus, if both κ and ν are zero, the magnetic field is frozen in to the $\mathbf{v}_*$ field. If only ν vanished, the field is frozen in to $\mathbf{v}_* + \kappa\mathbf{j}$. In the general case, no freezing in result holds. From Eqs. (34) and (35), we see that the $\mathbf{v}_*$ field satisfies the equation of motion:

$$\frac{\partial \mathbf{v}_*}{\partial t} + \mathbf{W} \times \mathbf{v}_* = -\nabla\sigma + \frac{1}{\rho_m}\mathbf{j} \times \mathbf{B} + \lambda^2(\nabla \times \mathbf{W}) \times \mathbf{W}. \quad (39)$$

It is interesting to observe that Eq. (38) for the electric field associated with this system may be written as,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi = -\mathbf{v}_* \times \mathbf{B} - \kappa\mathbf{j} \times \mathbf{B}_*, \quad (40)$$

$$\mathbf{B}_* = \left[\mathbf{B} - \frac{\nu}{\kappa}\mathbf{W}\right]. \tag{41}$$

It follows that the electrostatic potential Φ and the vector potential satisfy the "Ohm's law:"

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{B} \times \mathbf{v}_* = -\nabla\Phi + \kappa \mathbf{j} \times \mathbf{B}_*. \tag{42}$$

We may fix a suitable gauge (for example, the Coulomb gauge, $\nabla . \mathbf{A}=0$) to determine the potentials uniquely. The terms involving κ can be interpreted (with suitable sign) in two-fluid theory as arising from the Hall effect. The term involving ν has no apparent interpretation, except as a modification to the local magnetic field due to vorticity of the flow.

We can derive a generalized "helicity" conservation law for this extended set of equations. Thus from Eq. (33) and the vorticity equation, we obtain,

$$\frac{d}{dt}\int \mathbf{v} . \mathbf{W} dV = \frac{2}{\rho_m}\int \mathbf{W} . (\mathbf{j} \times \mathbf{B}) dV. \tag{43}$$

Then, using Eqs. (34) and (41), we obtain the integral relation:

$$\frac{d}{dt}\int \mathbf{A} . \mathbf{B} dV = 2\nu \int \mathbf{B} . (\mathbf{W} \times \mathbf{j}) dV.$$

It follows that, $K_* = \int[\mathbf{A} . \mathbf{B} - \nu\rho_m \mathbf{v} . \mathbf{W}]dV$ is a constant of the motion, neglecting the time dependence of the mass density, as before. This generalizes the well-known magnetic helicity invariant of ideal MHD.

Although our regularization does permit us to prevent the enstrophy and field energy from growing unboundedly during the evolution, it is not immediately clear that singularities in $\mathbf{j}$ will not form. Thus, it is not obvious that $\int \mathbf{j} . \mathbf{j} dV$ will be bounded for all time. If electron inertia (e.g., Refs. 15, 17, and 18) is included in the generalized Ohm's law, it is highly probable that current singularities can also be avoided. In any case, we have only considered incompressible motions, although these are relevant for many problems in plasma physics and in the theory of liquid metals. If compressibility is taken into account, we must allow the possibility of transformations of energy among kinetic, field, and internal energies. The regularization problem within a purely reactive, time-reversible frame-work becomes much more difficult. Indeed, physically, in compressible flow, one must always allow the possibility of shock formation. In general, shocks lead to entropy production and it seems difficult, if not impossible to conceive of avoiding a dissipative (or kinetically derived collisionless) regularization in such situations.

IV. DISCUSSION AND CONCLUSIONS

In this work, we have demonstrated the existence of some simple and unique (subject to the stated restrictions to be put on the spatial derivatives, locality, and symmetry of the regularizing terms) regularized extensions of incompressible, inviscid three-dimensional hydrodynamics and ideal MHD. The fact that such regularizations exist in dimensions higher than one is in itself of interest. The reactive "stresses" considered here have not been derived as in the KdV case from some exact dispersion relation in some asymptotic limit, but obtained from symmetry considerations. In the MHD case, the extra terms can be interpreted in terms of two-fluid effects such as the Hall effect.

It should be noted that incompressible hydrodynamics is essentially a phenomenological continuum model in the case of liquids, as opposed to gas dynamics, which is derivable in principle from kinetic theory using the standard Chapman–Enskog expansion in the short mean-free-path limit. Thus, modifications to the Euler system such as the one proposed here are physically meaningful and could, in principle be compared with experiments in suitable conditions when dissipation is truly negligible. If one considers the richness of dynamical structures (solitons, recurrence, etc.) shown by the solutions of the KdV equation as contrasted with those of its "preregularized" form, the KWE, one can expect the regularized-Euler equations to exhibit correspondingly richer vortical structure and at the same time be more tractable numerically.

It is of great interest to study the properties of the solutions to these equations and prove the existence to the Cauchy problem subject to suitable initial and boundary data. Such solutions may also reveal how the spectral distributions of energy and enstrophy are dynamically maintained in long-term stationary (generally time-dependent) conditions. As is known from previous results (cf. Ref. 12) the number of effective degrees of freedom involved in the dynamics is determined by the **initial data**, unlike dissipative systems where it is the balance between driving forces and dissipation which governs this key parameter (as discussed in Ref. 2). Computational solutions of typical problems such as flows between rotating cylinders/spheres or in tori and MHD problems involving rotating/self-gravitating magnetic fluids with strong, embedded dipole magnetic fields would be of great interest in this regularized frame-work. The open question here is the one of showing to what degree the results for the overall dynamics is independent of the regularizing constants. This is closely related to the following question of numerical simulations of nonlinear systems of evolutionary equations: which results of such a simulation can be considered truly "grid-independent" and which are artifacts? Such questions have only been answered for special systems (such as the KdV or Burgers equations) or in purely linear problems. The regularized equations can also be expected to throw new light on the dynamical development of nonlinear instabilities (cf. Refs. 21) in the systems considered and on reconnection problems (discussed in Refs. 23 and 16).

The extension of the present ideas to obtaining regularized solutions to collisionless kinetic equations is also of great interest in plasma physics and astrophysics. This can enable us to simulate the long-term asymptotic behavior of a mass of cold dark matter, for example, without running into purely numerical difficulties associated with unbounded enstrophy or finite-time blow up catastrophes. It will be noticed that the regularizations suggested here can be applied to both the Navier–Stokes and the viscoresistive MHD systems under incompressibilty conditions. The resultant *a priori* bounds involving the swirl energy strongly suggest that the Leray–Ladyzhenskaya proofs of existence and uniqueness of classical solutions could be extended from two to three dimensions for the regularized systems. As stated earlier, physically there is no reason why such regularized solutions should not be used as practical tools, just as the Navier–Stokes solutions to shock, blast, and deflagration waves are often used to describe the overall (macro- and mesoscale) dynamics, even when they are known to be actually invalid in the shock layer itself. An even better known example is that of quantum electrodynamics which, when regularized by the covariant "renormalization" process (invented by Feynman, Schwinger, and Tomonaga and developed by Dyson), leads to remarkable agreement with experiment. Future work will be devoted to investigations of these and related issues in greater detail than is possible here.

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[1]V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics*, (Springer, New York, 1998), Vol. 125, p. 303.
[2]U. Frisch, *Turbulence* (Cambridge University Press, New York, 1995), p. 37.
[3]C. R. Doering and J. D. Gibbon, *Applied Analysis of the Navier–Stokes Equations* (Cambridge University Press, Cambridge, 1995), p. 88.
[4]G. Gallavotti, *Foundations of Fluid Dynamics* (Springer, New York, 2002), p. 143.
[5]P. G. Saffman, J. Fluid Mech. **106**, 49 (1981).
[6]P. Constantin, SIAM Rev. **36**, 73 (1994).
[7]J. D. Gibbon, B. Galanti, and R. M. Kerr, in *Turbulence Structure and Vortex Dynamics*, edited by J. C. R. Hunt and J. C. Vassilicos (Cambridge University Press, Cambridge, 2000), p. 23.
[8]G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1973), p. 26.
[9]J. M. Burgers, Adv. Appl. Mech. **1**, 171 (1948).
[10]D. J. Korteweg and G. de Vries, Philos. Mag. **39**, 422 (1895).
[11]R. Davidson, *Methods in Nonlinear Plasma Theory* (Academic Press, New York, 1972), p. 15.
[12]A. Thyagaraja, Phys. Fluids **22**, 2093 (1979).
[13]A. Thyagaraja, in *Nonlinear Waves*, edited by L. Debnath, (Cambridge University Press, London, 1983), p. 308.
[14]P. D. Lax, in *Nonlinear Waves*, edited by S. Leibovich and A. R. Seebass (Cornell University Press, London, 1974), p. 291.
[15]B. Coppi, Phys. Fluids **8**, 2273 (1965).
[16]J. A. Wesson, Nucl. Fusion **30**, 2545 (1990).
[17]F. Porcelli, Phys. Rev. Lett. **66**, 425 (1991).
[18]K. G. McClements and A. Thyagaraja, Plasma Phys. Controlled Fusion **46**, 39 (2004).
[19]J. Leray, Acta Math. **63**, 193 (1934).
[20]O. A. Ladyzhenskaya, *Mathematical Theory of Viscous Incompressible Flow*, translated by R. A. Silverman (Gordon and Breach, London, 1963), p. 159.
[21]P. Drazin and W. Reid, *Hydrodynamic Stability* (Cambridge University Press, Cambridge, 1981), p. 370.
[22]G. I. Taylor and A. E. Green, Proc. R. Soc. London, Ser. A **158**, 499 (1937), also in 499 Collected Papers, Cambridge U.P., Cambridge, 1960, Vol. 2, 409.
[23]O. N. Boratav, R. B. Pelz, and N. J. Zabusky, in *Vortex Dynamics and Vortex Methods*, Lectures in Applied Mathematics, edited by C. R. Anderson and C. Greengard (American Mathematical Society, Providence, 1991), Vol. 28, p. 11.
[24]A. J. Chorin, in *Vortex Dynamics and Vortex Methods*, Lectures in Applied Mathematics, edited by C. R. Anderson and C. Greengard (American Mathematical Society, Providence, 1991), Vol. 28, p. 85.
[25]C. N. Lashmore-Davies, D. R. McCarthy, and A. Thyagaraja, Phys. Plasmas **8**, 5121 (2001).