Surface currents on ideal plasmas

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The surface (or "skin") current that can flow at a perturbed interface between plasma and vacuum is considered in the approximation where a surface marks a sharp transition from plasma to vacuum. A short magnetohydrodynamic calculation gives an exact and general expression for the component perpendicular to the average of the magnetic field either side of the surface, finding it proportional to the edge plasma pressure. A consequence is that for all plasmas with zero surface current at equilibrium, the surface current associated with *any* linear instability will flow parallel to the magnetic field. The surface current is calculated for a simple but realistic model of a cylindrical plasma, and found to depend on the type of instability, and consequently on the particular plasma equilibrium. This is illustrated for two well known cases. [doi:10.1063/1.3517096]

The boundary from a plasma to a vacuum is often approximated as a surface dividing plasma on one side and vacuum on the other. When such a surface moves due to a plasma instability for example, a current can flow along the surface. Recently such "surface currents" have been proposed as a mechanism by which plasma instabilities can drive "wall currents," leading to strong and potentially damaging forces on the vacuum vessel and tokamak components.[1] Surface currents have subsequently appeared in other disruption calculations,[2,3] making it increasingly important that we understand and are able to calculate them. An analytical calculation for the surface current was presented in Ref. 3, but was for the specific example of an equilibrium with constant toroidal current. Here the surface current is calculated with the ideal magnetohydrodynamic[4] model of plasma, is evaluated on the plasma's perturbed surface, and applies to any equilibrium consistent with the approximations stated herein. First, some general results for the surface current on *any* plasma are calculated, be it a tokamak,[5] stellarator,[5] or even an astrophysical[6] plasma. Then the surface current is calculated for the simplest realistic model of a tokamak plasma, a cylindrical plasma in the "tokamak approximation."[7] The simplifications are intended to keep the resulting calculation as transparent and accessible as possible.

The well known argument[8] of considering an arbitrarily narrow current loop and using Ampere's law leads to the general result for a surface current $\vec{\sigma}$ that[8]

$$\vec{\sigma} = \vec{n} \wedge (\vec{B}^V - \vec{B}), \tag{1}$$

where $\vec{n}$ is the unit normal to the perturbed surface, $\vec{B}^V$ is the magnetic field in the vacuum adjacent to the perturbed surface, and $\vec{B}$ is the magnetic field in the plasma adjacent to the plasma's surface.[9] From Eq. (1), we can immediately deduce that

$$\vec{n} \cdot \vec{\sigma} = 0, \tag{2}$$

i.e., the skin current flows in the surface (hence the term surface current). We also immediately have that

$$(\vec{B}^V - \vec{B}) \cdot \vec{\sigma} = 0, \tag{3}$$

with the consequence that $\vec{\sigma} \cdot \vec{B} = \vec{\sigma} \cdot \vec{B}^V$. Taking the cross product of $\vec{\sigma}$ with the unit normal $\vec{n}$ and simplifying gives

$$\vec{\sigma} \wedge \vec{n} = \vec{B}^V - \vec{B}, \tag{4}$$

which gives the "jump" in the magnetic field across the plasma's surface in terms of the surface current $\vec{\sigma}$ and the unit normal $\vec{n}$. If we furthermore take the dot product of $\vec{\sigma} \wedge \vec{n}$ with $(\vec{B}^V + \vec{B})$, we get

$$\vec{\sigma} \wedge \vec{n} \cdot (\vec{B}^V + \vec{B}) = B^{V^2} - B^2, \tag{5}$$

or rearranging terms and using the boundary condition between the plasma and vacuum of Ref. 4,

$$\left[\left| p + \frac{B^2}{2} \right| \right] = 0, \tag{6}$$

where $[|f|]$ denotes the difference in the value of f just inside the plasma surface, and just outside the plasma surface, then we get

$$\vec{\sigma} \cdot \vec{n} \wedge \left(\frac{\vec{B}^V + \vec{B}}{2} \right) = \frac{B^{V^2} - B^2}{2} = p. \tag{7}$$

This is an exact result for the magnitude of the skin current in the direction $\vec{n} \wedge (\vec{B}^V + \vec{B})$. Equation (7) shows that the component of surface current perpendicular to the average of the magnetic field either side of the surface $(\vec{B}^V + \vec{B})/2$, is proportional to the plasma pressure at the surface.

Now consider the simplest case of a cylindrical plasma with zero surface current at equilibrium ($\sigma_0 = 0$). Then Eq. (7) requires the equilibrium pressure to be zero at the plasma-vacuum boundary and Eq. (4) requires $[|\vec{B}_0|]$ to be zero. The pressure at the plasma boundary, which has been perturbed from $\vec{r}_0$ to $\vec{r}_0 + \vec{\xi}$, is

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$$p(\vec{r}_0+\vec{\xi}) = p(\vec{r}_0) + \vec{\xi}\cdot\nabla p(\vec{r}_0). \quad (8)$$

Writing $p(\vec{r})$ in terms of the equilibrium pressure p_0 and the (Eulerian) perturbation p_1 to it, we get

$$p(\vec{r}_0+\vec{\xi}) = p_0(\vec{r}_0) + p_1(\vec{r}_0) + \vec{\xi}\cdot\nabla p_0(\vec{r}_0) + O(\xi^2), \quad (9)$$

with for an adiabatic equation of state,[4]

$$p_1 = -\vec{\xi}\cdot\nabla p_0 - \gamma p_0 \nabla\cdot\vec{\xi}, \quad (10)$$

where γ is a constant usually taken to be 5/3. So we have

$$p(\vec{r}_0+\vec{\xi}) = p_0(\vec{r}_0) - \gamma p_0 \nabla\cdot\vec{\xi}(\vec{r}_0), \quad (11)$$

which is zero because $p_0(\vec{r}_0)=0$ with the consequence that Eq. (7) requires

$$\vec{\sigma}\cdot\vec{n}\wedge(\vec{B}^V+\vec{B}) = O(\xi^2), \quad (12)$$

or linearizing the above equation and using $[|\vec{B}_0|]=0$, then

$$\vec{\sigma}\cdot\vec{n}_0\wedge\vec{B}_0 = O(\xi^2), \quad (13)$$

where $\vec{n}_0$ and $\vec{B}_0$ are the unperturbed unit normal and magnetic field, respectively. Hence if σ_0 is zero, then we have the general result that to order ξ^2 at least, $\vec{\sigma}\cdot\vec{n}_0\wedge\vec{B}_0=0$, so that any induced surface currents must flow in a direction parallel to the unperturbed magnetic field. Subject to the assumptions stated above, this is a general result, and has not required any assumptions about the plasma's geometry.

Continuing to take $\vec{\sigma}_0=0$, we have that

$$\vec{\sigma} = \vec{n}_0\wedge[|\delta\vec{B}|], \quad (14)$$

where $[|\delta\vec{B}|]$ is the difference in the perturbed magnetic field either side of the perturbed plasma-vacuum boundary with

$$\delta\vec{B} = \vec{B}_1|_{r_0} + \vec{\xi}\cdot\nabla\vec{B}_0|_{r_0}. \quad (15)$$

We have found that for $\vec{\sigma}_0=0$ any nonzero $\vec{\sigma}$ must be parallel to $\vec{B}$, so considering $\vec{\sigma}\cdot\vec{B}$ that with Eq. (14) is found to be

$$\vec{\sigma}\cdot\vec{B} = [|\delta\vec{B}|]\cdot\vec{B}_0\wedge\vec{n}_0. \quad (16)$$

For $\vec{\sigma}_0=0$, Eq. (4) requires that $[|\vec{B}_0|]=0$. Using this plus the equilibrium vacuum field $\vec{B}_0^V = B_\theta|_{r_0}(r_0/r)\vec{e}_\theta + B_z\vec{e}_z$, we get

$$[|\vec{\xi}\cdot\nabla\vec{B}_0|]|_{r_0}\cdot\vec{B}_0\wedge\vec{n}_0 = -(\vec{B}\cdot\vec{J})\xi_r. \quad (17)$$

We also have that

$$[|\vec{B}_1|]\cdot\vec{B}_0\wedge\vec{n}_0 = B_z[|b_\theta|] - B_\theta[|b_z|], \quad (18)$$

where $b_\theta=\vec{e}_\theta\cdot\vec{B}_1$ and $b_z=\vec{e}_z\cdot\vec{B}_1$. Using the tokamak approximation of $B_\theta/B_z \ll 1$, then gives

$$\frac{\vec{\sigma}\cdot\vec{B}_0}{B_0} \simeq -J_z\xi_r + [|b_\theta|]. \quad (19)$$

Maxwell requires that $\nabla\cdot\vec{B}_1=0$, which when evaluated either side of the plasma surface requires that

$$0 = \left[\left|\frac{1}{r}\frac{\partial}{\partial r}(rb_r)\right|\right] + \frac{im}{r}[|b_\theta|] + ik[|b_z|], \quad (20)$$

with $b_r=\vec{e}_r\cdot\vec{B}_1$. Using the tokamak approximation of $kr=-nr/R=-nq(B_\theta/B_z)\ll 1$, with n an integer, R is the major radius, and $q=rB_z/RB_\theta$ is the tokamak "safety factor,"[4] this simplifies to

$$[|b_\theta|] = -\frac{1}{im}\left[\left|\frac{\partial}{\partial r}(rb_r)\right|\right], \quad (21)$$

where we used $[|b_r|]=0$, as required from $\nabla\cdot\vec{B}_1=0$. Hence $\sigma=\vec{\sigma}\cdot\vec{B}_0/B_0$ has

$$\sigma = -J_z\xi_r - \frac{b_r}{im}\left[\left|\frac{1}{b_r}\frac{\partial}{\partial r}(rb_r)\right|\right]. \quad (22)$$

In the tokamak approximation, the normal mode equations for a cylinder[4] give the radial equation for the radial component of $\vec{\xi}$ as

$$\frac{d}{dr}\left[(\rho\gamma^2+F^2)r\frac{d}{dr}(r\xi_r)\right] - \left[\frac{m^2}{r}(\rho\gamma^2+F^2) + \frac{dF^2}{dr}\right](r\xi_r) = 0, \quad (23)$$

with ρ the plasma density and $F=mB_\theta/r+kB_z$. Writing ξ_r in terms of the perturbed radial magnetic field component b_r with $\xi_r=b_r/iF$ allows a clear interpretation of the equation when it is integrated from just inside to just outside the plasma-vacuum boundary. Doing this integration, denoting the vacuum magnetic field perturbations as b_r^V, we get

$$-(\rho\gamma^2+F^2)\left(\frac{rb_r}{iF}\right)' + F^2\left(\frac{rb_r^V}{iF}\right)' = 0, \quad (24)$$

where primes denote derivatives with respect to r that may alternately be written as

$$(\rho\gamma^2+F^2)\left[\left|\left(\frac{rb_r}{iF}\right)'\right|\right] - \rho\gamma^2\left(\frac{rb_r^V}{iF}\right)' = 0. \quad (25)$$

This procedure is equivalent to using the pressure balance boundary condition equation (6) (that is found by integrating the momentum equation across the plasma-vacuum boundary), linearizing it at the perturbed plasma-vacuum boundary, and using the cylindrical normal mode equations to write it solely in terms of ξ_r.

Note that

$$F' = \frac{m}{r}J_z - kJ_\theta - 2\frac{mB_\theta}{r^2}, \quad (26)$$

and that the currents are taken to be zero in the vacuum. Using this, along with $b_r=iF\xi_r$, $[|b_r|]=0$, and a cylindrical vacuum solution with $b_r^V\to 0$ as $r\to\infty$ for which $(rb_r^V)'=-miF\xi$, a little algebra then gives

$$\left[\left|\frac{(rb_r)'}{b_r}\right|\right] = \frac{1}{F}\left\{-mJ_z + \frac{\rho\gamma^2}{\rho\gamma^2+F^2}\left[-mF + 2\frac{mB_\theta}{r}\right]\right\}. \quad (27)$$

Substituting this into Eq. (22), and using $b_r=iF\xi_r$, we get

$$\sigma = \frac{\rho\gamma^2}{\rho\gamma^2 + F^2}[Fr - 2B_\theta]\frac{\xi_r}{r}, \tag{28}$$

or alternately,

$$\hat{\sigma} = \frac{\hat{\gamma}^2}{\hat{\gamma}^2 + (m-nq)^2}[m - nq - 2]\frac{\xi_r}{r}, \tag{29}$$

with $\hat{\gamma}^2 = \rho\gamma^2 r^2/B_\theta^2$ and $\hat{\sigma} = \sigma/B_\theta$. For $(m-nq)=0$, we get a result that is independent of the equilibrium and the growth rate with $\hat{\sigma} = -2\xi_r/r$ as in Ref. 1 (that considers $q=1$). In general, however, the evaluation of σ depends upon the exact form of equilibrium, the mode numbers, and the resulting solution of the radial normal mode equation. Taking for example that q and ρ are constant, then solving Eq. (23), we find $(r\xi_r) \propto (r/r_0)^m$. Using this solution in Eq. (24) along with a vacuum solution with $b_r^V \to 0$ as $r \to \infty$, which has $(rb_r^V)' = -miF\xi_r$, we may solve for the growth rate to find

$$\hat{\gamma}^2 = -2(m-nq)(m-nq-1). \tag{30}$$

This is a maximum (has a most unstable solution) for a choice of m and n for which $(m-nq)$ most closely approximates 1/2 at which $\hat{\gamma}^2$ has its maximum value of 1/2 and

$$\hat{\sigma} = \frac{1/2}{1/2 + (1/2)^2}\left(\frac{1}{2} - 2\right)\frac{\xi_r}{r} = -\frac{\xi_r}{r}. \tag{31}$$

For marginal stability ($\hat{\gamma}^2=0$), but $(m-nq) \neq 0$, Eq. (29) simply gives $\hat{\sigma}=0$.

Alternately, we can write $\hat{\gamma}^2 = \tilde{\gamma}^2 m^2 q^2$ with $\tilde{\gamma}^2 = \rho\gamma^2 R^2/B_z^2 m^2$ (this normalization is more convenient for a numerical calculation later), then for a given m and n, we can maximize Eq. (30) with respect to q finding a maximum at $q = 2m(m-1)/(2m-1)n$. This gives $\tilde{\gamma}^2 = n^2/2(m-1)m^3$, which for the example considered later with $m=3$ and $n=1$, has $\tilde{\gamma}^2 \simeq 0.009$.

To illustrate the surface current's dependence on the equilibrium and the particular instability causing it, we solve for the surface current $\hat{\sigma}$ when q has a profile that quadratically depends upon r/r_0, with constant B_z, and with the density ρ taken for simplicity to be constant in the plasma before sharply dropping to zero at the plasma edge. For the numerical integration, Eq. (23) is written as

$$\frac{d}{dr}\left[(\tilde{\gamma}^2 + \Delta^2) r \frac{d}{dr}(r\xi_r)\right] - \left[\frac{m^2}{r}(\tilde{\gamma}^2 + \Delta^2) + \frac{d\Delta^2}{dr}\right](r\xi_r) = 0, \tag{32}$$

with $\Delta = 1/q(r) - n/m$. This form keeps the number of terms with a radial dependence to a minimum. With this notation, the boundary condition equation (24) becomes

$$-(\tilde{\gamma}^2 + \Delta^2)\frac{(r\xi_r)'}{\xi_r} + \Delta^2\left(-m + \frac{2}{\Delta q}\right) = 0. \tag{33}$$

For the calculation, whose sole purpose is to illustrate that σ is not in general independent of the equilibrium, we take $m=3$ and $n=1$. The growth rates are obtained by numerically integrating Eq. (32) for a particular growth rate, then plotting the left hand side of Eq. (33) as a function of γ, with solutions obtained from where the function passes through zero [and Eq. (33) is satisfied]. A typical plot is shown in Fig. 1 that also illustrates how the most unstable value of $\tilde{\gamma}^2$ depends on the edge-q value. The calculation is repeated for an increasingly sheared q-profile both for a constant edge-q value of $q=2.5$ $(m-nq=1/2)$ and for an edge q value that is varied until the largest growth rate is found. The q profile is taken to be

$$q = q_0 + q''(r/r_0)^2, \tag{34}$$

for which a constant edge q can be kept by simultaneously reducing q_0 while increasing q''. Alternately, the edge q can be changed at a fixed shear q'' by varying q_0. For $m=3$ and $n=1$, we obtain Fig. 2 for the growth rate and Fig. 3 for the surface current.

To summarize, we have calculated the surface current that will arise from a plasma instability within the ideal magnetohydrodynamic model of plasma in the approximation of the plasma-vacuum boundary as a sharp surface dividing plasma and vacuum. We have found the general result of Eq. (7) that for the commonly expected situation with zero equilibrium surface currents simply requires the surface current

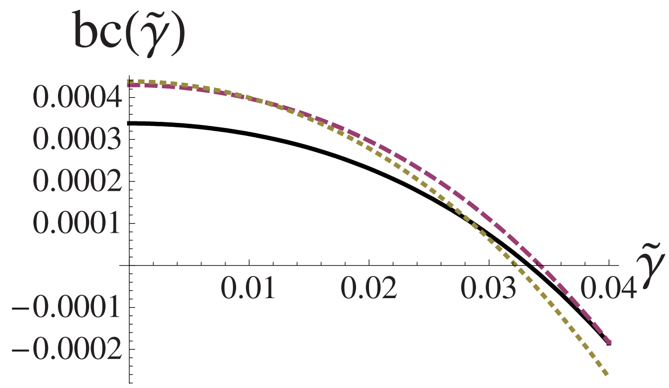

FIG. 1. (Color online) The left hand side of Eq. (33), which is $bc(\tilde{\gamma})$, is plotted as a function of the growth rate $\tilde{\gamma}$ for the example of $m=3$ and $n=1$. Where the function passes through zero are the values of $\tilde{\gamma}$ for which Eq. (33) holds. Plotted are for $q''=0.15$ and $q_0=2.28$, 2.24, and 2.20; the solid black, dashed red, and dotted yellow lines, respectively, these correspond to $(m-nq)=0.57$, 0.61, and 0.65 at the plasma's edge. The largest growth rate is found for $q_0=2.24$ for which $\tilde{\gamma} \simeq 0.034$, $(\tilde{\gamma}^2 \simeq 0.001)$.

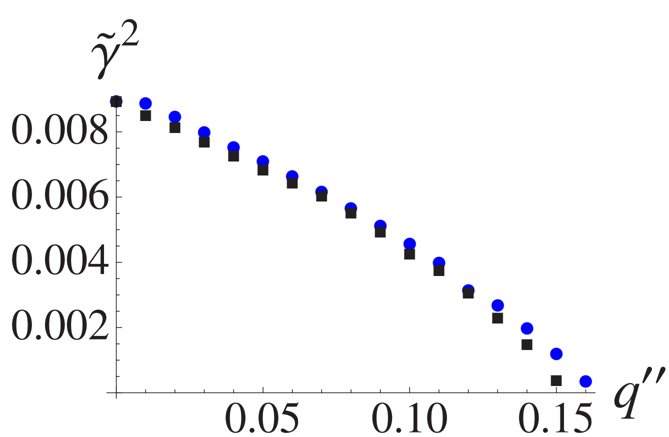

FIG. 2. (Color online) The square of the growth rate $\tilde{\gamma}$ is plotted for the example with $m=3$ and $n=1$, as a function of q'' for the two cases of edge $q=2.5$ (black squares) and edge q chosen to maximize $\tilde{\gamma}$ (blue circles).

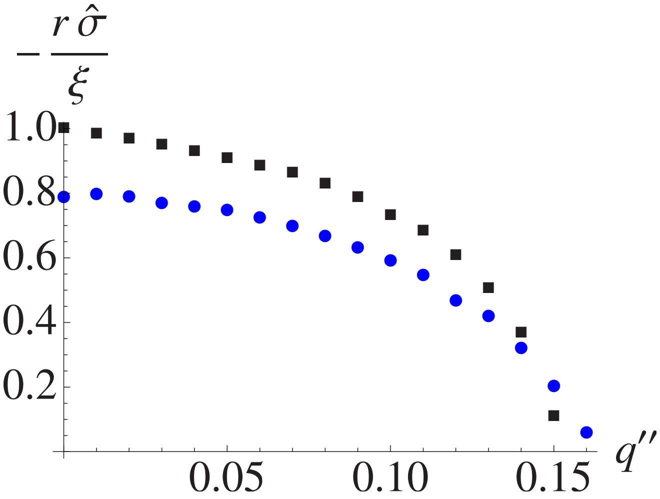

FIG. 3. (Color online) The surface current is plotted for the example with $m=3$ and $n=1$, as a function of q'' for constant edge $q=2.5$ (black squares) and edge q chosen to maximize $\tilde{\gamma}$ (blue circles).

to flow parallel to the equilibrium magnetic field. Whereas Eq. (7) is entirely general, the latter result is restricted to linearized perturbations but is otherwise generally true for all plasma geometries, within the assumptions stated above. The magnitude of the surface current has also been calculated for a cylindrical plasma in the "tokamak approximation."[7] The surface current is found to depend upon the type of instability, and hence the particular equilibrium. This is illustrated for the well known examples of a cylindrical equilibrium with a constant q-profile, and a q-profile that quadratically increases with the plasma cylinder's radius. A companion to this paper will consider how to evaluate the surface current from an instability in a general toroidal plasma as will be required for realistic calculations for present and future tokamaks.

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[1] L. E. Zakharov, Phys. Plasmas **15**, 062507 (2008).
[2] R. Fitzpatrick, Phys. Plasmas **16**, 012506 (2009).
[3] H. R. Strauss, R. Paccanella, and J. Breslau, Phys. Plasmas **17**, 082505 (2010).
[4] J. P. Freidberg, *Ideal Magnetohydrodynamics* (Plenum, New York, 1987).
[5] J. Sheffield, Rev. Mod. Phys. **66**, 1015 (1994).
[6] A. R. Choudhuri, *The Physics of Fluids and Plasmas: An Introduction for Astrophysicists* (Cambridge University Press, Cambridge, 1998).
[7] H. R. Strauss, Phys. Fluids **19**, 134 (1976).
[8] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York, 1975).
[9] Strictly speaking, Eq. (1) should be modified when the interface is moving, but the correction is of the order of the speed of the interface squared divided by the speed of light squared (Ref. 8) and may be neglected.