AIP | Physics of Plasmas

The role of surface currents in plasma confinement

Anthony J. Webster

Citation: Phys. Plasmas **18**, 112507 (2011); doi: 10.1063/1.3659486
View online: http://dx.doi.org/10.1063/1.3659486
View Table of Contents: http://pop.aip.org/resource/1/PHPAEN/v18/i11
Published by the American Institute of Physics.

The role of surface currents in plasma confinement

Anthony J. Webster[a)]
EURATOM/CCFE Fusion Association, Culham Science Centre, Abingdon OX14 3DB, Oxfordshire, United Kingdom

(Received 25 July 2011; accepted 13 October 2011; published online 22 November 2011)

During plasma instabilities, "surface currents" can flow at the interface between the plasma and the surrounding vacuum, and in most cases, they are a harmless symptom of the instability that is causing them. Large instabilities can lead to "disruptions," an abrupt termination of the plasma with the potential to damage the machine in which it is contained. For disruptions, the correct calculation of surface currents is thought to be essential for modelling disruptions properly. Recently, however, there has been debate and disagreement about the correct way to calculate surface currents. The purpose of this paper is to clarify as simply as possible the role of surface currents for plasma confinement and to show that a commonly used representation for surface currents $\vec{\sigma}$ with $\vec{\sigma} = \nabla I \wedge \vec{n}$, I a scalar function, and $\vec{n}$ the unit normal to the plasma surface, is only appropriate for the calculation of surface currents that are in magnetohydrodynamic equilibrium. Fortunately, this is the situation thought to be of most relevance for disruption calculations. [doi:10.1063/1.3659486]

I. INTRODUCTION

When the plasma in a tokamak abruptly terminates in a so-called "disruption," there is a potential to damage the tokamak, especially in large machines such as JET and ITER. During a disruption, surface currents can flow at the interface between the toroidal plasma and the surrounding vacuum. The surface currents are believed to be important for calculating the consequences of disruptions, and consequently for determining the potential damage (or not), to be expected from any given disruption. For such calculations to be reliable, it is probably essential that the surface currents are calculated correctly, and there has been some discussion[1–7] about how to do this. A commonly used procedure[1,4,7] is to write the surface current $\vec{\sigma}$ as $\vec{\sigma} = \nabla I \wedge \vec{n}$, where $\vec{n}$ is the unit normal to an idealised surface that marks a sharp transition from plasma to vacuum, and I is a scalar function. Reference 6 used Ampere's law and momentum balance to derive some simple but general requirements for all surface currents that are repeated here for convenience. Consideration of a narrow current loop at the plasma-vacuum surface and integrating Ampere's law around the loop leads to the well-known[8] general result for surface currents that $\vec{\sigma} = \vec{n} \wedge \left(\vec{B}^V - \vec{B}\right)$ where $\vec{B}$ and $\vec{B}^V$ are the magnetic fields inside and outside the plasma surface, respectively, and $\vec{n}$ is the unit normal to the surface.[9] As noticed in Ref. 6, if we first take the cross product of $\vec{\sigma}$ with $\vec{n}$ and then the dot product with $\left(\vec{B}^V + \vec{B}\right)$, then we find, $\vec{\sigma} \wedge \vec{n}.\left(\vec{B}^V + \vec{B}\right) = (B^V)^2 - B^2$. Combining this with the well-known ideal magnetohydrodynamic (MHD) momentum-balance boundary condition of Ref. 10 $[|p + B^2/2|] = 0$, where $[|A|]$ denotes taking the difference between the value of A just inside and just outside the plasma surface, then rearranging the left hand side of the resulting equation gives the general result that

$$\vec{\sigma}.\vec{n} \wedge \left(\vec{B} + \vec{B}^V\right) = 2p, \tag{1}$$

where p is the plasma pressure just inside the plasma-vacuum surface. For linear plasma stability calculations, it is usual[10] to take the plasma-vacuum boundary condition of $p = 0$ (but $\nabla p \neq 0$) that requires $\vec{\sigma}.\vec{n} \wedge (\vec{B} + \vec{B}^V) = 0$. It is also usually assumed that there are zero equilibrium surface currents, for which case a linearised perturbation will have[6]

$$0 = 2p = \vec{\sigma}.\vec{n} \wedge \left(\vec{B} + \vec{B}^V\right) = 2\vec{\sigma}.\vec{n}_0 \wedge \vec{B}_0 + O\left(\xi^2\right), \tag{2}$$

where $\vec{B}_0$ and $\vec{n}_0$ are the equilibrium magnetic field and equilibrium unit normal to the surface, respectively, and ξ is a small linearised displacement of the plasma surface. Reference 7 subsequently noticed some apparent paradoxes resulting from writing $\vec{\sigma} = \nabla I \wedge \vec{n}$, which it attempted to resolve. The present article addresses the concerns of Ref. 7 by: (i) clarifying the role of surface currents in plasma confinement, emphasising that their only direct role in plasma confinement is to balance any pressure jump at the plasma surface and (ii) to show that for linear ideal MHD stability calculations, it is not appropriate to write $\vec{\sigma} = \nabla I \wedge \vec{n}$, and that this is the source of the "paradox" discussed in Ref. 7. As with the other references mentioned here,[1–7] we will use the ideal MHD model throughout.

II. GENERAL REMARKS

First, we consider the role of surface currents in confining the plasma. We start from the ideal MHD equation

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \vec{J} \wedge \vec{B}, \tag{3}$$

take the dot product with the unit normal $\vec{e}_r$ to the flux surfaces (this could be the unit normal to flux surfaces of an arbitrarily shaped equilibrium or perturbed plasma), and integrate

a)Electronic mail: anthony.webster@ccfe.ac.uk.

in the direction $\vec{e}_r$, from just inside the plasma to just outside the plasma surface, giving

$$-p(a_-) = \int_{a_-}^{a_+} \nabla p.\vec{e}_r dr = \int_{a_-}^{a_+} \vec{J}.\vec{B} \wedge \vec{e}_r dr + O(\epsilon), \quad (4)$$

where dr is an infinitesimal distance normal to the flux surfaces, $\epsilon = O((a_+ - a_-)\rho d\vec{v}/dt)$ is expected to be negligible, and the "pressure" in the vacuum $p(a_+)$ is taken to be zero. Combining this with Eq. (1), we find

$$-\frac{1}{2}\vec{\sigma}.\vec{n} \wedge \left(\vec{B} + \vec{B}^V\right) = \int_{a_-}^{a_+} \vec{J}.\vec{B} \wedge \vec{e}_r, \quad (5)$$

with $\vec{e}_r = \vec{n}$ at the plasma's surface. Now integrate Eq. (3) in the direction $\vec{e}_r$ normal to the magnetic surfaces, from the magnetic axis at the centre of the plasma to just outside the plasma surface, giving

$$\int_0^{a_+} \rho \frac{d\vec{v}}{dt}.\vec{e}_r dr + \int_0^{a_+} \nabla p.\vec{e}_r dr = \int_0^{a_+} \vec{J} \wedge \vec{B}.\vec{e}_r dr. \quad (6)$$

Splitting the integral of the right hand side into an integral from 0 to a_- plus an integral from a_- to a_+ gives

$$\int_0^{a_+} \vec{J} \wedge \vec{B}.\vec{e}_r dr = \int_{a_-}^{a_+} \vec{J} \wedge \vec{B}.\vec{e}_r dr + \int_0^{a_-} \vec{J} \wedge \vec{B}.\vec{e}_r dr \quad (7)$$

and

$$\int_0^{a_+} \nabla p.\vec{e}_r dr = -p(0). \quad (8)$$

Combining Eqs. (4), (6), (7), and (8) gives

$$\int_0^{a_+} \rho \frac{d\vec{v}}{dt}.\vec{e}_r dr - p(0) = -p(a_-) + \int_0^{a_-} \vec{J} \wedge \vec{B}.\vec{e}_r dr. \quad (9)$$

If there are zero surface currents, then Eq. (1) requires $p(a_-) = 0$, and the plasma is confined solely by currents within the plasma. If $p(a_-) \neq 0$, then Eqs. (1) and (4) show that this pressure jump is balanced by surface currents and Eq. (9) shows that the rest of the plasma's pressure $(p(0) - p(a_-))$ is balanced by currents within the plasma. Whereas larger surface currents allow higher total plasma pressures to be confined, and consequently higher total stored energy within the plasma, from the perspective of momentum balance, the surface currents only balance pressure jumps at the plasma-vacuum surface and any other pressure gradients are balanced by currents *within* the plasma.

III. LINEAR STABILITY

Now we consider the concern raised in Ref. 7 that if $\vec{\sigma} = \nabla I \wedge \vec{n}$, then the requirement Eq. (2) that is usually (implicitly) assumed in linear stability calculations is not generally satisfied. We again start from the ideal MHD equation, Eq. (3), and take the dot product with the current $\vec{J}$ to find

$$\vec{J}.\rho \frac{d\vec{v}}{dt} = -\vec{J}.\nabla p. \quad (10)$$

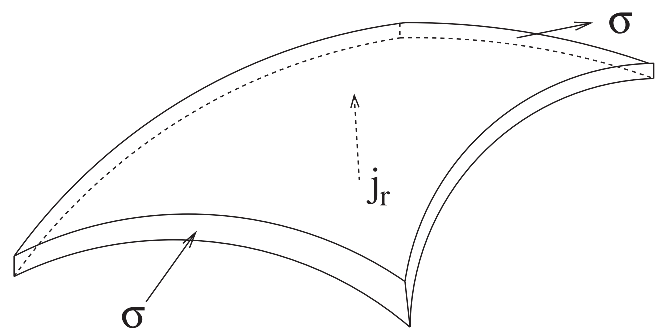

FIG. 1. The plasma surface is defined by a narrow radial region bounded on the outside by the first contour of constant pressure p at which the pressure p is zero. If $\nabla p.\vec{J} \neq 0$, then currents can flow radially to or from this surface layer so that the surface current $\vec{\sigma}$ that flows within the surface layer need not be conserved. Hence, if $\nabla p.\vec{J} \neq 0$, then $\nabla.\vec{\sigma} \neq 0$ also.

There are two distinctly different cases, $\rho = 0$ and $\rho \neq 0$, at the plasma-vacuum boundary. For $\rho = 0$, Eq. (10) requires $0 = \vec{J}.\nabla p$, requiring that $\vec{J}$ flows in surfaces of constant pressure p. Therefore we must have $\nabla.\vec{\sigma} = 0$, and for this case we can write $\vec{\sigma} = \nabla I \wedge \vec{n}$. This case is discussed in more detail shortly.

If $\rho \neq 0$ at the plasma surface, instead of being treated as a step function, then $\vec{J}.\nabla p \neq 0$. For this case with $\vec{J}.\nabla p \neq 0$, current can flow across surfaces of constant pressure onto and away from the plasma-vacuum surface. This is illustrated in Figure 1. Therefore, if $\rho \neq 0$, then we can have $\nabla.\vec{\sigma} \neq 0$. However, if $\nabla.\vec{\sigma} \neq 0$, we can no longer write $\vec{\sigma} = \nabla I \wedge \vec{n}$, and we no longer have the problem discussed by Ref. 7—instead, we need to solve for $\vec{\sigma}$ directly as in Ref. 6. A formal mathematical derivation of these remarks is in the Appendix.

If $\rho = 0$, then as noted above, we can write $\vec{\sigma} = \nabla I \wedge \vec{n}$, and we have Ref. 7's concern that because linear stability calculations usually take $0 = p = \vec{n} \wedge (\vec{B} + \vec{B}^V).\vec{\sigma}$, then we will also need $(\vec{B} + \vec{B}^V).\nabla I = 0$, which is not true in general. Now, consider the specific case of a perturbed cylindrical plasma, as considered in Ref. 6. If $\rho = 0$ at the plasma's edge, then the analysis of Ref. 6 changes, with Eq. (24) of Ref. 6 becoming

$$0 = \left[\left|\left(\frac{rb_r}{iF}\right)'\right|\right] = \left[\left|\frac{(rb_r)'r}{iF}\right|\right] - \left[\left|\frac{(rb_r)}{iF^2}F'\right|\right], \quad (11)$$

where primes denote differentiation with respect to r, and the notation of Ref. 6 is used (r is the cylinder's minor radius, b_r is the perturbation to the magnetic field in the cylinder's radial direction, $F = mB_\theta/r + kB_z$ with B_θ and B_z the poloidal and longitudinal equilibrium magnetic fields, respectively, and m, k are the poloidal mode number and longitudinal wave vectors, respectively). Reference 6 and the analysis of this section only consider $F \neq 0$; $F = 0$ is considered in Ref. 7. The case with $\rho = 0$ is in the tokamak approximation considered by Ref. 6, mathematically equivalent to marginal stability of a linear perturbation. Rearranging Eq. (11) and using $F' = mJ_z/r + kJ_\theta \simeq mJ_z/r$ in the tokamak approximation, gives

$$\left[\left|(rb_r)'\right|\right] = \frac{rb_r}{F}\left(-\frac{mJ_z}{r}\right). \quad (12)$$

Combining Eq. (12) with Eq. (22) of Ref. 6 then gives

$$\sigma = -J_z\xi_r - \frac{rb_r}{imF}\left(-\frac{mJ_z}{r}\right), \quad (13)$$

with in leading order in the tokamak approximation $\vec{\sigma}$ parallel to the longitudinal direction $\vec{e}_z$. Noting that $b_r = iF\xi_r$, this simplifies to

$$\sigma = -J_z\xi_r + \frac{iF\xi_r}{imF}\left(\frac{mJ_z}{r}\right) = 0. \quad (14)$$

Therefore, if we start from a situation with zero equilibrium surface current and in addition we take $\rho = 0$ at the plasma's edge, then following a linearisable perturbation of a cylindrical plasma in the tokamak approximation at least, $\sigma = 0$.

More generally, Ref. 12 reasons that if $\rho = 0$ at the plasma surface, then so also must $p = 0$, and $[|B^2|] = 0$ then requires $\vec{\sigma} = 0$. Strictly speaking, $[|B^2|] = 0$ requires $\vec{B}^V = \pm\vec{B}$, with either $\vec{\sigma} = 0$ or $\vec{\sigma} = -2\vec{n} \wedge \vec{B}$, both solutions correctly having $[|B^2|] = 0$ and $\vec{\sigma}.\vec{n} \wedge \left(\vec{B} + \vec{B}^V\right) = 2p = 0$. It is unclear whether the latter solution with $\vec{\sigma} = -2\vec{n} \wedge \vec{B}$ is important or not. These results hold generally, not just for equilibrium or perturbed plasmas, and regardless of whether or not $F = 0$ at the plasma's edge. As the above calculation confirmed, a full calculation for the surface current, as outlined in Ref. 6, is completely consistent with these remarks.

IV. PLASMA DISRUPTIONS

To this point we have considered surface currents that are induced by an ideal MHD instability, for which the inertial term $\rho d\vec{v}/dt$ can be comparable with the ∇p and $\vec{J} \wedge \vec{B}$ terms. Now, we briefly consider the equilibrium case with $\rho d\vec{v}/dt = 0$, and as a consequence, both $\vec{J}.\nabla p = 0$ and $\nabla.\vec{\sigma} = 0$, allowing us to write $\vec{\sigma} = \nabla I \wedge \vec{n}$. The equilibrium case will be relevant to disruption calculations whenever the plasma disruption occurs on a time scale that is long compared to the MHD timescale. For disruptions, it will often be reasonable to model the plasma-vacuum interface with a sharp jump in pressure, because for plasma motion that is fast compared with the current diffusion timescale, induced currents will not have time to diffuse into the plasma, instead flowing on the plasma's surface as "surface currents." This appears to be the approach adopted in Ref. 2 and is likely to be the situation of interest for Refs. 1, 4, 7. Whereas ideal MHD (Eq. (1)) continues to determine the surface current needed to balance any given jump in plasma pressure at the plasma-vacuum surface, ideal MHD does not appear to constrain the component of surface current that flows parallel to the magnetic field, beyond the observation of Ref. 6 that $\vec{\sigma}.\vec{B} = \vec{\sigma}.\vec{B}^V$.

V. SUMMARY

The role of surface currents in plasma confinement is considered within the ideal MHD model of plasma, with the interface between plasma and vacuum modelled as a sharp transition from plasma to vacuum at the plasma-vacuum "surface." It is shown that the only direct role of surface currents in plasma confinement is to balance any pressure jump at the plasma's surface. If there is no pressure jump, the currents play no direct role in balancing the pressure and confining the plasma. Regarding the concerns of Ref. 7—for linear stability calculations, there are two distinctly different cases, depending on whether the plasma density ρ is taken to be $\rho = 0$ or $\rho \neq 0$ at the plasma's edge. For $\rho = 0$, we require that $\vec{J}.\nabla p = 0$, and consequently, we require that $\nabla.\vec{\sigma} = \nabla.\vec{J} = 0$. This would allow us to write $\vec{\sigma} = \nabla I \wedge \vec{n}$, as in Ref. 7. However, if the pressure at the plasma surface is taken to be zero, then we require[6] $\vec{\sigma}.(\vec{B} + \vec{B}^V) \wedge \vec{n}_0 = 0$, raising the concern noticed by Ref. 7, that for $\vec{\sigma} = \nabla I \wedge \vec{n}$, this would not generally be the case. To resolve this, the linearised MHD calculation of Ref. 6 is repeated with $\rho = 0$, finding $\vec{\sigma} = 0$, and consequently $\vec{\sigma}.(\vec{B} + \vec{B}^V) \wedge \vec{n}_0 = 0$ as required. Therefore, for this case, there is no paradox, only the requirement that if we take $\rho = 0$ at the plasma surface, then $\vec{\sigma} = 0$. More generally, as argued in Ref. 12, if we take $\rho = 0$ at the plasma's surface, then we must also have $\vec{\sigma} = 0$. For $\rho \neq 0$, then $\vec{J}.\nabla p \neq 0$, and consequently $\nabla.\vec{\sigma} \neq 0$, and we therefore can no longer write $\vec{\sigma} = \nabla I \wedge \vec{n}$. The author's conclusion is that for linear stability calculations, we cannot generally write $\vec{\sigma} = \nabla I \wedge \vec{n}$ but must instead solve for $\vec{\sigma}$ directly, as done for example in Ref. 6. For equilibrium surface current calculations, the inertial term in Eq. (3) is taken to be zero, ensuring that $\vec{J}.\nabla p = \nabla.\vec{\sigma} = 0$ and allowing us to write $\vec{\sigma} = \nabla I \wedge \vec{n}$. This situation where the inertial term can be neglected will often be most relevant for disruption calculations.

Clearly real physical situations will never have either ρ or p exactly zero, and there is only a boundary between plasma and vacuum in the sense of Eqs. (4) and (5), a narrow region with a sharp change in one or more of pressure, current, or density—approximated as becoming zero within the idealised model. This paper has discussed (within ideal MHD) the requirements for different modelling assumptions to be consistent with a commonly used mathematical representation for surface currents, describing those situations where it may be used and those situations requiring a full detailed solution.

ACKNOWLEDGMENTS

Thanks to the referees for their time and helpful comments, including directing me to Ref. 12. Thanks especially to Greg Colyer for helping refine the mathematical calculation presented in the Appendix and Sarah Newton for pointing me to Ref. 11. This paper arose from trying to work through Ref. 7, which has been thought provoking (as is usual with the author's papers). Thanks to Ian Chapman and Tim Hender for reading and commenting on this paper and for the comments of Yueqiang Liu, Wayne Arter, and the MAST team. This work was funded by the RCUK Energy Programme under Grant No. EP/I501045 and the European Communities under the contract of Association between EURATOM and CCFE.

APPENDIX: MATHEMATICAL FORMULATION

The physically intuitive remarks of the main text can be given a formal mathematical derivation. First, $\nabla \wedge \vec{B}^V = 0$ allows $\vec{B}^V$ to be written as the gradient of a scalar with, e.g., $\vec{B}^V = \nabla V$. Using the notation of Ref. 11 in which *Div* and *Rot* refer to divergence and rotation within the surface on which the function is defined, then

$$\vec{\sigma} = \vec{n} \wedge \left(\nabla V|_{S_+} - \vec{B}|_{S_-} \right), \tag{A1}$$

with $\vec{B}^V = \nabla V$, and S_+ and S_- referring to surfaces just above and below the idealised plasma-vacuum surface. Then, using[11] $Div(\vec{n} \wedge \vec{f}) = -\vec{n}.Rot(\vec{f}) = -\vec{n}.\left(\nabla \wedge \vec{f}\right)$

$$\begin{aligned} Div(\vec{\sigma}) &= -\vec{n}.Rot(\nabla V)|_{S_+} + \vec{n}.Rot\left(\vec{B}\right)|_{S_-} \\ &= \vec{n}.\nabla \wedge \vec{B}|_{S_-} \\ &= \vec{n}.\vec{J}, \end{aligned} \tag{A2}$$

a result we might reasonably have stated directly without derivation, as discussed in the main text. If we had written $\vec{\sigma} = \nabla I \wedge \vec{n}$, then we would have

$$Div(\vec{\sigma}) = -Div(\vec{n} \wedge \nabla I) = \vec{n}.Rot(\nabla I) = 0. \tag{A3}$$

Confirming mathematically the physically intuitive remarks of the main text, namely that if $\vec{n}.\vec{J} \neq 0$, then we cannot write $\vec{\sigma} = \nabla I \wedge \vec{n}$, but for $\vec{n}.\vec{J} = 0$, it appears consistent to do so. Furthermore, if we take $\vec{n}.\vec{J} = 0$ and assume we can write $\vec{\sigma} = \nabla I \wedge \vec{n}$, then substituting this into Eq. (A1) and taking the cross product with $\vec{n}$, we find

$$\vec{B} = \nabla(I + V) + \vec{n}f \tag{A4}$$

for some function f. Requiring that $\vec{n}.\vec{B} = 0$ then gives

$$\vec{B} = \nabla(I + V) - (\vec{n}.\nabla(I + V))\vec{n}. \tag{A5}$$

Using $\vec{n}.Rot(\vec{f}) = \vec{n}.\left(\nabla \wedge \vec{f}\right)$ and the usual rules of vector algebra plus either $Rot(\vec{n}) = 0$ or $\vec{n}.\nabla \wedge \vec{n} = 0$ for the unit normal $\vec{n}$ to a surface, then it is easily shown that $\vec{n}.Rot\left(\vec{B}\right) = 0$, which combines with Eq. (A2) to confirm that $Div(\vec{\sigma}) = 0$. However, $\vec{J} = \nabla \wedge \vec{B} = -\nabla \wedge (\vec{n}.\nabla(I + V)\vec{n})$, which for general I and V is non-zero, and hence allows non-zero plasma currents. Therefore, for $\vec{n}.\vec{J} = 0$, it appears to be entirely consistent to write $\vec{\sigma} = \nabla I \wedge \vec{n}$, as was previously suggested. All the above conclusions result solely from Ampere's law and $\nabla.\vec{B} = 0$.

[1] L. E. Zakharov, Phys. Plasmas **15**, 062507 (2008).
[2] R. Fitzpatrick, Phys. Plasmas **16**, 012506 (2009).
[3] H. R. Strauss, R. Paccagnella, and J. Breslau, Phys. Plasmas **17**, 082505 (2010).
[4] L. E. Zakharov, Phys. Plasmas **17**, 124703 (2010).
[5] H. R. Strauss, R. Paccagnella, and J. Breslau, Phys. Plasmas **17**, 124704 (2010).
[6] A. J. Webster, Phys. Plasmas **17**, 110708 (2010).
[7] L. E. Zakharov, Phys. Plasmas **18**, 062503 (2011).
[8] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York, 1975).

[9] Because the definition of $\vec{\sigma}$ involves an integral of $\nabla \wedge \vec{B}$, then unlike the current $\vec{J} = \nabla \wedge \vec{B}$, $\vec{\sigma}$ is not generally equal to the curl of a vector quantity, and consequently, $\nabla.\vec{\sigma}$ is not generally zero.

[10] J. P. Freidberg, *Ideal Magnetohydrodynamics* (Plenum, New York, 1987).
[11] L. Brand, *Vector and Tensor Analysis* (John Wiley & Sons, New York, 1947).
[12] T. J. M. Boyd and J. J. Sanderson, *The Physics of Plasmas* (Cambridge University, Cambridge, 2003).