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Equivalence of two independent calculations of the higher order guiding center Lagrangian

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The difference between the guiding center phase-space Lagrangians derived in J. W. Burby *et al.* [Phys. Plasmas **20**, 072105 (2013)] and F. I. Parra and I. Calvo [Plasma Phys. Controlled Fusion **53**, 045001 (2011)] is due to a different definition of the guiding center coordinates. In this brief communication, the difference between the guiding center coordinates is calculated explicitly. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4897317]

A new automated procedure to calculate the phase-space Lagrangian of a guiding center has been developed.[1] This procedure was used to compute a phase-space Lagrangian with the same symplectic part as the Lagrangian calculated in Ref. 2, but unfortunately the result of the procedure described in Ref. 1 did not give the Hamiltonian calculated in Ref. 2. There are two reasons for the difference: (i) there was a typographical error in Eq. (135) of Ref. 2, now corrected,[3] and (ii) the guiding center coordinates in Ref. 1 are different from the guiding center coordinates in Ref. 2. As noted in both Refs. 1 and 2, when comparing guiding center equations, it is important to remember that guiding center transformations are not unique. It is then not surprising that two different procedures that lead to different coordinates give different Hamiltonians, even if the symplectic part of the phase-space Lagrangian is the same in both procedures. In this brief communication, we calculate the difference between the gyrokinetic coordinates in Refs. 1 and 2 by deriving the form of the transformations between guiding center coordinates that leave the symplectic part of the Lagrangian unchanged.

We use the notation and normalization of Ref. 2. By setting the electrostatic potential φ to zero, the phase-space Lagrangian that corresponds to the coordinates calculated by Parra and Calvo,[2] $\{\mathbf{R}, u, \mu, \theta\}$, is

$$\mathcal{L}_{PC} = \left[\frac{1}{\epsilon}\mathbf{A} + u\hat{\mathbf{b}} + \epsilon\left(\mu\nabla_{\mathbf{R}}\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 - \frac{\mu}{2}\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\times\hat{\mathbf{b}}\right)\right]\cdot\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} - \epsilon\mu\frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{1}{2}u^2 - \mu B - \epsilon^2\bar{H}^{(2)}_{PC}, \tag{1}$$

where

$$\bar{H}^{(2)}_{PC} = \mu^2\left[\frac{1}{4B}\left(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right):\nabla_{\mathbf{R}}\nabla_{\mathbf{R}}\mathbf{B}\cdot\hat{\mathbf{b}} - \frac{3}{4B^2}|\nabla_{\mathbf{R}\perp}B|^2 + \frac{1}{8}\nabla_{\mathbf{R}\perp}\hat{\mathbf{b}}:(\nabla_{\mathbf{R}\perp}\hat{\mathbf{b}})^{\mathrm{T}} - \frac{1}{16}(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}})^2 - \frac{1}{16}(\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\times\hat{\mathbf{b}})^2\right]$$
$$+ u^2\mu\left[-\frac{3}{2B^2}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{R}}B + \frac{1}{2B}\nabla_{\mathbf{R}}\hat{\mathbf{b}}:\nabla_{\mathbf{R}}\hat{\mathbf{b}} - \frac{1}{4B}\nabla_{\mathbf{R}\perp}\hat{\mathbf{b}}:(\nabla_{\mathbf{R}\perp}\hat{\mathbf{b}})^{\mathrm{T}} - \frac{3}{8B}(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}})^2 + \frac{3}{2B}|\boldsymbol{\kappa}|^2 + \frac{1}{8B}(\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\times\hat{\mathbf{b}})^2\right] - \frac{u^4}{2B^2}|\boldsymbol{\kappa}|^2. \tag{2}$$

Here $\boldsymbol{\kappa} = \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}$ is the curvature of the magnetic field line.

A Lagrangian with the same symplectic part as the Lagrangian (1) is given in Eqs. (33)–(35) of Ref. 1. The latter equations correspond to the phase-space Lagrangian for guiding center coordinates $\{\mathbf{R}', u', \mu', \theta'\}$ that are slightly different from $\{\mathbf{R}, u, \mu, \theta\}$, as we will show shortly. The Lagrangian for these variables is (see Eqs. (33)–(35) of Ref. 1)

$$\mathcal{L}_{BSQ} = \left[\frac{1}{\epsilon}\mathbf{A}' + u'\hat{\mathbf{b}}' + \epsilon\left(\mu'\nabla_{\mathbf{R}'}\hat{\mathbf{e}}'_2\cdot\hat{\mathbf{e}}'_1 - \frac{\mu'}{2}\hat{\mathbf{b}}'\hat{\mathbf{b}}'\cdot\nabla_{\mathbf{R}'}\times\hat{\mathbf{b}}'\right)\right]\cdot\frac{\mathrm{d}\mathbf{R}'}{\mathrm{d}t} - \epsilon\mu'\frac{\mathrm{d}\theta'}{\mathrm{d}t} - \frac{1}{2}(u')^2 - \mu'B' - \epsilon^2\bar{H}^{(2)}_{BSQ}, \tag{3}$$

where

$$\bar{H}^{(2)}_{BSQ} = (\mu')^2\left[\frac{15}{16}\left(\nabla_{\mathbf{R}'}\cdot\hat{\mathbf{b}}'\right)^2 + \frac{3}{16}|\boldsymbol{\kappa}'|^2 + \frac{1}{4}\hat{\mathbf{b}}'\cdot\nabla_{\mathbf{R}'}\left(\nabla_{\mathbf{R}'}\cdot\hat{\mathbf{b}}'\right) + \frac{1}{16}\nabla_{\mathbf{R}'}\hat{\mathbf{b}}' : \nabla_{\mathbf{R}'}\hat{\mathbf{b}}' - \frac{3}{16}\nabla_{\mathbf{R}'}\hat{\mathbf{b}}' : \left(\nabla_{\mathbf{R}'}\hat{\mathbf{b}}'\right)^{\mathrm{T}} - \frac{3}{4(B')^2}|\nabla_{\mathbf{R}'}B'|^2 \right.$$
$$\left. + \frac{1}{4B'}\boldsymbol{\kappa}'\cdot\nabla_{\mathbf{R}'}B' + \frac{1}{4B'}\nabla^2_{\mathbf{R}'}B'\right] + (u')^2\mu'\left[\frac{3}{8B'}\nabla_{\mathbf{R}'}\hat{\mathbf{b}}' : \nabla_{\mathbf{R}'}\hat{\mathbf{b}}' - \frac{1}{8B'}\nabla_{\mathbf{R}'}\hat{\mathbf{b}}' : \left(\nabla_{\mathbf{R}'}\hat{\mathbf{b}}'\right)^{\mathrm{T}} + \frac{1}{8B'}\left(\nabla_{\mathbf{R}'}\cdot\hat{\mathbf{b}}'\right)^2 \right.$$
$$\left. + \frac{1}{2B'}\hat{\mathbf{b}}'\cdot\nabla_{\mathbf{R}'}\left(\nabla_{\mathbf{R}'}\cdot\hat{\mathbf{b}}'\right) + \frac{13}{8B'}|\boldsymbol{\kappa}'|^2 - \frac{3}{2(B')^2}\boldsymbol{\kappa}'\cdot\nabla_{\mathbf{R}'}B'\right] - \frac{(u')^4}{2(B')^2}|\boldsymbol{\kappa}'|^2. \quad (4)$$

Here, the prime indicates that the function depends on the variables $\{\mathbf{R}', u', \mu', \theta'\}$, e.g., $\mathbf{A}' = \mathbf{A}(\mathbf{R}')$ and $\mathbf{B}' = \mathbf{B}(\mathbf{R}')$. Importantly, Lagrangians (1) and (3) are not exact. The Hamiltonian and the terms that multiply $\mathrm{d}\mathbf{R}/\mathrm{d}t$ are calculated to order ϵ^2, and the terms that multiply $\mathrm{d}u/\mathrm{d}t$, $\mathrm{d}\mu/\mathrm{d}t$, and $\mathrm{d}\theta/\mathrm{d}t$ are calculated to order ϵ^3 (the terms that multiply $\mathrm{d}u/\mathrm{d}t$ and $\mathrm{d}\mu/\mathrm{d}t$ are zero to order ϵ^3).

We calculate the difference between Hamiltonians (2) and (4) using that $\mathbf{R}' = \mathbf{R} + O(\epsilon^2)$ and $u' = u + O(\epsilon^2)$ (see Eqs. (11) and (12) below), and that there is no difference in the definition of the magnetic moment

$$\mu' = \mu. \quad (5)$$

Employing

$$(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla_{\mathbf{R}}\nabla_{\mathbf{R}}\mathbf{B}\cdot\hat{\mathbf{b}} = \nabla^2_{\mathbf{R}}B + B\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}) - B(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}})^2 + \boldsymbol{\kappa}\cdot\nabla_{\mathbf{R}}B - B\nabla_{\mathbf{R}}\hat{\mathbf{b}} : (\nabla_{\mathbf{R}}\hat{\mathbf{b}})^{\mathrm{T}} + B|\boldsymbol{\kappa}|^2, \quad (6)$$

$$|\nabla_{\mathbf{R}\perp}B|^2 = |\nabla_{\mathbf{R}}B|^2 - B^2(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}})^2, \quad (7)$$

$$\nabla_{\mathbf{R}\perp}\hat{\mathbf{b}} : (\nabla_{\mathbf{R}\perp}\hat{\mathbf{b}})^{\mathrm{T}} = \nabla_{\mathbf{R}}\hat{\mathbf{b}} : (\nabla_{\mathbf{R}}\hat{\mathbf{b}})^{\mathrm{T}} - |\boldsymbol{\kappa}|^2, \quad (8)$$

and

$$\nabla_{\mathbf{R}}\hat{\mathbf{b}} : (\nabla_{\mathbf{R}}\hat{\mathbf{b}})^{\mathrm{T}} - \nabla_{\mathbf{R}}\hat{\mathbf{b}} : \nabla_{\mathbf{R}}\hat{\mathbf{b}} = |\boldsymbol{\kappa}|^2 + (\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\times\hat{\mathbf{b}})^2, \quad (9)$$

we find that

$$\bar{H}^{(2)}_{BSQ} - \bar{H}^{(2)}_{PC} = \frac{u^2\mu}{2B}\left[\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\left(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}\right) + \left(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}\right)^2\right] + \frac{\mu^2}{2}\left(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}\right)^2 + O(\epsilon^2) = \left(u\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}} - \mu\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}B\frac{\partial}{\partial u}\right)\left(\frac{u\mu}{2B}\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}\right) + O(\epsilon^2). \quad (10)$$

Note that the difference between $\bar{H}^{(2)}_{BSQ}$ and $\bar{H}^{(2)}_{PC}$ can be written as the derivative of the quantity $(u\mu/2B)\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}$ along the lowest order trajectories.

In this brief communication, we show that the variables $\mathbf{R}'$, u', and θ' differ from the variables $\mathbf{R}$, u, and θ by corrections of order ϵ and higher,

$$\mathbf{R}' = \mathbf{R} + \epsilon^2\mathbf{R}_2 + \epsilon^3\mathbf{R}_3, \quad (11)$$

$$u' = u + \epsilon^2 u_2, \quad (12)$$

and

$$\theta' = \theta + \epsilon\theta_1 + \epsilon^2\theta_2, \quad (13)$$

and that this explains the difference (10).

The corrections $\mathbf{R}_2$, $\mathbf{R}_3$, u_2, θ_1, and θ_2 do not depend on the gyrophase θ or the time t. By substituting relations (11), (12), and (13) into the Lagrangian (3), and adding the time derivative of the function

$$F = \epsilon^2 S_2 + \epsilon^3 S_3 - \left[\epsilon\mathbf{A} + \epsilon^2 u\hat{\mathbf{b}} + \epsilon^3\left(\mu\nabla_{\mathbf{R}}\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 - \frac{\mu}{2}\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\times\hat{\mathbf{b}}\right)\right]\cdot\mathbf{R}_2 - (\epsilon^2\mathbf{A} + \epsilon^3 u\hat{\mathbf{b}})\cdot\mathbf{R}_3 - \frac{\epsilon^3}{2}\mathbf{R}_2\cdot\nabla_{\mathbf{R}}\mathbf{A}\cdot\mathbf{R}_2 + \epsilon^2\mu\theta_1 + \epsilon^3\mu\theta_2, \quad (14)$$

we find

$$\mathcal{L}_{BSQ} + \frac{\mathrm{d}F}{\mathrm{d}t} = \left[\frac{1}{\epsilon}\mathbf{A} + u\hat{\mathbf{b}} + \epsilon\left(\mathbf{B}\times\mathbf{R}_2 + \mu\nabla_{\mathbf{R}}\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 - \frac{\mu}{2}\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\times\hat{\mathbf{b}}\right) + \epsilon^2\left(\mathbf{B}\times\mathbf{R}_3 + u_2\hat{\mathbf{b}} + u\left(\nabla_{\mathbf{R}}\times\hat{\mathbf{b}}\right)\times\mathbf{R}_2 + \nabla_{\mathbf{R}}S_2\right)\right]\cdot\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}$$
$$+ \left[\epsilon^2\left(-\hat{\mathbf{b}}\cdot\mathbf{R}_2 + \frac{\partial S_2}{\partial u}\right) + \epsilon^3\left(-\hat{\mathbf{b}}\cdot\mathbf{R}_3 + \frac{1}{2}(\mathbf{B}\times\mathbf{R}_2)\cdot\frac{\partial\mathbf{R}_2}{\partial u} + \frac{\partial S_3}{\partial u}\right)\right]\frac{\mathrm{d}u}{\mathrm{d}t}$$
$$+ \left[\epsilon^2\left(\theta_1 + \frac{\partial S_2}{\partial\mu}\right) + \epsilon^3\left(\theta_2 - \mathbf{R}_2\cdot\nabla_{\mathbf{R}}\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 + \frac{1}{2}\mathbf{R}_2\cdot\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\times\hat{\mathbf{b}} + \frac{1}{2}(\mathbf{B}\times\mathbf{R}_2)\cdot\frac{\partial\mathbf{R}_2}{\partial\mu} + \frac{\partial S_3}{\partial\mu}\right)\right]\frac{\mathrm{d}\mu}{\mathrm{d}t} - \epsilon\mu\frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{1}{2}u^2 - \mu B$$
$$-\epsilon^2\left(\bar{H}^{(2)}_{BSQ} + uu_2 + \mu\mathbf{R}_2\cdot\nabla_{\mathbf{R}}B\right). \quad (15)$$

Here, we have assumed that S_2 and S_3 do not depend on the gyrophase θ or the time t, we have used that $\mathbf{R}_2$, $\mathbf{R}_3$, u_2, θ_1, and θ_2 are independent of the gyrophase θ and the time t, and we have neglected terms of order ϵ^3 in the Hamiltonian and in the terms multiplying $\mathrm{d}\mathbf{R}/\mathrm{d}t$, and terms of order ϵ^4 in the terms multiplying $\mathrm{d}u/\mathrm{d}t$, $\mathrm{d}\mu/\mathrm{d}t$, and $\mathrm{d}\theta/\mathrm{d}t$. We can set the symplectic part of the Lagrangian in (15) equal to the symplectic part of (1), giving

$$\mathbf{R}_2 = \frac{\partial S_2}{\partial u}\hat{\mathbf{b}}, \tag{16}$$

$$\mathbf{R}_3 = \frac{\partial S_3}{\partial u}\hat{\mathbf{b}} + \frac{u}{B}\frac{\partial S_2}{\partial u}\hat{\mathbf{b}}\times\boldsymbol{\kappa} + \frac{1}{B}\hat{\mathbf{b}}\times\nabla_{\mathbf{R}}S_2, \tag{17}$$

$$u_2 = -\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}S_2, \tag{18}$$

$$\theta_1 = -\frac{\partial S_2}{\partial \mu}, \tag{19}$$

and

$$\theta_2 = \frac{\partial S_2}{\partial u}\left(\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 - \frac{1}{2}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\times\hat{\mathbf{b}}\right) - \frac{\partial S_3}{\partial \mu}. \tag{20}$$

With these results, Eq. (15) becomes

$$\mathcal{L}_{BSQ} + \frac{\mathrm{d}F}{\mathrm{d}t} = \left[\frac{1}{\epsilon}\mathbf{A} + u\hat{\mathbf{b}} + \epsilon\left(\mu\nabla_{\mathbf{R}}\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 - \frac{\mu}{2}\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\times\hat{\mathbf{b}}\right)\right]\cdot\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} - \epsilon\mu\frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{1}{2}u^2 - \mu B - \epsilon^2\left[\bar{H}^{(2)}_{BSQ} - \left(u\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}} - \mu\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}B\frac{\partial}{\partial u}\right)S_2\right]. \tag{21}$$

We can choose S_2 such that the Hamiltonians of (21) and (1) are equal

$$S_2 = \frac{u\mu}{2B}\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}, \tag{22}$$

where we have used the result in (10). Then, the corrections $\mathbf{R}_2$, u_2, and θ_1 are

$$\mathbf{R}_2 = \frac{\mu}{2B}\left(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}}, \tag{23}$$

$$u_2 = -\frac{u\mu}{2B}\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\left(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}\right) - \frac{u\mu}{2B}\left(\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}\right)^2, \tag{24}$$

and

$$\theta_1 = -\frac{u}{2B}\nabla_{\mathbf{R}}\cdot\hat{\mathbf{b}}. \tag{25}$$

The parallel component of $\mathbf{R}_3$ and the correction θ_2 are undetermined because we are free to choose S_3 to this order.

To summarize, the difference between the Lagrangians given in Refs. 1 and 2 is due to the difference between the guiding center coordinates used in Refs. 1 and 2. In the procedures to determine the guiding center Lagrangian described in Refs. 1 and 2, the choice of guiding center coordinates is not set just by fixing the symplectic part. Using the notation in Ref. 1, in the equation $\langle\alpha_1\rangle + \gamma_l = \mathrm{d}f_l$, the function f_l can be chosen to be anything (see the discussion under Eq. (31) of Ref. 1). In Ref. 2, we are free to choose the gyrophase independent piece of the generating functions $S_P^{(n)}$ (see Eq. (63) of Ref. 1). In this brief communication, we have shown explicitly that the two procedures in Refs. 1 and 2 can give exactly the same equations if the right choices are made.

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