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Theory of isolated, small-scale magnetic islands in a high temperature tokamak plasma

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A theory for the existence of noninteracting small-scale, "drift" magnetic islands in a high temperature tokamak plasma is presented. This situation contrasts with that discussed by Rebut and Hugon [Plasma Phys. Controlled Fusion **33**, 1085 (1991)] which involves a background "sea" of magnetic turbulence caused by island overlap. The islands are driven by the effect of finite ion Larmor radius on the particle drifts and they propagate with a velocity comparable to the diamagnetic velocity. In contrast with the work of Smolyakov [Plasma Phys. Controlled Fusion **35**, 657 (1993)] collisions are assumed to be rare. Although the saturated island size is independent of the collision frequency in the model discussed here, collisions play a crucial role in determining the frequency of the magnetic islands. An estimate is made of the anomalous heat transport which results from the fluctuations in the electrostatic potential associated with these magnetic islands. The predicted thermal diffusivity has several, but not all, of the characteristics of the Rebut-Lallia-Watkins transport model.

I. INTRODUCTION

Heat and particle transport in tokamaks is generally "anomalous," exceeding the predictions of neoclassical theory by up to two orders of magnitude. This is thought to be the result of fine-scale turbulence in the tokamak, though the origin of the turbulence is not yet fully understood.¹ In this report we investigate the possibility of small-scale magnetic islands existing in the tokamak² and look at some of the consequences of such islands for the radial electron thermal transport. It is interesting to note that a correlation between magnetic fluctuations and confinement has been observed in Tore Supra.³

Rutherford theory for a simple resistive magnetohydrodynamic (MHD) plasma⁴ predicts that small-scale magnetic islands evolve at a rate proportional to the tearing mode parameter, Δ' , with $\Delta' > 0$ corresponding to island growth. Clearly, at large poloidal mode number, m , when $\Delta' = -2m/r$ (r represents the minor radius of the mode rational surface), this theory predicts no island growth. However, for more complex plasma models, extra drives for the island growth can appear from the island region itself and high m magnetic islands are predicted to exist with a width w determined by a balance between this drive and the damping of the Δ' term. The drive arises from a perturbed parallel current and there are theories for this based on the bootstrap current,⁵⁻⁷ drift effects,⁸⁻¹³ and current diffusion.¹⁴ In this report we concentrate on drift magnetic islands which have a width comparable to the ion Larmor radius, when the island drive is a consequence of finite ion Larmor radius modifications to the perpendicular drift velocities. The different responses of electrons and ions result in a perpendicular current which, through $\nabla \cdot \mathbf{J} = 0$, implies a parallel current, and this can sustain the island against the Δ' damping.

Growth of these drift islands depends on the island rota-

tion frequency, ω , so that this must be determined as well. An equation for this is deduced from toroidal torque balance, which in turn is related to entropy production (i.e., dissipation) so that the island properties typically depend on the collisionality regime or the level of plasma turbulence (which can lead to an "anomalous" viscosity). Some previous authors^{8,13,15} have considered the plasma turbulence to be the dominant effect. In a self-consistent theory, in which this turbulence is itself a result of the magnetic islands, the evolution of a single island chain cannot be deduced independently of the others. In other work^{9,10} the finite resistivity of the plasma provides the dissipation and then the island chains evolve independently. A collisional electron plasma was assumed in this latter work so that Braginskii fluid theory was appropriate. In this paper we address the situation for an isolated island chain in a low collisionality plasma, when the Landau resonance, $\omega = k_{\parallel} v_{\parallel}$, has an important effect on the electron distribution function, particularly for low parallel velocity electrons.

Actually, three collisionality regimes can be identified which are reminiscent of those found in neoclassical transport theory.

- (1) At the lowest collision frequency the electron orbits form "drift islands" around the Landau resonance which lead to plateaus in the distribution function.¹⁵ The result of this is that there is no energy transfer from the particles to the wave due to Landau damping. However, when a low level of collisions is introduced, the plateaus partially relax to a Maxwellian which then restores a residual Landau damping, giving rise to dissipation proportional to the collision frequency, ν_e .¹⁶ This collision frequency regime is analogous to the banana collisionality regime of neoclassical theory and is relevant when

$$\nu_e < \frac{k_{\theta} \rho_i}{16} \frac{v_{thi}}{L_n} \left(\frac{L_n}{L_s} \right)^3 \left(\frac{m_i}{m_e} \right)^{3/2} \left(\frac{w}{\rho_i} \right)^3,$$

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where ρ_i is the ion Larmor radius, L_n is the density scale length, $v_{thi} = \sqrt{2T_i/m_i}$ is the ion thermal velocity, ν_e is the electron collision frequency, $L_s = Rq/s$ is the shear length, w is the island width, q is the safety factor, R is the major radius, $k_\theta = m/r$, $s = rq'/q$ is the magnetic shear, and m_j is the mass of the species labeled by the subscript j . This regime has not been investigated previously and is the subject of this paper.

At an “intermediate” collision frequency, which is sufficiently large that the drift islands do not form, but sufficiently small that collisional fluid dissipation is not important, the linear Landau dissipation is fully restored and is independent of collision frequency. This is analogous to the plateau regime of neoclassical theory and is relevant when $\nu_e < |\bar{k}_\parallel|v_{the}$ and $\nu_{eff} > |\bar{k}_\parallel|v_{the}$ where ν_{eff} is an effective electron collision frequency close to the Landau resonance and $\bar{k}_\parallel = -k_\theta w/L_s$ is an effective parallel wave number. In terms of the equilibrium parameters the frequency regime is

$$\frac{k_\theta \rho_i}{16} \frac{v_{thi}}{L_n} \left(\frac{L_n}{L_s}\right)^3 \left(\frac{m_i}{m_e}\right)^{3/2} \left(\frac{w}{\rho_i}\right)^3 < \nu_e < \frac{k_\theta w v_{the}}{L_s}.$$

This can only be satisfied for very thin islands, in which case the Landau resonance of electrons with $v_\parallel \sim v_{the}$ is typically outside the island. Such a case has been considered by Smolyakov and Hirose,¹¹ who showed that islands can exist with a width of the order the collisionless skin depth and a propagation frequency $\omega \sim \omega_{*e}(1 + \eta_e/2)$, which is the same as that obtained in the linear theory of collisionless tearing modes.¹⁷ This is consistent with the observations of Dubois *et al.*² who argue that when the Landau resonance is a long way from the magnetic island, then the electrons cannot reach an equilibrium over length scales of the order of the parallel wavelength at the resonance. The current perturbation which drives the island is then insensitive to the island topology and rather results from this lack of equilibrium, which can be determined by the linear theory.

At the highest collision frequency collisional dissipation dominates and a fluid theory is appropriate. This is analogous to the Pfirsch–Schlüter regime of neoclassical theory and is relevant when $\nu_e > |\bar{k}_\parallel|v_{the}$, or

$$\nu_e > \frac{k_\theta w v_{the}}{L_s}$$

and is the regime which has been considered by Smolyakov.^{9,10}

The relevant collisionality regime for given equilibrium parameters is determined by the island width and length as illustrated in Fig. 1.

We stress that the theory of magnetic island evolution in a “collisionless” plasma presented here assumes isolated chains of islands, so that they evolve independently of each other. A theory of small-scale magnetic islands in a collisionless plasma has also been developed in Ref. 15, where the interaction of the electron Landau resonances associated with adjacent magnetic island chains breaks up the drift islands and results in an “anomalous” electron viscosity taking a similar form to the linear Landau dissipation. This interac-

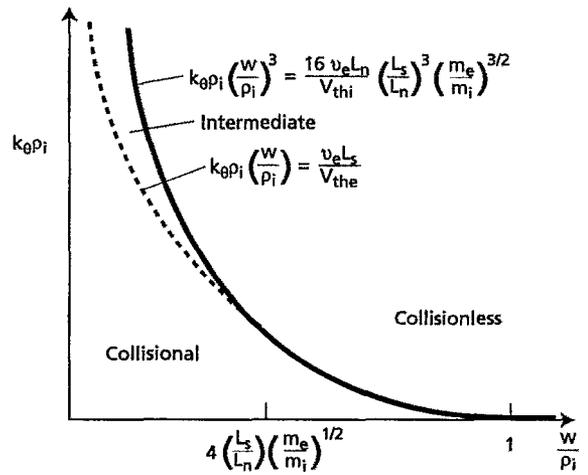


FIG. 1. Sketch of $k_\theta \rho_i (w/\rho_i)^3 = (16 \nu_e L_n / v_{thi}) (L_s/L_n)^3 (m_e/m_i)^{1.5}$ (full curve) and $k_\theta \rho_i (w/\rho_i) = (\nu_e L_s / v_{the})$ (dashed curve) to illustrate the three collision frequency regimes.

tion occurs when the electron Landau resonances of adjacent island chains overlap; for the isolated magnetic island chains considered here, this process is neglected.

The model which we adopt in our calculation is as follows. Island drive mechanisms from magnetic curvature are neglected so a sheared slab magnetic geometry is adequate to represent the standard nested surfaces of the tokamak plasma, which we shall refer to as the “equilibrium” plasma. We then impose on this a helical magnetic perturbation which results in a chain of magnetic islands propagating in the flux surface, perpendicular to the equilibrium magnetic field, with a rotation frequency, ω . The electrons and ions respond differently to this perturbation because of finite Larmor radius (FLR) effects and different parallel dynamics and therefore a local electrostatic potential exists in the vicinity of the magnetic island to satisfy quasineutrality. The responses to the magnetic and electrostatic potential perturbations are calculated using the drift-kinetic and gyrokinetic equations, respectively. The perturbed current which results is used in Ampère’s law to deduce the self-consistent island width, while the rotation frequency is found from toroidal torque balance, with collisions restoring a “residual” Landau damping to provide the dissipation. Here, we are interested in stationary solutions for the saturated island width. The nonlinear evolution of tearing modes in a collisionless plasma has been developed for a simpler model in Ref. 18, where electron inertia provides the dissipation necessary for the tearing of field lines which occurs during island growth. This dissipation plays a quite distinct role from the Landau damping which we appeal to in the steady-state torque balance equation.

The paper is structured as follows. In Sec. II we describe the calculation of the island width and rotation frequency. We introduce the magnetic geometry and dispersion relation in Sec. II A and calculate the ion distribution function from the gyrokinetic equation in Sec. II B. In Sec. II C we calculate the electron response and mode frequency. This rather

involved calculation is described as follows. In Sec. II C 1 the drift-kinetic equation for the electrons is simplified and the concept of the “drift” islands is introduced. The leading-order solution for the electron distribution function is calculated in Sec. II C 2. This depends on a free function which is constant on the drift orbits and is to be determined from a constraint equation, which is also derived in this section. In Sec. II C 3 this constraint equation is shown to be a stationary point of a functional, K where K can be related to the entropy production. Thus the constraint equation is solved by a variational procedure, which has the physical meaning that it minimizes the entropy production. A Lorentz collision operator is employed for the calculation presented here. In Sec. II C 4 we demonstrate that the equation for torque balance, which determines the island frequency, can be expressed as $K=0$. The expression for K obtained in Sec. II C 3 then yields the island frequency. Finally, we derive the electron density in Sec. II C 5 and apply quasineutrality to determine the electrostatic potential; this provides all the information required to calculate the saturated island width. Analytic results are obtained in the limits $w \gg \rho_i$ and $w \ll \rho_i$. In Sec. III we address the thermal transport which might be expected to result from the island structures. The model for the transport which is introduced here neglects island interaction effects, which are outside the scope of this work; thus we do not consider transport resulting from magnetic stochasticity. Instead we consider a particular model in which the transport results from the electrostatic perturbations associated with the islands, in which case we argue that interaction between adjacent island chains is weak and can be neglected. The thermal diffusivity which we calculate has several, but not all, of the characteristics of the widely used, but semi-empirical, Rebut-Lallia-Watkins law.¹⁹ We close with a summary in Sec. IV.

II. SATURATED ISLAND WIDTH AND FREQUENCY

A. Magnetic topology and island “dispersion” relation

We represent the equilibrium magnetic field (we shall refer to the plasma without the island as the “equilibrium”) using a slab geometry with the shear in the magnetic field represented by a radial-dependent equilibrium flux. A helical magnetic perturbation is imposed with a parallel vector potential $A_{\parallel} = -\tilde{\psi} \cos \xi$ which is symmetric about the rational surface $m = nq$, where m and n are the poloidal and toroidal mode numbers of the perturbation, respectively. Here ξ can be expressed in terms of the tokamak poloidal and toroidal angles (θ and ϕ respectively) and the rotation frequency, ω :

$$\xi = m \left(\theta - \frac{\phi}{q} \right) - \int^t \omega(t') dt'. \quad (1)$$

Taylor expanding the equilibrium magnetic field about the rational surface $m = nq$, we can express the total magnetic field as

$$\mathbf{B} = B_0 \nabla z - \nabla \psi \times \nabla z, \quad (2)$$

where B_0 is the equilibrium field and

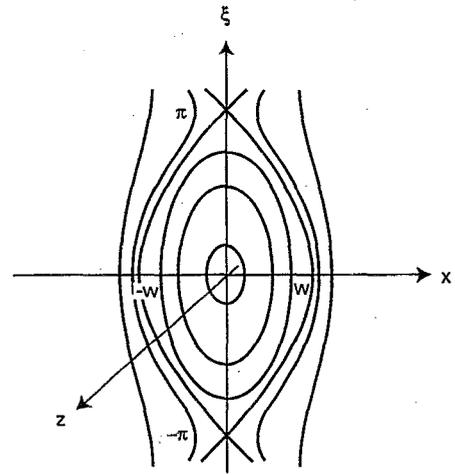


FIG. 2. Magnetic island geometry.

$$\psi = -\frac{B_0 x^2}{2L_s} + \tilde{\psi} \cos \xi. \quad (3)$$

The radial distance from the rational surface has been represented by x . The equilibrium magnetic field at the rational surface is in the z direction and the shear length, L_s , measures the rate of variation of this field direction with radius. Then $\nabla \xi$, ∇x , and ∇z , form a convenient orthogonal coordinate system, with

$$\nabla x \times \nabla \xi = k_{\theta} \nabla z. \quad (4)$$

The expression for ψ can be written in terms of the island width, w

$$-\frac{\psi}{\tilde{\psi}} = \Omega = \frac{2x^2}{w^2} - \cos \xi, \quad (5)$$

where

$$w = \sqrt{4L_s \tilde{\psi} / B_0} \quad (6)$$

and we have defined a convenient new dimensionless flux function, Ω . The solutions for the constant ψ surfaces then depend on the value of Ω . For $\Omega > 1$ the constant ψ surfaces are open. For the special case $\Omega = 1$ the ψ surface is described by $x = \pm w \cos(\xi/2)$ and forms the separatrix of the island with X-points at $\xi = \pm \pi$. For $-1 < \Omega < 1$ the solutions are the closed elliptical surfaces of the island, with $\Omega = -1$ corresponding to the island O-point. The geometry is shown in Fig. 2, together with the orthogonal coordinate system used, where we recall that x represents the radial distance from the rational surface, ξ is along the length of the island chain and z is the direction of the equilibrium magnetic field and is a direction of symmetry for the full system.

We shall be concerned with the “steady-state” solution in which ψ and ω are independent of time so that Eq. (5) represents an island chain which propagates in the ξ direction at frequency ω . We consider ψ to have a slow radial variation (compared with the island width): the so-called “constant ψ ” approximation. However, a narrow current sheet exists near the rational surface $m = nq$ and this leads to a discontinuity in the radial derivative of A_{\parallel} , which is denoted by Δ' :

$$\left(\frac{\partial A_{\parallel}}{\partial r} \Big|_{r=r_s^{(+)}} - \frac{\partial A_{\parallel}}{\partial r} \Big|_{r=r_s^{(-)}} \right) = \Delta' A_{\parallel}, \quad (7)$$

where $r_s^{(\pm)}$ indicate the asymptotic limits as r approaches the rational surface from either side. For large m , $\Delta' = -2m/r$ and then the constant ψ approximation corresponds to

$$2k_{\theta} w = |\Delta'| w \ll 1. \quad (8)$$

The island saturation condition is determined from the parameter Δ' by asymptotically matching the island solution to the external solution. Using Ampère's law to relate A_{\parallel} to the perturbed parallel current J_{\parallel} and projecting out the $\cos \xi$ and $\sin \xi$ components we obtain the following matching conditions which determine the saturated island width and rotation frequency, respectively,

$$\int_{-\pi}^{\pi} \int_{-\infty}^{\infty} J_{\parallel} \cos \xi \, dx \, d\xi = \frac{c \Delta'}{4} \tilde{\psi}, \quad (9)$$

$$\int_{-\pi}^{\pi} \int_{-\infty}^{\infty} J_{\parallel} \sin \xi \, dx \, d\xi = 0. \quad (10)$$

We assume the current is localized around the island and so are justified in taking the limits on the x integration to be $\pm\infty$.

It is interesting to note that Eq. (10) results from toroidal torque balance as follows.²⁰ The toroidal component of torque, T_{ϕ} , can be expressed as

$$T_{\phi} = \int (\delta \mathbf{J} \times \delta \mathbf{B}) \cdot \nabla \phi \, dx \, d\xi, \quad (11)$$

where $\delta \mathbf{J}$ and $\delta \mathbf{B}$ are the perturbed current and magnetic field. The perturbed magnetic field is dominated by the parallel component of the vector potential, so that Ampère's law requires that $\delta \mathbf{J}$ be parallel to the magnetic field. Thus $\delta \mathbf{J} = J_{\parallel} \mathbf{b}$, where \mathbf{b} is a unit vector in the direction of the magnetic field, and

$$T_{\phi} = \int J_{\parallel} [\mathbf{b} \times (\nabla A_{\parallel} \times \nabla z)] \cdot \nabla \phi \, dx \, d\xi, \quad (12)$$

which reduces to

$$T_{\phi} = \frac{m \tilde{\psi}}{R^2 q} \int J_{\parallel} \sin \xi \, dx \, d\xi. \quad (13)$$

Neglecting any external torques with the same helicity, torque balance then requires $T_{\phi} = 0$ and we obtain Eq. (10).

B. Ion response

In this section we calculate the ion density response assuming the standard drift wave ordering, in which the ions satisfy $\omega \sim \omega_{*} \gg k_{\parallel} v_{\parallel}$. This ordering implies that we are considering islands of width

$$\frac{w}{\rho_i} \ll \frac{L_s}{L_n} \quad (14)$$

and is consistent with the assumption that the ions carry no parallel current. However, the ion dynamics do play a role in determining the perpendicular current which is not divergence free and therefore drives a parallel current in the elec-

trons. Because the finite size of the ion Larmor radius relative to the island width is important, the ion response is calculated from the nonlinear ion gyrokinetic equation:²¹

$$\begin{aligned} \omega \frac{\partial g_i}{\partial \xi} \Big|_x - k_{\parallel} v_{\parallel} \frac{\partial g_i}{\partial \xi} \Big|_{\psi} - \frac{c}{B_0^2} (\mathbf{B}_0 \times \nabla \langle \varphi \rangle_{\alpha}) \cdot \nabla g_i \\ = \frac{q_i}{T_i} F_{Mi} (\omega - \omega_{*i}^T) \frac{\partial}{\partial \xi} \Big|_x \left(\langle \varphi \rangle_{\alpha} - \frac{v_{\parallel}}{c} A_{\parallel} \right), \end{aligned} \quad (15)$$

where g_i is the nonadiabatic part of the perturbed ion distribution δf_i . We have also defined $k_{\parallel} = -k_{\theta} x / L_s$, $\omega_{*j} = k_{\theta} c T_j / q_j B L_n$, and $\omega_{*j}^T = \omega_{*j} [1 + (v^2 / v_{thj}^2 - 3/2) \eta_j]$ with q_j the species charge, T_j the temperature and $\eta_j = L_n / L_{Tj}$, where $L_n^{-1} = d \ln n / dr$ and $L_{Tj}^{-1} = d \ln T_j / dr$.

The particle thermal velocity is defined as $v_{thj} = \sqrt{2T_j / m_j}$. Here F_{Mi} is the leading order ion distribution function which is taken to be a Maxwellian in velocity space and with a spatial dependence on the "radial" variable x only [note, this differs from other calculations^{9,10,13} in which the leading-order distribution function is taken to be a function of ψ , but is equivalent provided the "drive" terms on the right-hand side of Eq. (15) are retained]. The angled brackets $\langle \dots \rangle_{\alpha}$ represent an average of the enclosed quantity around the ion Larmor orbit where α is the gyroangle and g_i is independent of this quantity. It is therefore convenient to introduce the guiding center coordinate, $X = x - \rho \cos \alpha$ where $\rho = v_{\perp} / \omega_{ci}$ and ω_{ci} is the ion cyclotron frequency. The ion density perturbation $\delta n_i(x)$ is easily derived in terms of g_i by Fourier transforming

$$g_i(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}_i(k) e^{-ikX} \, dk \quad (16)$$

so that

$$\begin{aligned} \delta n_i(x) = -n_0 \frac{q_i \varphi}{T_i} + \sqrt{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dv_{\parallel} \\ \times \int_0^{\infty} v_{\perp} \, dv_{\perp} \tilde{g}_i(k) J_0(k\rho) e^{-ikx}. \end{aligned} \quad (17)$$

To evaluate the ion density response we integrate Eq. (15) over the parallel velocity variable, neglecting the small parallel derivative term since $\omega \gg k_{\parallel} v_{\parallel}$, to obtain

$$\begin{aligned} \omega \frac{\partial G_i}{\partial \xi} \Big|_x - \frac{c}{B_0^2} (\mathbf{B}_0 \times \nabla \langle \varphi \rangle_{\alpha}) \cdot \nabla G_i \\ = \frac{q_i}{T_i} \int_{-\infty}^{\infty} F_{Mi} (\omega - \omega_{*i}^T) dv_{\parallel} \frac{\partial}{\partial \xi} \Big|_x \langle \varphi \rangle_{\alpha}, \end{aligned} \quad (18)$$

where $G_i = \int_{-\infty}^{\infty} g_i \, dv_{\parallel}$ and terms odd in v_{\parallel} have disappeared on integrating. Clearly a solution of the linear equation,

$$\omega \frac{\partial G_i}{\partial \xi} \Big|_x = \frac{q_i}{T_i} \int_{-\infty}^{\infty} F_{Mi} (\omega - \omega_{*i}^T) dv_{\parallel} \frac{\partial}{\partial \xi} \Big|_x \langle \varphi \rangle_{\alpha}, \quad (19)$$

is a solution of the nonlinear equation (18) if the arbitrary function of x which results from integrating Eq. (19) is set to

zero. Thus Fourier analyzing the scalar potential φ and G_i with respect to x and integrating over the gyrophase angle, we obtain the solution

$$\omega \tilde{G}(k) = \frac{q_i \tilde{\phi}(k)}{T_i} J_0(k\rho) \int_{-\infty}^{\infty} F_{Mi}(\omega - \omega_{*i}^T) dv_{\parallel}. \quad (20)$$

The ion density response is then calculated from $\tilde{G}(k)$

$$\begin{aligned} \frac{\delta n_i}{n_0} = & -\frac{q_i \varphi(x)}{T_i} + \sqrt{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} v_{\perp} dv_{\perp} \\ & \times \frac{q_i \tilde{\phi}(k)}{T_i} \frac{F_{Mi}}{n_0} \left(1 - \frac{\omega_{*i}^T}{\omega}\right) [J_0(k\rho)]^2 e^{-ikx}. \end{aligned} \quad (21)$$

The velocity integrals can be performed analytically so that, after expressing $\tilde{\phi}(k)$ in terms of $\varphi(x)$ we obtain a final expression for the ion density perturbation:

$$\begin{aligned} \delta n_i(x) = & -n_0 \frac{q_i \varphi(x)}{T_i} + \frac{n_0}{2\pi^{3/2}} \frac{q_i}{T_i} \int_{-\infty}^{\infty} \frac{dx'}{\rho_i} \varphi(x') \\ & \times G(x-x'), \end{aligned} \quad (22)$$

where we have defined the Larmor radius at the ion thermal velocity, $\rho_i = v_{thi}/\omega_{ci}$ and the kernel

$$\begin{aligned} G(u) = & \left\{ \frac{(\omega - \omega_{*i})}{\omega} K_0\left(\frac{u^2}{8\rho_i^2}\right) + \frac{\omega_{*i}\eta_i}{2\omega} \right. \\ & \times \left[\left(1 - \frac{u^2}{4\rho_i^2}\right) K_0\left(\frac{u^2}{8\rho_i^2}\right) - \frac{u^2}{4\rho_i^2} K_1\left(\frac{u^2}{8\rho_i^2}\right) \right] \\ & \left. \times \exp\left(-\frac{u^2}{8\rho_i^2}\right) \right\}. \end{aligned} \quad (23)$$

Here the functions $K_n(z)$ are the Bessel functions which have the integral representation

$$K_n(z) = \int_0^{\infty} e^{-z \cosh t} \cosh(nt) dt. \quad (24)$$

This agrees with the result obtained in Ref. 8 for $\eta_i=0$. Two limits are of interest: (i) $\rho_i \gg w$ and (ii) $\rho_i \ll w$. Since $G(x-x')$ is small unless $|x-x'| \lesssim \rho_i$, and $\varphi(x')$ is small unless $x' \lesssim w$, we find that if $w/\rho_i \ll 1$ the integral is negligible and the ion density perturbation is simply the Boltzmann response:

$$\delta n_i = -n_0 \frac{q_i \varphi}{T_i}. \quad (25)$$

In the opposite limit, $\rho_i \ll w$, the only contribution to the integral comes from $x' \approx x$; thus we can approximate $\varphi(x') \approx \varphi(x) + (x-x')\varphi'(x) + (x-x')^2\varphi''(x)/2$ so that

$$\begin{aligned} \delta n_i = & -n_0 \frac{q_i \varphi(x)}{T_i} + \frac{n_0}{2\pi^{3/2}} \frac{q_i}{T_i} \int_{-\infty}^{\infty} \frac{dx'}{\rho_i} \left(\varphi(x) \right. \\ & \left. + \frac{(x-x')^2}{2} \frac{d^2\varphi}{dx^2} \right) G(x-x'). \end{aligned} \quad (26)$$

The x' integral can be performed analytically with the result

$$\begin{aligned} \delta n_i = & -n_0 \frac{q_i \varphi}{T_i} \frac{\omega_{*i}}{\omega} + \frac{n_0}{2} \frac{q_i}{\omega T_i} \\ & \times [\omega - \omega_{*i}(1 + \eta_i)] \rho_i^2 \frac{d^2\varphi}{dx^2}, \end{aligned} \quad (27)$$

where the second term represents the leading finite ion Larmor radius correction which must be retained in order to derive the island evolution equation. Physically, this term results from the ion polarization drift. Both of the above limiting forms will be used later in order to derive analytic expressions for the conditions necessary for island growth and the corresponding saturated island width. In general the full expression in Eq. (22) should be used for the ion density perturbation.

C. Electron response and mode frequency

In this section we derive the electron response and the island rotation frequency, ω , in a low collision frequency plasma. We shall see that the calculation of ω requires knowledge of the electron dissipation which, in the linear theory of low collisionality plasmas, is dominated by the Landau resonance and leads to $\omega = \omega_{*e}(1 + \eta_e/2)$.¹⁷ The nonlinear equilibrium theory described here predicts that the electron distribution is distorted around the resonance to form "drift" islands, so that energy transfer between the electrons and the wave does not occur and no "linear" electron Landau damping results. However, a low level of collisions attempts to restore a Maxwellian distribution and so a residual Landau damping exists which is proportional to the electron collision frequency, ν_e ;¹⁶ this is the dominant dissipation mechanism here. It is therefore important to retain the effects of the drift islands on the Landau resonance to correctly calculate the dissipation, though we will regard the parameter $\omega/k_{\parallel}v_{\parallel e}$ as small when appropriate.

The calculation of the electron dynamics is described within this section as follows. In Sec. II C 1 we simplify the electron drift-kinetic equation and introduce the concept of the drift islands. A leading-order expression for the nonadiabatic electron response, g_e , is derived in Sec. II C 2 by adopting an expansion in small $c\varphi/v_{\parallel}A_{\parallel}$. This result for the response is expressed in terms of a function, h , which is constant on the drift surfaces and is determined by a constraint equation, also derived in Sec. II C 2. A variational approach which exploits some of the mathematical techniques used in the banana regime calculation of neoclassical transport²² is developed to solve this constraint equation in Sec. II C 3. The constraint equation for h can be shown to result from the stationary point of a self-adjoint functional $K(g, g)$ which can be related to entropy production. Thus, physically the free function, h , adjusts so as to minimize entropy production. In Sec. II C 4 we show that $K(g, g) = 0$ corresponds to the island frequency equation of the dispersion relation; this allows us to determine ω . Finally the electron density and saturated island width are derived in Sec. II C 5.

1. Electron drift equation

The electron Larmor radius is assumed to be much smaller than the island width so the electron distribution function can be calculated from the drift kinetic equation. The nonadiabatic response, g_e , where

$$\delta f_e = -\frac{q_e F_{Me}}{T_e} + g_e \quad (28)$$

then satisfies

$$\begin{aligned} \omega \frac{\partial g_e}{\partial \xi} \Big|_x - k_{\parallel} v_{\parallel} \frac{\partial g_e}{\partial \xi} \Big|_{\psi} + \frac{c}{B_0^2} (\mathbf{B} \times \nabla g_e) \cdot \nabla \varphi - \frac{q_e E_{\parallel}}{m_e} \frac{\partial g_e}{\partial v_{\parallel}} \\ = \frac{q_e F_{Me}}{T_e} (\omega - \omega_{*e}^T) \frac{\partial}{\partial \xi} \Big|_x \left(\varphi - \frac{v_{\parallel} A_{\parallel}}{c} \right) - C(g_e). \end{aligned} \quad (29)$$

The term in $\partial g_e / \partial v_{\parallel}$ is smaller than all other terms by a factor $\sim w/L_n$ and will be ignored in the following. It is convenient to define a variable, y

$$y = \left(\frac{L_s}{k_{\theta} w v_{\parallel}} \right)^2 (\omega - k_{\parallel} v_{\parallel})^2 - \frac{1}{2} \cos \xi, \quad (30)$$

which can be interpreted as a new radial coordinate. The drift equation then simplifies to

$$\begin{aligned} \sigma \frac{k_{\theta} w v_{\parallel}}{L_s} p(y, \xi) \frac{\partial g_e}{\partial \xi} \Big|_y + \frac{c}{B_0^2} (\mathbf{B} \times \nabla g_e) \cdot \nabla \varphi \\ = \frac{q_e F_{Me}}{T_e} (\omega - \omega_{*e}^T) \frac{\partial}{\partial \xi} \Big|_x \left(\varphi - \frac{v_{\parallel} A_{\parallel}}{c} \right) - C(g_e), \end{aligned} \quad (31)$$

where we have defined

$$p(y, \xi) = \sqrt{y + \frac{1}{2} \cos \xi} \quad (32)$$

and $\sigma = \pm 1$ is the sign of $(\omega - k_{\parallel} v_{\parallel})/v_{\parallel}$. When the nonlinear $\mathbf{E} \times \mathbf{B}$ term is neglected the characteristics of Eq. (31) are the surfaces of constant y . These surfaces are the same as the constant ψ surfaces in shape and size, but are shifted in x to form the so-called drift islands, centered on the Landau resonance for the particular parallel velocity under consideration. It is interesting to note that the presence of these drift islands, illustrated in Fig. 3, modifies the electron distribution function in the vicinity of the Landau resonance so that the collisionless dissipation, which would be present in the linear theory, is absent. Collisional relaxation tries to restore the Landau resonance and leads to a dissipation proportional to collision frequency;¹⁶ this is the dominant dissipation mechanism for isolated magnetic islands. In the limit $\omega/\bar{k}_{\parallel} v_{\parallel} \rightarrow 0$, $y \rightarrow \Omega/2$ and the drift islands are coincident with the magnetic island. However, it is important to retain the small, but finite, difference between y and Ω to obtain the dissipation.

2. Form of electron solution

We solve for the electron response g_e assuming $c\varphi/v_{\parallel} A_{\parallel} \ll 1$ for the electrons so that, to leading order, Eq. (31) becomes

$$(\omega - k_{\parallel} v_{\parallel}) \frac{\partial g_e}{\partial \xi} \Big|_y = -\frac{q_e F_{Me}}{c T_e} v_{\parallel} \tilde{\psi} (\omega - \omega_{*e}^T) \sin \xi. \quad (33)$$

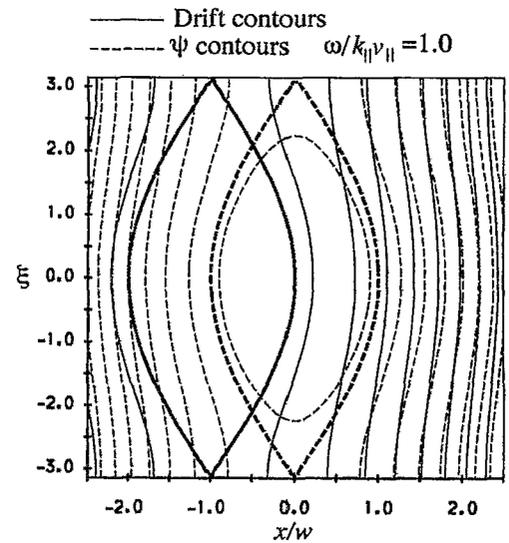


FIG. 3. Constant y "drift" surfaces (full lines) for $\omega/\bar{k}_{\parallel} v_{\parallel} = 1$ shown relative to the constant ψ magnetic surfaces (dashed lines).

Thus

$$g_e^0 = \frac{w}{L_n} \frac{F_{Me}}{\omega_{*e}} (\omega - \omega_{*e}^T) [\sigma p(y, \xi) - h(y)], \quad (34)$$

where $h(y)$ is, as yet, an arbitrary function of y . Outside the drift island $h(y)$ may depend on σ ; however, inside the drift island $h(y)$ must be independent of σ so that g_e^0 is single valued at $p=0$ (i.e., the Landau resonance position). At constant y , v_{\parallel} labels the positions of the different drift islands. Since the electrons mainly respond to the magnetic perturbation, all these drift islands are equivalent in the constant y approximation and we are justified in taking h to be independent of v_{\parallel} at constant y (it has a weak v_{\parallel} dependence at constant x).

The function $h(y)$ must be chosen so as to satisfy a certain constraint equation which can be derived from Eq. (31). This takes different forms according to the topology of the constant y surfaces, for which two regions can be identified. First, for $y > 1/2$, the surfaces are "open" and cover the full range $-\pi < \xi < \pi$. Thus, dividing Eq. (31) by p and integrating over a period in ξ at constant y and with the periodicity boundary condition on g_e , we obtain the constraint equation for g_e for this case. When $-1/2 < y < 1/2$ the constant y surfaces are closed so we integrate Eq. (31) over $-\xi_0 < \xi < \xi_0$, where $\cos \xi_0 = -2y$, and sum over σ . The constraint equation for g_e in this region results from the boundary condition that g_e be single valued at $\xi = \xi_0$. Thus the constraint equations in the two regions are

$$\begin{aligned} \oint_y \frac{C(g_e)}{p} d\xi = 0, \quad y > \frac{1}{2}, \\ \sum_{\sigma=\pm 1} \int_{-\xi_0}^{\xi_0} \frac{C(g_e)}{p} d\xi = 0, \quad -\frac{1}{2} < y < \frac{1}{2}, \end{aligned} \quad (35)$$

where the integrals over ξ are to be taken at constant y . By assuming an explicit form for the collision operator, Eq. (35)

can be solved to derive the form for $h(y)$. A variational approach to this calculation is described in the following section.

Before we discuss the solution to Eq. (35) we first comment on the validity of the electron distribution function given by Eq. (34). Its derivation relies on the smallness of the parameter $c\varphi/v_{\parallel}A_{\parallel}$, where we have the ordering $e\varphi/T_e \sim w/L_n$. Thus, for $v_{\parallel} \sim v_{\text{the}}$ this is easily satisfied, while for resonant particles, with $v_{\parallel} \sim \omega/k_{\parallel}$ this is marginal at $x=w$. However, these slow electrons are only important for determining the dissipation, which is dominated by the region far from the magnetic island. Here, φ decays sufficiently fast with x that the assumption remains valid and the form for g_e^0 which we have derived can be used in the calculation of the dissipation below.

3. Variational solution for electrons

The constraint equation can be solved by a variational technique by defining a functional $K(f, g)$

$$K(f, g) = \int d^3\mathbf{v} \int d^2\mathbf{x} \frac{f}{b(v)} C(g), \quad (36)$$

where $b(v) = (w/L_n)F_{Me}(\omega - \omega_{*e}^T)/\omega_{*e}$, so that $g_e^0 = b(v)(\sigma p - h)$. Requiring K to be stationary with respect to variations in the adjoint function f reproduces Eqs. (35) which determine the form for h . [An equation for f is derived from Eq. (36) by requiring $K(f, g)$ to be stationary with respect to variations in g .] In fact, because $K(f, g)$ is self-adjoint, $f=g$ so that we need only consider $K(g, g)$ and choose h so that $K(g, g)$ is stationary with respect to variations in h .

In order to solve explicitly for the "arbitrary" function $h(y)$ and the island frequency, ω , it is necessary to assume a specific model collision operator; we retain only $e-i$ collisions so that we can use the Lorentz model:

$$\begin{aligned} C(g_e) &= \nu(v) \left. \frac{\partial}{\partial v_{\parallel}} \right|_{v,x} (v^2 - v_{\parallel}^2) \left. \frac{\partial g_e}{\partial v_{\parallel}} \right|_{v,x} \\ &= \nu(v) \left(\left. \frac{\partial}{\partial v_{\parallel}} \right|_{v,y} - \frac{\sigma a p}{v_{\parallel}^2} \left. \frac{\partial}{\partial y} \right|_{v_{\parallel},v} \right) (v^2 - v_{\parallel}^2) \\ &\quad \times \left(\left. \frac{\partial}{\partial v_{\parallel}} \right|_{v,y} - \frac{\sigma a p}{v_{\parallel}^2} \left. \frac{\partial}{\partial y} \right|_{v_{\parallel},v} \right) g_e, \end{aligned} \quad (38)$$

where $\nu(v) = \nu(v_{\text{the}})(v_{\text{the}}/v)^3$ and $a = 2L_s\omega/k_{\theta}w$. In the variational treatment it is sufficient to retain pitch angle scattering terms only in the collision term because of the small width of the drift islands. Thus, as in neoclassical theory, it is unnecessary to retain the effects of the mean ion flow in the electron-ion collision operator. Using the identity

$$2 \int_{-\infty}^{\infty} \frac{dx}{w} \int_{-\pi}^{\pi} d\xi = \sum_{\sigma=\pm 1} \int_{-1/2}^{\infty} dy \int_{-\xi_0}^{\xi_0} \frac{d\xi}{p(y, \xi)}, \quad (39)$$

where

$$\begin{aligned} \cos \xi_0 &= -2y, \quad -1/2 < y < 1/2, \\ \xi_0 &= \pi, \quad y > 1/2; \end{aligned} \quad (40)$$

the functional then becomes

$$\begin{aligned} K(g, g) &= \pi w \sum_{\sigma} \int_0^{\infty} dv \nu(v) b(v) \int_{-v}^v dv_{\parallel} \int_{-1/2}^{\infty} dy \\ &\quad \times \int_{-\xi_0}^{\xi_0} d\xi \frac{(v^2 - v_{\parallel}^2)}{p} \left[\left. \frac{\partial}{\partial v_{\parallel}} \right|_{v,y} - \frac{\sigma a p}{v_{\parallel}^2} \left. \frac{\partial}{\partial y} \right|_{v_{\parallel},v} \right] \frac{g}{b(v)} \right]^2 \end{aligned} \quad (41)$$

The method for determining $h(y)$ and the island frequency is now clear. We evaluate the velocity integrals assuming that $\partial/\partial y$ derivatives dominate those involving $\partial/\partial v_{\parallel}$ (recall that this is a consequence of the fact that all drift islands are equivalent for the leading-order electron response). However, in the constant $\tilde{\psi}$ approximation the velocity integrals are divergent at low v_{\parallel} , which corresponds to the drift islands which are farthest from the magnetic island. In performing the velocity integrals it is therefore necessary to retain the decay of $\tilde{\psi}$ with x in order to obtain convergence. Thus we have

$$\begin{aligned} K(g, g) &= \pi w \sum_{\sigma} \int_0^{\infty} dv \nu(v) b(v) \int_{-v}^v dv_{\parallel} \int_{-1/2}^{\infty} dy \\ &\quad \times \int_{-\xi_0}^{\xi_0} d\xi \frac{v^2 - v_{\parallel}^2}{p} \left(\frac{a p}{v_{\parallel}^2} \left. \frac{\partial}{\partial y} (\sigma p - h) \right)^2 \right. \\ &\quad \times \exp\left(-\frac{k_{\theta} w a}{|v_{\parallel}|}\right), \end{aligned} \quad (42)$$

where the exponential factor takes account of the slow $\tilde{\psi}$ radial decay of the form $\sim e^{-k_{\theta}|x|}$. Defining the angled brackets by

$$\langle \dots \rangle = \frac{1}{2\xi_0} \int_{-\xi_0}^{\xi_0} \dots d\xi \quad (43)$$

and performing the velocity integrals we obtain

$$\begin{aligned} K(g, g) &= \frac{2w}{(k_{\theta}w)^3} \frac{w}{L_n} \left(\frac{\bar{k}_{\parallel} v_{\text{the}}}{\omega} \right) \frac{n_0 \nu_e}{\omega_{*e}} \\ &\quad \times [\omega - \omega_{*e}(1 - \eta_e)] \sum_{\sigma} \int_{-1/2}^{\infty} dy 2\xi_0 \\ &\quad \times \left\{ \langle p \rangle^{1/2} \frac{dh}{dy} - \frac{\sigma}{2\langle p \rangle^{1/2}} \right\}^2 \\ &\quad + \frac{1}{4} \left[\left\langle \frac{1}{p} \right\rangle - \frac{1}{\langle p \rangle} \right], \end{aligned} \quad (44)$$

where $\nu_e = \nu(v_{\text{the}})$. The value of $K(g, g)$ is minimized when the first square bracket is closest to zero. Outside the drift islands h can depend on σ so the condition on h is that the first square bracket be zero. However, inside the island such a solution violates the condition that h be independent of σ [see below Eq. (34)] so that inside the island we must choose $h(y) = 0$. Thus we have

$$\frac{dh(y)}{dy} = \Theta\left(y - \frac{1}{2}\right) \frac{\sigma}{2\langle p \rangle}, \quad (45)$$

where $\Theta(x)$ is the Heaviside function.

4. Equation for ω

Expressions (29) and (31) can be used to derive the $\cos \xi$ and $\sin \xi$ components of the parallel current perturbation. First, an equation for the $\cos \xi$ component is obtained by multiplying Eq. (29) by q_e and integrating over velocity space:

$$k_{\parallel} \frac{\partial J_{\parallel}}{\partial \xi} \Big|_{\psi} = \frac{n_0 q_e^2}{T_e} \omega_{*e} \frac{\partial \varphi}{\partial \xi} \Big|_x + q_e \omega \frac{\partial}{\partial \xi} \Big|_x (\delta n_i) - \frac{c q_e}{B_0^2} (\mathbf{B} \times \nabla \varphi) \cdot \nabla (\delta n_i), \quad (46)$$

where we have neglected $J_{\parallel i}$ and assumed quasineutrality; this equation is essentially $\nabla \cdot \mathbf{J} = 0$. The current is then determined once the electrostatic potential perturbation φ is known; this is derived from the quasineutrality condition. Since φ is symmetric in ξ , Eq. (46) only yields a $\cos \xi$ component for the current perturbation. To derive an expression for the $\sin \xi$ component of current we multiply Eq. (31) by $g_e / (\omega - \omega_{*e}^T) F_{Me}$ and integrate over space and velocity. It is convenient to transform from x to y as the radial variable, in which case the spatial integrals can be carried out using the identity in Eq. (39). We then find that Eq. (10) for the frequency becomes

$$\sum_{\sigma} \int d^3 v \int_{-1/2}^{\infty} dy \int_{-\xi_0}^{\xi_0} d\xi \frac{g_e}{b(v)} \frac{C(g_e)}{p(y, \xi)} = 0. \quad (47)$$

Thus we see from Eq. (36) that Eq. (47) is equivalent to $K(g, g) = 0$. It follows from Eq. (44) for $K(g, g)$ that the island frequency

$$\omega = \omega_{*e} (1 - \eta_e). \quad (48)$$

Clearly the collision frequency does not appear explicitly in our final result, though the velocity dependence of the collision operator influenced the coefficient of η_e . The result which we obtain here in the low collisionality regime differs from that deduced by Smolyakov^{9,10} in the collisional fluid regime. Since we shall see that Eq. (9) for the island width depends on ω , we derive different criteria for the existence of microislands in a hot plasma.

5. Electron density and saturated island width

In principle the saturated island width could be calculated numerically from Eqs. (9), (22), (23), (46), (48), and (34) without expanding Eq. (34) in $\omega/k_{\parallel} v_{\parallel e} \ll 1$ for the electrons and without assumptions on the size of the ion Larmor radius relative to the island width. However, in order to demonstrate the existence of microislands it is convenient to expand Eq. (34) for small $\omega/k_{\parallel} v_{\parallel e}$ and consider the two limiting cases for the island width discussed in Sec. II B. While it was essential to retain $\omega/k_{\parallel} v_{\parallel e}$ effects in the electron equation

in order to derive the island frequency, these are not important for the island width. The condition under which they can be ignored is

$$\tau^{1/2} \frac{\rho_i}{w} \left(\frac{m_e}{m_i}\right)^{1/2} \frac{L_s}{L_n} (1 - \eta_e) \ll 1, \quad (49)$$

where $\tau = T_e/T_i$. This is generally easily satisfied except for very small island widths. Thus the electron density response is

$$\frac{\delta n_e}{n_0} = -\frac{q_e \varphi}{T_e} + \frac{w}{L_n} \frac{(\omega - \omega_{*e})}{\omega_{*e}} \times \left(\frac{x}{w} - \frac{\Theta(\Omega - 1)}{2\sqrt{2}} \int_1^{\Omega} \frac{d\Omega'}{P(\Omega')} \right), \quad (50)$$

where

$$P(\Omega) = \frac{1}{2\pi} \oint_{\Omega} \sqrt{\Omega + \cos \xi} d\xi. \quad (51)$$

The full system of equations to be solved can now be summarized as

$$k_{\parallel} \frac{\partial J_{\parallel}}{\partial \xi} \Big|_{\Omega} = \frac{n_0 q_e^2}{T_e} \omega_{*e} \frac{\partial \varphi}{\partial \xi} \Big|_x + \omega q_e \frac{\partial}{\partial \xi} \Big|_x (\delta n_i) - \frac{c q_e}{B^2} (\mathbf{B} \times \nabla \varphi) \cdot \nabla (\delta n_i), \quad (52)$$

$$\delta n_i = -\frac{n_0 q_i \varphi(x)}{T_i} + \frac{n_0 q_i}{2\pi^{3/2} T_i} \int_{-\infty}^{\infty} \frac{dx'}{\rho_i} G(x - x') \varphi(x'), \quad (53)$$

$$(1 + \tau) \frac{q_e \varphi}{T_e} - \frac{\tau}{2\pi^{3/2}} \frac{q_e}{T_e} \int_{-\infty}^{\infty} \frac{dx'}{\rho_i} G(x - x') \varphi(x') = \frac{w}{L_n} \frac{\omega - \omega_{*e}}{\omega_{*e}} \left(\frac{x}{w} - \sigma \frac{\Theta(\Omega - 1)}{2\sqrt{2}} \int_1^{\Omega} \frac{d\Omega'}{P(\Omega')} \right), \quad (54)$$

$$\omega = \omega_{*e} (1 - \eta_e), \quad (55)$$

where Eq. (54) results from quasineutrality.

Analytic expressions can be obtained in the limits $\rho_i/w \gg 1$ and $\rho_i/w \ll 1$ as follows. For the case $\rho_i/w \ll 1$, $\varphi(x')$ can be expanded for small $(x - x')$ as outlined in Sec. II B, with the result that the perturbed current is

$$J_{\parallel} = -8 \frac{n_0 q_e \tau}{k_{\theta}} \left(\frac{\rho_i}{w}\right)^2 \frac{L_s}{L_n} [\omega - \omega_{*i} (1 + \eta_i)] \times \frac{\omega}{\omega_{*e}} h' [h' + 2(\Omega + \cos \xi) h''], \quad (56)$$

where $h' \equiv d[h(\Omega/2)]/d\Omega$ and $h'' \equiv d^2[h(\Omega/2)]/d\Omega^2$. The spatial variation of J_{\parallel} is illustrated in Fig. 4, where it can be seen that all the current perturbation lies outside the magnetic island. Substituting this expression for the current into the dispersion relation in Eq. (9) yields the following equation for the saturated island width in the limit $\rho_i/w \ll 1$:

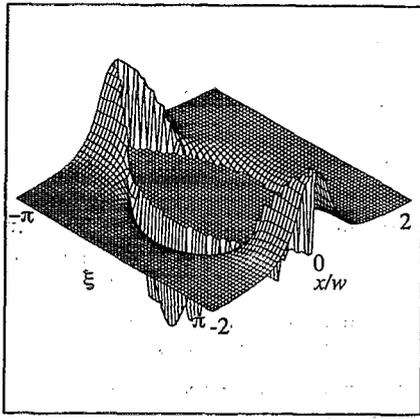


FIG. 4. Current perturbation around a magnetic island for the case $w \gg \rho_i$.

$$\Delta' w - 8 C_1 \tau \beta \left(\frac{L_s}{L_n} \right)^2 \left(\frac{\rho_i}{w} \right)^2 \frac{\omega}{\omega_{*e}} \frac{[\omega - \omega_{*e}(1 + \eta_i)]}{\omega_{*e}} = 0. \quad (57)$$

Here we have defined $\beta = 2n_0 T_e / B^2$ and evaluated

$$C_1 = 8\sqrt{2} \pi \int_1^\infty Q^2 (2\sqrt{2}SQW - T) d\Omega = 2.24, \quad (58)$$

where

$$Q = \frac{1}{2\sqrt{2}} \frac{1}{\langle \sqrt{\Omega + \cos \xi} \rangle}, \quad T = \frac{1}{2\pi} \oint \frac{\cos \xi}{\sqrt{\Omega + \cos \xi}} d\xi, \quad (59)$$

$$W = \frac{1}{2\pi} \oint \cos \xi \sqrt{\Omega + \cos \xi} d\xi,$$

$$S = \frac{1}{2\pi} \oint \frac{d\xi}{\sqrt{\Omega + \cos \xi}}.$$

Equation (57) agrees with that derived by Smolyakov^{9,10} in the fluid limit. However, we have seen that kinetic effects are important in the calculation of the island frequency; ω , as low collision frequency and this influences the criteria for island growth as follows. Using Eq. (55) for ω and $\Delta' = -2m/r$ for large m , Eq. (57) predicts that drift islands with $w \gg \rho_i$ can only exist in a low collisionality plasma when the electron temperature gradient satisfies

$$1 < \eta_e < 1 + \tau^{-1}(1 + \eta_i). \quad (60)$$

The condition for the existence of small islands, with $w \ll \rho_i$, is different. The ion response is then Boltzmann and the current perturbation is

$$J_{||} = -4n_0 q_e \frac{\tau}{(1 + \tau)} \frac{L_s}{L_n} \frac{1}{k_\theta} \frac{(\omega - \omega_{*e})(\omega - \omega_{*i})}{\omega_{*e}} \times h' \left(\frac{x}{w} - h(\Omega) \right). \quad (61)$$

The dispersion relation for the saturated island width in this case is

$$\Delta' w + 8 C_2 \beta \left(\frac{L_s}{L_n} \right)^2 \frac{\tau}{1 + \tau} \frac{(\omega - \omega_{*e})(\omega - \omega_{*i})}{\omega_{*e}^2} = 0, \quad (62)$$

where the numerical factor C_2 is given by the integral

$$C_2 = -4\sqrt{2} \pi \int_1^\infty h \frac{dh}{d\Omega} T d\Omega = 1.2, \quad (63)$$

with T defined in Eq. (59). This again gives the same relation between the island width and its frequency that Smolyakov obtained using fluid theory, but once more the criteria for the existence of islands differ as a consequence of the different expression for the island frequency in the two cases. Thus again using Eq. (55) for ω we find small islands ($w \ll \rho_i$) can only exist if one of the following criteria on the electron temperature gradient is satisfied

$$\eta_e < 0 \quad \text{or} \quad \eta_e > 1 + \tau^{-1}. \quad (64)$$

III. ELECTRON HEAT TRANSPORT

In this section we consider the effect of the island chain on the electron heat transport. The conventional view is that island chains on adjacent rational surfaces overlap, so that they are separated by regions of stochastic magnetic field. This will then result in anomalous transport as argued by Réchester and Rosenbluth.²³ Our calculation does not encompass this situation because we consider the case when the islands evolve independently of each other, neglecting any drive or damping mechanisms which result from coupling of adjacent island chains. However, an interesting transport mechanism which we can address is the possibility that it is the electrostatic fluctuations associated with the island structures which determine the transport. According to arguments given by Hegna and Callen²⁴ the anomalous transport is negligible unless the regions containing the fluctuations touch, essentially because of the "series" nature of the contributions of adjacent radial transport regions to the overall transport. As the electrostatic fluctuations extend a distance $\sim w$ beyond the magnetic islands the region between adjacent chains will be filled with a fluctuating electrostatic field when the magnetic island chains are still a distance $\sim 4w$ apart. In this situation the transport in a real tokamak is expected to be dominated by the effect of these electrostatic fluctuations on the trapped particles.²⁵ However, in calculating this effect, we assume that toroidicity, implicit in the presence of trapped particles, will not qualitatively alter the conclusions of our slab-like island analysis, above.

The mechanism which we consider has been described by Connor *et al.*²⁵ The electrostatic fluctuations have a long parallel wavelength (of the order of the connection length $\sim Rq$) so that, while the passing particles experience a full variation of the fluctuation as they travel around the torus, a (magnetically) trapped electron will see a steady perturbation during many banana orbits if the fluctuation frequency $\omega \sim \omega_* \ll \omega_{be}$, where ω_{be} is the electron bounce frequency. Thus during a half-period of the electrostatic oscillation the passing particles will have a negligible radial drift while trapped electrons will travel a radial distance

$$\Delta r \sim \frac{v_{Er}}{\omega} \sim \frac{ck_{\phi}\varphi}{\omega B_{\theta}}. \quad (65)$$

In the absence of a mechanism to decorrelate the trapped particle from its orbit, it will experience a drift in the opposite direction over the second half-period of the fluctuation and return to its initial starting position. However, in the presence of collisions the trapped particle can be decorrelated from its orbit and thereby acquire a net radial drift. We consider the particles which are within a narrow pitch angle band $\sim \delta$ of the trapped-passing boundary; there are a fraction $\delta^{1/2}$ of these. They experience a large effective collisional scattering frequency $\sim \nu_e/\delta$. Their contribution to the electron thermal diffusivity is

$$\chi_e \sim \delta^{1/2} (\Delta r)^2 (\nu_e/\delta), \quad (66)$$

which is clearly dominated by small δ . A cutoff in δ arises from the condition that the effective collision frequency cannot exceed the fluctuation frequency, otherwise the trapped electrons do not perform a full radial excursion and less transport results: hence the transport is dominated by those trapped particles with $\delta \sim \nu_e/\omega$. This results in a thermal diffusivity

$$\chi_e \sim \left(\nu_e \frac{k_{\theta} \rho_s c_s}{L_n} \right)^{1/2} \left(\frac{q_e \varphi}{T_e} L_n \right)^2. \quad (67)$$

We have commented above that the global confinement will not be significantly affected unless the electrostatic structures touch. Considering a fixed toroidal mode number, this condition provides an expression relating the island width to the poloidal mode number:

$$k_{\theta} w \sim 1/4s, \quad (68)$$

where we have assumed that the electrostatic perturbation extends a distance $\sim 2w$ from the rational surface. (Strictly, we assumed $k_{\theta} w \ll 1$ in our island theory, which requires $s > 1/4$.) Equations (68) and (57) imply $w \gg \rho_i$ and we can use the result of Eq. (57) to determine the island width:

$$w \sim \sqrt{s} \beta (L_s/L_n) \rho_i, \quad (69)$$

where numerical factors and τ and η_j dependencies have been ignored for simplicity. Using Eq. (54), the electrostatic perturbation has the form $q_e \varphi/T_e \sim w/L_n$. Combining this with the expressions for w and k_{θ} in Eqs. (68) and (69), we obtain an expression for the electron thermal diffusivity which results from the "isolated drift islands" considered in this paper:

$$\chi_e \sim \epsilon^{3/4} \left(\frac{\nu_{*e}}{q} \right)^{1/2} \left(\frac{m_i}{m_e} \right)^{1/4} s^{1/4} \beta^{3/4} \left(\frac{L_n}{R} \right)^{1/2} \left(\frac{L_s}{L_n} \right)^{3/2} \frac{\rho_s^2 c_s}{L_n}. \quad (70)$$

Of course, the condition (60) on η_e must be satisfied for this transport process to be operative. It is interesting to note some similarities between the expression for χ_e derived here and the semiempirical result derived by Rebut, Lallia, and Watkins¹⁹ which can be cast in the form

$$\chi_e \sim \epsilon^{3/4} \left(\frac{L_s}{L_n} \right) \left(\frac{L_n}{R} \right) \beta^{-1/2} \left(\frac{\nu_{*e}}{q} \right)^{1/2} \frac{\rho_s^2 c_s}{L_n}. \quad (71)$$

However, the two formulas have significantly different β scalings.

IV. SUMMARY

The evolution of isolated islands involves coupled equations for the island width and frequency. The principal new result presented here is a calculation of the rotation frequency of an isolated island in a low collisionality plasma; this complements the collisional fluid treatment of Smolyakov.^{9,10} This frequency has been determined from the electron response to the magnetic island and results from the effect of the small, but finite, electron-ion collision frequency on electron orbits near the Landau resonance $\omega = k_{\parallel} v_{\parallel}$. Although the collision frequency does not appear explicitly, the result does depend on the particular form (i.e., the velocity dependence) of the collision operator. We have employed a Lorentz model collision operator which neglects electron-electron collisions and obtained the result $\omega = \omega_{*e} (1 - \eta_e)$. Saturated islands are then predicted to exist if $1 < \eta_e < 1 + \tau^{-1} (1 + \eta_i)$ if $\rho_i \ll w$ and $\eta_e < 0$ or $\eta_e > 1 + \tau^{-1}$ if $\rho_i \gg w$. (A system of equations [(52)–(55)] has been presented which allows the existence of islands to be investigated for arbitrary ion Larmor radius, though these require a numerical treatment.) These results can be compared with those of Smolyakov,^{9,10} who finds that (for $\tau = 1$, $Z_{\text{eff}} = 2$) islands can exist provided $0.49 < \eta_e < 0.49(2 + \eta_i)$ for $\rho_i \ll w$ and $\eta_e > 0.98$ or $\eta_e < 0$ for $\rho_i \gg w$.

We have presented a model for the transport which results from the electrostatic fluctuations associated with the magnetic islands, and have argued that this does not require a strong interaction of islands on adjacent rational surfaces. Therefore we expect that our model, which describes the evolution of isolated magnetic islands in a collisionless plasma, provides an adequate description of the island dynamics for this case. It is interesting to note that the expression which we obtain [Eq. (70)] shows some similar characteristics to the Rebut-Lallia-Watkins semiempirical result (e.g., the scalings with ν_{*} and geometry) but not the inverse β scaling. It remains to be established whether the transport from the stochastic magnetic fluctuations associated with somewhat larger islands, when there is a stronger island interaction, dominates this electrostatic contribution. A further point which could be addressed using a transport code is the self-consistency of the island existence criteria involving η_e with the values of η_e which arise from the anomalous transport resulting from the islands.

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