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Scaling laws for two-dimensional divertor modeling

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To gain insight into divertor operation, similarity techniques are employed to investigate whether model systems of equations plus boundary conditions admit scaling transformations that lead to useful divertor scaling laws. These can be used to perform similarity experiments or more fully exploit large computer simulations. Fluid plasma models of the divertor region are adopted that ignore anomalous processes. We consider neutral descriptions in both the short and long mean-free path limits. As usual, the more approximations that are made, the more scaling transformations are allowed, leading to fewer independent dimensionless parameters that need to be considered, thereby imposing fewer divertor similarity constraints. The simplest model considered balances electron heat conduction with impurity radiation and places the fewest constraints on similarity. To be able to model the onset of detached divertor operation in short mean-free path regimes, a fluid neutral description is employed that balances plasma pressure by neutral pressure. In this model the constraints on divertor similarity are most severe. A less constrained long mean-free path or Knudsen neutral model is also considered. It models the onset of detached divertor operation by balancing plasma pressure by momentum transfer to the neutrals that are randomized by collisions with the deep slot sidewalls. The simpler models have relaxed divertor similarity constraints, but all models remain severely restricted by the collisionality constraints. © *1996 American Institute of Physics.* [S1070-664X(96)01003-6]

I. INTRODUCTION

Two-dimensional numerical models of divertors employing fluid descriptions of the plasma and either short (fluid) or long (Knudsen) mean-free path descriptions of the neutrals contain large numbers of dimensionless parameters that must be varied to investigate all operating regimes expected to be of interest. Given the complexity of the descriptions, the task is a rather daunting one. The question arises, therefore, as to whether useful information can be obtained and the number of independent parameters reduced by considering the scaling transformation properties[1,2] of the system of differential equations and boundary conditions. The retention of boundary conditions is a new and necessary feature that must be considered when determining the allowed scaling transformations. If scale transformations can be found for a particular system, then by the invariance principle any quantities evaluated from the same system must satisfy the same scalings. To this end, we consider various divertor models and show that the techniques introduced by Connor and Taylor[1] and reviewed by Connor[2] can be employed on boundary conditions as well as the accompanying differential equations to find the constraints on divertor similarity.

The sections that follow consider fluid and Knudsen neutral models with fluid plasma equations and boundary conditions appropriate for complete recycling. Both the short and long mean-free path neutral descriptions adopted are capable of modeling the observed drops in temperature, particle flux, and energy flux at the target. The models assume all perpendicular transport is due to the neutrals since they ignore anomalous transport processes. They are best viewed as models of the divertor between the X point and target. More sophisticated models retaining anomalous transport are possible only if explicit perpendicular transport models are assumed. In order to avoid such *ad hoc* assumptions about the anomalous transport coefficients we investigate cases in which they enter only through the scrape-off layer width, which we assume specified.

In Sec. II, we present the fluid neutral and fluid plasma equations and boundary conditions, as obtained from Refs. 3 and 4. The equations and boundary conditions are made dimensionless in Sec. III, using the ionization energy of hydrogen, the peak upstream plasma pressure, and an appropriate neutral penetration length. In Sec. IV the scaling transformations and similarity constraints of various fluid descriptions are considered. The use of the technique of Connor and Taylor[1,2] when boundary conditions must be treated is illustrated in detail in Sec. IV A for the simple case in which the neutrals are neglected and electron heat conduction balances impurity radiation. The scaling law for the power to the target plates P divided by the major radius R is derived, and several possible ways to consider similarity are noted. In Sec. IV B, we consider a reduced two-dimensional (2-D) fluid neutral model based on the one-dimensional (1-D) model shown in Ref. 5 to exhibit many of the key features of divertor detachment. In two dimensions we recover Lackner's[6] P/R=constant scaling, while in the 1-D limit P/R need not be held constant for similarity. Moreover, for this fluid neutral model it is found that $P/R \propto \Delta_p / l_n \gg 1$, with Δ_p and l_n the scrape-off layer (SOL) width and neutral penetration length. In Sec. IV C it is shown that the general 2-D fluid neutral model that does not allow any scaling transformations leads to essentially the same conclusions as the reduced 2-D fluid model, but involves more parameters. In Sec. V we describe the Knudsen neutral model and its limitations. We then investigate the scaling transformations for the dimensionless equations and boundary conditions to show that $P/R \propto (\Delta_p/l_n)^{3/2} \ll 1$ for the Knudsen model and that it need not be held constant for similar devices. In Sec. V we discuss the implications of our results.

II. FLUID NEUTRAL AND PLASMA EQUATIONS AND BOUNDARY CONDITIONS

We adopt a simplied version of the full fluid neutral and plasma equations in the SOL, as given in Refs. 3 and 4. Most of the simplifications correspond to those normally employed in 2-D SOL codes and are based in part on the 1-D model solved in Ref. 5. Neglecting recombination here and elsewhere, the steady-state ion and neutral continuity equations are

$$\nabla \cdot (N_i \mathbf{V}_i) = \langle \sigma v \rangle_z N_e N_n \tag{1}$$

and

$$\nabla \cdot (N_n \mathbf{V}_n) = -\langle \sigma v \rangle_z N_e N_n , \tag{2}$$

where the subscripts e, i, and n denote electrons, ions, and neutrals; N_j and V_j denote the density and mean velocity of species j; and $\langle \sigma v \rangle_z$ is the rate constant for electron impact ionization. We assume singly charged ions and employ quasineutrality and local ambipolarity to obtain

$$N_e = N_i \quad \text{and} \quad \mathbf{V}_e = \mathbf{V}_i . \tag{3}$$

We assume that the mean ion velocity perpendicular to the magnetic field $\mathbf{V}_{i\perp}$ vanishes,

$$\mathbf{V}_{i\perp} = 0, \tag{4}$$

and for the perpendicular mean neutral velocity $\mathbf{V}_{n\perp}$ we use the perpendicular neutral momentum balance equation to obtain

$$N_n \mathbf{V}_{n\perp} = \frac{-\nabla_\perp (N_n T) + 0.24 N_n \nabla_\perp T}{M N_i \langle \sigma v \rangle_x + M N_e \langle \sigma v \rangle_z}, \tag{5}$$

where M and T are the mass and temperature of the ions and neutrals and $\langle \sigma v \rangle_x$ is the charge exchange rate constant. The $0.24 N_n \nabla T$ term is the thermal force found in Ref. 4 and the inertial terms are neglected since we assume that the perpendicular mean flows are small. By taking $\mathbf{V}_{i\perp}=0$, we are neglecting the anomalous perpendicular particle flux normally retained in the codes. Were explicit expressions available for the anomalous transport coefficients, the boundary conditions could be suitably modified and similarity techniques employed on the generalized system of equations. However, divertor simulators and codes need not necessarily model the details of anomalous transport in the SOL. We avoid making *ad hoc* assumptions about the anomalous transport coefficients in the SOL by specifying the SOL width as an input parameter and by considering only the divertor region between the X point and the target.

To find the equation for the parallel ion velocity $V_{i\parallel}$, we add the parallel ion and electron momentum equations to eliminate electron–ion friction and obtain the parallel plasma momentum balance equation,

$$\begin{aligned} \nabla \cdot [(M N_i V_{i\parallel}^2 + N_i T + N_e T_e + \pi_{i\parallel}) \hat{\mathbf{n}}] \\ = \langle \sigma v \rangle_z M N_e N_n V_{n\parallel} \\ - \langle \sigma v \rangle_x M N_n N_i (V_{i\parallel} - V_{n\parallel}) - 0.24 N_n \hat{\mathbf{n}} \cdot \nabla T, \end{aligned} \tag{6}$$

where $\hat{\mathbf{n}} = \mathbf{B}/B$ is the unit vector along the magnetic field and T_e is the electron temperature. Because of charge exchange coupling the parallel ion viscosity $\pi_{i\parallel}$ is $(N_n + N_i)/N_i$ times the Braginskii[7] value[3,4] and therefore given by

$$\pi_{i\parallel} = -1.3 (N_i + N_n) T_i \tau_i \hat{\mathbf{n}} \cdot \nabla V_{i\parallel} , \tag{7}$$

with τ_i the ion–ion collision time,

$$\tau_i = \frac{3 M^{1/2} T^{3/2}}{4 (\pi)^{1/2} N_i e^4 \ln \Lambda} . \tag{8}$$

The parallel neutral velocity $V_{n\parallel}$ is found from the parallel neutral momentum equation,

$$\begin{aligned} \nabla \cdot [(M N_n V_{n\parallel}^2 + N_n T) \hat{\mathbf{n}} + + \boldsymbol{\pi}_n \cdot \hat{\mathbf{n}}] \\ = -\langle \sigma v \rangle_z M N_e N_n V_{n\parallel} \\ + \langle \sigma v \rangle_x M N_n N_i (V_{i\parallel} - V_{n\parallel}) + 0.24 N_n \hat{\mathbf{n}} \cdot \nabla T, \end{aligned} \tag{9}$$

where, in the presence of charge exchange coupling to the ions, the neutral viscosity $\boldsymbol{\pi}_n \cdot \hat{\mathbf{n}}$ is given by[3,4]

$$\boldsymbol{\pi}_n \cdot \hat{\mathbf{n}} = \left(\frac{N_n}{N_i} \right) \pi_{i\parallel} \hat{\mathbf{n}} - \frac{N_n T}{N_i \langle \sigma v \rangle_x} \left(\nabla V_{n\parallel} + \frac{1}{3} \hat{\mathbf{n}} \hat{\mathbf{n}} \cdot \nabla V_{n\parallel} \right). \tag{10}$$

Charge exchange causes the ion and neutrals temperatures to be equal to lowest order, as well as coupling the ion and neutral flows via Eq. (6). To close our system of equations we need separate equations for the ion=neutral and electron temperatures. Ion plus neutral energy conservation with the viscosity and electric field terms neglected gives the equation for T to be

$$\begin{aligned} \nabla \cdot \left[\left(\frac{5}{2} T + \frac{1}{2} M V_\parallel^2 \right) (N_i + N_n) \mathbf{V} + \mathbf{q}_i + \mathbf{q}_n \right] \\ = \frac{3 m N_e (T_e - T)}{M \tau_{ei}}, \end{aligned} \tag{11}$$

where $\mathbf{V}$ and $V_\parallel$ are mean mass velocities defined by

$$(N_i + N_n) \mathbf{V} = N_i \mathbf{V}_i + N_n \mathbf{V}_n , \tag{12}$$

$V^2 \approx V_\parallel^2$, and τ_{ei} is the electron–ion collision time,

$$\tau_{ei} = \frac{3 m^{1/2} T_e^{3/2}}{4 (2\pi)^{1/2} N_i e^4 \ln \Lambda} . \tag{13}$$

Charge exchange makes the parallel ion heat flux $\mathbf{q}_i$ larger than its Braginskii[7] value by[3,4] $(N_n + N_i)/N_i$, giving

$$\mathbf{q}_i = -3.9 [(N_i + N_n) T \tau_i / M] \hat{\mathbf{n}} \hat{\mathbf{n}} \cdot \nabla T. \tag{14}$$

The neutral heat flux $\mathbf{q}_n$ in the presence of charge exchange coupling to the ions is given by[3,4]

$$\mathbf{q}_n=\left(\frac{N_n}{N_i}\right)\mathbf{q}_i-\frac{2.4N_nT}{MN_i\langle\sigma v\rangle_x}\,\nabla T-0.24N_nT(\mathbf{V}_n-\mathbf{V}_i), \tag{15}$$

where the last term is of the form referred to as a diffusion thermo-effect according to Chapman and Cowling.[8]

The final equation for the electron temperature T_e follows from the electron energy conservation equation,

$$\nabla\cdot\left(\frac{5}{2}\,T_eN_e\mathbf{V}_e+\mathbf{q}_e\right)=-\frac{3mN_e(T_e-T)}{M\tau_{ei}}-(I\langle\sigma v\rangle_z+E_{\rm H}\langle\sigma v\rangle_{\rm H})N_eN_n-E_I\langle\sigma v\rangle_IN_eN_I, \tag{16}$$

where I is the ionization potential for hydrogen (13.6 eV), $E_{\rm H}$ is the excitation energy of hydrogen (10.2 eV for Lyman α), with an excitation rate constant of $\langle\sigma v\rangle_H$, N_I is the impurity density, and E_I is the energy of the relevant excited impurity state with an excitation rate constant of $\langle\sigma v\rangle_I$. The parallel electron heat flux $\mathbf{q}_e$ is just the Braginskii[7] result,

$$\mathbf{q}_e=-3.2(N_eT_e\tau_{ei}/m)\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\nabla T_e. \tag{17}$$

The preceding system of equations requires ten boundary conditions since it is tenth order in the eight unknowns N_e, N_n, $V_{i\parallel}$, $\mathbf{V}_n$, T_e, and T. For upstream boundary conditions we employ

$$N_n|_{\rm up}=0, \tag{18}$$

$$N_e(T_e+T)|_{\rm up}=P_{\rm up}, \tag{19}$$

$$-3.2(N_eT_e\tau_{ei}/m)\hat{\mathbf{n}}\cdot\nabla T_e|_{\rm up}=q_{e\parallel}^{\rm up}, \tag{20}$$

and

$$-3.9[(N_i+N_n)T\tau_i/M]\hat{\mathbf{n}}\cdot\nabla T|_{\rm up}=q_{i\parallel}^{\rm up}, \tag{21}$$

where the upstream pressure $P_{\rm up}$ and upstream parallel electron (ion) heat flux $q_{e\parallel}^{\rm up}(q_{i\parallel}^{\rm up})$ are all specified functions at the upstream entrance to the divertor. The neutral density boundary condition corresponds to considering complete recycling in the divertor region since it, ambipolarity, and the sum of the two continuity equations require that there be no plasma entering through the upstream boundary.

The remaining six boundary conditions are applied downstream at the divertor target plates and sidewalls, where wall values of density, temperature, etc. will be denoted by a subscript w. The outward directed unit vector normal to the wall is defined as $\hat{\mathbf{w}}$, where, for definiteness, we assume $\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}>0$ ($\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}<0$) when $V_{i\parallel}<0$ ($V_{i\parallel}>0$). The parallel ion flow into the walls must satisfy a generalized Bohm sheath criterion,

$$\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}V_{i\parallel}|_w=-\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}\alpha(T_w/M)^{1/2}, \tag{22}$$

where $\alpha\sim0.5$–1. To maintain a steady state, complete recycling is imposed at the walls by demanding that the outgoing normal neutral flux equal the incoming normal ion flux,

$$\hat{\mathbf{w}}\cdot(\hat{\mathbf{n}}N_iV_{i\parallel}+N_n\mathbf{V}_n)|_w=0. \tag{23}$$

The parallel plasma momentum flux normal to the wall is specified by

$$\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}(MN_iV_{i\parallel}^2+N_iT+N_eT_e+\pi_{i\parallel})|_w=\gamma_\parallel\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}N_{iw}T_w, \tag{24}$$

where $\gamma_\parallel\sim2$–3 is the plasma momentum transmission coefficient. Similarly, the normal neutral momentum flux normal to the wall is taken to be

$$|[\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}(MN_nV_{n\parallel}^2+N_nT)+\hat{\mathbf{w}}\cdot\boldsymbol{\pi}_n\cdot\hat{\mathbf{n}}]|_w=\gamma_mN_{nw}T_w, \tag{25}$$

with $\gamma_m\sim1$–2 the neutral momentum transmission coefficient.

The final wall boundary conditions are on the energy fluxes. The normal ion plus neutral energy flux onto the wall is taken as

$$|\hat{\mathbf{w}}\cdot[(5T/2+MV^2/2)(N_n+N_i)\mathbf{V}+\mathbf{q}_i+\mathbf{q}_n]|_w=-(\alpha|\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}|\gamma_iN_{iw}+\gamma_nN_{nw})T_w(T_w/M)^{1/2}, \tag{26}$$

where $\gamma_i\sim2$–3 and $\gamma_n\sim0.1$–0.3 are the ion and neutral heat transmission coefficients, respectively. The $\alpha\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}$ in the ion contribution on the right side accounts for the ion heat flux moving along the magnetic field and ions hitting the wall with a parallel speed satisfying the Bohm condition. A similar expression holds for the normal electron energy flux onto the wall,

$$\hat{\mathbf{w}}\cdot[\hat{\mathbf{n}}(5T_e/2)N_eV_{e\parallel}+\mathbf{q}_e]|_w=-\alpha\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}\gamma_eN_{ew}T_{ew}(T_{ew}/M)^{1/2}, \tag{27}$$

with $\gamma_e\sim2$–3 the electron heat transmission coefficient. The various transmission coefficients are discussed in Refs. 9.

The preceding equations and boundary conditions form the basic system of equations that we will investigate for fluid neutrals. In the following sections we will make further assumptions that allow us to combine the preceding equations to obtain simpler systems. Moreover, In Sec. V we will make modifications that allow us to consider Knudsen neutrals.

III. DIMENSIONLESS FORM OF THE EQUATIONS AND BOUNDARY CONDITIONS

To determine if the system of equations and boundary conditions (and, therefore, the heat loads on target, etc.) of Sec. II are invariant under particular sets of scaling transformations, it is convenient to first make the entire system dimensionless. To do so, we will adopt a 2-D Cartesian model, with x and y denoting the radial and poloidal coordinates and z corresponding to the ignorable toroidal direction ($\partial/\partial z=0$). Writing the unit vector along the magnetic field as

$$\hat{\mathbf{n}}=(B_T/B)\hat{\mathbf{z}}+b\hat{\mathbf{y}},$$

with $b=B_P/B$ and $B_P(B_T)$ the poloidal (toroidal) magnetic field components, gives

$$\hat{\mathbf{n}}\cdot\nabla=\frac{b\,\partial}{\partial y},\quad \hat{\mathbf{z}}\cdot\hat{\mathbf{V}}_{n\perp}=0,\quad V_{iy}=bV_{i\parallel},$$
$$V_{iz}=\left(\frac{B_T}{B}\right)V_{i\parallel}\quad\text{and}\quad V_{j\parallel}=bV_{jy}+\left(\frac{B_T}{B}\right)V_{jz}, \tag{28}$$

where $j=i$, n, or e and we will assume that $B=|\mathbf{B}|$ is a constant.

We normalize temperatures and velocities to the hydrogen ionization potential I by defining

$$\tau=T/I, \quad \tau_e=T_e/I, \quad v_\parallel=(M/I)^{1/2}V_{i\parallel},$$
$$u=(M/I)^{1/2}V_{ny}, \quad w=(M/I)^{1/2}V_{nx}, \quad \text{and} \tag{29}$$
$$v=(M/I)^{1/2}V_{n\parallel}.$$

Densities are normalized by introducing the peak upstream plasma pressure P_u so that the known function $P_{\rm up}$ in the upstream pressure boundary condition may be written as

$$P_{\rm up}=P_uS_p(x/\Delta_p), \tag{30}$$

where $S_p(x/\Delta_p)$ is a specified order unity SOL shape function. Using P_u and I the normalized plasma and neutral densities are then defined as

$$n=IN_e/P_u=IN_i/P_u \quad \text{and} \quad \eta=IN_n/P_u. \tag{31}$$

We desire to normalize lengths to the neutral penetration scale length l_n at the temperature I and density P_u/I. As a result, it is convenient to introduce another set of shape functions $S_x(T)$ and $S_z(T_e)$ by defining

$$\langle\sigma v\rangle_x=K_xS_x(T) \quad \text{and} \quad \langle\sigma v\rangle_z=K_zS_z(T_e), \tag{32}$$

where K_x and K_z are true constants equal to the appropriate peak values of the charge exchange and ionization rate constants, so that the functions $S_x(T)$ and $S_z(T_e)$ are of order unity. Using the preceding definitions, l_n is defined as

$$l_n=\frac{I(I/M)^{1/2}}{P_u(K_xK_z)^{1/2}}, \tag{33}$$

and the normalized poloidal (β) and radial (ρ) variables may be defined as

$$\beta=y/l_n \quad \text{and} \quad \rho=x/l_n. \tag{34}$$

Using the preceding definitions and introducing the definition

$$\sigma=(K_z/K_x)^{1/2}, \tag{35}$$

the continuity equations and the perpendicular neutral flux equations become

$$b\,\frac{\partial}{\partial\beta}\,(nv_\parallel)=\sigma n\eta S_z(\tau_e), \tag{36}$$

$$\frac{\partial}{\partial\beta}\,(\eta u)+\frac{\partial}{\partial\rho}\,(\eta w)=-\sigma n\eta S_z(\tau_e), \tag{37}$$

$$\eta(u-bv)=\frac{-(\partial/\partial\beta)(\eta\tau)+0.24\eta(\partial\tau/\partial\beta)}{n[\sigma S_z(\tau_e)+\sigma^{-1}S_x(\tau)]}, \tag{38}$$

and

$$\eta w=\frac{-(\partial/\partial\rho)(\eta\tau)+0.24\eta(\partial\tau/\partial\rho)}{n[\sigma S_z(\tau_e)+\sigma^{-1}S_x(\tau)]}. \tag{39}$$

To make the parallel plasma momentum equation dimensionless, we need to introduce a new dimensionless parameter μ that is the Coulomb mean-free path λ divided by the neutral penetration length at temperature I,

$$\mu=\frac{0.55(MK_xK_z)^{1/2}I^{3/2}}{e^4\ln\Lambda}\approx\left.\frac{\lambda}{l_n}\right|_{T=I}. \tag{40}$$

Using this definition in $\pi_{i\parallel}$, parallel plasma momentum balance becomes

$$b\,\frac{\partial}{\partial\beta}\left(nv_\parallel^2+n(\tau+\tau_e)-\mu b\tau^{5/2}\,\frac{(n+\eta)}{n}\,\frac{\partial v_\parallel}{\partial\beta}\right)+0.24b\,\eta\,\frac{\partial\tau}{\partial\beta}$$
$$=n\eta[\sigma vS_z(\tau_e)-\sigma^{-1}(v_\parallel-v)S_x(\tau)]. \tag{41}$$

No further new parameters are introduced by neutral momentum balance, which, upon inserting the neutral viscosity, may be written as

$$b\,\frac{\partial}{\partial\beta}\left(\eta v^2+\eta\tau-\mu b\tau^{5/2}\,\frac{\eta(n+\eta)}{n^2}\,\frac{\partial v_\parallel}{\partial\beta}\right)-0.24b\,\eta\,\frac{\partial\tau}{\partial\beta}$$
$$-\frac{\partial}{\partial\beta}\left(\frac{\sigma(1+\frac{1}{3}b^2)\eta\tau}{nS_x(\tau)}\,\frac{\partial v}{\partial\beta}\right)-\frac{\partial}{\partial\rho}\left(\frac{\sigma\eta\tau}{nS_x(\tau)}\,\frac{\partial v}{\partial\rho}\right) \tag{42}$$
$$=-n\eta[\sigma vS_z(\tau_e)-\sigma^{-1}(v_\parallel-v)S_x(\tau)].$$

To write electron energy balance in dimensionless form, it is convenient to first introduce two new shape functions $S_{\rm H}(T_e)$ and $S_I(T_e)$ for the hydrogen and impurity radiation by letting

$$\langle\sigma v\rangle_{\rm H}=K_{\rm H}S_{\rm H}(T_e) \quad \text{and} \quad \langle\sigma v\rangle_I=K_IS_I(T_e), \tag{43}$$

where $K_{\rm H}$ and K_I are true constants equal to the appropriate peak values of the hydrogen and impurity rate constants, such that the functions $S_{\rm H}(T_e)$ and $S_I(T_e)$ are order unity functions. Next, we define the dimensionless impurity density n_I and dimensionless constants $\sigma_{\rm H}$ and σ_I corresponding, respectively, to energy and impurity density times energy weighted ratios of radiation to neutral penetration scale lengths,

$$n_I=\frac{IN_I}{P_u}, \quad \sigma_{\rm H}=\frac{E_{\rm H}K_{\rm H}}{I(K_xK_z)^{1/2}}, \quad \text{and} \quad \sigma_I=\frac{n_IE_IK_I}{I(K_xK_z)^{1/2}}. \tag{44}$$

Notice that we have implicitly assumed that the shape function for n_I depends only on T_e, but we could have allowed additional spatial shape functions to model radial and poloidal impurity profiles to remove this restriction. A final dimensionless constant parameter k, proportional to $(M/m)^{1/2}$ times the Coulomb mean-free path λ divided by the neutral penetration length at temperature I, is defined by

$$k=\frac{0.96M(K_xK_z)^{1/2}I^{3/2}}{m^{1/2}e^4\ln\Lambda}\approx\left(\frac{M}{m}\right)^{1/2}\left.\frac{\lambda}{l_n}\right|_{T_e=I}. \tag{45}$$

Notice that k is also a measure of the temperature equilibration length divided by the neutral penetration length. In terms of the preceding parameters, the dimensionless electron energy equation becomes

$$b\,\frac{\partial}{\partial\beta}\left(\frac{5}{2}\,\tau_env_\parallel-kb\tau_e^{5/2}\,\frac{\partial\tau_e}{\partial\beta}\right)=-\frac{9.6n^2(\tau_e-\tau)}{k\tau_e^{3/2}}$$
$$-\sigma n\eta S_z(\tau_e)-\sigma_{\rm H}n\eta S_{\rm H}(\tau_e)-\sigma_InS_I(\tau_e). \tag{46}$$

The ion plus neutral energy balance introduces no new parameters. With the heat fluxes inserted it becomes the rather complicated expression

$$\frac{\partial}{\partial\beta}\left\{\left[\frac{5}{2}\tau+\frac{1}{2}\left(\frac{nv_\parallel+\eta v}{n+\eta}\right)^2\right](bnv_\parallel+\eta u)\right.$$
$$\left.-3\mu b^2\tau^{5/2}\frac{(n+\eta)^2}{n^2}\frac{\partial\tau}{\partial\beta}-0.24\eta\tau(u-bv_\parallel)\right\}$$
$$+\frac{\partial}{\partial\rho}\left\{\left[\frac{5}{2}\tau+\frac{1}{2}\left(\frac{nv_\parallel+\eta v}{n+\eta}\right)^2\right]\eta w-0.24\eta\tau w\right\}$$
$$-\frac{\partial}{\partial\beta}\left(\frac{2.4\sigma\eta\tau}{nS_x(\tau)}\frac{\partial\tau}{\partial\beta}\right)-\frac{\partial}{\partial\rho}\left(\frac{2.4\sigma\eta\tau}{nS_x(\tau)}\frac{\partial\tau}{\partial\rho}\right)$$
$$=\frac{9.6n^2(\tau_e-\tau)}{k\tau_e^{3/2}}, \tag{47}$$

where to simplify this result we have assumed for the first time that B_P (and, therefore, b) is a constant.

To obtain the boundary conditions in dimensionless form it is convenient to introduce the shape functions $S_e(x/\Delta_p)$ and $S_i(x/\Delta_p)$ for known upstream poloidal electron ($bq_{e\parallel}^{\mathrm{up}}$) and ion ($bq_{i\parallel}^{\mathrm{up}}$) heat fluxes by defining

$$bq_{e\parallel}^{\mathrm{up}}=-Q_eP_u(I/M)^{1/2}S_e(x/\Delta_p)\quad\text{and}$$
$$bq_{i\parallel}^{\mathrm{up}}=-Q_iP_u(I/M)^{1/2}S_i(x/\Delta_p). \tag{48}$$

The dimensionless parameters Q_e and Q_i times $P_u(I/M)^{1/2}$ are the peak upstream values of the poloidal electron and ion heat fluxes, respectively. Letting $y=0$ and $y=L$ denote the target and the location at which the upstream boundary conditions are applied, respectively, and employing the preceding definitions, the dimensionless forms of the upstream boundary conditions for the neutral density, plasma pressure, and electron and ion heat fluxes are as follows:

$$\eta(\beta=L/l_n)=0, \tag{49}$$

$$n(\tau+\tau_e)|_{\beta=L/l_n}=S_p(\rho l_n/\Delta_p), \tag{50}$$

$$kb^2\tau_e^{5/2}\left.\frac{\partial\tau_e}{\partial\beta}\right|_{\beta=L/l_n}=Q_eS_e\left(\frac{\rho l_n}{\Delta_p}\right), \tag{51}$$

and

$$3\mu b^2\tau^{5/2}\left.\frac{\partial\tau}{\partial\beta}\right|_{\beta=L/l_n}=Q_iS_i\left(\frac{\rho l_n}{\Delta_p}\right). \tag{52}$$

We assume that the magnetic field is parallel to the sidewalls located at $x=\pm\Delta$ and denote by a subscript d the target plate ($y=0$). Then, the dimensionless forms of the boundary conditions for the parallel ion flow, parallel plasma momentum, and parallel electron energy flow (which do not involve the sidewalls) follow:

$$v_{\parallel d}=-\alpha\tau_d^{1/2}, \tag{53}$$

$$\alpha^2n_d\tau_d+n_d(\tau_d+\tau_{ed})-\mu b\tau_d^{5/2}\left.\frac{(n_d+\eta_d)\partial v_\parallel}{n_d\quad\partial\beta}\right|_d$$
$$=\gamma_\parallel n_d\tau_d, \tag{54}$$

and

$$\frac{5}{2}\alpha n_d\tau_{ed}\tau_d^{1/2}+kb\tau_{ed}^{5/2}\left.\frac{\partial\tau_e}{\partial\beta}\right|_d=\alpha\gamma_en_d\tau_{ed}^{3/2}. \tag{55}$$

The remaining boundary conditions involve the neutrals so we must distinguish between the sidewalls (at which $\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}=\hat{\mathbf{x}}\cdot\hat{\mathbf{n}}=0$ and whose location we denote by the subscript s meaning $x=\pm\Delta/2$) and target (at which $\hat{\mathbf{w}}\cdot\hat{\mathbf{n}}=\hat{\mathbf{y}}\cdot\hat{\mathbf{n}}=b$). As a result, for the remaining dimensionless boundary conditions for the ion plus neutral flows, the neutral momentum, and the ion plus neutral energy we find the following forms:

$$\eta_du_d=\alpha bn_d\tau_d^{1/2},\quad \eta_sw_s=0, \tag{56}$$

$$b\eta_d(v_d^2+\tau_d)-\mu b^2\tau_d^{5/2}\left.\frac{\eta_d(n_d+\eta_d)\partial v_\parallel}{n_d^2\quad\partial\beta}\right|_d$$
$$-\left.\frac{\sigma(1+\frac{1}{3}b^2)\eta_d\tau_d\partial v}{n_dS_x(\tau_d)\quad\partial\beta}\right|_d=\gamma_m\eta_d\tau_d, \tag{57a}$$

$$-\left.\frac{\sigma\eta_s\tau_s\ \partial v}{n_sS_x(\tau_s)\partial\rho}\right|_s=\gamma_m\eta_s\tau_s, \tag{57b}$$

and

$$\left(\frac{2.4\sigma\eta_d\tau_d}{n_dS_x(\tau_d)}+3\mu b^2\tau_d^{5/2}\frac{(n_d+\eta_d)^2}{n_d^2}\right)\left.\frac{\partial\tau}{\partial\beta}\right|_d+0.24\eta_d\tau_d$$
$$\times(u_d-bv_{\parallel d})=(\alpha b\gamma_in_d+\gamma_n\eta_d)\tau_d^{3/2}, \tag{58a}$$

$$-\left.\frac{2.4\sigma\eta_s\tau_s\partial\tau}{n_sS_x(\tau_s)\,\partial\rho}\right|_s=\gamma_n\eta_s\tau_s^{3/2}. \tag{58b}$$

Our fluid description in its most general form is now complete. In the next section we will consider the properties and simplifications of our system of equations.

IV. SCALING TRANSFORMATIONS AND SIMILARITY FOR FLUID DESCRIPTIONS

The full 2-D system of differential equations and boundary conditions as given by Eqs. (36)–(39), (41), (42), (46), (47), (49)–(52), (53)–(55), and (56)–(58) involves the eight unknowns n, η, $v_\parallel$, w, u, v, τ, and τ_e and the 17 dimensionless parameters σ, b, μ, k, σ_{H}, σ_I, α, Q_e, Q_i, $\gamma_\parallel$, γ_m, γ_e, γ_i, γ_n, L/l_n, Δ_p/l_n, and Δ/l_n. It is important to notice that P_u only appears in the definitions n, η, l_n, σ_I, and through the normalizations for the upstream heat fluxes, as given by Eq. (48). As a result, for similar configurations the densities (recall $\sigma_I\propto n_I\propto N_I/P_u$) must scale linearly with P_u, while the depth (L) and width (Δ_p and Δ) scaling must be inversely proportional to P_u. Recall also that $Q_e\propto bq_{e\parallel}^{\mathrm{up}}/P_u$ because of the normalization of the upstream heat fluxes. Notice that the usual collisionality scaling of density times scale length is embedded in L/l_n, Δ_p/l_n, and Δ/l_n, which are proportional to density through P_u.

Fortunately, most of the dimensionless parameters are reasonably well known, but rather wide ranges of b, σ_I, Q_e, Q_i, L/l_n, Δ_p/l_n, and Δ/l_n are of interest since they depend on the upstream plasma pressure and heat fluxes, the impurities, and the divertor configuration. Typically $\sigma\sim0.1$–1,

$\sigma_I \sim 1$, and for a tokamak $b \sim 0.05$; while for $P_u \sim 10^{14}$ cm$^{-3} \times 100$ eV, $\lambda \sim l_n \sim 0.3$ cm, giving $\mu \sim 1$ and $k \sim 50$.

To reduce the number of dimensionless parameters we can seek scale transformations of this system of equations. It will turn out that the full system does not allow any scale transformations, but that by simplifying the system, scale transformations can be found that allow us to reduce the number of dimensionless parameters that need be considered.

A. Model without neutrals

We first employ the Connor and Taylor[1,2] procedure on a simple limit to illustrate the need to retain boundary conditions and how they are easily included by their technique. If we adopt the simplest model possible by neglecting the neutrals completely ($\eta \to 0$), letting $v_\parallel \to 0$ and $\tau = \tau_e$, adding the energy equations to eliminate electron–ion energy exchange, and neglecting the ion heat conduction compared to that of the electrons, we find that plasma pressure balance becomes $2n\tau_e = S_p(\rho l_n/\Delta_p)$, and we need only consider

$$2b\frac{\partial}{\partial\beta}\left(kb\tau_e^{5/2}\frac{\partial\tau_e}{\partial\beta}\right) = \sigma_I\tau_e^{-1}S_I(\tau_e)S_p\left(\frac{\rho l_n}{\Delta_p}\right). \quad (59)$$

In this limit the upstream boundary condition is given by Eq. (51) and the target boundary condition is given by Eq. (55) with the convection term neglected,

$$2kb\tau_{ed}^{5/2}\frac{\partial\tau_e}{\partial\beta}\bigg|_d = \alpha\gamma_e\tau_{ed}^{1/2}S_p\left(\frac{\rho l_n}{\Delta_p}\right). \quad (60)$$

In this model the parameter Δ/l_n does not enter because sidewall boundary conditions are not needed since $\Delta_p < \Delta$.

We seek scale transformations by scaling the dependent variable τ_e, the two independent variables ρ and β, and the seven dimensionless parameters b, k, σ_I, Q_e, $\alpha\gamma_e$, L/l_n, and Δ_p (but only the arguments of shape functions, since their coefficients are accounted for by scaling the dimensionless parameters). Letting $\tau_e \to \omega_1\tau_e$, $\beta \to \omega_2\beta$, $\rho \to \omega_3\rho$, $b \to \omega_4 b$, $k \to \omega_5 k$, $\sigma_I \to \omega_6\sigma_I$, $Q_e \to \omega_7 Q_e$, $a\gamma_e \to \omega_8 a\gamma_e$, $L/l_n \to \omega_9 L/l_n$, and $\Delta_p/l_n \to \omega_{10}\Delta_p/l_n$, we find four independent scale transformations:

$$\begin{aligned}
&\text{(i)}\quad \beta \to \omega_2\beta, \quad L/l_n \to \omega_2 L/l_n, \quad k \to \omega_2 k,\\
&\qquad \sigma_I \to \sigma_I/\omega_2;\\
&\text{(ii)}\quad \rho \to \omega_3\rho, \quad \Delta_p/l_n \to \omega_3\Delta_p/l_n; \qquad (61)\\
&\text{(iii)}\quad b \to \omega_4 b, \quad \sigma_I \to \omega_4\sigma_I, \quad k \to k/\omega_4,\\
&\qquad Q_e \to \omega_4 Q_e;
\end{aligned}$$

and

$$\begin{aligned}
&\text{(iv)}\quad Q_e \to \omega_7 Q_e, \quad \sigma_I \to \omega_7\sigma_I, \quad k \to \omega_7 k,\\
&\qquad a\gamma_e \to \omega_7 a\gamma_e;
\end{aligned}$$

where ω_j denotes the scaling constants and for each transformation we do not indicate quantities not scaled (notice that τ_e can never be scaled because of the complicated dependence of S_I). The second transformation is needed to keep the argument of ρ-dependent shape functions fixed.

We started with seven dimensionless parameters and two dimensionless independent variables and have found four transformations. Any physical quantity of interest derivable from this system of equations must be invariant under the four scaling transformations. In particular, we can find the five invariant combinations of the seven dimensionless parameters and the two dimensionless variables by considing the scaling properties of the invariant product,

$$Z_{ABC\cdots} = b^A k^B \sigma_I^c Q_e^D(\alpha\gamma_e)^E(L/l_n)^F(\Delta_p/l_n)^G\rho^H\beta^I. \quad (62)$$

For both sides of Eq. (62) to be invariant ($Z_{ABC\cdots} \to Z_{ABC\cdots}$) under the scale transformations of Eq. (61) requires

$$1 = \omega_2^{B-C+F+I}\omega_3^{G+H}\omega_4^{A-B+C+D}\omega_7^{B+C+D+E}$$

or

$$F = C-B-I, \quad G = -H, \quad A = B-C-C, \quad \text{and}$$

$$E = -B-C-D,$$

which, when used in Eq. (62), gives

$$Z_{BCDHI} = \left(\frac{bkl_n}{L\alpha\gamma_e}\right)^B\left(\frac{\sigma_I L}{l_n b\alpha\gamma_e}\right)^C\left(\frac{Q_e}{b\alpha\gamma_e}\right)^D\left(\frac{\rho l_n}{\Delta_p}\right)^H\left(\frac{\beta l_n}{L}\right)^I. \quad (63)$$

As a result, only the three independent dimensionless parameters: $bkl_n/La\gamma_e$, $\sigma_I L/l_n ba\gamma_e$, and $Q_e/ba\gamma_e$, as well as the independent variable combinations $\rho l_n/\Delta_p$ and $\beta l_n/L$ satisfy the four scaling transformations, reducing the number of dimensionless quantities by four.

We are particularly interested in the form of the normalized poloidal energy flux on the target plates $Q_t \equiv q_t/P_u(I/M)^{1/2}$, which depends on $\rho l_n/\Delta_p$, but not $\beta l_n/L$, since $\beta = 0$. We first note that Q_t must contain a coefficient having the same scaling as $Q_e(Q_t \to \omega_4\omega_7 Q_t)$ times an unknown function of these three independent parameters and $\rho l_n/\Delta_p$:

$$q_t = Q_e P_u\left(\frac{I}{M}\right)^{1/2} f\left(\frac{bkl_n}{L\alpha\gamma_e}, \frac{\sigma_I L}{l_n b\alpha\gamma_e}, \frac{Q_e}{b\alpha\gamma_e}, \frac{\rho l_n}{\Delta_p}\right). \quad (64)$$

Here and elsewhere f is used to denote an unknown function of the arguments listed. The arguments of f can be written in various equivalent ways since products of the invariant dimensionless quantities are invariant. A convenient form for our purposes is

$$q_t = b\alpha\gamma_e P_u\left(\frac{I}{M}\right)^{1/2} f\left(\frac{bkl_n}{L\alpha\gamma_e}, \frac{k\sigma_I}{\alpha^2\gamma_e^2}, \frac{Q_e}{b\alpha\gamma_e}, \frac{\rho l_n}{\Delta_p}\right), \quad (65)$$

where Q_e and $ba\gamma_e$ scale the same way.

Since the hydrogen ionization potential does not enter this simple model, we can make the replacement $I \to E_I$ in all quantities to replace I by the excitation energy of interest. Notice that none of the dimensionless parameters that enter Eqs. (64) or (65) depend on $K_x K_z$ since the neutral penetration length, which no longer enters, is replaced by b times an appropriate Coulomb mean-free path ($0.96E_I^3/P_u e^4 \ln\Lambda$). In this simple model, these same dimensionless parameters can be found by direct integration and application of the boundary conditions. Then, it can be seen that for a sufficiently

TABLE I. Divertor similarity: Constraints (key arguments of f) and P/R scaling (coefficient of f).

Similarity constraint	No neutrals (three arguments)	Fluid neutrals (six key arguments)	Knudsen neutrals (five key arguments)
Parallel heat flux (determines Q_e)	$Q_e/b\alpha\gamma_e$ ($\alpha\gamma_e\approx$const)	Q_e/b	Q_e/b
Impurity radiation (determines σ_I)	$\sigma_I/(\alpha\gamma_e)^2$ ($\alpha\gamma_e\approx$const)	σ_I	σ_I
Collisionality (determines P_u)	$bl_n/\alpha\gamma_e L$ ($\alpha\gamma_e\approx$const)	bl_n/L	bl_n/L
Chamber width (determines Δ)	none	Δ/L	d/b^2 ($d\propto cV\Delta/l_n$)
B field ratio (determines b)	none	b	none
SOL width (determines Δ_p)	none	Δ_p/Δ	Δ_p/Δ
Other constants	k	$k,\sigma,\alpha,\sigma_{\rm H},\gamma_n,\gamma_e+\gamma_i$	$k,\sigma,\alpha,\sigma_{\rm H},\gamma_e+\gamma_i$
P/R scaling (free to adjust)	$b^2\Delta_p/L$ (Δ_p,b,B)	constant (B)	$b^2\Delta_p/L$ (cV,b,B)

localized shape function $S_I(\tau_e)$ with a τ_e dependence permitting no impurity radiation losses for $y>L$, the function f in Eq. (65) will be insensitive to the first parameter, which describes the Coulomb collisionality. Moreover, for a sufficiently low impurity density, f will be insensitive to the second parameter.

If we form the power to the plates P by integrating x over the SOL width Δ_p, then for a single null divertor, $P=4\pi R\int dx\ q_t$ gives

$$\frac{P}{R}=b\alpha\gamma_e\ \Delta_p P_u\left(\frac{I}{M}\right)^{1/2} f\left(\frac{bkl_n}{L\alpha\gamma_e},\ \frac{k\sigma_I}{\alpha^2\gamma_e^2},\ \frac{Q_e}{b\alpha\gamma_e}\right), \quad (66)$$

where R is the major radius of the tokamak and, of course, f is a different unknown function. Notice that the only dependence on the SOL width enters as an explicit multiplier in Eq. (66). For similar divertors we must keep the unknown function f fixed. Since $bkl_n/La\gamma_e\propto b/a\gamma_e LP_u$, we may use $P_u\propto b/a\gamma_e L$ in Eq. (66) to obtain

$$P/R\propto ba\gamma_e P_u\Delta_p\propto b^2\Delta_p/L, \quad (67)$$

for similar devices in which $bkl_n/La\gamma_e$, $k\sigma_I/a^2\gamma_e^2$, and $Q_e/ba\gamma_e$ are held constant to keep the unknown function f constant. If f is independent of its first argument (localized S_I), then the final form of Eq. (67) is not relevant.

In Lackner's[6] treatment all lengths and magnetic fields in the divertor must scale the same way for similar tokamaks or similar divertors, so that Δ_p/L and b, and, therefore, P/R are constants. It should be mentioned that Lackner actually takes P to be the power from the core that crosses the separatrix, rather than the power entering the divertor. In our restricted model Δ_p, P_u, b, Q_e, and $k\sigma_I\propto n_I E_I K_I$, are viewed as independent control parameters (k is a constant and $a\gamma_e$ does not vary significantly), so for a given L we can adjust P_u, $k\sigma_I\propto n_I E_I K_I$, and Q_e to keep the arguments of f fixed and still be free to adjust Δ_p and b. The constraints and P/R scaling for this model are summarized in the first column of Table I.

Similarity constraints are significantly relaxed by considering this substantially reduced description with no neutrals. In the International Thermonuclear Experimental Reactor (ITER)[10] $P\sim 300$ MW, $R\sim 8$ m, $b\sim\frac{1}{6}$, $\Delta_p\sim 1$ cm, and $P_u\sim 10^{14}$ cm$^{-3}\times 500$ eV, so using Eq. (67), a similar device with $P\sim 30$ MW and $R\sim 4$ m gives $P_{\rm ITER}/R_{\rm ITER}\sim 5P/R$ and is possible for a P_u about five times smaller (but L five times *larger*) than the ITER value, assuming roughly the same b, Δ_p, and $a\gamma_e$ (if Δ_p in ITER turns out to be larger than 1 cm, then the power of the similar device could be lowered by the same amount). If we employ the anticipated Alcator C-MOD[11] numbers ($P\sim 8$ MW, $R\sim 0.7$ m, $\Delta_p\sim 1$ cm, and $P_u\sim 10^{14}$ cm$^{-3}\times 100$ eV), we obtain a value of $P/RP_u\Delta_p$ larger than the ITER value by almost a factor of 2 so that it could provide good similarity to ITER in the fluid neutral limit at the onset of detachment if the collisionality constraint ($bkl_n/La\gamma_e\propto b/a\gamma_e LP_u$=const) does not need to be satisfied because the impurity radiation losses are sufficiently localized[11–14] that the depth of the divertor becomes unimportant.

In addition to Δ_p and b, we are also free to adjust B (since only the ratio b enters in our equations). These three adjustments can be used to satisfy other constraints, such as gyroradius over scale length (for example, constant $B\Delta_p$ or $B_p\Delta_p$) and/or plasma beta (constant $\beta\propto P_u/B^2$) constraints. Some possibilities are shown in Table II and include a $P/R\propto\alpha\gamma_e B_p\propto B_p$ scaling similar to that found In Ref. 15.

TABLE II. Some additional constraints and resulting scalings for no neutral model.

Case	Additional constraints	Resulting P/R scaling
(a)	b=const, Δ_p/L=const	constant
(b)	$b\Delta_p/L$=const, $B\Delta_p$=const	b
(c)	$\beta\propto P_u/B^2$=const, $B\Delta_p$=const	$\alpha\gamma_e B_p$
(d)	b=const, $B\Delta_p$=const	$1/BL$
(e)	b=const, $\beta\propto P_u/B^2$=const	$\alpha\gamma_e B^2\Delta_p$

B. Reduced fluid neutral model

If we assume that $\sigma \ll 1$ and $b \ll 1$, and neglect inertial ($v^2 \approx v_\parallel^2 \ll \tau \sim 1$) and ion ($\sigma \mu b^2 \ll 1$) and neutral viscosity ($\sigma \ll 1$) effects, then the ionization terms in Eqs. (38) and (39) may be neglected and Eq. (41) or (42) employed to find $v \approx v_\parallel$. As a result, the perpendicular neutral flux equations, Eqs. (38) and (39), become

$$n\eta(u - bv_\parallel)S_x(\tau) = \sigma\left(-\frac{\partial}{\partial \beta}(\eta\tau) + 0.24\eta \frac{\partial \tau}{\partial \beta}\right) \tag{68}$$

and

$$n\eta w S_x(\tau) = \sigma\left(-\frac{\partial}{\partial \rho}(\eta\tau) + 0.24\eta \frac{\partial \tau}{\partial \rho}\right). \tag{69}$$

If we also assume that the ions and neutrals have equilibrated with the electrons, the sum of Eqs. (41) and (42) with inertial and viscous terms neglected gives total parallel momentum balance to be

$$\frac{\partial}{\partial \beta}[(2n+\eta)\tau] = 0. \tag{70}$$

Equilibration between electrons and ions and neutrals follows from Eq. (47) if convection and conduction are made small by taking $k\sigma \ll 1$, since we have already assumed perpendicular flows to be small compared to parallel flows and $b \ll 1$. We obtain the equation for τ under the same assumptions, by adding Eqs. (46) and (47) and using $k \gg \mu$ to form the total energy conservation equation,

$$\begin{aligned} & b\frac{\partial}{\partial \beta}\left(5\tau n v_\parallel + 0.24\tau\eta v_\parallel - kb\tau^{5/2}\frac{\partial \tau}{\partial \beta}\right) + \frac{\partial}{\partial \beta}(2.3\tau\eta u) \\ &\quad + \frac{\partial}{\partial \rho}(2.3\tau\eta w) - \frac{\partial}{\partial \beta}\left(\frac{2.4\sigma\eta\tau}{nS_x(\tau)}\frac{\partial \tau}{\partial \beta}\right) \\ &\quad - \frac{\partial}{\partial \rho}\left(\frac{2.4\sigma\eta\tau}{nS_x(\tau)}\frac{\partial \tau}{\partial \rho}\right) \\ &\quad = -\sigma n\eta S_z(\tau) - \sigma_{\mathrm{H}} n\eta S_{\mathrm{H}}(\tau) - \sigma_I n S_I(\tau). \end{aligned} \tag{71}$$

Equations (36), (37), and (68)–(71) are the reduced fluid neutral equations that consist of a fifth-order system of six equations for the six unknowns n, η, $v_\parallel$, u, w, and τ. For the orderings $\partial/\partial\beta \sim 1 \sim \partial/\partial\rho$, $n \sim 1 \sim \eta$, $v_\parallel^2 \ll \tau \sim 1$, $\sigma \ll 1$, $b \ll 1$, $k\sigma \ll 1$, and $\mu \sim 1$, we see that $bv_\parallel \sim u \sim w \sim kb^2 \sim \sigma_{\mathrm{H}} \sim \sigma_1 \ll 1$. The 1-D version of this reduced system of equations with the thermal force and impurity radiation neglected is investigated in Ref. 5. To complete the reduced fluid neutral description, we need the five boundary conditions given by the Bohm sheath criterion, Eq. (53); complete recycling, Eq. (56); the upstream pressure and heat flux,

$$2n\tau|_{\beta = L/l_n} = S_p(\rho l_n/\Delta_p), \tag{72}$$

$$kb^2\tau^{5/2}\frac{\partial \tau}{\partial \beta}\bigg|_{\beta = L/l_n} = Q_e S_e\left(\frac{\rho l_n}{\Delta_p}\right); \tag{73}$$

and the energy flux into the walls,

$$\begin{aligned} &\frac{5}{2}\alpha b n_d \tau_d^{3/2} + \left(kb^2\tau_d^{5/2} + \frac{2.4\sigma\eta_d\tau_d}{n_d S_x(\tau_d)}\right)\frac{\partial \tau}{\partial \beta}\bigg|_d + 0.24\eta_d\tau_d(u_d \\ &\quad - bv_{\parallel d}) = \alpha b(\gamma_e + \gamma_i) n_d \tau_d^{3/2} + \gamma_n \eta_d \tau_d^{3/2}, \end{aligned} \tag{74a}$$

$$-\frac{2.4\sigma\eta_s\tau_s \partial\tau}{n_s S_x(\tau_s)\,\partial\rho}\bigg|_s = \gamma_n\eta_s\tau_s^{3/2}. \tag{74b}$$

Notice that Eqs. (72) and (73) follow from (50) and (51), while Eq. (74) follows from (55) and (58). Equation (49) is no longer needed, since it follows from integrating (70) and using (72), and Eqs. (54) and (57) are not needed because of the neglect of viscous effects.

Equations (36), (37), (68)–(71), (53), (56), and (72)–(74) permit only two independent scale transformations:

$$\begin{aligned} \text{(i)}\quad & \sigma \to \omega_1\sigma, \quad b \to \omega_1 b, \quad u \to \omega_1 u, \quad w \to \omega_1 w, \\ & k \to k/\omega_1, \quad \sigma_{\mathrm{H}} \to \omega_1\sigma_{\mathrm{H}}, \quad \sigma_I \to \omega_1\sigma_I, \\ & Q_e \to \omega_1 Q_e, \quad \gamma_n \to \omega_1\gamma_n; \end{aligned}$$

and

$$\text{(ii)}\quad v_\parallel \to \omega_2 v_\parallel, \quad b \to b/\omega_2, \quad k \to \omega_2^2 k, \quad \alpha \to \omega_2\alpha. \tag{75}$$

In this case the poloidal energy flux onto the plates, q_t, depends on ρ and the 12 parameters: b, k, σ, Q_e, l_n/L, σ_{H}, σ_I, α, γ_n, $\gamma_e + \gamma_i$, Δ_p/l_n, and Δ/l_n. The parameter Δ/l_n enters because of the sidewall boundary condition of Eq. (74b). Recalling that q_t scales as Q_e and proceeding as before, gives

$$\begin{aligned} q_t = b\alpha P_u\left(\frac{I}{M}\right)^{1/2} f\bigg(&\frac{l_n}{L}, \frac{bk}{\alpha}, \frac{Q_e}{b\alpha}, \frac{\sigma_1}{\sigma}, \frac{\sigma}{b\alpha}, \frac{\sigma_{\mathrm{H}}}{\sigma}, \frac{\gamma_n}{b\alpha}, \\ &\gamma_e + \gamma_i, \frac{\Delta}{L}, \frac{\Delta}{\Delta_p}, \frac{\rho l_n}{\Delta_p}\bigg). \end{aligned} \tag{76}$$

The dimensionless parameters that enter q_t are consistent (apart from notation) with those found in the one-dimensional model of Ref. 5, which neglects thermal force effects (and modifications required by Onsager symmetry), impurity radiation in its recycling region, and the parameters l_n/L, Δ/L, and Δ/Δ_p that depend on the depth and width of the divertor channel and scrape-off layer to find

$$\tau_d = f\left(\frac{b^2 k}{\sigma}, \frac{Q_e}{b\alpha}, \frac{\sigma}{b\alpha}, \frac{\sigma_{\mathrm{H}}}{\sigma}, \frac{\gamma_n}{b\alpha}, \gamma_e + \gamma_i\right), \tag{77a}$$

which is then solved to obtain the following form of the global energy balance equation:

$$Q_e = b\alpha f\left(\tau_d, \frac{b^2 k}{\sigma}, \frac{\sigma}{b\alpha}, \frac{\sigma_{\mathrm{H}}}{\sigma}, \frac{\gamma_n}{b\alpha}, \gamma_e + \gamma_i\right). \tag{77b}$$

Forming the power to the plates P for a single null divertor using $P = 4\pi R \int dx\, q_t$ to integrate Eq. (76) over the scrape-off layer width δ_p gives

$$\frac{P}{R}=b\alpha\ \Delta_p P_u\left(\frac{I}{M}\right)^{1/2} \times f\left(\frac{l_n}{L},\frac{bk}{\alpha},\frac{Q_e}{b\alpha},\frac{\sigma_I}{\sigma},\frac{\sigma}{b\alpha},\frac{\sigma_{\rm H}}{\sigma},\frac{\gamma_n}{b\alpha},\gamma_e+\gamma_i,\frac{\Delta}{L},\frac{\Delta_p}{\Delta}\right). \tag{78}$$

For similar devices described by this reduced fluid neutral model, we need to keep f fixed by keeping each of the dimensionless parameters constant. To see to what extent this is possible, we note that the parameters k, σ, $\sigma_{\rm H}$, and $l_n P_u$ are constants and that α, γ_n/α, and $\gamma_e+\gamma_i$ do not vary significantly. For a specified depth L we adjust P_u, b, Q_e, and σ_I to hold $l_n/L\propto 1/LP_u$, bk/α, $Q_e/b\alpha$, and σ_I/σ fixed, then we must adjust Δ and Δ_p to keep Δ/L and Δ/Δ_p constant. Therefore, we must keep

$$P/R\propto b\alpha\ \Delta_p P_u\propto\Delta_p/l_n=\text{const.} \tag{79}$$

Consequently, similarity for this fluid neutral model recovers Lackner's[6] result and requires the same b, Q_e (recall that $Q_e\propto q_{e\|}^{\rm up}/P_u$), $l_n/L\propto 1/LP_u$, σ_I, Δ/L, and Δ/Δ_p, as well as the same α, k, σ, $\sigma_{\rm H}$, γ_n, and $\gamma_e+\gamma_i$. For the fluid neutral model, B (or B_p) is the only adjustable quantity that can be used to satisfy either a gyroradius or beta scaling. The second column of Table I summarizes the constraints and P/R scaling (which, of course, can be reexpressed using the constraints) for this model.

It is important to note that the constant Δ_p/l_n must be much larger than unity for the fluid neutral model since the SOL thickness must be large compared to the neutral penetration scale length. As a result, the larger P/R, the more the neutrals behave like a fluid. Except for the Alcator-C-MOD,[11] current machines[12–14] are in the opposite limit of Knudsen neutrals. However, there is recent experimental evidence from the Alcator-C-MOD that the detached divertor operation observed is insensitive to the depth[16] L. If this observation is confirmed it would mean that f is insensitive to L in detached regimes (as in the 1-D model of Ref. 5), then the parameters l_n/L and Δ/L in f could be replaced by the single L independent parameter Δ/l_n or Δ_p/l_n. Recall that in the model without neutrals, the insensitivity to L occurred whenever the radiation losses were sufficiently localized.

We can relax the P/R scaling of this neutral fluid model by considering the one-dimensional limit of Eqs. (36), (37), (68)–(71), (53), (56), and (72)–(74), in which $\partial/\partial\rho=0=w$ in Eqs. (37), (69), and (71) for the SOL region $|\rho|<\Delta_p/2l_n$ ($|x|<\Delta_p/2$) and the sidewalls and the sidewall boundary condition (74b) do not enter. In this case, which includes the model of Ref. 5, Δ cannot enter q_t and P/R. Moreover, the scaling transformation (ii) of Eq. (61) is allowed, since ρ enters only through shape functions, which means that Δ_p does not enter the unknown function in P/R and only enters q_t via the combination $\rho l_n/\Delta_p$. In addition, the divertor is assumed infinitely deep ($L\rightarrow\infty$) and impurity radiation is assumed to occur upstream so that L and σ_I do not enter. As a result, for the one-dimensional neutral fluid case,

$$q_t=b\alpha P_u\left(\frac{I}{M}\right)^{1/2} f\left(\frac{bk}{\alpha},\frac{Q_e}{b\alpha},\frac{\sigma}{b\alpha},\frac{\sigma_{\rm H}}{\sigma},\frac{\gamma_n}{b\alpha},\gamma_e+\gamma_i,\frac{\rho l_n}{\Delta_p}\right), \tag{80a}$$

$$\frac{P}{R}=b\alpha\ \Delta_p P_u\left(\frac{I}{M}\right)^{1/2} f\left(\frac{bk}{\alpha},\frac{Q_e}{b\alpha},\frac{\sigma}{b\alpha},\frac{\sigma_{\rm H}}{\sigma},\frac{\gamma_n}{b\alpha},\gamma_e+\gamma_i\right), \tag{80b}$$

and, upon choosing b and Q_e to hold the unknown function f fixed (recall k, σ, $\sigma_{\rm H}$, and $l_n P_u$ are constants, and α and the γ's do not vary significantly),

$$P/R\propto\Delta_p P_u\propto\Delta_p/l_n. \tag{80c}$$

Unlike the 2-D case, Δ_p/l_n need not be a constant for similar devices in the 1-D case. Therefore, a 1-D model of a SOL of width Δ_p allows a device similar to ITER to have a much smaller P/R and $\Delta_p P_u\propto\Delta_p/l_n$ as in the discussion following Eq. (67).

If we include the inertial terms in our reduced neutral fluid model, then we lose the second scaling transformation in Eq. (75) since $v_\|^2$ and τ must scale in the same way. As a result, α must be held constant for similarity since it appears as a separate parameter in the argument of f. Since $\alpha\approx$const to satisfy the Bohm sheath criterion, the modified f is effectively the same as Eq. (78) and, therefore, leads to the same conclusions.

If, in addition, we keep ion heat conduction and neutral and ion viscosity, but assume $b^2\ll 1$, we need only supplement the first scaling transformation in Eq. (75) by $\mu\rightarrow\mu/\omega_1$, $\gamma_m\rightarrow\omega_1\gamma_m$, and $Q_i\rightarrow\omega_1 Q_i$. Then the only allowed scaling transformation gives Eq. (78) with the additional dimensionless parameters α, $b\mu$, γ_m/b, $\gamma_\|$, and Q_i/b appearing in the argument of the unknown function. Since μ is a constant and γ_m and $\gamma_\|$ do not vary significantly, and b had to be held constant to keep f fixed (note that bk is one of the arguments of f and k is a constant), viscosity and ion heat conduction only alter our conclusions from Eqs. (76) and (78) by requiring that each of the dimensionless upstream electron and ion heat fluxes, Q_e and Q_i, respectively, be held fixed in similar devices.

Finally, if we attempt to keep the ionization terms as well as the charge exchange terms in the perpendicular neutral momentum balance equations, (38) and (39), we will also lose the first scaling transformation in Eq. (75) because $\sigma\rightarrow\sigma$. Then the unknown function in q_t will depend on the 16 dimensionless parameters σ, b, μ, k, $\sigma_{\rm H}$, σ_I, α, Q_e, Q_i, $\gamma_\|$, γ_m, $\gamma_e+\gamma_i$, γ_n, Δ/L, Δ/Δ_p, and L/l_n, as well as $\rho l_n/\Delta_p$. Since σ, μ, k, $\sigma_{\rm H}$, α, $\gamma_\|$, γ_m, $\gamma_e+\gamma_i$, and γ_n are either constants or are unable to vary significantly, the additional ionization terms have no significant impact on similarity, since only the same b, σ_I, Q_e, Q_i, Δ/L, Δ/Δ_p, and L/l_n are required.

C. General fluid neutral model

The general fluid neutral model system of equations described in Sec. III cannot allow any scaling transformations since none are permitted for the reduced system mentioned at the end of Sec. IV B. Neglecting ion and neutral viscosity, inertial terms, and ionization in the perpendicular neutral flow equations does not help because of the neutral–ion momentum exchange due to charge exchange and the ion–electron energy equilibration terms. Consequently, for the general neutral fluid system no simplification occurs, so

$$\frac{P}{R}=\Delta_p P_u\left(\frac{I}{M}\right)^{1/2} f\left(\sigma,b,\mu,k,\sigma_{\rm H},\sigma_I,\alpha,Q_e,Q_i,\right.$$

$$\left.\gamma_\parallel,\gamma_m,\gamma_e,\gamma_i,\gamma_n,\frac{\Delta}{L},\frac{\Delta}{\Delta_p},\frac{L}{l_n}\right), \tag{81}$$

with Q_e and Q_i and γ_e and γ_i appearing separately rather than as sums. In this case similarity requires the same b, σ_I, Δ/L, Δ/Δ_p, $L/l_n \propto P_u L$, Q_e, and Q_i (with σ, μ, k, $\sigma_{\rm H}$, α, $\gamma_\parallel$, γ_m, γ_e, γ_i, and γ_n either constants or unable to vary significantly), and, not surprisingly, leads to Lackner's[8] $P/R=$const result. Therefore, the general fluid neutral model introduces additional parameters, but leads to essentially the same conclusions as the reduced fluid neutral model.

V. SCALING TRANSFORMATIONS AND SIMILARITY FOR KNUDSEN NEUTRALS

Reference 5 also considers a deep divertor slot geometry with a complete recycling model of the neutrals in which the neutral mean-free path is long compared to the divertor width and the Coulomb mean-free path is assumed small compared to parallel scale lengths. This idealized limit, in which charge exchange is retained and the long mean-free path neutrals are randomized by collisions with the walls, is referred to as the Knudsen neutral model or Knudsen flow approximation in Ref. 5. For this model the depth of the narrow divertor slot L must be much larger than its width Δ so that nearly all of the neutrals created at the target by the recombining ions can stream to the sidewalls to be randomized within a few Δ's of the target and well before they reach the upstream divertor entrance. As a result, the neutrals are assumed to be uniformly distributed in x (as well as z) for $\Delta \ll y < L$, with no poloidal or toroidal flow. The randomizing wall collisions are assumed to result in a diffusive poloidal neutral flow,

$$N_n \mathbf{V}_n=-\hat{y} D\,\frac{\partial N_n}{\partial y}, \tag{82a}$$

where the diffusion coefficient D is given by

$$D=cV\Delta, \tag{82b}$$

with Δ and V the characteristic step in y and speed between randomizing collisions at the walls and c an order unity numerical coefficient, which depends on the properties and conditioning of the walls. The speed V is set by the neutral temperature, which in this model is small compared to the ion and electron temperatures. Equation (82) is then inserted into Eq. (2) to obtain the neutral continuity equation. To make D dimensionless we introduce the dimensionless diffusivity d, defined by

$$d=\frac{cV\Delta}{l_n(I/M)^{1/2}}. \tag{82c}$$

The remaining equations for the Knudsen neutral model are ion continuity, Eq. (1) with Eq. (4) inserted; parallel plasma momentum balance as obtained from Eq. (6) by setting $V_{n\parallel}=0$, neglecting inertia, and dropping the thermal force term; and total plasma energy conservation as obtained by assuming $T_e \approx T$, neglecting inertia, and adding Eqs. (11) and (16) together with the neutral flow and neutral heat flux terms ignored. If we neglect ion heat conduction and viscosity the dimensionless form of the Knudsen neutral model equations is as follows:

$$d\,\frac{\partial^2 \eta}{\partial \beta^2}=\sigma n\,\eta S_z(\tau), \tag{83}$$

$$b\,\frac{\partial}{\partial \beta}\,(n v_\parallel)=\sigma n\,\eta S_z(\tau), \tag{84}$$

$$\sigma b\,\frac{\partial}{\partial \beta}\,(2n\tau)=-n\,\eta v_\parallel S_x(\tau), \tag{85}$$

$$b\,\frac{\partial}{\partial \beta}\left(5\tau n v_\parallel - k b \tau^{5/2}\,\frac{\partial \tau}{\partial \beta}\right)$$

$$=-\sigma n\,\eta S_z(\tau)-\sigma_{\rm H} n\,\eta S_{\rm H}(\tau)-\sigma_1 n S_I(\tau). \tag{86}$$

This sixth-order system of equations in the four unknowns n, η, $v_\parallel$, and τ is employed with the three upstream boundary conditions given by Eqs. (49), (72), and (73), and the three downstream boundary conditions given by Eq. (53) and the appropriately modified versions of (56) and (74a):

$$-d\,\left.\frac{\partial \eta}{\partial \beta}\right|_d \Delta=\alpha b\int_0^{\Delta_p} dx\; n_d \tau_d^{1/2}$$

$$=\alpha b \Delta_p \int_0^1 d\left(\frac{\rho l_n}{\Delta_p}\right) n_d \tau_d^{1/2}, \tag{87}$$

$$5\alpha b n_d \tau_d^{3/2}+k b^2 \tau_d^{5/2}\left.\frac{\partial \tau}{\partial \beta}\right|_d=\alpha b(\gamma_e+\gamma_i) n_d \tau_d^{3/2}. \tag{88}$$

The three downstream boundary conditions are applied a few Δ's from the target in order for the diffusive model of Eqs. (82) to be valid. Equation (87) is obtained by demanding complete recycling, and recalling that the neutral distribution is uniform across the entire divertor channel of width Δ, while the plasma is localized to the SOL of width Δ_p. Notice that Δ enters through d, as well as Eq. (87), but not through any sidewall boundary conditions, and recall that $\beta = L/l_n$ at the upstream entrance.

The Knudsen neutral model, consisting of Eqs. (49), (53), (72), (73), and (83)–(88), contains the 12 dimensionless parameters d, σ, b, k, $\sigma_{\rm H}$, σ_I, L/l_n, Q_e, α, $\gamma_e+\gamma_i$, Δ_p/l_n, and Δ_p/Δ, and permits the following three scale transformations:

$$\text{(i)}\quad \sigma\to\omega_1\sigma,\quad v_\parallel\to\omega_1 v_\parallel,\quad k\to\omega_1 k,$$
$$\sigma_{\rm H}\to\omega_1\sigma_{\rm H},\quad \sigma_I\to\omega_1\sigma_I,\quad d\to\omega_1 d,$$
$$Q_e\to\omega_1 Q_e,\quad \alpha\to\omega_1\alpha;$$
$$\text{(ii)}\quad b\to\omega_2 b,\quad \beta\to\omega_2\beta,\quad d\to\omega_2^2 d,$$
$$Q_e\to\omega_2 Q_e,\quad L/l_n\to\omega_2 L/l_n;$$

and

$$\text{(iii)}\quad \rho\to\omega_3\rho,\quad \Delta_p/l_n\to\omega_3\Delta_p/l_n. \tag{89}$$

These transformation reduce the number of independent dimensionless parameters by three and the procedure of Sec. IV A gives

$$q_t=b\alpha P_u\left(\frac{I}{M}\right)^{1/2} f\left(\frac{bl_n}{L},\frac{k}{\alpha},\frac{Q_e}{b\alpha},\frac{\sigma_I}{\sigma},\frac{\sigma}{\alpha},\frac{\sigma_{\rm H}}{\sigma},\frac{d}{\alpha b^2},\gamma_e+\gamma_i,\frac{\Delta_p}{\Delta},\frac{\rho l_n}{\Delta_p}\right) \tag{90}$$

and

$$\frac{P}{R}=b\alpha\,\Delta_p P_u\left(\frac{I}{M}\right)^{1/2} f\left(\frac{bl_n}{L},\frac{k}{\alpha},\frac{Q_e}{b\alpha},\frac{\sigma_I}{\sigma},\frac{\sigma}{\alpha},\frac{\sigma_{\rm H}}{\sigma},\frac{d}{\alpha b^2},\gamma_e+\gamma_i,\frac{\Delta_p}{\Delta}\right). \tag{91}$$

The new parameter in the unknown function is the effective diffusivity $d/\alpha b^2$ (or $l_n d/\alpha bL\propto cV\Delta/\alpha bL$), which replaces the $\gamma_n/b\alpha$ (or $\gamma_n\Delta/b\alpha L$) parameter of the fluid neutral model (cV replaces γ_n). Moreover, for the deep slot Knudsen model, Δ/L no longer enters and the b dependence is altered (β scales the same way as b) from the fluid neutral model, since the neutrals are no longer strongly coupled to the ions by charge exchange. The other parameters σ/α, $\sigma_{\rm H}/\sigma$, k/α, and $\gamma_e+\gamma_i$ in the argument of f for P/R must also be held constant. Therefore, if we adjust P_u, Δ, Δ_p, Q_e, and σ_I to keep bl_n/L, $d/\alpha b^2\propto(cV\,\Delta P_u/\alpha b^2)$, Δ_p/Δ, $Q_e/b\alpha$, and σ_I/α fixed for a specified L, we obtain the scaling

$$P/R\propto b\alpha\,\Delta_p P_u\propto b^2\Delta_p/L\propto(cV)^2(\Delta_p/L)^3 \propto(cV)^{1/2}(\Delta_p/l_n)^{3/2}, \tag{92}$$

where we have used $P_u\propto 1/l_n\propto b/L$ and $b\propto(cV\Delta_p/\alpha l_n)^{1/2}$, and α must be kept constant since σ and k are constants. As a result, P/R is not a constant for the Knudsen fluid model, which requires $\Delta_p/l_n\ll 1$, and we are still free to adjust cV and b, as well as B, to satisfy other constraints. The constraints and P/R scaling for the Knudsen model are summarized in the third column of Table I. Keeping the inertial corrections in the Knudsen neutral model results in the loss of scaling transformation (ii) and makes it necessary to keep α fixed for similarity, so results in no significant change.

VI. DISCUSSION

Based on the model without neutrals and the fluid neutral models considered here, the collisionality, upstream parallel heat flux, and P/R constraints on ITER divertor similarity are difficult to simultaneously satisfy in present tokamaks. The model without neutrals has the least restrictive P/R scaling and the fluid neutral model the most constraints. The less restrictive and more favorable scaling of the Knudsen model means that the constant P/R constraint found for fluid neutrals is relaxed, as the neutrals make the transition from $\Delta_p/l_n\gg 1$ to $\Delta_p/l_n\ll 1$, where Δ_p and l_n are the scrape-off layer width and the characteristic neutral penetration depth [recall Eq. (33)], respectively. Consequently, lower-density machines may have a less constrained P/R scaling than higher-density ones. In either limit, however, a severe similarity constraint arises from collisionality. For a given $b=B_p/B$, similar collisionality requires P_uL=const and therefore deep divertor chambers at the lower upstream pressures of present tokamaks. However, in light of recent experimental results from the Alcator-C-MOD[16] indicating that divertor operation during detachment is insensitive to the divertor depth L, it is tempting to speculate that it may be possible to ignore the L dependence of f in Eq. (91) and replace the l_n/L and Δ/L dependences of the unknown function f in Eq. (78) by Δ/l_n. Even if this is not the case, the database from the present machines, coupled with the key parameters found here and in divertor modeling codes, might be used to construct a power law form for the unknown function in the P/R scaling law. The constraint on the upstream parallel heat flux is less severe because it is divided by P_u [recall the definition of Q_e from Eq. (48)]. The techniques employed herein can be used to determine the key parameters for the equations and boundary conditions solved in the modeling codes.

Generally speaking, recombination only plays an important role in plasma–neutral interactions when detachment is achieved.[17] Therefore, the threshold for detachment can be obtained by neglecting recombination as in the models considered here (and as shown in detail in one dimension in Ref. 5). In very low-temperature regimes when recombination must be retained, there are two possible mechanisms: two body via the molecular channel and three body. When two-body recombination is substantially enhanced by the presence of molecular hydrogen at moderate plasma densities,[18] our scalings can be recovered, since an additional dimensionless parameter proportional to the effective rate constant of the new reactions appears in the arguments of our unknown functions. At high densities three-body recombination must be retained and we can no longer scale densities to the upstream pressure. Therefore, we must require the same density, as well as temperature, profiles along and across the magnetic field in similar devices. Such a requirement on the density would be overly restrictive when recombination is negligible and when it does not alter the plasma and neutral behavior upstream of the localized, low ($<$2 eV) temperature, high plasma density region in which it must be retained. Consequently, three-body recombination was neglected in the interest of simplicity, even though there are indications that it plays an important role in the low-temperature, high-density regions downstream of the ionization front.[17]

The list of dimensionless parameters given for each of the models is not intended to be exhaustive. For more complicated divertor geometries, additional geometric parameters must be introduced. Moreover, in modeling impurity radiation in the electron energy balance equation, we assumed for simplicity that the impurity density could be written as a specified function of the plasma temperature, thereby restricting impurity similarity. This assumption is appropriate when the impurity pressure gradient along the magnetic field is balanced by the thermal force between the impurities and background plasma. In a more general model, the impurity density would be governed by a complicated set of impurity conservation equations that would introduce a large number of additional dimensionless parameters associated with the impurities, which would have to be matched to ensure full impurity similarity.

The upstream heat flux entering the divertor also places a severe constraint on the similarity of present tokamak divertors to ITER. However, for the various models considered, only the parallel heat flux must be matched. Since a divertor simulator need not be a conventional tokamak, the normalized parallel heat flux, Eq. (48), might be matched by adjusting the field line angle in a toroidal device to make $b=B_p/B$ smaller for a fixed poloidal heat flux. For a nonconventional tokamak simulator there is much more flexibility in making the collisionality and other divertor and geometrical parameters similar (for a simulator of SOL length £ the replacement $R\rightarrow £/4\pi$ is made in P/R).

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