

Influence of higher-order harmonics on the saturation of the tearing mode

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Abstract. The nonlinear saturation of the tearing mode is revisited in slab geometry by taking into account higher-order harmonics in the outer solution. The general formalism for tackling this problem in the case of a vanishing current gradient at the resonant surface is derived. It is shown that, although the higher-order harmonics lead to corrections in the final saturation equation, they are of higher order in the perturbation parameter, which provides a formal proof that the standard one-harmonic approach is asymptotically correct.

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1. Introduction

The tearing mode [1] is a resistive magnetohydrodynamic (MHD) instability that resonates on magnetic surfaces where the mode's wave vector is perpendicular to the magnetic field. It leads to the formation of so-called magnetic islands around the resonant surface, resulting in a local change of magnetic topology and a greater radial transport on a length scale of the order of the islands' full width, w . Therefore, these structures are of great interest for nuclear fusion, and it is especially important to have a keen insight into their stability properties as well as their maximum achievable amplitude.

While the linear theory of the tearing mode is rather well understood, the nonlinear one, pioneered by Rutherford thirty-five years ago [2], is taking longer to unravel. Recently, a series of works have derived solutions for the saturation of the tearing mode in the framework of simple physical models [3–8], somewhat rekindling interest in this subject. Although rigorous in their mathematical details, they share a common feature based on an assumption first made by Rutherford, i.e. they neglect higher-order (poloidal) harmonics in the magnetic perturbation. However, this is not really justified *a priori*, since nonlinearities naturally couple all harmonics, and one may therefore question the validity of the afore-mentioned references.

In this work, we investigate this problem using the simplest of physical models, namely that of reduced MHD in slab geometry in the so-called symmetric case (i.e. no current gradient at the resonant surface), which is the same as that used in [3, 4]. We first derive the general solution to this problem and then explicitly solve the case with two harmonics for two different types of equilibrium. Last, we compare our results with those obtained from previous theories and draw conclusions.

2. Model equations

We first introduce the following normalizations:

$$t = \tau_\eta \tilde{t}; \quad x = L\tilde{x}; \quad y = L\tilde{y}; \quad J = J_N \tilde{J}; \quad \psi = \mu_0 J_0 L^2 \tilde{\psi}; \quad \varphi = \frac{\eta}{\mu_0} \tilde{\varphi}, \quad (1)$$

where t is the time variable, $\tau_\eta = \mu_0 L^2 / \eta$ is the resistive diffusion time, μ_0 is the permeability of free space, η is the (uniform) resistivity, x and y are the radial and poloidal variables respectively, L is a characteristic radial length, $J_N = J_{eq}(0)$, J (resp. J_{eq}) is the (resp. equilibrium) current density, ψ is the magnetic flux function (i.e. $\mathbf{B} \equiv B_z \mathbf{e}_z + \nabla \times (\psi \mathbf{e}_z)$, where \mathbf{B} is the magnetic field and \mathbf{e}_z is a unit vector perpendicular to the xy plane) and φ is the electric potential and plays the role of the (ion) stream function (i.e. $\mathbf{v} \equiv \mathbf{e}_z \times \nabla \varphi$, where \mathbf{v} is the velocity field). Note that these normalizations are such that it is the equilibrium current density, and *not* the equilibrium magnetic field, that is normalized to unity at the resonant surface. We then use the reduced MHD equations in slab geometry [9] which, taking into account the normalizations while omitting the “ \sim ” for clarity, read:

$$\partial_t \Delta_\perp \varphi + [\varphi, \Delta_\perp \varphi] = S^2 [J, \psi] + \frac{S}{Re} \Delta_\perp^2 \varphi \quad (2)$$

$$\partial_t \psi + [\varphi, \psi] = J_{eq} - J \quad (3)$$

$$J = -\Delta_\perp \psi, \quad (4)$$

where $S = v_A L \mu_0 / \eta$ is the Lundquist number, $Re = v_A L / \nu$ is the Reynolds number, $v_A = J_N L \sqrt{\mu_0 / \rho}$ is the Alfvén speed, ν is the viscosity and ρ is the mass density (ν and ρ are assumed to be constant). The Poisson brackets are given by $[f, g] \equiv \partial_x f \partial_y g - \partial_x g \partial_y f$. Finally, the resonant surface is conveniently set at the origin by letting $\psi'_{eq}(0) = 0$, where the equilibrium magnetic flux function satisfies $\psi''_{eq} = -J_{eq}$.

3. Perturbed equilibrium

Provided there are no equilibrium flows in the xy plane and given the fact that $S, Re \gg 1$, as is the case in present-day tokamak plasmas, the equation of motion (2) simply yields:

$$[J, \psi] = 0. \quad (5)$$

We then look for a perturbed equilibrium of the form:

$$\psi \sim \psi_{eq}(x) + \sum_{n \geq 1} \delta_n^2(t) \psi_n(x) \cos [n\chi + \beta_n(t)], \quad (6)$$

where we have defined $\chi \equiv ky$, k being the mode's wave-number and, throughout this paper, ‘ \sim ’ has the meaning ‘equals plus higher order terms’. Note that, without loss of generality, it is possible to choose $\beta_1 = 0$, which will henceforth be the case. Substituting this expression into (5), the ψ_n satisfy:

$$\psi_n'' + \left[\frac{J'_{eq}}{\psi'_{eq}} - (nk)^2 \right] \psi_n = 0. \quad (7)$$

In the following, we suppose that the first harmonic dominates the others at all times, i.e. that $\delta_1 \geq \delta_{n>1}$. This assumption certainly makes sense in the linear regime,

since the first harmonic is the most unstable one [1], but it also has to be consistent with the final saturation result, which will be shown to be the case later on. Based on the work already done in e.g. [7], one can then infer from (3) that there is a boundary layer of width δ_1 centered on $x = 0$ and that one thus has to resort to the technique of asymptotic matching. Writing $J_{eq} \sim 1 + a_2 x^2$, choosing $\psi_{eq}(0) = 0$ for convenience and solving (7), we derive the following expansion for the *outer* solution:

$$\begin{aligned} \zeta_{out} \sim & \frac{\xi^2}{2} - \sum_{n \geq 1} \alpha_n \cos(n\chi + \beta_n) \\ & - \delta_1 |\xi| \sum_{n \geq 1} \alpha_n \frac{\Delta'_n}{2} \cos(n\chi + \beta_n) \\ & + \delta_1^2 \left\{ a_2 \frac{\xi^4}{12} - \sum_{n \geq 1} \alpha_n \left[a_2 + \frac{(nk)^2}{2} \right] \xi^2 \cos(n\chi + \beta_n) \right\}, \end{aligned} \quad (8)$$

where we have defined $\zeta_{out} \equiv -\psi/\delta_1^2$, $\alpha_n \equiv (\delta_n/\delta_1)^2$ and the inner variable $\xi \equiv x/\delta_1$ in order to make the ordering in the small parameter δ_1 explicit. As to the linear stability parameters, Δ'_n , they are related to the logarithmic jump of the ψ_n around the resonant surface [1], namely

$$\Delta'_n = \lim_{\epsilon \rightarrow 0^+} [\psi'_n(\epsilon) - \psi'_n(-\epsilon)]/\psi_n(0), \quad (9)$$

and depend on both the equilibrium current density profile and the boundary conditions.

The solution (8) holds in the so-called outer region of the plasma where ideal MHD is a good approximation to our set of equations, but breaks down around the resonant surface where resistivity has to be taken into account. The solution that is valid in this resistive boundary layer is the *inner* one which we derive in the next section. It ultimately has to be matched to (8) that, in effect, is analogous to a boundary condition at infinity (i.e. $|\xi| \rightarrow \infty$). At this stage, it is perhaps worthwhile to stress once more that it is the $n > 1$ terms in the sums appearing in (8) that were neglected in previous works.

4. Solution in the inner region

4.1. Inner equations

The inner equations are obtained by re-writing (2)-(4) with respect to the inner variable ξ :

$$[\zeta, J] = 0 \quad (10)$$

$$-\delta_1^2 \partial_t \zeta + \delta_1 \partial_t \delta_1 (\xi \partial_\xi \zeta - 2\zeta) + k \delta_1 [\zeta, \varphi] \sim 1 + a_2 \delta_1^2 \xi^2 - J \quad (11)$$

$$J = \partial_\xi^2 \zeta + k^2 \delta_1^2 \partial_\chi^2 \zeta, \quad (12)$$

where the Poisson brackets are now taken with respect to the (ξ, χ) variables. Note that (10) is valid in the nonlinear regime only, where the island is supposed to be larger than the resistive and visco-resistive layer widths [2]. This set of equations is then classically solved using perturbation expansions in powers of δ_1 , i.e. writing $\zeta = \sum_{l \geq 0} \delta_1^l \zeta_l$, $\varphi = \sum_{l \geq 0} \delta_1^l \varphi_l$, and $J = \sum_{l \geq 0} \delta_1^l J_l$.

4.2. Order δ_1^0

Ohm's law simply gives $J_0 = 1$, and the integration of (12) together with the matching condition provided by (8) at this order yields:

$$\zeta_0 = \xi^2/2 - \sum_{n \geq 1} \alpha_n \cos(n\chi + \beta_n). \quad (13)$$

4.3. Order δ_1

Equation (10) implies that the order 1 component of J is a function of ζ_0 only on either side of the resonant surface, i.e. $J_1(\xi, \chi) = j_1(\zeta_0; \pm)$, where we have defined $\pm \equiv \text{sign}(\xi)$. It is therefore easier to work in $(\zeta_0, \chi; \pm)$ variables, which will be the case in the following. Ohm's law then yields:

$$2 \sum_{n \geq 1} \partial_t \delta_n \sqrt{\alpha_n} \cos(n\chi + \beta_n) + k\xi \partial_\chi \varphi_0|_{\zeta_0} = -j_1(\zeta_0; \pm). \quad (14)$$

In order to solve this equation, it is convenient to define, for any function f , its flux average $\langle f \rangle$ as:

$$\left\{ \begin{array}{l} \int_{-\pi}^{\pi} d\chi f(z, \chi; \pm) / \xi \quad \text{if } z \geq \zeta_{sep} \equiv \zeta_0(0, \pi) \\ 1/2 \sum_{\sigma = \pm} \sigma \int_{-\chi_0}^{\chi_0} d\chi f(z, \chi; \sigma) / \xi \quad \text{if } z \leq \zeta_{sep} \end{array} \right., \quad (15)$$

where $\chi_0 \in [0, \pi]$ is the turning point of the corresponding flux surface, i.e. it satisfies $\zeta_0(0, \chi_0) = z$, and, in this expression, ξ has to be taken as a function of $(z, \chi; \pm)$, i.e. $\xi = \pm \sqrt{2} [z + \sum_{n \geq 1} \alpha_n \cos(n\chi + \beta_n)]^{1/2}$. With this definition in mind, it is then easy to show that the solution to (14) reads:

$$j_1 = -2 \sum_{n \geq 1} \partial_t \delta_n \sqrt{\alpha_n} \langle \cos(n\chi + \beta_n) \rangle / \langle 1 \rangle. \quad (16)$$

4.4. Order δ_1^2

The calculation is similar to that of the previous section except that we now neglect all terms dependant on $\partial_t \delta_1$ since they would only lead to higher order corrections in the final result. Consequently, (3) gives:

$$- \sum_{n \geq 1} \alpha_n \partial_t \beta_n \sin(n\chi + \beta_n) + k\xi \partial_\chi \varphi_0|_{\zeta_0} = a_2 \xi - j_2(\zeta_0; \pm), \quad (17)$$

which, after applying the bracket $\langle \dots \rangle$ operator on both sides, yields:

$$j_2 = \left[a_2 \langle \xi^2 \rangle + \sum_{n \geq 1} \alpha_n \partial_t \beta_n \langle \sin(n\chi + \beta_n) \rangle \right] / \langle 1 \rangle. \quad (18)$$

We stop the inner calculation here since it is the lowest relevant order. Indeed, we shall see in the next section that the asymptotic matching conditions on the Δ'_n terms in (8) already provide a saturation theory for the tearing mode at that order.

5. Asymptotic matching conditions

When taking the asymptotic expansion of the inner solution derived in Section 4 for $|\xi| \rightarrow \infty$, one shows that the matching with the outer solution given by (8) provides the following conditions (see, e.g., [7] for more details on the asymptotic matching procedure):

$$-\frac{2}{\pi} \int_{\zeta_{min}}^{+\infty} \frac{d\zeta_0}{\langle 1 \rangle} \langle \cos(m\chi + \beta_m) \rangle E(\zeta_0) \sim \alpha_m \Delta'_m \quad (19)$$

$$\int_{\zeta_{min}}^{+\infty} \frac{d\zeta_0}{\langle 1 \rangle} \langle \sin(m\chi + \beta_m) \rangle E(\zeta_0) \sim 0, \quad (20)$$

where $\zeta_{min} \equiv \zeta_0(0, 0)$ and $E(\zeta_0)$ is given by:

$$E = j_1 + \delta_1 \sum_{n \geq 1} \alpha_n \langle 2a_2 \cos(n\chi + \beta_n) + \partial_t \beta_n \sin(n\chi + \beta_n) \rangle. \quad (21)$$

Since we want to focus on the saturation of the mode, it is possible to simplify this set of equations. Indeed, (20) implies that, at saturation, $\beta_n \in \{0, \pi\}$ for all $n > 1$ (recall that $\beta_1 = 0$ by assumption). Therefore, we can do away with the β_n 's altogether provided we allow the α_n 's to be either positive or negative (except for α_1 that is always equal to 1). If we do this, (20) is automatically met and, letting $\partial_t = 0$, (19) can be re-written as:

$$\alpha_m \Delta'_m + a_2 \delta_1 \sum_{n \geq 1} \alpha_n F_{mn}(\{\alpha_i\}) \sim 0, \quad (22)$$

where we have defined

$$F_{mn}(\{\alpha_i\}) = \frac{4}{\pi} \int_{\zeta_{min}}^{+\infty} \frac{d\zeta_0}{\langle 1 \rangle} \langle \cos m\chi \rangle \langle \cos n\chi \rangle. \quad (23)$$

Note that, trivially, $F_{mn} = F_{nm}$. The set of equations given by (22) provides a general theory for the nonlinear saturation of the full island width $w_s \equiv 4\delta_1 (\sum_{n \geq 0} \alpha_{2n+1})^{1/2}$ when taking into account any number of harmonics in the outer solution.

Although it is not possible to solve it analytically, we make an important comment about this result. Indeed, if we relax Rutherford's original assumption (namely $-\Delta'_n \gg 1$ for $n > 1$) [2], and instead (more reasonably) assume $\Delta'_{n>1}$ to be of order one, then (22) implies $\alpha_{n>1} = O(a_2 \delta_1 / \Delta'_n) = O(\delta_1)$. Consequently, the higher-order harmonics only lead to a higher order correction in the saturation equation, i.e.:

$$\Delta'_1 + a_2 \delta_1 \left(F_{11} + \sum_{n>1} \alpha_n F_{1n} \right) = \Delta'_1 + a_2 \delta_1 F_{11} + O(\delta_1^2) \sim 0, \quad (24)$$

showing that the standard one-harmonic approach is asymptotically correct in the limit of vanishing δ_1 . Incidentally, the fact that $\alpha_{n>1} = O(\delta_1)$ also ensures that our original assumption, namely that the first harmonic dominates over the higher ones, is consistent. In order to demonstrate this result quantitatively, we now illustrate our theory through the simplest case of two harmonics and compare the outcome with that of [3, 4].

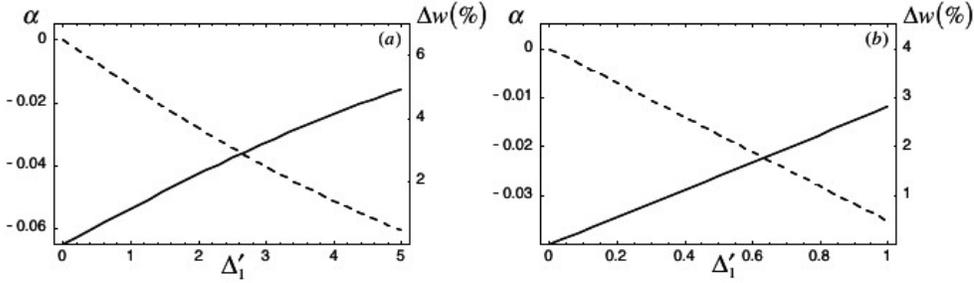


Figure 1. Plot of Δw (—) and α (---) as a function of Δ'_1 for the cosh (a) and sheet pinch (b) equilibria.

6. Corrections due to the second harmonic for two specific equilibria

Taking into account the first two harmonics only, (22) gives:

$$4\Delta'_1 + a_2 w_s (F_{11} + \alpha F_{12}) = 0 \quad (25)$$

$$4\alpha \Delta'_2 + a_2 w_s (F_{21} + \alpha F_{22}) = 0, \quad (26)$$

where $w_s = 4\delta_1$ and we have written $\alpha = \alpha_2$ for the sake of clarity. Since α is expected to be small, it makes sense to Taylor expand the F coefficients to first order as $F_{mn} \sim A_{mn} + \alpha B_{mn}$. It is then straightforward to solve for w_s and α :

$$w_s = \frac{-4\Delta'_1}{a_2 [A_{11} + \alpha(B_{11} + A_{12})]} ; \alpha = \frac{A_{12}\Delta'_1}{\Delta'_2 A_{11} - \Delta'_1 (B_{12} + A_{22})}, \quad (27)$$

where $A_{11} \simeq 3.29$, $B_{11} \simeq 2.22$, $A_{12} \simeq 0.34$, $B_{12} \simeq 2.03$ and $A_{22} \simeq 1.59$ have been computed numerically.

These expressions can be evaluated provided Δ'_1 , Δ'_2 and a_2 are given. To this end, we consider two specific equilibria often used in the literature: the cosh (i.e. $\psi_{eq} = 1/(2 \cosh^2 x)$ [10], where the 1/2 factor has been included to comply with our normalization scheme) and sheet pinch (i.e. $\psi_{eq} = -\log(\cosh x)$ [11]) equilibria. The first one has $a_2 = -4$ and the other $a_2 = -1$, while the Δ'_n are given by:

$$\Delta'_n = \frac{2(5 - n^2 k^2)(3 + n^2 k^2)}{n^2 k^2 \sqrt{4 + n^2 k^2}} \text{ and } \Delta'_n = 2 \left(\frac{1}{nk} - nk \right) \quad (28)$$

respectively. The results so-obtained are then compared with the one given by standard theory, namely $\hat{w}_s = -4\Delta'_1/a_2 A_{11}$ [3,4], and the outcome is shown in figure 1, where we have plotted $\Delta w \equiv (w_s - \hat{w}_s)/\hat{w}_s$ and α for both equilibria. We see that the island is found to be slightly bigger than was predicted by previous theory, but the corrections are very small, namely up to 5% (resp. 3%) larger for the cosh (resp. sheet pinch) equilibrium with $\Delta'_1 \lesssim 5$ (resp. $\Delta'_1 \lesssim 1$). Of course, the greater Δ'_1 , the bigger these corrections can get, and, indeed, they reach up to 10% (resp. 15%). However the constant- ψ approximation breaks down for too large a Δ'_1 , so that the ranges shown in figure 1 are, in effect, appropriate as far as our asymptotic matching theory is concerned. Therefore, we conclude that the standard theory of [3,4] gives a correct result in its region of validity, which was expected since it has been confirmed numerically in [12].

We now make one last remark concerning these results. One may naturally wonder whether the neglect of the third and higher harmonics is justified, since (22) only

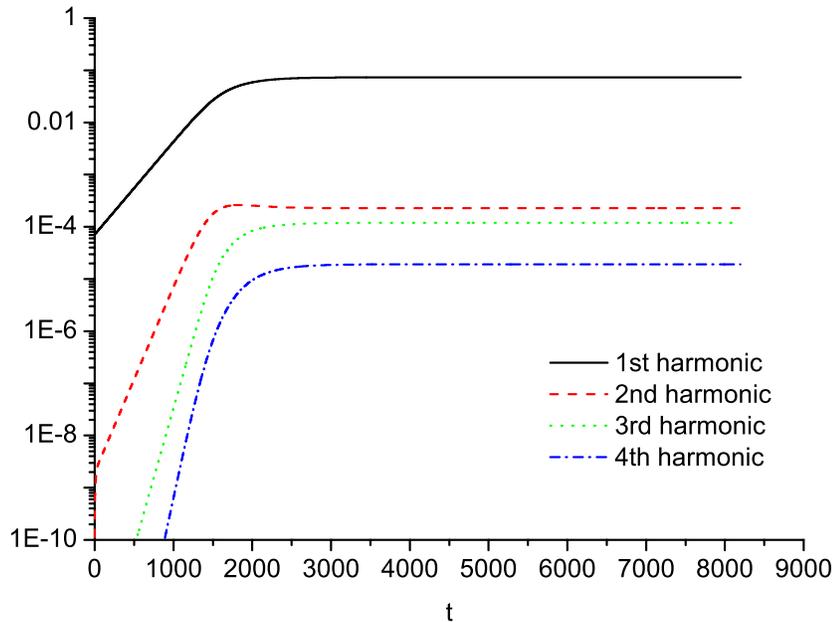


Figure 2. (colour online). Plot of the first four harmonics' amplitude versus time for the cosh equilibrium with $\Delta'_1=1.0$.

implies that all $n > 1$ harmonics are $O(\delta_1/\Delta'_n)$. To test that this is the case, we have run a full numerical simulation for the cosh equilibrium with the pseudospectral code already used in [12], taking $\Delta'_1 = 1.0$. The results are shown in figure 2, where we see that the second harmonic always has a greater amplitude than the third and fourth (as well as the higher-order ones, not shown). The reason for that is twofold: first, the nonlinear coupling to the first harmonic can be inferred to grow weaker for higher-order ones, and second, since $\Delta'_n \propto n$ for large n , the amplitude of the higher-order harmonics scales (at most) as $1/n$. Incidentally, we also see in figure 2 that the first harmonic largely dominates at all times, which gives further evidence that our original assumption (namely $\delta_1 \geq \delta_{n>1}$) is correct, at least in the regime (Δ'_1 not too large) we investigate.

7. Conclusion

In this paper, we have examined the impact of higher-order harmonics on the nonlinear evolution of the tearing mode in (symmetric) slab geometry. We have provided a general set of dynamical equations as well as a compact formula for the saturation of the mode. We have shown that the contribution due to the higher-order harmonics in the saturation equation led to a higher order correction in the perturbation parameter. Hence, we have justified the standard one-harmonic calculation as asymptotically valid. To make this claim more tangible, we have

computed numerically the contribution due to the second harmonic for two types of equilibrium and shown that it indeed led to small corrections in a relevant range of values for Δ_1' . Therefore, we believe this work usefully complements previous results in the theory of tearing mode saturation.

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