Torque on an Ideal Plasma

J. B. Taylor

EURATOM/UKAEA Fusion Association, Culham Science Centre, Abingdon, Oxon OX14 3DB, United Kingdom

(Received 30 May 2003; published 10 September 2003)

Ripples in the confining field may exert a torque on a rotating plasma. Time reversal symmetry implies that this torque should vanish for an ideal plasma. However, even in an apparently ideal plasma, singularities can give rise to a nonzero torque. This torque is evaluated for a simple configuration. Although the primary force is magnetic, an essential contribution arises from other nonlinear terms in the equations of motion. The net force is confined to the singular layer, in the direction of the ripple wave vector and related to the energy absorbed in the layer.

DOI: 10.1103/PhysRevLett.91.115002 PACS numbers: 52.30.–q, 52.55.Fa

In a toroidal confinement system, ripples in the toroidal magnetic field may exert a torque that slows down a rotating plasma. The reaction to this torque is, of course, taken up by the wall or the field coils. It is generally accepted [1–5] that this torque should vanish in an "ideal" plasma. Here we examine this property and its limitations.

An underlying basis for the "zero torque" result is the fact that the equations governing an ideal plasma, i.e., one without dissipation, are time reversible. Thus, in the simple model of a perfectly conducting inviscid fluid the equations are

$$d\rho/dt + \rho\nabla\cdot\mathbf{v} = 0, \quad (1)$$

$$\rho d\mathbf{v}/dt + \nabla(p + B^2/2) - (\mathbf{B}\cdot\nabla)\mathbf{B} = 0, \quad (2)$$

$$d\mathbf{B}/dt - (\mathbf{B}\cdot\nabla)\mathbf{v} + \mathbf{B}(\nabla\cdot\mathbf{v}) = 0, \quad (3)$$

$$dp/dt + \gamma p\nabla\cdot\mathbf{v} = 0, \quad (4)$$

and it is clear that these are unchanged by the transformation $t\to -t$, $\mathbf{v}\to -\mathbf{v}$ or by the transformation $t\to -t$, $\mathbf{v}\to -\mathbf{v}$, $\mathbf{B}\to -\mathbf{B}$, $\mathbf{J}(=\nabla\times\mathbf{B})\to -\mathbf{J}$. Invariance under the first transformation applies only in a fluid description of the plasma, but invariance under the second, more fundamental, transformation is valid for any nondissipative plasma model, whether based on a fluid or a particle description.

Now consider a toroidal plasma in a reference frame in which the plasma is stationary and the ripples are rotating at an angular speed Ω. Then the perturbed field and current can be expressed as

$$\mathbf{b} = \sum \mathbf{b}_n \exp[i(n\phi - \Omega t)],$$
$$\mathbf{j} = \sum \mathbf{j}_n \exp[i(n\phi - \Omega t)]. \quad (5)$$

The period-averaged force on the plasma is then

$$\mathbf{F} = \mathrm{Re}\frac{1}{2}\Big[\sum(\mathbf{j}_n^* \times \mathbf{b}_n)\Big]. \quad (6)$$

The important point is that under time reversal this force is unchanged. However, the phase of the perturbation $(n\phi - \Omega t)$ becomes $(n\phi + \Omega t)$; that is, the direction of rotation of the field ripples is reversed. But, if clockwise and anticlockwise rotating ripples produce equal torque in the same direction, then symmetry implies that this torque must be zero.

It should be noted that this null result ceases to apply if the plasma is nonideal in any respect, not just if it is resistive. Even in an apparently ideal plasma, governed by equations such as (1)–(4), the null result may be misleading. This is because of the possibility of singularities— leading to resonance or continuum damping. As is well known, this effectively introduces dissipation despite the fact that the underlying equations remain invariant under time reversal.

To illustrate the torque produced by continuum damping, we consider an ideal, low β, fluid plasma in a plane slab configuration. That is, $\rho = \rho(x)$, $\mathbf{B} = [0,\ B_y(x), B_z(x)]$, and $dB^2/dx = 0$. This model was used by Chen and Hasegawa [6] in a study of energy absorption by continuum damping. We first summarize some features of this model that are required for the calculation of the torque.

For a linearized perturbation

$$\boldsymbol{\xi} = \boldsymbol{\xi}(x)\exp(ik_y y + ik_z z - i\omega t), \quad (7)$$

Eqs. (1)–(4) reduce to

$$\varepsilon\boldsymbol{\xi} = \nabla\tilde{p} + \mathbf{B}(i\mathbf{k}\cdot\mathbf{B})(\nabla\cdot\boldsymbol{\xi}), \quad (8)$$

$$\mathbf{b} = i(\mathbf{k}\cdot\mathbf{B})\boldsymbol{\xi} - \mathbf{B}(\nabla\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\nabla)\mathbf{B}, \quad (9)$$

with $\varepsilon(x) = \omega^2\rho(x) - k_\parallel^2 B^2$, $\tilde{p} = \mathbf{b}\cdot\mathbf{B}$.

The displacement and perturbed field can be expressed entirely in terms of ξ_x. In a coordinate system aligned with the local magnetic field ($\mathbf{e}_\parallel = \mathbf{B}/B$, $\mathbf{e}_\perp = \mathbf{e}_\parallel \times \mathbf{e}_x$) the perpendicular displacement is

$$\xi_\perp = \frac{-ik_\perp B^2}{(\varepsilon - k_\perp^2 B^2)}\frac{d\xi_x}{dx}, \quad (10)$$

while the parallel displacement vanishes unless $\omega^2 = 0$.

 © 2003 The American Physical Society

(Here $k_{\|}B = k_z B_z + k_y B_y$, $k_{\perp}B = k_y B_z - k_z B_y$.) In the fixed rectangular coordinate system (x, y, z)

$$\xi_y = \xi_{\perp} B_z/B, \qquad \xi_z = -\xi_{\perp} B_y/B, \tag{11}$$

and the perturbed field is

$$b_x = i(\mathbf{k} \cdot \mathbf{B})\xi_x, \tag{12}$$

$$b_y = -\frac{d(B_y \xi_x)}{dx} + i(\mathbf{k} \cdot \mathbf{B})\xi_y - i(\mathbf{k} \cdot \boldsymbol{\xi})B_y, \tag{13}$$

$$b_z = -\frac{d(B_z \xi_x)}{dx} + i(\mathbf{k} \cdot \mathbf{B})\xi_z - i(\mathbf{k} \cdot \boldsymbol{\xi})B_z. \tag{14}$$

Finally, the displacement ξ_x satisfies the (Alfvén) wave equation

$$\frac{d}{dx}\left(\frac{\varepsilon B^2}{\varepsilon - k_{\perp}^2 B^2}\right)\frac{d\xi_x}{dx} + \varepsilon \xi_x = 0. \tag{15}$$

As is well known, this is singular wherever $\varepsilon(x) = 0$ and in the neighborhood of such a point

$$\xi_x(x) = c_1 K_0(k_{\perp} x) + c_2 I_0(k_{\perp} x), \tag{16}$$

and

$$\lim_{x \to 0} \xi_x(x) = C \ln x + D. \tag{17}$$

The singularity is resolved by introducing causality, replacing ω by $\omega + i\delta$, $\delta \to 0$, or by analytical continuation of the Bessel functions I_0, K_0 around the singularity. This introduces a discontinuity in ξ_x on the real axis, at which

$$(\ln x)_{x<0} \to [\ln(x) + i\pi]_{x>0}. \tag{18}$$

We can now calculate the torque. For perturbations of the form (7), the period-averaged Maxwell stress tensor is

$$\Pi_{ij}(x) = (b_i b_j - \varepsilon_{ij} b^2/2). \tag{19}$$

Without loss of generality, we may take $\mathbf{k} = (0, 0, k)$. Then the magnetic force in the z direction (i.e., parallel to $\mathbf{k}$) is

$$F_z^{\text{mag}} = \frac{1}{2}\text{Re}\left[\frac{d}{dx}(b_x^* b_z)\right]. \tag{20}$$

In previous calculations of torque, where plasma inertia was neglected by setting $\omega = 0$, this was the only contribution considered. In the present calculation, this term alone would lead to an infinite result after integration over the singular layer. However, when $\omega^2 \neq 0$ there is another contribution arising from nonlinear velocity terms in the equation of motion

$$\frac{\partial(\rho \mathbf{v})}{\partial t} = (\mathbf{j} \times \mathbf{b}) - \rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \mathbf{v}(\mathbf{v} \cdot \nabla\rho) - \rho\mathbf{v}(\nabla \cdot \mathbf{v}). \tag{21}$$

The first term on the right is just the magnetic force F_z^{mag}; the other terms yield a "Reynolds stress" contribution

$$F_z^{\text{kin}} = -\frac{d(\rho v_x v_z)}{dx}. \tag{22}$$

Hence, the total force in the z direction is

$$F_z^{\text{tot}} = \frac{1}{2}\text{Re}\left[\frac{d}{dx}(b_x^* b_z - \rho\omega^2 \xi_x^* \xi_z)\right]. \tag{23}$$

This can be expressed in terms of $\xi_x(x)$ as

$$F_z^{\text{tot}} = \frac{1}{2}\text{Re}\left[ikB^2 \frac{d}{dx}\left(\frac{\varepsilon B^2}{\varepsilon - k_{\perp}^2 B^2}\xi_x^* \frac{d\xi_x}{dx}\right)\right], \tag{24}$$

and after integration over the singular layer this leads to a finite force G,

$$G = \frac{1}{2}\text{Re}\left[ikB^2 \frac{\varepsilon}{\varepsilon - k_{\perp}^2 B^2}\xi_x^* \frac{d\xi_x}{dx}\right]_{x_0-\delta}^{x_0+\delta} = \frac{\pi k}{2k_{\perp}^2}\left(\frac{d\varepsilon}{dx}\right)_{x_0}. \tag{25}$$

Outside the singular layer F_z^{tot} vanishes since there, using the wave equation, it can be expressed as

$$F_z = \frac{1}{2}\text{Re}\left[ik\left(-\varepsilon|\xi_x|^2 + \frac{\varepsilon B^2}{\varepsilon - k_{\perp}^2 B^2}\left|\frac{d\xi_x}{dx}\right|^2\right)\right] = 0. \tag{26}$$

A similar calculation shows that in the y direction (i.e., perpendicular to $\mathbf{k}$) the kinetic force F_y^{kin} exactly cancels the magnetic force F_y^{mag} and the net force in that direction is zero. Thus, G is the only force acting on the plasma.

In conclusion, we have confirmed that in general the torque due to magnetic field ripple should vanish for an ideal plasma as a consequence of time reversal invariance. However, any form of dissipation can invalidate this conclusion, not just resistivity as might be thought from the magnetic origin of the force. Furthermore, even in a plasma described by ideal, nondissipative, equations, singularities may lead to continuum or resonance damping that can invalidate the zero torque result.

Although the force arising from such resonance damping is ultimately magnetic, it is essential to take account of an indirect kinetic, or Reynolds stress, contribution. This did not appear in earlier calculations as these considered only situations in which $\omega^2 \to 0$. (In this event two Alfvén singularities coalesce and ξ_x is then continuous but $d\xi_x/dx$ is not. The torque can then, but only then, be related to the standard Δ' parameter of the "constant ψ" approximation [1–5].)

In the direction orthogonal to the ripple wave vector $\mathbf{k}$, the indirect kinetic force cancels the direct magnetic force and the net force in this direction is zero. (This is understandable as, e.g., it is clearly impossible for a circular coil to exert any force around its axis.) Outside the resonant layer, the net force parallel to $\mathbf{k}$ also vanishes. (Note that this does not immediately follow from the general zero torque argument as this requires the whole system to be reversible, not just a part of it.) Thus, the force due to continuum damping acts only in the resonant layer and only parallel to $\mathbf{k}$. Integrated over the resonant layer the total force is

$$G = \frac{\pi k}{2k_\perp^2}\left(\frac{d\varepsilon}{dx}\right)_{x_0}. \tag{27}$$

Comparing this with the energy W absorbed in the layer, as calculated by Chen and Hasegawa [6], we have $G = (k/\omega)W$. As one might expect, this can be interpreted as the result of absorption of "quanta" of energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$, or as the effect of a force moving at the wave velocity ω/k. However, it is noteworthy that this interpretation exists despite the fact that the energy is calculated solely from the electromagnetic stress whereas the momentum involves both this electromagnetic stress and a mechanical stress.

The author thanks colleagues Per Helander, Chris Gimblett, and Jim Hastie for their help. This work was funded by the United Kingdom Engineering and Physical Sciences Research Council and by EURATOM.

[1] P. H. Rutherford, in *Basic Processes of Toroidal Plasmas: Proceedings of the Workshop held at Varenna, Italy, 1985* (CEC, Brussels, 1986), Vol. II, p. 531.
[2] R. Fitzpatrick, in *Theory of Fusion Plasmas: Proceedings of the Varenna-Lausanne Workshop, Varenna, Italy, 1992* (Editrice Compositori, Bologna 1992), p. 147.
[3] R. Fitzpatrick, Nucl. Fusion **33**, 1049 (1993).
[4] R. Fitzpatrick, R. J. Hastie, T. J. Martin, and C. M. Roach, Nucl. Fusion **33**, 153 (1993).
[5] C. Gimblett and R. J. Hastie, Phys. Plasmas **7**, 258 (2000).
[6] L. Chen and A. Hasegawa, Phys. Fluids **17**, 1399 (1974).