Role of Flow Shear in the Ballooning Stability of Tokamak Transport Barriers

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A tokamak's confinement time is greatly increased by a transport barrier (TB), a region having a high pressure gradient and usually also a strongly sheared plasma flow. The pressure gradient in a TB can be limited by ideal magnetohydrodynamic instabilities with a high toroidal mode number n ("ballooning modes"). Previous studies in the limit $n \to \infty$ showed that arbitrarily small (but nonzero) flow shears have a large stabilizing influence. In contrast, the more realistic *finite n* ballooning modes studied here are found to be insensitive to sub-Alfvénic flow shears, provided the magnetic shear $s \sim 1$ (typical for TBs near the plasma's edge). However, for the lower magnetic shears that are associated with internal transport barriers, significantly lower flow shears will influence ballooning mode stability, and flow shear should be retained in the analysis of their stability.

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Introduction.—According to ideal magnetohydrodynamics (MHD), the most unstable short-wavelength perturbations minimize their bending of magnetic field lines by restricting their short-wavelength component to be perpendicular to the magnetic field [1]. Studies of these ballooning modes [1] have found that they can limit the maximum pressure gradient that may be sustained in a toroidal magnetically confined plasma [1–5].

The highest pressure gradients in tokamak plasmas [6] are found in transport barriers (TBs) [7], which are also associated with strong flow shears. Here we study ballooning modes in an axisymmetric tokamak plasma that has nested, isothermal magnetic flux surfaces, and an axisymmetric flow about the toroidal axis. Such flows have a constant angular velocity on a given magnetic flux surface, but this may be sheared, changing from one flux surface to the next.

An equilibrium with toroidal flow $\vec{v}_0$ complicates the study of ballooning modes. Using axisymmetry to study perturbations ξ with a single toroidal mode number n (that will be taken to be large), $\xi \sim e^{-in\phi}$ (where ϕ is the angle about the toroidal axis), and both convective derivatives $\vec{v}_0 \cdot \nabla$ and derivatives parallel to the magnetic field $\vec{B} \cdot \nabla$, lead to terms of order n. The most unstable short-wavelength (high n) perturbations must be constructed so as to minimize both the magnetic field-line bending *and* the stream-line bending. An eikonal representation has previously been introduced to reflect this [8], which uses the fact that the $\vec{v}_0 \cdot \nabla$ operator occurs in conjunction with the $\partial/\partial t$ operator, so that

$$\left(\frac{\partial}{\partial t} + \vec{v}_0 \cdot \nabla\right) F(\psi, \chi, t) e^{inS(\psi,\chi,t)} = e^{inS(\psi,\chi,t)} \frac{\partial F}{\partial t}$$

and

$$\vec{B} \cdot \nabla\{F(\psi, \chi, t) e^{inS(\psi,\chi,t)}\} = e^{inS(\psi,\chi,t)} \frac{1}{J} \frac{\partial F}{\partial \chi},$$

with $\partial F/\partial t$ and $\partial F/\partial \chi$ of order 1. Here the eikonal is $S = (\phi - \int_{\chi_0}^{\chi} \nu d\chi' - \Omega t)$, the angular flow velocity is $\Omega(\psi)$ with the toroidal flow $\mathbf{v}_0 = \Omega(\psi) R e_\phi$ and, as in [3], R is the major radius [6], ν is a measure of the local field-line pitch [3], and (ψ, ϕ, χ) form an orthogonal coordinate system with ψ the poloidal magnetic flux function (used as the radial coordinate) and χ the poloidal angle. Then, constructing a periodic perturbation by transforming to ballooning space [3], one can use this representation to expand in $1/n$ as $n \to \infty$, while both $\partial F/\partial t$ and $\partial F/\partial \chi$ remain $O(1)$ [8,9].

A stability analysis based on this procedure starts with the linearized MHD equations and expands in $1/n$. However, radial derivatives acting on S lead to terms in $\Omega' t$. So despite being linearized MHD, the time behavior is not simply $\xi \sim e^{\gamma t}$. Thus the problem involves time as an extra coordinate, and stability is determined by the solution of four coupled first-order 2D partial differential equations [9]. This contrasts with the zero flow shear case for which, as $n \to \infty$, the problem reduces to a 1D second-order ordinary differential equation in ballooning space [2,3].

Miller *et al.* [9] have used an eikonal formalism in an initial value code (in the $n \to \infty$ limit) to study a circular-flux-surfaces ("s-α" type) model of the equilibrium, that includes toroidal flow. In that model, TBs are represented by a narrow annular region of high pressure gradient and strong flow shear, but centrifugal forces from the flow are neglected, and a low $\beta = 2p/B^2$ ordering is taken. This enabled the study to focus solely on the effects of flow shear, which they showed can stabilize ballooning modes. Furukawa *et al.* [10] have applied the same eikonal formalism to realistic, arbitrary aspect ratio tokamak geometries, finding similar results.

In contrast, the present analysis uses an *eigenmode* formalism to study ballooning modes in the presence of flow shear. This determines the growth rate as an eigenvalue, and significantly extends the work of Miller *et al.* [9], by providing mode structures, enabling the study of finite n, and by discerning multiple solution branches. More importantly, calculations with $n = \infty$ have led to the widespread belief that ballooning modes are significantly stabilized by *any* nonzero flow shear, however small. Here we find this result to be misleading. In particular, we find that unless the magnetic shear is small, ballooning modes with an n that is large but finite are insensitive to experimentally relevant flow shears.

Our starting point is the linearized MHD equations, projected in the $\vec{B}$, $\nabla\psi$, and $\vec{B} \wedge \nabla\psi$ directions (for example, see [11]). We study the internal transport barrier (ITB) model of Miller *et al.* [9], neglecting the flow's centrifugal effects but retaining terms in the flow shear. We use the high-n ballooning mode ordering of $\vec{B} \cdot \nabla\vec{\xi} \sim 1$, $\nabla\psi \cdot \nabla\vec{\xi} \sim n$, and $\nabla\phi \cdot \nabla\vec{\xi} \sim n$ to reduce the three original equations to two simpler ones. A rapid radial coordinate $x = n(q_0 - q)$ is used, in terms of which $in\Omega(x) = in\Omega_0 - i(d\Omega/dq)x + O(1/n)$, where q is the safety factor and q_0 its value on a reference flux surface. The constant $in\Omega_0$ term is incorporated into a Doppler shifted growth rate $(\gamma - in\Omega_0)$.

Writing $\Gamma = \gamma + i\frac{d\Omega}{dq}x$, and Fourier expanding in straight field-line coordinates θ, with $\nabla\psi \cdot \vec{\xi} = \Gamma\sum_m u_m(x)\exp\{-in\phi + im\theta\}$ and $\nabla \cdot \vec{\xi} = \Gamma\sum_m g_m(x)\times \exp\{-in\phi + im\theta\}$ results in leading order equations (in $1/n$), that are free from any explicit dependence upon n.

We study the same equilibrium as Miller *et al.* [9](described above), and consider the limit $\beta \ll 1$, $\frac{d\Omega}{dq}R/c_s \gg \sqrt{\beta}$, for which terms in the flow shear are retained while those in β may be neglected. In this limit the number of equations is further reduced from two to one. Then taking our final equation and projecting out the Fourier amplitudes gives

$$s^2\frac{d}{dx}\left\{[(x-M)^2+\Gamma^2]\frac{du_m}{dx}\right\} - [(x-M)^2+\Gamma^2]u_m + \alpha\Big(-\frac{s}{2}\frac{d}{dx}\{[(x-M)^2+\Gamma^2](u_{m+1}-u_{m-1})\} - \frac{s}{2}[(x-M)^2+\Gamma^2+1]\frac{d}{dx}[u_{m+1}-u_{m-1}] - s(x-M)\frac{d}{dx}[u_{m+1}+u_{m-1}] + \frac{1}{2}[u_{m+1}+u_{m-1}]\Big) - \frac{\alpha^2}{2}\left\{[(x-M)^2+1+\Gamma^2]\left[u_m - \frac{1}{2}[u_{m+2}+u_{m-2}\right] - (x-M)[u_{m+2}-u_{m-2}]\right\} = 0 \tag{1}$$

as the equations that we solve for a range of m. The equation is normalized such that $\Gamma^2(x)$ replaces $[\Gamma^2(x)R^2q^2/c_s^2](p/B^2)$, u_m replaces $\frac{u_m}{R^2B_p}$, with $s = \frac{r}{q}\frac{dq}{dr}$, and $\alpha = -\frac{2r^2}{RB_p^2}\frac{dp}{dr}$ being the usual normalized magnetic shear and pressure gradient parameters, respectively [2]. B_p is the poloidal field and r is the minor radius. Note that our normalization is such that a change in flow speed Δv over a TB of width Δr will have $\frac{d\Omega}{dq} \sim \frac{r}{s}\frac{\Delta v}{\Delta r}\frac{1}{v_A}$, where v_A is the Alfvén speed.

Zero flow ballooning theory [3] indicates that a finite radial mode width is not possible for constant equilibrium parameters s and α. Therefore we introduce slow equilibrium variations, taking the magnetic shear s as constant, but representing the TB's enhanced pressure gradient by a Gaussian pressure gradient profile

$$\alpha(x) = \alpha_0 \exp\left[-\frac{1}{2}\left(\frac{r}{\Delta r}\right)^2\frac{(x-x_0)^2}{n^2q_0^2s^2}\right] \tag{2}$$

[where the factors $(\frac{r}{\Delta r})^2\frac{1}{n^2q^2s^2}$ arise from normalizing the rapid radial coordinate x so that the α profile extends across a transport barrier width of order Δr]. Then the zero flow ballooning theory of Connor *et al.* [3] predicts a radial mode envelope width of order $\Lambda \equiv \sqrt{nq\Delta r/r}$ and a growth rate with an order $1/\Lambda^2$ correction to the $n = \infty$ growth rate. The equations, which now depend weakly upon n, are solved with the boundary condition that the mode amplitudes tend to zero as $x \rightarrow \pm\infty$.

Results.—Two features are expected to be characteristic of ITBs: low magnetic shear s and finite flow shear. We first consider zero flow shear, but low magnetic shear, with $s = 0.1$ and $\alpha = 0.5$. The mode structures are highly localized in x (Fig. 1) and are insensitive to changes in Λ, but when $\Lambda = \infty$ they become repeated on adjacent rational surfaces (but with an alternately positive and negative amplitude). These results agree with analytic calculations for $s \sim \alpha \rightarrow 0$ [12].

We now take $s = 1$ and consider the effect of flow shear. In the absence of flow shear, the Fourier mode structure will be infinitely extended when $\Lambda = \sqrt{nq\Delta r/r} = \infty$ [3]. However, we find that flow shear will radially localize the mode, even when $\Lambda = \infty$, a result in agreement with analytic calculations for $d\Omega/dq \ll 1$ that indicate a mode width $\sim\sqrt{1/(d\Omega/dq)}$. So for finite flow shear we can take $\Lambda = \infty$ and compare directly with Miller *et al.*, finding good agreement between the results (Fig. 2). The Fourier modes' envelope has a symmetric shape [Fig. 3(a)] and narrows as the flow shear is increased. At high flow shears the mode envelopes are similar in shape, though narrower in width and reduced in amplitude. The growth rates are real throughout. Figure 3(d) shows the mode structure in the poloidal cross section. The modes are peaked on a radial surface $r = r_0$, for which the flow is zero. There is positive toroidal flow for $r > r_0$ and negative toroidal flow for $r < r_0$. Positive flow causes the mode to rotate clockwise relative to the zero flow mode in Fig. 3(e), whereas negative flow causes counterclockwise rotation of the mode. The net effect is that the mode's peaks are elongated in the upper half of Fig. 3(d), but narrowed in the lower half of the figure.

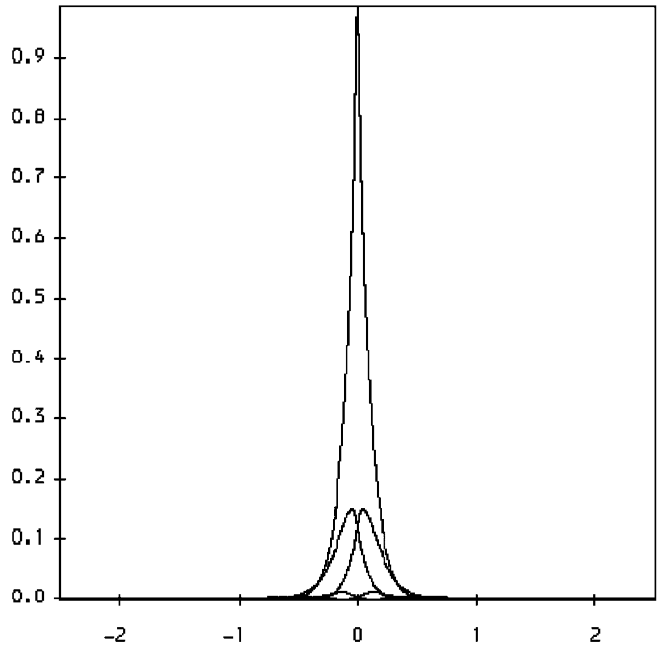

FIG. 1. Fourier mode amplitudes as a function of $x = n(q_0 - q)$, for low magnetic shear $s = 0.1$ and $\alpha = 0.5$.

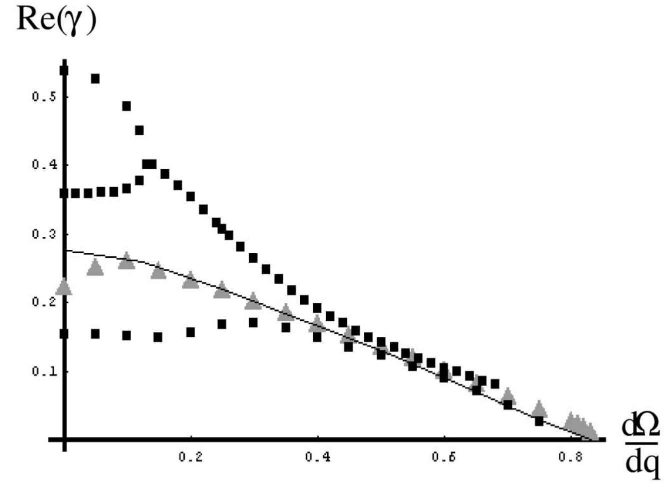

FIG. 2. Growth rates as a function of flow shear $d\Omega/dq$, for $\Lambda = \infty$ (gray triangles) and $\Lambda = 1.35$ (black squares, upper and lower curves). The continuous line shows the $n = \infty$ growth rates calculated by Miller *et al.* [9] ($s = 1.0$, $\alpha = 2.0$).

We turn now to finite Λ (finite n), taking $\Lambda = 1.35$ (corresponding to $\Delta r/r \sim 0.1$, $n = 10$, and $q = 2$, for example). For zero flow shear there are three unstable modes (Fig. 2). These modes correspond to the three most unstable solutions of the mode envelope equation [Eq. (36) of [3]]. As the flow shear is increased the two most unstable branches coalesce. After coalescence γ becomes complex and there exist two complex conjugate solutions, whose Fourier mode structures are mirror images of one another [Fig. 3(c)]. There are important differences between the growth rates for $\Lambda = \infty$ and $\Lambda = 1.35$. At low flow shears the finite-n growth rates are much larger than those for $\Lambda = \infty$. This is partially explained by analytic calculations at low flow shears and $n \to \infty$ of the growth rate in terms of the zero flow growth rate γ_0. For zero flow, γ_0 is a function of a free parameter θ_0 that is chosen to maximize $\gamma_0(\theta_0)$ [3]; whereas for $0 \neq d\Omega/dq \ll 1$ and $n \to \infty$, $\gamma = \frac{1}{2\pi}\oint \gamma_0(\theta_0)d\theta_0$ [13], predicting a discontinuous reduction in γ for $d\Omega/dq \neq 0$. However, Fig. 2 shows that for $\Lambda = 1.35$ the growth rates only tend to those for $\Lambda = \infty$ when $\frac{d\Omega}{dq}$ is sufficiently large. So for realistic instabilities with finite n, ballooning modes are insensitive to small flow shears, instead requiring flow shears $\Delta v/\Delta r \sim \frac{d\Omega}{dq} s v_A/r$ to noticeably affect their stability.

Similar results are found for low magnetic shears such as $s = 0.2$, with normalized flow shears $\frac{d\Omega}{dq} \sim 1$ being required to stabilize the mode. However, because the non-normalized flow shears $\Delta v/\Delta r \sim \frac{d\Omega}{dq} s v_A/r$, regions with a low magnetic shear (as found in ITBs) may be affected by flow shears that are orders of magnitude smaller than those required near the plasma's edge. Hence it is important to include flow shear into stability analyses of internal transport barriers.

The mode structures for $\Lambda = 1.35$ are different from those for $\Lambda = \infty$. Figure 3(c) shows the Fourier mode structure for $d\Omega/dq = 0.5$, which has a nonsymmetric structure that narrows in radial extent as the flow shear $d\Omega/dq$ is increased. In poloidal cross section the mode's peak is found to be torn into two pieces [Fig. 3(f)], unlike for $\Lambda = \infty$ where the peak remains intact but its edges are torn.

Conclusions.—We have used an eigenmode (as opposed to "eikonal") approach to study ballooning modes in the presence of flow shear, applying it to a simple circular flux surfaces model of TBs.

In the absence of flow shear, we find that at sufficiently low magnetic shear the mode structure is insensitive to the equilibrium α profile, and the calculated mode structures are in good agreement with analytic calculations of this limit [12].

The ballooning modes in the presence of flow shear studied by Miller *et al.* [9] using a time-dependent eikonal in the limit $n \to \infty$ differ from the finite-n ballooning modes studied by Connor *et al.* [2,3]. At low flow shears the growth rate is significantly increased when n is finite and slow equilibrium variations in $\alpha(x)$ are included. Then the top two solution branches from ballooning theory can couple to produce nonsymmetric mode structures that are quite different from the symmetric structures found for $\Lambda = \infty$. At high flow shears the growth rates become independent of Λ, but the mode structures remain distinct.

Previously it was believed from calculations with $n = \infty$ that arbitrarily small flow shears will have a significant stabilizing effect on ballooning modes [13].

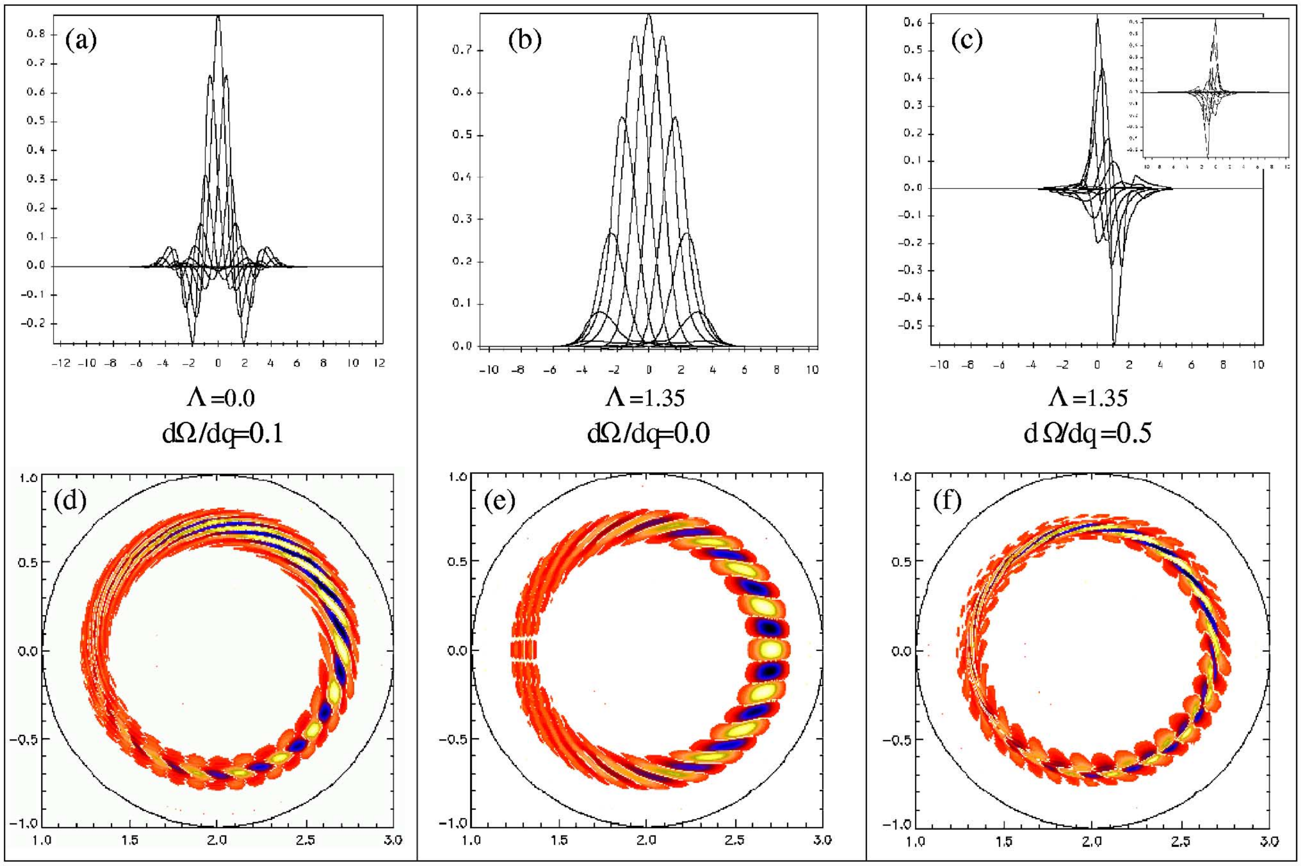

FIG. 3 (color online). Fourier mode amplitudes as a function of $x = n(q_0 - q)$ (a)–(c) and the mode structures in poloidal cross section (d)–(f). (c) contains the mode structures of the pair of solutions with complex conjugate growth rates ($s = 1$, $\alpha = 2$).

Here we find this is not so, because the more relevant ballooning modes have a finite toroidal mode number n, for which we find that flow shears $\Delta v/\Delta r \sim s v_A/r$ are required to stabilize them. We do note, however, that *internal* transport barriers may have very low magnetic shears s(that may even be zero) [14], and ballooning stability may then be affected by flow shears that are much smaller in size. Experimentally observed flow shears of order $0.15(v_A/r)$ are not uncommon at ITBs in the Joint European Torus (JET), for example [14], and therefore should be incorporated into studies of their stability.

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