

UKAEA-CCFE-PR(19)09

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## Abstract

Kinetic treatments of drift tearing modes that match an inner, resonant layer solution to an external MHD solution, characterised by  $\Delta'$ , can fail to match the ideal MHD boundary condition on the parallel electric field,  $E_{\parallel} = 0$ . In this paper we demonstrate how consideration of ion sound and ion Landau damping effects achieves this, placing the theory on a firm footing. These effects are found to modify the effective critical  $\Delta'$  for instability of drift tearing modes, in particular for weak electron temperature gradients. The implications for a realistic hot plasma resonant layer model - involving large ion Larmor radius and semi-collisional electron physics (Connor et al., 2012) - are determined.

## 1. Introduction

Tearing modes are driven by current gradients and pressure gradients in the plasma whose destabilising effect is characterised by a quantity  $\Delta'$ . The associated tearing of a tokamak magnetic field occurs at the resonant radius,  $r = r_0$ , where  $m = nq(r_0)$ ,  $m$  and  $n$  being the poloidal and toroidal mode numbers of the helical instability and  $q$  is the safety factor. In the vicinity of  $r = r_0$  complex kinetic effects can occur and to account for these it is convenient to treat a narrow ‘inner’ region in which these effects are included but the equations are simplified because of the localised nature of this inner layer. Stability is then determined by matching solutions of these inner equations to outer ones, valid across the regions away from the layer (and satisfying appropriate boundary conditions at the plasma boundary and magnetic axis) where a simple ideal MHD description of the plasma suffices. The solution of the inner equations provides a quantity  $\Delta(\omega)$ , where  $\omega$  is the complex mode frequency, while the solution of the outer equations yields the afore-mentioned quantity  $\Delta'$ : the matching condition leads to a dispersion relation and a critical value of  $\Delta'$  for instability.

For a simple resistive MHD model (or even a cold ion model containing electron diamagnetic effects) of the inner region, the corresponding solutions match satisfactorily to the outer ones, with the perturbed parallel electric field,  $E_{\parallel}$ , vanishing properly as one reaches the outer ideal MHD region. However, for a

kinetic or fluid treatment of the ion physics when the ions have finite temperature, so that the effects of finite ion Larmor radius (FLR) and their parallel thermal motion or transport are included, this matching becomes more complex (Coppi et al., 1979, Cowley et al., 1986, Fitzpatrick, 1989, Connor et al., 2012). Even though one can match the longitudinal component of the perturbed vector potential,  $A_{\parallel}$ , from the inner region to its form in the ideal region, where  $A_{\parallel} \propto 1 + (r - r_0)\Delta'$ , the expression for  $E_{\parallel}$  does not necessarily tend to zero. To accomplish this (i.e. decay of  $E_{\parallel}$ ) within the matching region, one must consider an intermediate region where ion sound and ion Landau damping become significant at the characteristic frequency of a drift tearing mode (Coppi et al., 1979). Consequently, the overall matching condition is modified, with an impact on the stability of the drift tearing mode. The investigation of this effect for the realistic situation in a hot plasma where the ion Larmor orbit width exceeds the reconnecting layer width associated with a semi-collisional electron model, is the subject of this paper. In this work, the intermediate region is described by a cylindrical, collision-less plasma model.

We note in passing that the stabilising effects of ion sound on the ‘drift collisional’ tearing mode were also presented by Bussac et al. (1978) and a simple model for the effects of ion Landau damping on the semi-collisional neoclassical drift tearing mode was given by Fitzpatrick (1989), while a more fluid-like treatment of this problem was presented by Connor et al. (Connor et al. 2017).

In Section 2 we develop the governing equations, namely the vorticity equation and Ampere’s equation, that pertain in the intermediate region. To complete these equations, we solve the collision-less kinetic equations for the perturbed parallel electron and ion currents. We obtain these solutions in Section 3 by calculating in the space of the radial co-ordinate. However, we wish to connect this region to inner region models which feature large ion Larmor radii (as compared to the electron scales of the reconnecting layer). Since such inner region models are better dealt with in Fourier space (Pegoraro & Schep, 1986), it is convenient to operate with Fourier transforms of the radially dependent perturbed fields (as in the formalism of Connor et al. (Connor et al., 2012)), deducing the corresponding real space behaviour from their long wavelength limit in Fourier transformed variables. The resulting intermediate region solution is then inserted into Ampere’s law in Section 4 to calculate the effect of the intermediate region on  $\Delta'$ .

The results are characterised by a quantity  $a = \frac{\omega\tau L_S}{2\omega_{*e}L_n} \frac{(\omega - \omega_{*e})}{(\omega\tau + (1 + \eta_i)\omega_{*e})}$ . Here  $\omega$  is the mode frequency,  $\omega_{*j} = mT_j / (e_j B r L_n)$  are diamagnetic frequencies with  $j$

labelling the species ( $j = e, i$  for electrons and ions, respectively),  $1/L_n = d(\ln n)/dx$  with  $n$  the plasma density,  $\eta_j = d(\ln T_j)/d(\ln n)$  with  $T_j$  the temperature of species  $j$ ,  $\tau = T_e/T_i$ ,  $r$  is a minor radius of the cylinder and  $L_s$  is the magnetic shear length. Since drift tearing modes typically have a real frequency  $\omega = \omega_{*e}(1 + \alpha\eta_e)$ , where  $\alpha$  is  $O(1)$  (for the collisionless case,  $\alpha = 0.5$  (Drake & Lee, 1977; Coppi et al., 1979; Cowley et al., 1986), while for semi-collisional electrons,  $\alpha = 0.74$  (Connor et al., 2012)),  $\alpha$  is usually large, but for weak electron temperature gradients it too can become  $O(1)$ . This is the situation addressed by Coppi et al. (1979) and in the present work.

In Section 5, we discuss the implications for a realistic hot plasma resonant layer model - involving large ion Larmor radius and semi-collisional electron physics (Connor et al., 2012). In contrast to the calculation in (Coppi et al., 1979), we now need both real and imaginary parts of the modified  $\Delta'$ . A final section discusses the results and draws conclusions.

## 2. Intermediate Region Plasma Model

We consider cylindrical geometry with co-ordinates  $r, \theta, z$  and magnetic fields  $(0, B_\theta(r), B_z(r))$ ; the total field strength is  $B$ . The perturbed fields of the tearing mode involve the parallel vector potential,  $A_\parallel$ , and electrostatic potential,  $\varphi$ , with frequency  $\omega$  and structure  $A_\parallel \sim A(x)e^{-i\omega t + ikz - im\theta}$  for example. Here  $x = r - r_0$ , where the resonant surface,  $r_0$ , is given by  $kB_z(r_0) - k_\theta B_\theta(r_0) = 0$ , with  $k_\theta = m/r$ , so that the wave-number parallel to the magnetic field is  $k_\parallel = mx/rL_s$ , with  $L_s$  being the magnetic shear length. In the toroidal geometry of a tokamak,  $L_s = Rq/s$ , with  $R$  the major radius,  $s = rd(\ln q)/dr$  and  $q$  the safety factor.

The electrons and ions satisfy the linearized, collision-less, kinetic equation:

$$\left(\omega - \frac{mxv_\parallel}{rL_s}\right) g_j = (\omega - \omega_{*j}^T) \frac{e_j F_{Mj}}{T_j} J_0\left(\frac{k_\perp v_\perp}{\omega_{cj}}\right) (\varphi - v_\parallel A_\parallel) \quad (1)$$

Here the perturbed electron distribution,  $\delta f_j$  is written  $\delta f_j = -\frac{e_j F_{Mj}}{T_j} \varphi + g_j$ , where  $F_{Mj}$  is the Maxwellian distribution for species  $j$ ,  $v_\parallel$  is the parallel velocity and  $v_\perp$  the perpendicular velocity, while  $\omega_{*j}^T = \omega_{*j} \left(1 + \eta_j \left(\frac{m_j v_\perp^2}{2T_j} - \frac{3}{2}\right)\right)$ , with  $v^2 = v_\parallel^2 + v_\perp^2$ .  $J_0$  is the Bessel function of zero order with  $k_\perp$ , where  $k_\perp^2 = k_x^2 +$

$\frac{m^2}{r^2}$ , being the wave-number perpendicular to the magnetic field, and  $\omega_{cj} = \frac{e_j B}{m_j}$ , with  $m_j$  the mass of species  $j$ , is the cyclotron frequency for species  $j$ .

The stability of tearing modes can be investigated from two equations: the vorticity equation and Ampere's law. The former is obtained by multiplying eqn. (1) by  $e_j$ , integrating over velocity space, summing over the two species,  $j$ , and applying the quasi-neutrality condition. The result is:

$$k_{\parallel} J_{\parallel} = \frac{e^2 n \varphi}{T_i} \left[ (1 - \Gamma_0(b)) (\omega - \omega_{*i}) - \omega_{*i} \eta_i b (\Gamma_0(b) - \Gamma_1(b)) \right] \equiv \frac{e^2 n \varphi}{T_i} F(b) \quad (2)$$

Here  $J_{\parallel} = J_{\parallel,i} + J_{\parallel,e}$  is the perturbed parallel current density,  $e$  the proton charge and  $\Gamma_l(b) = I_l(b) e^{-b}$ , where  $b = k_{\perp}^2 T_i / (m_i \omega_{ci}^2)$ , with  $I_l(b)$  the modified Bessel functions of order  $l$ . Ampere's equation is simply:

$$\frac{d^2 A_{\parallel}}{dx^2} = -J_{\parallel}, \quad (3)$$

In general, one must consider several regions in  $x$ : (i) the innermost layer,  $x \sim \delta_e$ , where  $\delta_e$  is the width of the very narrow region where electron physics such as electron Landau damping or semi-collisional effects play a role in reconnection; (ii) at somewhat larger values of  $x$ , namely  $x \sim \rho_i > \delta_e$ , where  $\rho_i = v_{Ti} / \omega_{ci}$ , with  $v_{Ti}$  the ion thermal velocity,  $v_{Ti} = (2T_i/m_i)^{1/2}$ , is the ion Larmor radius, one must retain full ion Larmor radius effects; (iii) the 'intermediate region',  $\omega \sim k_{\parallel} v_{Ti}$ , where full ion parallel dynamics operates (ion sound at  $x \sim (L_s/L_n)^{1/2} \rho_i$  and ion Landau damping at  $(L_s/L_n) \rho_i$ ; and finally; (iv) the outer ideal MHD region. These regions are shown schematically in Fig. 1. Solutions of eqns. (2) and (3) must be matched through all these four regions. In earlier work Connor et al. (2012) considered this problem, but omitted the intermediate region required to ensure  $E_{\parallel} \rightarrow 0$  as one approaches the ideal region. We now develop the equations to describe this region.

Expressions for the parallel current densities,  $J_{\parallel,e}$  and  $J_{\parallel,i}$ , valid in the ion sound region and where  $b \ll 1$ , are derived from solving the respective kinetic equations for electrons and ions, eqn. (1):

$$J_{\parallel,e} = -i \frac{e^2 n E_{\parallel}}{k_{\parallel}^2 T_e} (\omega - \omega_{*e}) \quad (4)$$

$$J_{\parallel,i} = -i \frac{e^2 n E_{\parallel}}{k_{\parallel}^2 T_i} \left[ (\omega - \omega_{*i}) (1 + \zeta_i Z(\zeta_i)) - \omega_{*i} \eta_i \left( \zeta_i^2 (1 + \zeta_i Z(\zeta_i)) - \frac{1}{2} \zeta_i Z(\zeta_i) \right) \right], \quad (5)$$

where  $E_{\parallel} = -i(k_{\parallel}\phi - \omega A_{\parallel})$ ,  $\zeta_i = \omega L_s/k_y x v_{Ti}$  and  $Z(\zeta_i)$  is the plasma dispersion function. Making use of the long-wavelength approximation,  $b \sim |\rho_i^2 d^2/dx^2| \ll 1$ , for the ions in eqn. (2), eliminating  $J_{\parallel,e,i}$  and introducing the dimensionless variable,  $X = 1/\zeta_i$ , so that  $X = (2L_n/L_s)(\omega_{*i}/\omega)(x/\rho_i)$ , we obtain the following fourth order system describing the potentials  $\phi$  and  $A_{\parallel}$ , replacing the former by  $\phi = \phi/v_{Ti}$  and denoting the latter by  $A$ : (so that  $E_{\parallel} \propto (X\phi - A)$ ):

$$X \frac{d^2 A}{dX^2} = \beta_i \left(1 - \frac{\omega_{*i}(1+\eta_i)}{\omega}\right) \frac{d^2 \phi}{dX^2}, \quad (6)$$

and

$$X \frac{d^2 A}{dX^2} = \frac{\beta_e}{2} \left(\frac{L_s}{L_n} \frac{\omega_{*i}}{\omega}\right)^2 \left(\phi - \frac{A}{X}\right) \left[1 - \frac{\omega_{*e}}{\omega} + \tau \left(1 - \frac{\omega_{*i}}{\omega}\right) \left(1 + \frac{Z}{X}\right) - \tau \frac{\omega_{*i}}{\omega} \frac{\eta_i}{X^2} \left(1 + \frac{Z}{X} - \frac{XZ}{2}\right)\right], \quad (7)$$

where  $Z = Z(1/X)$  and the  $\beta_j$  are the ratios of plasma to magnetic pressure for each species. These two equations can be combined to yield:

$$\frac{d^2 \phi}{dX^2} = Q(X) \left(\phi - \frac{A}{X}\right), \quad (8)$$

where

$$Q(X) = \frac{1}{2} \left(\frac{L_s}{L_n}\right)^2 \frac{(\hat{\omega}\tau)^2}{(\hat{\omega}\tau+1+\eta_i)} \left[\hat{\omega} - 1 + (\hat{\omega}\tau + 1) \left(1 + \frac{Z}{X}\right) + \eta_i \left(\frac{1}{X^2} + \frac{Z}{X^3} - \frac{Z}{2X}\right)\right], \quad (9)$$

with  $\hat{\omega} = \omega/\omega_{*e}$ . The quantity  $Q(X)$  contains the effects of ion sound when  $X \ll 1$  and ion Landau damping in the region  $X \sim 1$ . Introducing the ion sound approximation,

$$Q(X) \approx \frac{1}{2} \left(\frac{L_s}{L_n}\right)^2 \frac{(\hat{\omega}\tau)^2}{(\hat{\omega}\tau+1+\eta_i)} \left[\hat{\omega} - 1 - (\hat{\omega}\tau + 1 + \eta_i) \frac{X^2}{2}\right]. \quad (10)$$

The validity of the ion finite Larmor radius (FLR) and ion sound expansions requires these terms both to be small so that the balance of terms in eqn. (10) implies  $\hat{\omega} - 1 \sim \eta_e \sim \frac{L_n}{L_s} < 1$

We shall find it to be sufficient to consider only the ion sound approximation, in order to ensure  $E_{\parallel} \rightarrow 0$ , the physics of ion Landau damping only playing a conceptual role that justifies the use of the ‘outward carrying energy’ wave boundary conditions, as in the theory of the electron drift wave (Pearlstein & Berk, 1969).

### 3. Solution in the Intermediate Region

At low  $\beta_i$ , eqn. (6) allows us to write  $A \equiv \text{constant}$ , since  $A_{\parallel}$  is an even function of  $x$ , which we choose to be unity:  $A = 1$ . Introducing the following scaled variables:

$$y = \left(\frac{\hat{\omega}\tau L_s}{L_n}\right)^{1/2} X, \quad \hat{\phi} = \left(\frac{\hat{\omega}\tau L_s}{L_n}\right)^{-1/2} \phi, \quad a = \frac{\hat{\omega}\tau L_s}{2L_n} \frac{(\hat{\omega}-1)}{(\hat{\omega}\tau+1+\eta_i)}, \quad (11)$$

eqn. (8) can then be written as

$$\frac{d^2}{dy^2} \hat{\phi} + \left(\frac{y^2}{4} - a\right) \hat{\phi} = R(y), \quad R(y) = -\frac{a}{y} + \frac{y}{4}. \quad (12)$$

The validity of the ion FLR and ion sound expansions means that eqn. (12) is only valid on the domain  $(L_s/L_n)^{1/2} > y > (L_n/L_s)^{1/2}$ , requiring  $(L_n/L_s) \ll 1$ .

To solve eqn. (12) we adopt two different techniques, corresponding to the two components of the inhomogeneous term,  $R(y)$ . For the first term,  $R(y) = R_1(y) = -a/y$ , we employ a Fourier transform for its solution,  $\hat{\phi}_1$ . For the second one,  $R(y) = R_2(y) = y/4$ , we generalise a technique due to Basu and Coppi (1977), involving an integral representation of its solution,  $\hat{\phi}_2$ .

#### (i) Solution $\hat{\phi}_1(y)$

Fourier transforming eqn. (12) with  $R(y) = -a/y$  we obtain an equation for

$$\tilde{\phi}_1(k) = \int_{-\infty}^{\infty} dk \exp(iky) \hat{\phi}_1(y), \quad (13)$$

namely

$$\frac{1}{4} \frac{d^2}{dk^2} \tilde{\phi}_1 + \left(\frac{k^2}{4} + a\right) \tilde{\phi}_1 = i \pi a \operatorname{sgn}(k), \quad (14)$$

where the transform of the right-hand side is given by Pegoraro and Schep (1986), utilising the theory for Fourier transforms of generalised functions (Gel'fand & Shilov, 1964).

This can be cast in the canonical form for parabolic cylinder functions by introducing  $\kappa = 2k$ . The solution of eqn. (14) can then be written as:

$$\begin{aligned} \tilde{\phi}_1(\kappa, a) = & AE(-a, \kappa) + BE^*(-a, \kappa) + \\ & i \pi a \frac{E(-a, \kappa)}{W} \int_0^{\kappa} d\kappa' E^*(-a, \kappa') \operatorname{sgn}(\kappa') - \\ & i \pi a \frac{E^*(-a, \kappa)}{W} \int_0^{\kappa} d\kappa' E(-a, \kappa') \operatorname{sgn}(\kappa') \end{aligned} \quad (15)$$

where  $E(-a, \kappa)$  and  $E^*(-a, \kappa)$  are the wave-like parabolic cylinder functions (Abramowitz & Stegun, 1972), the Wronskian,  $W = -2i$  and  $A$  and  $B$  are constants to be determined by applying appropriate boundary conditions.

The acceptable solution in configuration space (i.e.  $x$ -space) is a wave carrying energy outwards from the resonant layer, eventually decaying due to ion Landau damping (Pearlstein & Berk, 1969). Since the electron drift wave is a ‘backwards’ wave with group velocity opposed to its phase velocity, this translates into suppressing the wave solution with positive phase velocity in  $x$ -space. In  $k$ -space this implies suppressing the one with an apparent negative phase velocity, so that

$$B = \frac{i\pi a}{W} \int_0^\infty d\kappa' E(-a, \kappa') \operatorname{sgn}(\kappa'). \quad (16)$$

We also require  $\hat{\varphi}_1(\kappa, -a)$  to be odd in  $\kappa$  with  $\hat{\varphi}_1(0, a) = 0$ . This requires

$$A = -\frac{i\pi a E^*(-a, 0)}{WE(-a, 0)} \int_0^\infty d\kappa' E(-a, \kappa') \operatorname{sgn}(\kappa') \quad . \quad (17)$$

Thus

$$\begin{aligned} \tilde{\varphi}_1(\kappa, a) = & \frac{i\pi a}{W} \left[ E^*(-a, \kappa) \int_\kappa^\infty d\kappa' E(-a, \kappa') \operatorname{sgn}(\kappa') - \right. \\ & \frac{E^*(-a, 0)}{E(-a, 0)} E(-a, \kappa) \int_0^\infty d\kappa' E(-a, \kappa') \operatorname{sgn}(\kappa') + \\ & \left. E(-a, \kappa) \int_0^\kappa d\kappa' E^*(-a, \kappa') \operatorname{sgn}(\kappa') \right]. \quad (18) \end{aligned}$$

In calculating the contribution of  $\hat{\varphi}_1(y)$  to  $\Delta'$ , we shall find knowledge of  $\tilde{\varphi}_1(\kappa, a)$  is sufficient. Specifically, we shall require the quantity:

$$i \frac{d}{dk} \tilde{\varphi}_1(k, a)_{k=0} = -\frac{2a\pi}{E(-a, 0)} \int_0^\infty d\kappa' E(-a, \kappa') \operatorname{sgn}(\kappa'), \quad a > 0 \quad (19)$$

As shown in the appendix, eqn. (A.10), the integral over  $\kappa'$  can be performed analytically to yield:

$$i \frac{d}{dk} \tilde{\varphi}_1(k, a)_{k=0} = -\frac{2\pi e^{\frac{i\pi}{4}} a \Gamma\left(\frac{3-ia}{4}\right)}{\Gamma\left(\frac{5-ia}{4}\right)} F\left(\frac{1}{2}, \frac{1}{4} - \frac{ia}{2}; \frac{5}{4} - \frac{ia}{2}; -1\right) \equiv I_0(a) \quad (20)$$

Here  $F(a, b; c; z)$  is the hypergeometric function and  $\Gamma(z)$  the Gamma function (Abramowitz & Stegun, 1972).  $I_0(a)$  is displayed in Fig. 2.

### (ii) **Solution** $\hat{\varphi}_2(y)$

Introducing  $z = iy^2/4$  and  $\hat{\varphi}_2(y) = u \exp(-z)$  into eqn. (12) with  $R(y) = y/4$ , we obtain the equation:

$$z \frac{d^2u}{dz^2} + \left(\frac{1}{2} - 2z\right) \frac{du}{dz} + \left(-\frac{1}{2} + ia\right) u = -\frac{e^{i\pi/4} z^{1/2} e^z}{2} \quad (21)$$

for  $u(z)$ . Generalising the technique of Basu and Coppi (1977), we introduce the transform:

$$u(z) = z^\beta \int_C dtv(t)e^{zt} \quad (22)$$

where the contour  $C$  remains to be chosen for convenience. Inserting the substitution (22) in eqn. (21) leads to

$$z^{\beta-1} \int_C dtv(t)e^{zt} \left\{ z^2(t^2 - 2t) + z \left[ \left(2\beta + \frac{1}{2}\right)(t-1) + ia \right] + \beta(\beta - 1) + \beta/2 \right\} = -e^{i\pi/4} z^{\frac{1}{2}} e^z / 2. \quad (23)$$

Choosing  $\beta = 1/2$ , which eliminates the term independent of  $z$  on the right-hand side bracket, eqn. (23) simplifies to

$$\int_C dtv(t)e^{zt} \left\{ z^2(t^2 - 2t) + z \left[ \frac{3}{2}(t-1) + ia \right] \right\} = -e^{i\pi/4} e^z / 2 \quad (24)$$

Integrating by parts in  $t$ ,

$$\begin{aligned} & [v(t)e^{zt}(t^2 - 2t)]_{t_1}^{t_2} \\ & - \int_C dt e^{zt} \left\{ \frac{d}{dt} [(t^2 - 2t)v(t)] - \left[ \frac{3}{2}(t-1) + ia \right] v(t) \right\} = -e^{i\pi/4} e^z / 2, \end{aligned} \quad (25)$$

where  $t_1$  and  $t_2$  are the end-points of contour  $C$  and are to be chosen later for convenience.

Equation (25) is solved if

$$\frac{d}{dt} [(t^2 - 2t)v(t)] - \left[ \frac{3}{2}(t-1) + ia \right] v(t) = 0 \quad (26)$$

and

$$[v(t)e^{zt}(t^2 - 2t)]_{t_1}^{t_2} = -e^{i\pi/4} e^z / 2. \quad (27)$$

Equation (25) is satisfied by:

$$v(t) = \frac{\hat{c}}{t^{1/4+ia/2}(2-t)^{1/4-ia/2}} \quad (28)$$

and eqn. (27) by choosing  $t_1 = 0$ ,  $t_2 = 1$  and the constant  $\hat{c} = e^{i\pi/4} / 2$ .

## 4. The $\Delta'$ – Integral

Inserting the solution  $\hat{\varphi} = \hat{\varphi}_1 + \hat{\varphi}_2$  into Ampere's law (6) and integrating over  $X$  through the ion sound region yields the modification to  $\Delta'$  arising from the intermediate region (which represents ion sound and, implicitly, ion Landau damping physics and we designate it as such):

$$\Delta'_{ILD} = \frac{1}{A} \int_{-\infty}^{\infty} dx \frac{d^2 A}{dX^2} = \frac{1}{\rho_i} \int_{-\infty}^{\infty} \frac{dX}{X} \left[ 1 + \frac{(1+\eta_i)}{\hat{\omega}\tau} \right] Q(X) \left( \varphi - \frac{A}{X} \right) \quad (29)$$

where we have used eqn. (7). Introducing the normalisations (11), the contribution from the ion sound term in  $Q(X)$  becomes:

$$\Delta'_{ILD} = \frac{\beta_i(\hat{\omega}\tau+1+\eta_i)}{\rho_i} \left( \frac{L_s}{L_n \hat{\omega}\tau} \right)^{1/2} \int_0^{\infty} dy [1 - y\hat{\varphi}(y)] \quad (30)$$

We express this as

$$\Delta'_{ILD} = \Delta'^{(1)} + \Delta'^{(2)} \quad (31)$$

where

$$\Delta'^{(1)} = \frac{\beta_i(\hat{\omega}\tau+1+\eta_i)}{\rho_i} \left( \frac{L_s}{L_n \hat{\omega}\tau} \right)^{1/2} \int_0^{\infty} dy [1 - y\hat{\varphi}_1(y)] \quad (32)$$

and

$$\Delta'^{(2)} = \frac{\beta_i(\hat{\omega}\tau+1+\eta_i)}{\rho_i} \left( \frac{L_s}{L_n \hat{\omega}\tau} \right)^{1/2} \int_0^{\infty} dy [1 - y\hat{\varphi}_2(y)] \quad (33)$$

To evaluate  $\Delta'^{(1)}$  we introduce the Fourier transform for  $\hat{\varphi}_1(y, a)$ ,  $\tilde{\varphi}_1(k, a)$ , perform the integration over  $y$ , using the resulting delta-function,  $\delta(k)$ , to reduce the integral to:

$$\Delta'^{(1)} = \frac{\beta_i(\hat{\omega}\tau+1+\eta_i)}{2\rho_i} \left( \frac{L_s}{L_n \hat{\omega}\tau} \right)^{1/2} i \frac{d}{dk} \tilde{\varphi}_1(k, a)_{k=0} = \frac{\beta_i(\hat{\omega}\tau+1+\eta_i)}{2\rho_i} \left( \frac{L_s}{L_n \hat{\omega}\tau} \right)^{1/2} I_0(a) \quad (34)$$

on inserting the result (20).

In the case of  $\Delta'^{(2)}$  we insert solution (22) with  $v(t)$  given by eqn. (28), into integral (33), obtaining

$$\Delta'^{(2)} = \frac{\beta_i(\hat{\omega}\tau+1+\eta_i)}{\rho_i} \left( \frac{L_s}{L_n \hat{\omega}\tau} \right)^{1/2} \int_0^{\infty} dy \left[ 1 - \frac{iy^2}{4} \int_0^1 dt \exp\left(-\frac{iy^2 t}{4}\right) \left( 1 - t \right)^{-\frac{1}{4}-ia/2} (1+t)^{-\frac{1}{4}+ia/2} \right] \quad (35)$$

Using the identity

$$1 = \exp\left(-\frac{iy^2}{4}\right) + \frac{iy^2}{4} \int_0^1 dt \exp\left(-\frac{iy^2 t}{4}\right) \quad (36)$$

We can rewrite eqn. (35) as

$$\Delta'^{(2)} = \frac{\beta_i(\hat{\omega}\tau+1+\eta_i)}{\rho_i} \left(\frac{L_s}{L_n \hat{\omega}\tau}\right)^{1/2} \left\{ \int_0^\infty dy \exp\left(-\frac{iy^2}{4}\right) + \int_0^\infty dy \frac{iy^2}{4} \int_0^1 dt \exp\left(-\frac{iy^2 t}{4}\right) \left[1 - (1-t)^{-\frac{1}{4}-ia/2} (1+t)^{-\frac{1}{4}+ia/2}\right] \right\} \quad (37)$$

Now the integration over  $y$  can be readily performed to yield:

$$\Delta'^{(2)} = \sqrt{\pi} e^{-i\pi/4} \frac{\beta_i(\hat{\omega}\tau+1+\eta_i)}{\rho_i} \left(\frac{L_s}{L_n \hat{\omega}\tau}\right)^{1/2} \left\{ 1 + \frac{1}{2} \int_0^1 dt t^{-\frac{3}{2}} \left[1 - (1-t)^{-\frac{1}{4}-ia/2} (1+t)^{-\frac{1}{4}+ia/2}\right] \right\} \quad (38)$$

Thus, we require the integral

$$\hat{I}(a) = \frac{1}{2} \int_0^1 dt t^{-3/2} \left[1 - (1-t)^{-\frac{1}{4}-ia/2} (1+t)^{-\frac{1}{4}+ia/2}\right] \quad (39)$$

This can be evaluated analytically, as shown in eqn. (B.9) of Appendix B:

$$\hat{I}(a) = -1 + \frac{\sqrt{(2\pi)} \Gamma\left(\frac{3-ia}{4}\right)}{\Gamma\left(\frac{1-ia}{4}\right)} \quad (40)$$

Thus, we have

$$\Delta'_{ILD} = \frac{\beta_i(\hat{\omega}\tau+1+\eta_i)}{2\rho_i} \left(\frac{L_s}{L_n \hat{\omega}\tau}\right)^{1/2} (I_0(a) + I_1(a)) \quad (41)$$

where

$$I_1(a) = \sqrt{2} \pi e^{-i\pi/4} \frac{\Gamma\left(\frac{3-ia}{4}\right)}{\Gamma\left(\frac{1-ia}{4}\right)} \quad (42)$$

which is evaluated numerically and displayed in Fig. 3. The combination

$$I(a) = I_0(a) + I_1(a) \quad (43)$$

is plotted in Fig. 4. When  $a = 0$ , the expression (41) reduces to the analytic result for the real part of  $\Delta'$  quoted in Coppi et al. (1979) for the case  $\eta_e = 0$ .

## 5. The Effective $\Delta'$ for Stability

The result (41) for  $\Delta'_{ILD}$ , the modification to  $\Delta'$  due to the ion sound/ion Landau damping physics, can be substituted into expressions in the literature for the drift

tearing mode growth rate at finite ion temperature (Cowley et al., 1986, Connor et al., 2012) which have ignored the presence of these effects.

For example, the semi-collisional tearing mode in the absence of the ion sound intermediate layer has a growth rate  $\hat{\gamma}$  given by  $\text{Im}(\delta\hat{\omega})$ , where  $\hat{\omega} = \hat{\omega}_0 + \delta\hat{\omega}$ , and  $\delta\hat{\omega}$  satisfies the dispersion relation:

$$\delta\hat{\omega} \sim -\frac{\delta_{sc}\hat{\omega}_0^{1/2}}{\pi\hat{\beta}} e^{-i\pi/4} (\Delta'_{ILD} - \Delta'_{crit}). \quad (44)$$

Here  $\hat{\omega}_0 = (1 + 0.74\eta_e)$ , with  $\nu_e$  the electron collision frequency and  $v_{Te}$  the electron thermal velocity,  $\delta_{sc}$  is the semi-collisional width ( $\delta_{sc} \ll \rho_i$ ) and

$$\Delta'_{crit} = \frac{\sqrt{\pi}\hat{\beta}}{\rho_i} \frac{(\hat{\omega}_0 - 1)^2}{\hat{\omega}_0} \left(1 + \hat{\omega}_0\tau - \frac{\eta_i}{2}\right) \ln\left(\rho_i / (\delta_{sc}\hat{\omega}_0^{1/2})\right) - \frac{\pi\hat{\beta}}{\rho_i} \hat{\omega}_0(\hat{\omega}_0 - 1)\bar{I}, \quad (45)$$

with  $\hat{\beta} = \beta_e(L_s/L_n)^2/2$ .  $\bar{I}(\hat{\omega}_0(\eta_e), \eta_i, \tau)$  is an integral that is negative and numerical evaluation shows that it is well represented by  $\bar{I} \cong -1.3\eta_e^{1/2}$  (Connor et al., 2012)<sup>1</sup>. The first contribution to  $\Delta'_{crit}$  in eqn. (45) is due to ion finite Larmor radius effects (FLR), while the second term represents diamagnetic effects. The most sensitive dependence of  $\Delta'_{crit}$  is on  $\eta_e$ .

To introduce the effect of the intermediate ion sound layer, we replace  $\Delta'$  in eqn. (44) by  $(\Delta' - \Delta'_{ILD})$ . Using eqns. (41) and (43), with the result for  $I(a)$  displayed in Fig. 4, we show in Fig. 5,  $\Delta'_{eff}$ , the critical value of  $\Delta'$  (i.e. the value of  $\Delta'$  for which the growth rate, as calculated from eqn. (44), vanishes) as a function of  $\eta_e$  for typical values of the other parameters:  $\hat{\beta} = 0.05$ ,  $\frac{\delta_0}{\rho_i} = 10^{-3}$ ,  $\tau = 1$ ,  $\eta_i = 0$ ,  $\frac{L_s}{L_n} = 10$ . Because  $\Delta'_{ILD}$  is complex, both its real and imaginary parts contribute to the growth rate in eqn. (44). We see that the contribution from the ion sound/ion Landau damping layer is less important than those arising from  $\Delta'_{crit}$ , representing the FLR and diamagnetic terms, except near  $\eta_e = 0$  where it remains finite. There it provides a persistent stabilising contribution, unlike the others, which tend to zero.

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<sup>1</sup> We point out that eqn. (45) corrects an evident misprint in the corresponding eqn. (42) in Connor et al. (2012). The corresponding figure there, Fig.4, was, however, correctly calculated.

## 6. Discussion and Conclusions

We have investigated the impact of ion sound and ion Landau damping on the effective tearing mode stability parameter,  $\Delta'$ , in hot tokamak plasma. These effects are essential in ensuring the parallel electric field approaches zero as one attempts to match solutions of the inner resonant layer equations to those of the external ideal MHD model. Because the tearing mode frequency is effectively determined by the resonant layer physics and depends on  $\eta_e$ , it transpires that the effect is strongest for weak electron temperature gradients. In this situation one can introduce an intermediate region, between that where the key resonant layer physics lies and the ideal MHD region, where electron drift wave physics dominates (Coppi et al. 1979). Indeed, by recalling the ideas of Pearlstein and Berk (Pearlstein and Berk, 1969) one does not need to explicitly consider the ion Landau damping region, replacing it by the boundary condition that the drift waves must only carry energy outwards. Matching solutions through this layer provides a link between the true  $\Delta'$  and the effective one seen by the inner layer.

The effect of the intermediate layer is parametrised by the quantity  $a = \frac{\omega\tau L_S}{2\omega_{*e}L_n} \frac{(\omega - \omega_{*e})}{(\omega\tau + (1 + \eta_i)\omega_{*e})}$ . Recalling that a good fit to the drift tearing mode frequency is  $\hat{\omega}_0 = (1 + 0.74\eta_e)$  (Connor et al., 2012), we note that  $a$  is indeed essentially a function of  $\eta_e$  and  $L_S/L_n$ . However, the validity of the theory requires the electron temperature gradient to be rather weak, with  $\eta_e \sim L_n/L_S$ , i.e.  $L_{Te} \sim L_S$ , so that  $a \sim 0(1)$  (Coppi et al., 1979). Note that a further validity condition,  $a \gg L_n/L_S$ , required for our analysis of the intermediate layer, is satisfied provided  $L_n/L_S \ll 1$ . For the case  $\hat{\omega} = \hat{\omega}_0$ ,  $\eta_e = \eta_i$ ,  $\tau = 1$ , this requires  $\eta_e > 5.4(L_n/L_S)^2$ . Although the effects of the intermediate region increase as  $\eta_e$  increases, a valid treatment for  $\eta_e \sim 0(1)$  would require a numerical calculation involving the full effects of ion Larmor radius and ion Landau damping, as the approximations inherent in the simple electron drift wave model, namely the long wavelength ion Larmor radius and the ion sound expansion, no longer hold.

Given these constraints on  $\eta_e$ , we have applied the results to a realistic hot tokamak plasma model, though in cylindrical geometry, where the ion Larmor orbit exceeds the reconnecting layer width, which itself is described by semi-collisional electron physics (Cowley et al., 1986; Connor et al., 2012) and compared the stabilising effects from the ion sound / ion Landau damping physics with other stabilising effects, as a function of  $\eta_e$ . The sum of these effects then leads to a critical value,  $\Delta'_{eff}$  for tearing instability. Although this calculation is strictly valid for a collision-less plasma slab in a sheared magnetic field geometry, it only ignores possible curvature, collisional and trapped particle effects. These

are weak in the range of  $x$  ( $\rho_i < x < \rho_i L_s / L_n$ ) that is relevant for the ion dynamics involved in this calculation; it is therefore a meaningful calculation for more general geometries.

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## Appendix A: Derivation of eqn. (20) for $I_0(a)$

This appendix leans heavily on results concerning the Confluent Hypergeometric functions, the Hypergeometric functions and the Parabolic Cylinder functions, respectively from Chapters 13, 15 and 19 of Abramowitz and Stegun, 1972.

Firstly  $E(-a, z)$  can be related to the usual Parabolic Cylinder function  $D_{ia-1/2}(e^{-i\pi/4}z)$ , so that

$$I_0(a) = -2a\pi \int_0^\infty dz D_{ia-1/2}(e^{-i\pi/4}z) / D_{ia-1/2}(0) \quad (\text{A.1})$$

Now  $D_{ia-1/2}(e^{-i\pi/4}z)$  itself can be expressed in terms of a Confluent Hypergeometric function,  $U$ :

$$D_{ia-1/2}(e^{-i\pi/4}z) = 2^{(ia/2-1/4)} e^{iz^2/4} U\left(1/4 - \frac{ia}{2}, \frac{1}{2}; -iz^2/2\right) \quad (\text{A.2})$$

The Confluent Hypergeometric function,  $U(a, b; x)$ , has an integral representation; using this we can write:

$$U\left(\frac{1}{4} - \frac{ia}{2}, \frac{1}{2}; -\frac{iz^2}{2}\right) = \frac{e^{-iz^2/2}}{\Gamma\left(\frac{1}{4}-ia/2\right)} \int_1^\infty dt e^{itz^2/2} (t-1)^{-\left(\frac{ia}{2}+3/4\right)} t^{(ia/2-3/4)} \quad (\text{A.3})$$

This enables one to perform the integration over  $z$  in the numerator of eqn. (A.1), which becomes:

$$\frac{2^{(ia/2-1/4)} e^{i\pi/4}}{\Gamma\left(\frac{1}{4}-ia/2\right)} \left(\frac{\pi}{2}\right)^{1/2} \int_1^\infty dt (t-1)^{-\left(\frac{ia}{2}+3/4\right)} t^{(ia/2-3/4)} (t-1/2)^{-1/2} \quad (\text{A.4})$$

Finally, introducing  $u = 1/t$ , result (A.4) becomes

$$\frac{2^{\left(\frac{ia}{2}-1/4\right)} e^{i\pi/4}}{\Gamma\left(\frac{1}{4}-ia/2\right)} \left(\frac{\pi}{2}\right)^{1/2} \int_0^1 du (1-u)^{-\left(\frac{ia}{2}+3/4\right)} (1-u/2)^{-1/2} \quad (\text{A.5})$$

Now the Hypergeometric function,  $F(a, b; c; x)$  also has an integral representation and one observes

$$F\left(\frac{1}{2}, \frac{1}{4} - \frac{ia}{2}; \frac{5}{4} - \frac{ia}{2}; -1\right) = \frac{\Gamma\left(\frac{5-ia}{4}\right)}{\Gamma\left(\frac{1-ia}{4}\right)\Gamma(1)} \int_0^1 dt t^{-\left(\frac{ia}{2}+3/4\right)} (1+t)^{-1/2} \quad (\text{A.6})$$

so that, on substituting  $t = 1 - u$ , we obtain

$$\begin{aligned} & F\left(\frac{1}{2}, \frac{1}{4} - \frac{ia}{2}; \frac{5}{4} - \frac{ia}{2}; -1\right) \\ &= \frac{\Gamma\left(\frac{5-ia}{4}\right)}{\sqrt{2}\Gamma\left(\frac{1-ia}{4}\right)\Gamma(1)} \int_0^1 du (1-u)^{-\left(\frac{ia}{2}+3/4\right)} (1-u/2)^{-1/2} \end{aligned} \quad (\text{A.7})$$

Thus, result (A.5) can be rewritten as:

$$\frac{\sqrt{\pi} 2^{\left(\frac{ia}{2}-1/4\right)} e^{i\pi/4}}{\Gamma\left(\frac{5-ia}{4}\right)} F\left(\frac{1}{2}, \frac{1}{4} - \frac{ia}{2}; \frac{5}{4} - \frac{ia}{2}; -1\right) \quad (\text{A.8})$$

Concerning the denominator in eqn. (A.1), we have:

$$D_{ia-1/2}(0) = \frac{\sqrt{\pi} 2^{\left(\frac{ia}{2}-1/4\right)}}{\Gamma\left(\frac{3-ia}{4}\right)} \quad (\text{A.9})$$

so that, finally, on using properties of the Gamma function:

$$I_0(a) = -\frac{2\pi e^{i\pi/4} a \Gamma\left(\frac{3-ia}{4}\right)}{\Gamma\left(\frac{5-ia}{4}\right)} F\left(\frac{1}{2}, \frac{1}{4} - \frac{ia}{2}; \frac{5}{4} - \frac{ia}{2}; -1\right) \quad (\text{A.10})$$

Expression (A.10) provides eqn. (20) of the main text.

## Appendix B: Derivation of eqn. (40) for $\hat{I}(a)$

We first need to evaluate integral  $\hat{I}(a)$  in eqn. (39). This is integrable because the two terms in the denominator cancel as  $t \rightarrow 0$ . It is convenient to consider it as the difference of the two diverging integrals:

$$\begin{aligned} \hat{I}(a) &= \frac{1}{2} \lim_{\delta \rightarrow 0} \int_{\delta}^1 \frac{dt}{t^{\frac{3}{2}}} - \lim_{\delta \rightarrow 0} \frac{1}{2} \int_{\delta}^1 \frac{dt}{t^{\frac{3}{2}}} (1-t)^{-\frac{ia}{2}-\frac{1}{4}} (1+t)^{\frac{ia}{2}-\frac{1}{4}} \\ &\equiv J_1(\delta) - J_2(\delta, a) \end{aligned} \quad (\text{B.1})$$

The first is readily evaluated:

$$J_1(\delta) = -1 + \frac{1}{\delta^{1/2}} \quad (\text{B.2})$$

For the second we substitute,  $t = 1/(2v + 1)$ , to obtain

$$J_2(\delta, a) = 2^{-1/2} \int_0^{\delta} dv v^{-\frac{ia}{2}-1/4} (1+v)^{\frac{ia}{2}-1/4} \quad (\text{B.3})$$

where  $z = \frac{1-\delta}{2\delta}$ . Now  $J_2(\delta, a)$  can be expressed in terms of the Incomplete Beta function,  $B_{1-2\delta}\left(\frac{3}{4} - \frac{ia}{2}, -\frac{1}{2}\right)$  (see sections **1.5** and **2.53** of Erdelyi, A, 1953):

$$J_2(\delta, a) = 2^{-1/2} B_{1-2\delta}\left(\frac{3}{4} - \frac{ia}{2}, -\frac{1}{2}\right) \quad (\text{B.4})$$

The Incomplete Beta function can be expressed in terms of the Hypergeometric function,  $F(a, b; c; z)$  (Abramowitz and Stegun, 1972):

$$B_{1-2\delta}\left(\frac{3}{4} - \frac{ia}{2}, -\frac{1}{2}\right) = \frac{(1-2\delta)^{-\frac{ia}{2}+3/4}}{\left(-\frac{ia}{2}+3/4\right)} F\left(\frac{3}{4} - \frac{ia}{2}, \frac{3}{2}; \frac{7}{4} - \frac{ia}{2}; 1 - 2\delta\right), \quad (\text{B.5})$$

A suitable transformation for considering the limit  $\delta \rightarrow 0$  is the relation

$$\begin{aligned} F(a, b; c; z) &= \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b-c+1; 1-z) + \\ &(1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b; c-a-b+1; 1-z) \end{aligned} \quad (\text{B.6})$$

where we can then use the result

$$F(a, b; c; 0) = 1 \quad (\text{B.7})$$

(Abramowitz and Stegun, 1972) to take the limit  $\delta \rightarrow 0$ . The result is

$$J_2(\delta, a) = \frac{1}{\delta^{1/2}} - \sqrt{2\pi} \frac{\Gamma\left(\frac{3}{4} - \frac{ia}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{ia}{2}\right)} \quad (\text{B.8})$$

Combining with the result (B.2), we obtain

$$\hat{I}(a) = -1 + \sqrt{2\pi} \frac{\Gamma\left(\frac{3}{4} - \frac{ia}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{ia}{2}\right)} \quad (\text{B.9})$$

This is the result quoted in eqn. (40).

## Figure Captions

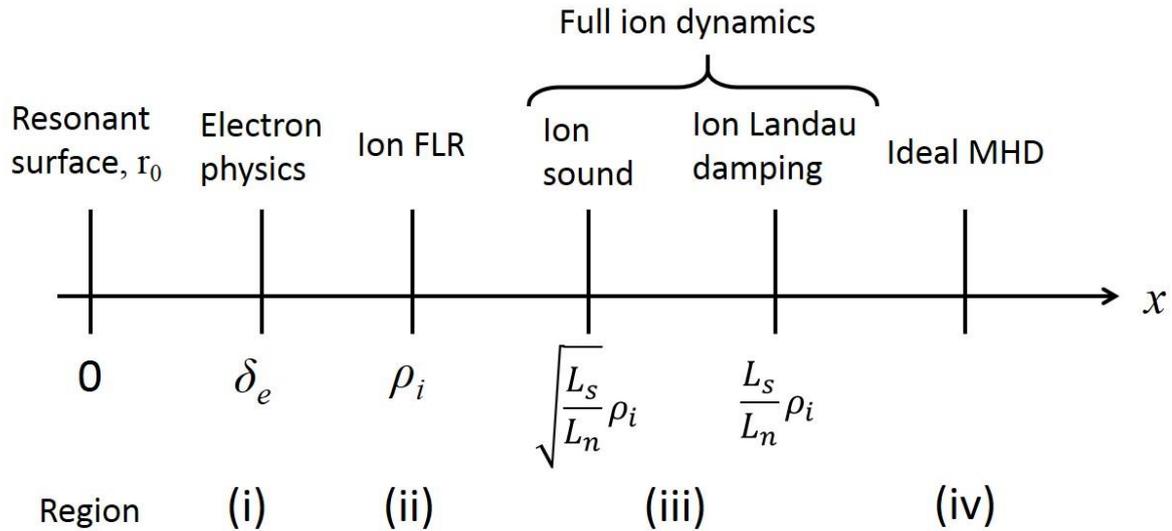
Fig. 1 The regions in which different physical processes occur as the distance,  $x = r - r_0$ , from the resonant surface,  $r_0$ , increases: (i) electron reconnecting physics at a scale  $\delta_e$ ; (ii) ion finite Larmor radius effects at a scale  $\rho_i$ ; (iii) full ion parallel dynamics, namely ion sound at a scale  $(L_s/L_n)^{1/2} \rho_i$  and ion Landau damping effects at a scale  $(L_s/L_n)\rho_i$ ; and (iv) the outer ideal MHD region.

Fig. 2 The real and imaginary parts of  $I_0(a)$  as a function of the parameter  $a = (L_s \hat{\omega} \tau / 2L_n)(\hat{\omega} - 1) / (\hat{\omega} \tau + 1 + \eta_i)$ .

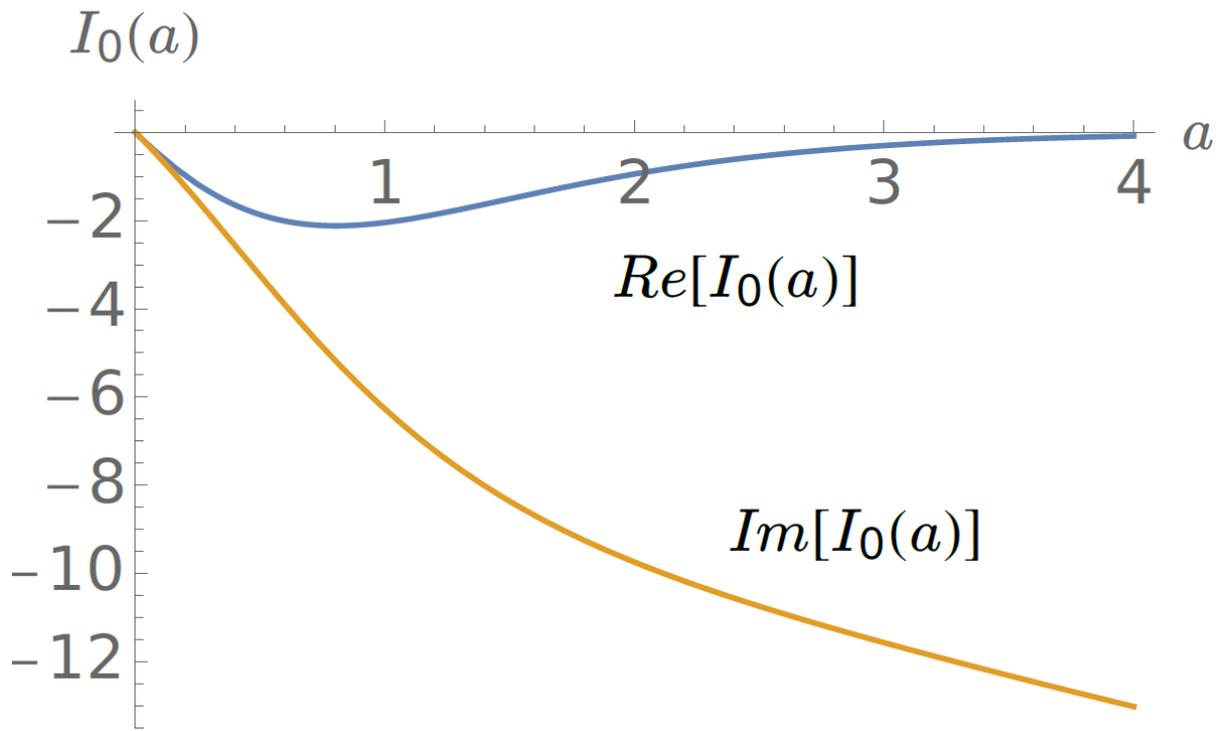
Fig. 3 The real and imaginary parts of  $I_1(a)$  as a function of the parameter  $a = (L_s \hat{\omega} \tau / 2L_n)(\hat{\omega} - 1) / (\hat{\omega} \tau + 1 + \eta_i)$ .

Fig. 4 The real and imaginary parts of  $I(a) = I_0(a) + I_1(a)$  as a function of the parameter  $a = (L_s \hat{\omega} \tau / 2L_n)(\hat{\omega} - 1) / (\hat{\omega} \tau + 1 + \eta_i)$ .

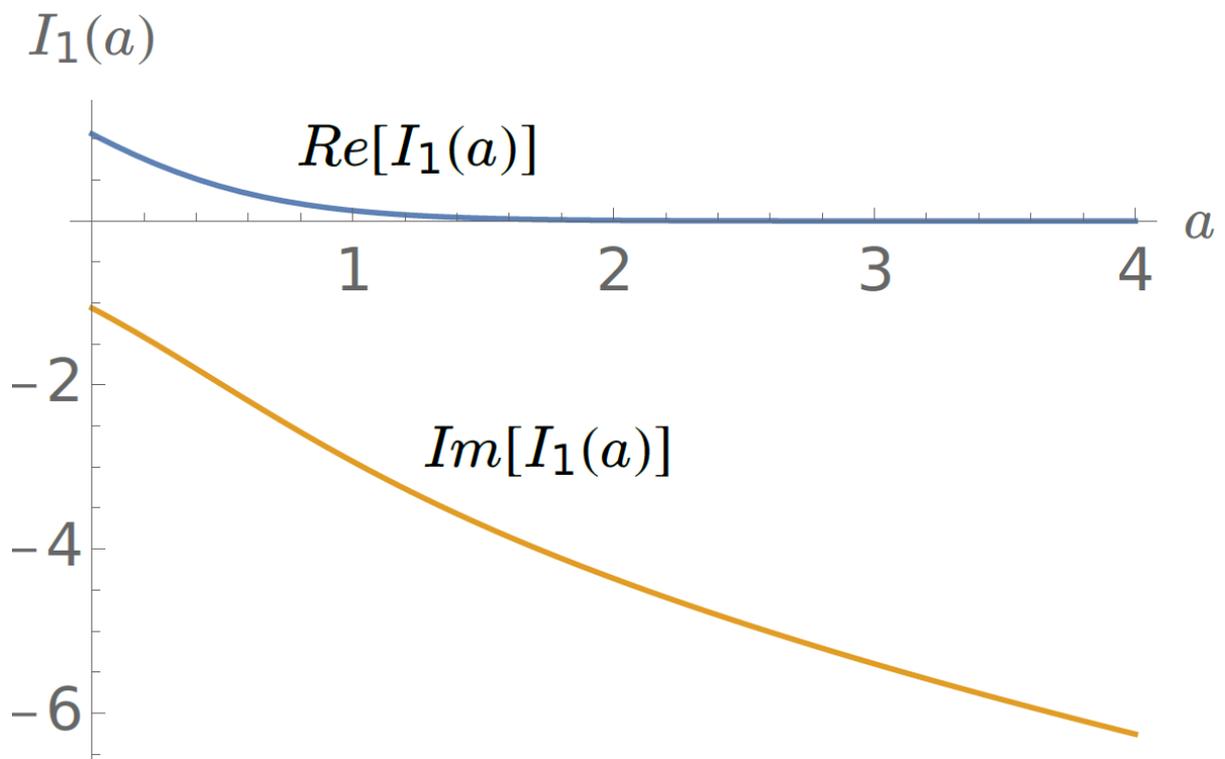
Fig. 5 The effective critical value of  $\Delta' \rho_i$ ,  $\Delta'_{eff} \rho_i$  (full (blue) line), as a function of  $\eta_e$  for typical values of the other parameters:  $\hat{\beta} = 0.05$ ,  $\frac{\delta_0}{\rho_i} = 10^{-3}$ ,  $\tau = 1$ ,  $\eta_i = 0$ ,  $\frac{L_s}{L_n} = 10$ .  $\Delta'_{crit}$  is the contribution from FLR and diamagnetic effects (long dashed (brown) line), while  $(Re[\Delta'_{ILD}] - Im[\Delta'_{ILD}])$  is the contribution from the real and imaginary parts of the ion Landau damping terms (dashed (green) line).



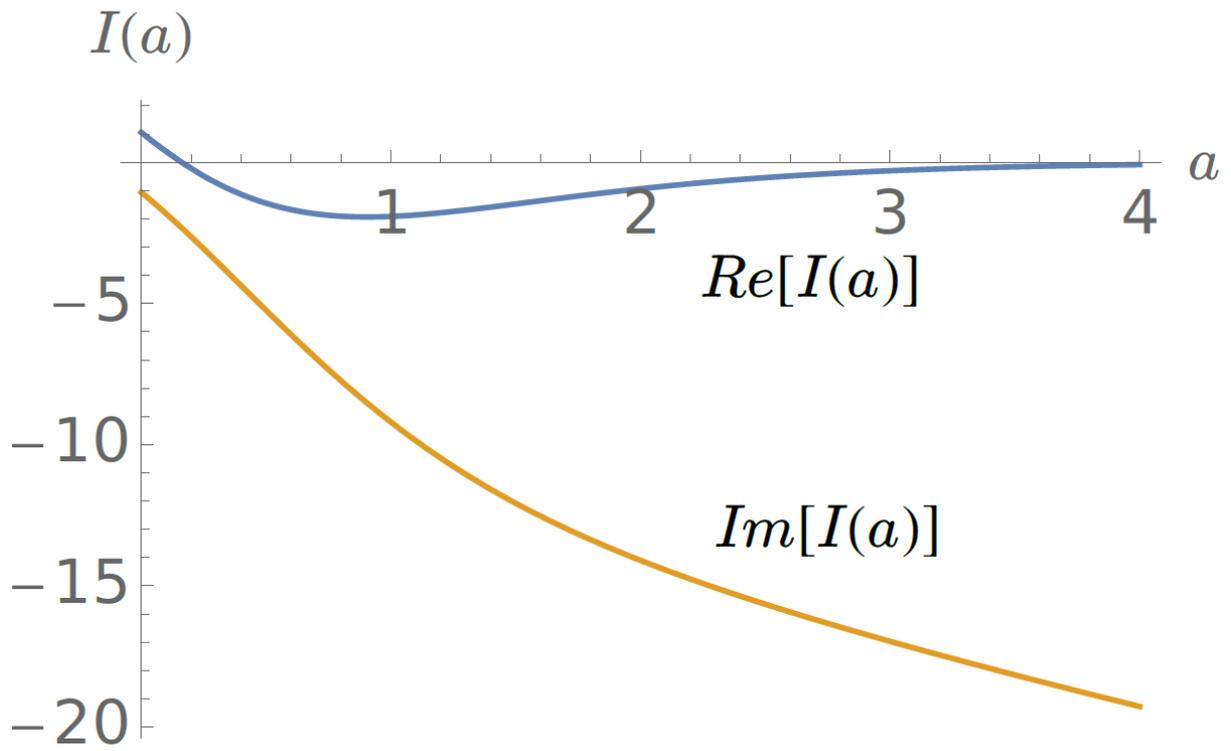
**Fig. 1**



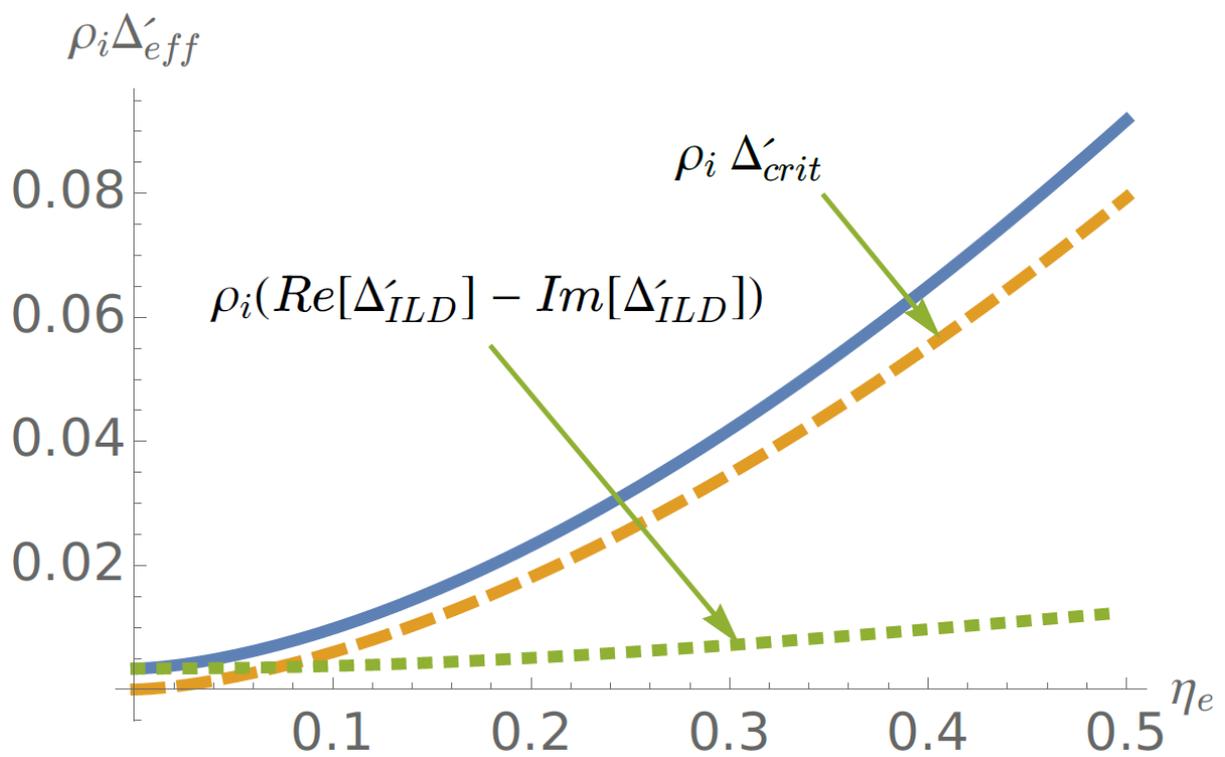
**Fig. 2**



**Fig. 3**



**Fig. 4**



**Fig. 5**

**J W Connor, Nov 1, 2018**