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Using dynamical mode decomposition to extract the limit cycle dynamics of modulated turbulence in a plasma simulation

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Abstract. The novel technique of dynamical mode decomposition (DMD) is applied to the outputs of a numerical simulation of Kelvin-Helmholtz turbulence in a cylindical plasma, so as to capture and quantify the time evolution of the dominant nonlinear structures. These structures comprise rotationally symmetric deformations together with spiral patterns, which are shown to be identifiable as DMD modes. A new method to calculate the time evolution of DMD mode amplitudes is proposed, based on convolution-type correlation integrals, and then applied to the simulation outputs in a limit cycle regime. The resulting time traces capture the essential physics far better than Fourier techniques applied to the same data.

1. Introduction

Strongly nonlinear phenomena are ubiquitous in plasma physics, both in experimental measurements and in the outputs from numerical simulations. The nonlinear phenomenology may be temporally transient [1] or sustained [2], spatially localised [3] or global [4]. Identifying the dominant dynamical features and their interactions, and quantifying their time evolution, is therefore a central task. Fourier decomposition has major limitations in this context, because the empirically identified key structures typically require a very large number of Fourier modes to represent them.

Here we consider the application of dynamical mode decomposition (DMD) [5, 6, 7] to this problem. DMD is attractive, in that it: assumes no functional form for the structures; is entirely data-driven, see Eq. (1) below; and is mathematically linear - mode identification and growth rates reduce to an eigenvector-eigenvalue procedure. When the time evolution of the mode amplitude is modulated, as in most cases of turbulence, the single DMD derived growth rate is insufficient to capture the dynamics. Here we therefore propose and develop a method to extract the modulation dynamics from the outputs of the DMD technique, as applied to a simulation of turbulence in a cylindrical plasma.

2. Extraction of nonlinear dynamics

The turbulence dataset is obtained from a direct numerical simulation, based on an extension of the Hasega-Wakatani reduced fluid model which includes ion-neutral collisions and electron parallel velocity evolution [8, 9]. Turbulent and nonlinear phenomena can be simulated, such as those arising from resistive drift waves and the Kelvin-Helmholtz instability in linear devices [10, 11]. The turbulence addressed here originates from the Kelvin-Helmholtz instability for the plasma parameters in MISTRAL [11]. Its phenomenology includes a limit cycle oscillation between the background plasma and turbulent fluctuations; for more detail, see [12]. The time evolution of the energy of each Fourier mode in a saturated state, and the two-dimensional patterns of the density at $t = 3050, 3150$ and 3200 are shown in Fig. 1, where time t is normalized by the ion gyrofrequency. The energies (squared amplitudes) of the background and the turbulence are modulated in time: the period of the limit cycle, $T_{LCO} \sim 100$, which is much longer than the timescale of turbulent oscillation, $T_{turb} = O(10)$. The computational time-step is much smaller, $\delta t = 2 \times 10^{-2}$. The spatial pattern in Fig. 1 changes on the timescale T_{LCO}. Let us now apply the DMD to the underlying dataset, and then propose and develop a novel method to extract the modulation dynamics.

We represent the system at time t by an array (state vector) $\mathbf{X} = \mathbf{X}(\mathbf{r}_1, \mathbf{r}_2, \cdots | t_1, t_2, \cdots)$, which is a matrix recording the value of the set of simulation outputs $\mathbf{X}$ (for example, density) at each point $\boldsymbol{r}_j$ and at each time t_j. The system transits to the state $\mathbf{X}' = \mathbf{X}(\mathbf{r}_1, \mathbf{r}_2, \cdots | t_1 + \Delta t, t_2 + \Delta t, \cdots)$, where Δt is the unit of time resolution chosen for DMD analysis. Here $\Delta t = 5$, which is large enough to reduce the computational cost, while remaining sufficient to resolve the turbulence evolution. In the DMD approach, we focus on the properties of the matrix A which generates the mapping

$$\mathbf{X}' = A\mathbf{X}. \tag{1}$$

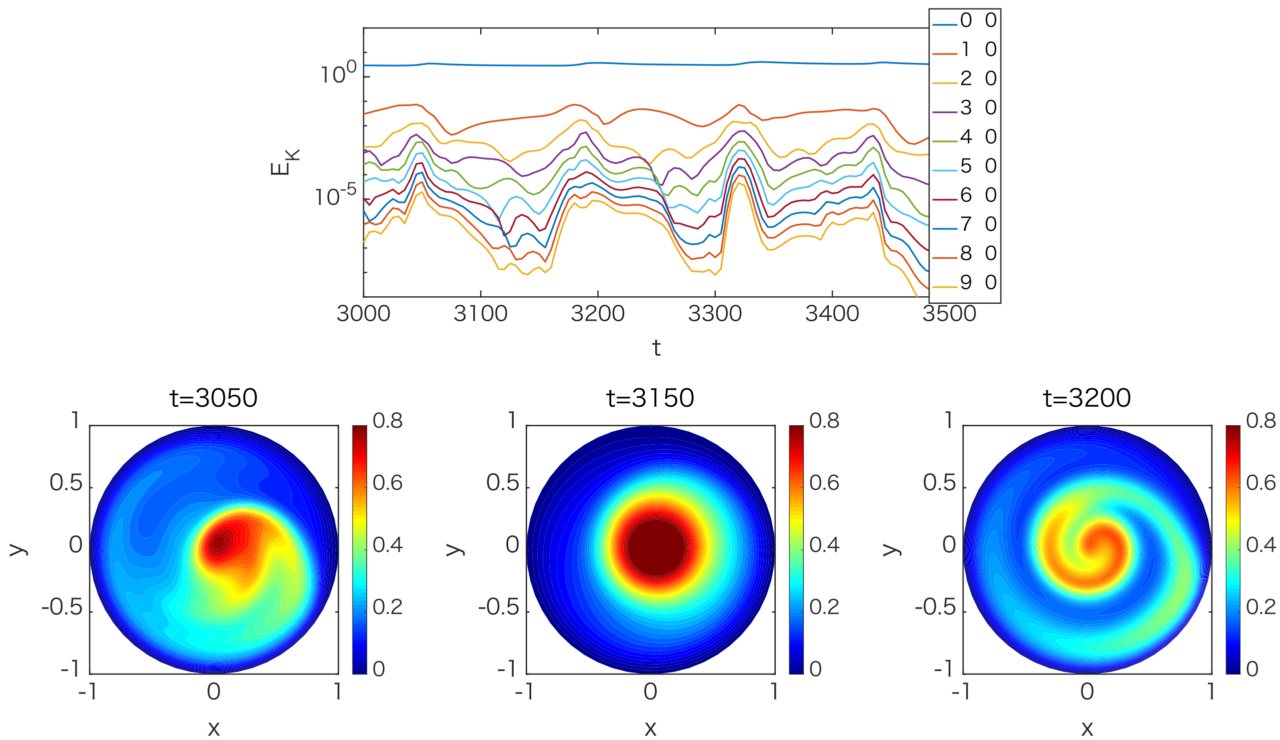

Figure 1. Top: Time evolution of the energy (logarithmic scale) in each cylindrical Fourier mode (m, n) of the simulation in its saturated phase, for integer $0 < m < 9$ and $n = 0$. Here m and n denote axial and azimuthal mode numbers. The background corresponds to the mode $(0, 0)$. Quasiperiodic energy flows are evident, and the flow into higher m-numbers indicates the formation of sharper spatial gradients associated with nonlinear structures at those times. Bottom: Full two-dimensional spatial patterns of the density from the direct numerical simulation at $t = 3050, 3150$ and 3200.

Whereas $\mathbf{X}$ and $\mathbf{X}'$ comprise datasets, A is taken to embody the physical dynamics of, in the present case, Kelvin-Helmoltz plasma turbulence. The challenge is, first, to reduce the rank of A to manageable level using singular value decomposition (SVD) [13], and then to identify the dominant eigenvalues and eigenvectors of A. The eigenvectors Ψ are the DMD modes: they correspond to the dominant nonlinear spatial structures, and represent their action in the time evolution of the data. The details of the DMD approach are summarized in the Appendix. In outline, mathematically,

$$\Psi = \mathbf{X}' V_r \Sigma_r^{-1} \boldsymbol{\xi}. \tag{2}$$

Here, the matrices V and Σ are obtained from the SVD of $\mathbf{X}$, and satisfy $\mathbf{X} = U\Sigma V^*$; U and V are unitary matrices, and Σ is the diagonal matrix consisting of the singular values of $\boldsymbol{X}$. The subscript r indicates the matrix is truncated to the rank r. $\boldsymbol{\xi}$ is the eigenvector of $U_r^* A U_r$, which is the projection of A on U. In this way, the key structures together with their frequency and growth rate are simultaneously obtained by DMD. This approach is model-independent and does not draw on knowledge of the underlying physical processes. The DMD eigenmodes are typically strongly nonlinear spatial structures, which would require numerous Fourier modes to represent them.

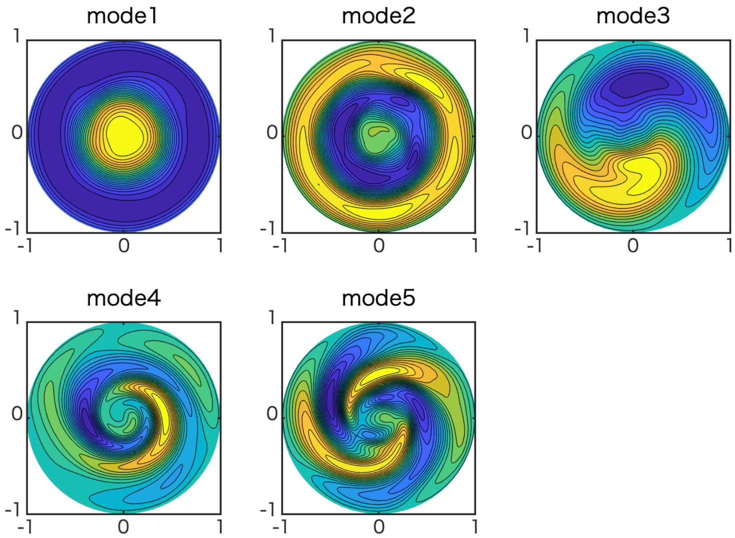

Figure 2. The five dominant eigenvectors, in the form of two-dimensional spatial patterns, derived from DMD analysis of the simulation outputs. These are the dominant nonlinear structures, Mode1 to Mode5, discussed in the main text.

Hence, the DMD approach greatly reduces the number of effective degree of freedom, compared to a Fourier-based approach.

Figure 2 illustrates the leading DMD modes obtained for the density fluctuation $\tilde{n}(r, \theta, t)$ in the turbulence simulation, where the rank r of A is truncated to $r = 9$. These DMD modes show the characteristic spatial structures: Mode1 and Mode2 correspond to the deformation of the background; Mode3 is the dominant fluctuation pattern; and Mode4 and Mode5 are the spiral structures, which transiently appear and disappear on the timescale of the limit cycle oscillation [12]. All the physical structures rotate in the azimuthal direction, so that each eigenvector has a counterpart complex conjugate pattern. Together, they represent each rotating mode, and each mode in the pair has the same eigenvalue as its complex conjugate. The real and imaginary parts of the DMD eigenvalue define the frequency and growth rate, respectively, of the corresponding DMD mode. However, when the turbulence is modulated, as in the case of the limit cycle oscillation here, a single growth rate cannot express the temporal dynamics. Thus, we must now develop a method to extract how the amplitude of each DMD mode changes with time.

We first propose a method to define the magnitude of each DMD mode. By calculating the instantaneous correlation coefficient between each DMD mode and the full turbulence dataset, the dynamical change of the amplitudes of a DMD mode can be deduced. This correlation can be estimated from the convolution integral

$$F_j(r, \theta, t) = \frac{\int \tilde{n}(r - r', \theta - \theta', t)\hat{\Psi}_j(r', \theta')r'dr'd\theta'}{\int \tilde{n}(r', \theta', t)r'dr'd\theta'}. \quad (3)$$

Here $\hat{\Psi}_j$ is the j-th DMD mode, normalized such that the two-dimensional spatial integral is unity. Recalling that the spatial pattern of the turbulence propagates in

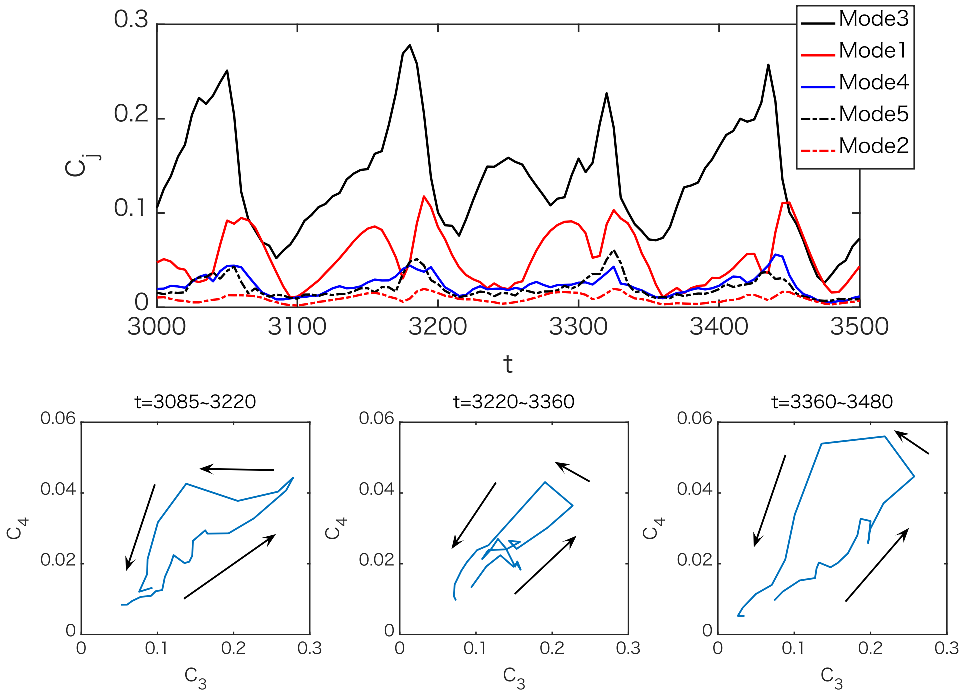

Figure 3. Top: Time evolution of the correlation coefficient C_j (see Eqs.(3) and (4)), tracing the changing relative amplitude of Mode1 to Mode5 (see Fig.2) in the simulation outputs. Bottom: Time evolution of the system plotted in the (C_3, C_4) plane, for three successive cycles identical from the upper panel. These closed Lissajous figures demonstrate the limit cycle dynamics that govern C_3 (KH instability) and C_4 (spiral structure) in combination.

the azimuthal direction, the correlation can be defined as

$$C_j(t) = \max[F_j((r, \theta, t))]. \tag{4}$$

C_j defines an effective amplitude for each DMD mode. The calculated time evolution of C_j is plotted in Fig. 3 for Mode1 to Mode5. This captures the changing contribution of each structure to the overall turbulence. The limit cycle between the deformation of the background and the dominant turbulence, with the appearance and disappearance of the spiral structure [12], is immediately evident. For example, the closed cycle Lissajous figures in (C_3, C_4) plane space shown in the lower panel of Fig. 3 clearly capture the causal relation between the KH instability and the spiral structure. They increase together, and then at a critical amplitude of the KH instability, the spiral structure becomes suddenly stronger, which leads to the suppression of the KH instability. Both amplitudes then decline to their starting point. Thus, the growth in the amplitude of the KH instability is constrained, and eventually reversed, by the excitation of the spiral structure, which itself finally decays. The next circulation on this limit cycle then commences. The approach presented here, of combining method DMD and the correlation integral, Eq. (3), enables one to create an approximation to the attractor for this strongly nonlinear and turbulent plasma system. We note that the correlation integral approach introduced here could also be used in the same way for the SVD [13, 14, 15] and proper orthogonal decomposition (POD) [16] methods. It is also potentially relevant to experimental imaging techniques such as those exploiting gas puffing [17, 18, 19], beam emission spectroscopy [20, 21], and visible light tomography [22, 23].

3. Conclusions

We have shown that the DMD technique, augmented by the correlation integral approach introduced in Eqs. (3) and (4), has great potential for the quantitative characterization of turbulent and strongly nonlinear phenomenology in plasmas. Using this method, we have systematically extracted the time evolution of the magnitude of each of the dominant, spatially coherent, global nonlinear structures (Fig. 3, upper), together with their coupled cyclic behavior (Fig. 3, lower). This would not be extremely difficult using Fourier mode decomposition. The method introduced here remains valid, even when the amplitude of the structure changes drastically on a timescale much longer than the typical fluctuation period; whereas the conventional DMD method applies on shorter timescales, comparable to the turbulence period. Hence, by combining conventional DMD with the present method, turbulence phenomenology that is multi-timescale (from the turbulence timescale to the transport timescale) can be systematically addressed and quantified.

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Appendix A. A brief summary of dynamical mode decomposition

In the DMD method [5]-[7], the dynamical system is expressed as $\mathbf{X}(\mathbf{r}_1, \mathbf{r}_2, \cdots | t_1, t_2, \cdots)$, where $\boldsymbol{r}_j$ and t_j are the measurement location and time, respectively. So, if one observes the system with grids that span space with N elements and time with M elements, the size of the matrix A is $N \times M$. The DMD method assumes that the system can be described by the linear combination of nonlinear dynamical states, as in Eq. (1), where the operator A governs the system evolution. The operator A is determined entirely from the observable data, as

$$A = \boldsymbol{X}'\boldsymbol{X}^{\dagger}. \tag{A.1}$$

The size of $\boldsymbol{X}$ is usually so large that a reduction of the dataset is necessary. To achieve this, we use the SVD technique [13]. Formally, we write

$$\begin{aligned} \boldsymbol{X} &= U\Sigma V^*, \\ &\approx U_r\Sigma_r V_r^*, \end{aligned} \tag{A.2}$$

where the subscript r denotes the r-rank truncation. Here, a general matrix is decomposed into two unitary matrices, U and V, which are combined with the diagonal matrix Σ, containing the singular values of the original matrix; $\Sigma_{ii} \neq 0, \Sigma_{ij} = 0$ $(i \neq j)$ [13]. It is necessary to construct the matrix $\tilde{A}$, which is the projection of A onto U_r:

$$\begin{aligned} \tilde{A} &= U_r^* A U_r \\ &= U_r^* \boldsymbol{X}' V_r \Sigma_r^{-1}. \end{aligned} \tag{A.3}$$

The eigenvalue problem for A is then recast as

$$\tilde{A}\boldsymbol{\xi} = \Lambda\boldsymbol{\xi}. \tag{A.4}$$

The eigenvalues Λ and eigenvectors $\boldsymbol{\xi}$ of $\tilde{A}$ are next obtained from Eq. (A.4). Because the eigenvalues of A and $\tilde{A}$ are the same, the eigenvector of A, Ψ, is given as

$$\Psi = U_r'\boldsymbol{\xi} = \boldsymbol{X}' V_r \Sigma_r^{-1}\boldsymbol{\xi}. \tag{A.5}$$

The eigenvector Ψ is called the DMD mode. Finally, the time evolution of the system $\boldsymbol{x}(t)$ is expressed by using DMD modes as

$$\boldsymbol{x}(t) = \Psi e^{\boldsymbol{\Omega} t}\Psi^{\dagger}\boldsymbol{x}(0), \tag{A.6}$$

where $\boldsymbol{x}(0)$ is the initial condition. We emphasize that this expression can be used only for the short timescale evolution, comparable to the fluctuation period, and given monotonic growth or damping. This is because the mode amplitude $\Psi^{\dagger}\boldsymbol{x}(0)$ in Eq. (A.6) is constant in time. Quantifying the time evolution of the modulated turbulence, where the amplitude changes dynamically, is therefore difficult using DMD alone; hence the present paper.

[1] J. W. Connor, Plasma Phys. Control. Fusion, **40**, 531 (1998).
[2] P. H. Diamond, et. al., Plasma Phys. Control. Fusion, **47**, R35 (2005).
[3] S. I. Krasheninnikov, et. al., Nucl. Fusion, **57**, 102010 (2017).
[4] F. Wagner, et. al., Phys. Rev. Lett., **49**, 1408 (1982).
[5] M. R. Jovanovic, et. al., Phys. Fluids, **26**, 024103 (2014).
[6] R. Taylor, et. al., Rev. Sci. Instruments, **89**, 053501 (2018).
[7] M. S. Hemati, et. al., Phys. Fluids, **26**, 111701 (2014).
[8] N. Kasuya, et. al., Phys. Plasmas, **15**, 052302 (2008).
[9] M. Sasaki, et. al., Phys. Plasmas, **24**, 112103 (2017).
[10] S. Inagaki, et. al., Sci. Rep. **6**, 22189 (2016).
[11] P. A. David, et. al., Phys. Plasmas, **23**, 103511 (2016).
[12] M. Sasaki, et. al., Phys. Plasmas, **26**, 042305 (2019).

[13] N. Nishino, et. al., J. Plasma Fusion Res.,**86**, 648 (2010).
[14] S. Ohdachi, et. al., Rev. Sci. Instruments,**74**, 2136 (2003).
[15] D. del-Castillo-Negrete, et. al., J. Comp. Phys,**222**, 265 (2007).
[16] P. Beyer, et. al., Phys. Rev. E, **61**, 813 (2000).
[17] S. J. Zweben, et. al., Phys. Plasmas, **9**, 1981 (2002).
[18] R. J. Maqueda, et. al., Rev. Sci. Instruments, **74**, 2020 (2003).
[19] J. R. Myra, et. al., Plasma Phys. Control. Fusion, **60**, 075015 (2018).
[20] G. R. McKee, et. al., Rev. Sci. Instruments, **75**, 3490 (2004).
[21] C. Fenzi, et. al., Rev. Sci. Instruments, **72**, 988 (2001).
[22] D. Iraji, et. al., Phys. Plasmas, **17**, 122304 (2010).
[23] A. Fujisawa, et. al., Plasma Phys. Control. Fusion, **58**, 025005 (2016).