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Solution of the linear wave-particle kinetic equation for global modes of arbitrary frequency in a tokamak

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Abstract

The linear response of a plasma to perturbations of arbitrary frequency and wavelength is derived for any axisymmetric magnetized toroidal plasma. An explicit transformation to action-angle coordinates is achieved using orthogonal magnetic coordinates and the Littlejohn Lagrangian, establishing the validity of this result to arbitrary order in normalized Larmor radius. The global resonance condition for compressional modes is clarified in more detail than in previous works, confirming that the poloidal orbit-average of the cyclotron frequency gives the desired result at lowest order in Larmor radius. The global plasma response to the perturbation at each resonance is captured by a poloidal and gyro average of the perturbing potential. A "global gyroaveraging" of the potential is a natural by-product of this analysis which takes into account the changing of the magnetic field over an orbit. The resonance condition depends on two arbitrary integers which completely separately capture the effects poloidal non-uniformity and finite Larmor radius in generating sidebands. We learn that poloidal sidebands generated for compressional modes are dominated by the change in gyrofrequency over the orbit, which is very different to shear modes where the gyrofrequency only contributes via a finite Larmor radius effect.

Introduction

The wave-particle interaction is of fundamental importance to predicting the stability of fusion plasma. It occurs because a minority of plasma particles have transit velocities comparable with the phase speed of the waves, leading to a situation where small perturbations in the electromagnetic fields will grow as particle energy is depleted through a coherent sustained interaction. This resonant power transfer process is in competition with other processes in the thermal plasma which remove energy from the waves. For sufficiently short timescales and small perturbations, linear kinetic theory provides the solution to the expected dynamics of the resonant particles, thereby predicting the drive of the instability. The linear drift-kinetic [1] and gyrokinetic [2,3] approximations to the Vlasov equation have been applied successfully to the prediction of shear Alfven eigenmodes (AEs) [4][5][6] in conventional aspect ratio tokamaks where the low frequency applicability criterion $\omega \ll \Omega_c$ is satisfied, with Ω_c a characteristic cyclotron frequency for any species in the plasma. Compressional Alfven eigenmodes (CAEs) observed on spherical tokamaks such as MAST [7] and NSTX [8], as well as conventional tokamaks such as DIII-D [9] exist at frequencies comparable the ion-cyclotron frequency range and are likely driven by resonant energetic particle interactions, as is ion-cyclotron emission (ICE) observed on many tokamaks [10][11].

It was understood quite early [12][13] that geometrical effects from non-uniform equilibrium are an important ingredient in the resonant wave-particle problem for shear Alfven waves, making the description of the linear kinetic solution necessarily global. Resonant transfer of energy between shear waves and particles in the drift and gyrokinetic theories requires synchronization between wave phase and the motion of the particle guiding centre such that

$$n\langle\dot{\phi}\rangle + p\langle\dot{\theta}\rangle - \omega = 0 \tag{1}$$

where $\dot{\theta}$ and $\dot{\phi}$ are the poloidal and toroidal angular velocities for the particle guiding centre, and $\langle \ldots \rangle$ denotes an average over the guiding centre orbit. This resonance condition can be derived by supposing how guiding centres, not particles, synchronize with waves over the entirety of their orbits. On the other hand, the theory of wave-particle interaction in uniform unmagnetized plasma (e.g.: [14]) gives the Cerenkov condition

$$\boldsymbol{k} \cdot \boldsymbol{v} - \omega = 0 \tag{2}$$

and particles in uniform magnetized plasmas are subject to the gyromagnetic resonance condition

$$k_{\parallel} v_{\parallel} - l\Omega_c - \omega = 0 \tag{3}$$

for arbitrary integer l. The electromagnetic quasilinear theories [15], relied on heavily for the study of radio-frequency heating in tokamaks, typically work with a geometrical optics approximation where the local resonance conditions are valid. Wave-particle encounters are localised and brief in these models, and particle gyrophase is deemed incoherent between encounters – these are random kicks that happen consistently at the same place along an orbit. This is a fundamentally different picture to the global resonance condition equation 1 where the wave phase information is retained at all times for the motion. A resonance condition which retains gyrophase coherently over the entire orbit is required for the wave-particle interaction with CAEs, particularly if the bounce time is becoming comparable with the cyclotron period as in MAST and NSTX, where as it happens, the cyclotron period is also ill-defined.

A plausible global gyromagnetic resonance condition could be arrived at by simply decomposing the velocity in the Cerenkov condition $\boldsymbol{k} \cdot \boldsymbol{v} - \omega = 0$ into parallel, drift, and Larmor components, but this does not account for the change in velocity over the course of the orbit, most notably in the Larmor component for CAEs. One would further have to perform an average as done in equation 1. These sorts of conclusions have been reached by previous authors [16][17], however important details on the strength of sidebands, definitions of the quantities, and reconciliation with existing Alfven wave theory, are not obvious. This information is important to extend existing resonance map methods to CAEs, which for AEs have shown a range of useful applications in understanding experimental stability and loss [18][19][20]. Estimating the size of the contribution from sideband resonances is a particularly important consideration, as it is often the case that a vast number of resonant particles can be identified whose wave-particle interaction is found negligible when later compared to the numerically obtained power transfer.

We wish to clarify the global linear kinetic theory and resonance condition for arbitrary mode frequency, and put this clearly alongside limiting cases identified previously. Our solution draws on a few different existing works; we use the action-angle approach based on Kaufman [21] which was repurposed successfully for analytical work on shear Alfven waves [22][23][24], where most of the important physics was captured by the drift-kinetic model. Rather than focus on the power transfer for say a TAE mode, we wish to clarify the form of the global resonance condition and linear solution [1] applicable to all waves including CAEs and ICE, where we de-mystify the role of particle gyration in global problems. We do this by retaining oft-neglected gyroangle information in the Littlejohn guiding centre model [25] which appears more ordinarily in the quasilinear theory of radio-frequency heating of plasma [26][27][28].

Guiding centre model retaining gyration for compressional modes

We wish to develop a collisionless theory for the resonant wave-particle interaction between fast particles and eigenmodes with frequencies that can compare with the cyclotron frequency and beyond. Such a theory must necessarily include the gyration of the particles.

Starting with the 6-D Vlasov phase space Lagrangian governing the full-orbit particle motion, we separate the fields into time varying and constant pieces

$$L(\boldsymbol{x},\boldsymbol{v},\dot{\boldsymbol{x}},\dot{\boldsymbol{v}},t)=\left[Ze\big(\boldsymbol{A}_0(\boldsymbol{x})+\delta\boldsymbol{A}(x,t)\big)+m\boldsymbol{v}\right]\cdot\dot{\boldsymbol{x}}-\left[\frac{1}{2}mv^2+Ze(\Phi_0(\boldsymbol{x})+\delta\Phi(\boldsymbol{x},t))\right] \tag{4}$$

To analyse the periodic motion of confined equilibrium particles, we wish to introduce the periods associated with gyration, toroidal and poloidal timescales. In general, the 6D motion of particles is not integrable, which is equivalent to saying that not all particles in all fields are confined to periodic orbits. For linear perturbation analysis, and to show confined particles, it will be convenient to express all quantities in terms of the periodic equilibrium motions. We therefore want a Hamiltonian structure separating the equilibrium and perturbed fields and consequent motion

$$L(\boldsymbol{x},\boldsymbol{v},\dot{\boldsymbol{x}},\dot{\boldsymbol{v}},t)=[Ze\boldsymbol{A}_0(\boldsymbol{x})+m\boldsymbol{v}]\cdot\dot{\boldsymbol{x}}-\left[\frac{1}{2}mv^2+Ze\Phi_0(\boldsymbol{x})\right]+Ze\delta\Phi(\boldsymbol{x},t)+Ze\delta\boldsymbol{A}(\boldsymbol{x},t)\cdot\dot{\boldsymbol{x}} \tag{5}$$

This is almost a Hamiltonian structure, except for the term $Ze\delta\boldsymbol{A}(\boldsymbol{x},t)\cdot\dot{\boldsymbol{x}}$. Previous variational treatments for the Alfven wave with guiding centre motion [22] have made the following non-canonical transformation which redefines the velocity

$$\boldsymbol{v}\rightarrow\boldsymbol{v}+\frac{Ze}{m}\delta\boldsymbol{A} \tag{6}$$

This re-definition of velocity is exact and can always be related to the actual velocity. However in Appendix A, we show that because the resonant particle velocity scales with the square root of perturbation amplitude, the distinction between ordinary velocity and this new quantity is unimportant for sufficiently small fields. We then find

$$L(\boldsymbol{x},\boldsymbol{v},\dot{\boldsymbol{x}},\dot{\boldsymbol{v}},t)=L_0+Ze\delta\Phi+Ze\boldsymbol{v}\cdot\delta\boldsymbol{A}-\frac{Z^2e^2\delta A^2}{2m} \tag{7}$$

where the equilibrium fields only are contained in L_0. Neglecting the last term in δA^2 ignores the nonlinear ponderomotive force associated with the perturbation, which we take as negligible for tokamak problems of practical interest.

To make the integrability of the equilibrium motion explicit and clear, we adopt the guiding centre variables following Littlejohn [25]

$$\begin{gathered}
\widehat{\boldsymbol{a}}(\boldsymbol{X},\xi)=\cos\xi\,\widehat{\boldsymbol{1}}(\boldsymbol{X})-\sin\xi\,\widehat{\boldsymbol{2}}(\boldsymbol{X})\\
\frac{\partial\widehat{\boldsymbol{a}}}{\partial\xi}(\boldsymbol{X},\xi)=\left(-\sin\xi\,\widehat{\boldsymbol{1}}(\boldsymbol{X})-\cos\xi\,\widehat{\boldsymbol{2}}(\boldsymbol{X})\right)=\widehat{\boldsymbol{c}}(\boldsymbol{X},\xi)\\
\boldsymbol{x}=\boldsymbol{X}+\boldsymbol{\rho}\\
\boldsymbol{\rho}=\frac{mv_\perp}{ZeB_0(\boldsymbol{X})}\widehat{\boldsymbol{a}}(\boldsymbol{X},\xi)\\
\boldsymbol{v}=V_\parallel\widehat{\boldsymbol{b}}(\boldsymbol{X})+v_\perp\widehat{\boldsymbol{c}}(\boldsymbol{X},\xi)\\
\widehat{\boldsymbol{b}}_0\times\widehat{\boldsymbol{1}}=\widehat{\boldsymbol{2}}\;\;\widehat{\boldsymbol{b}}_0\cdot\widehat{\boldsymbol{1}}=0\;\widehat{\boldsymbol{1}}\cdot\widehat{\boldsymbol{1}}=1
\end{gathered} \tag{8}$$

Taking advantage of the invariance of the Lagrangian under coordinate transformations and addition of derivative $\frac{dS}{dt}$ for any S, we may derive the exact particle Lagrangian in the new variables

$$
\begin{gathered}
L\big(\boldsymbol{X},V_\parallel,v_\perp,\xi,\dot{\boldsymbol{X}},\dot{V}_\parallel,\dot{v}_\perp,\dot{\xi},t\big) = \big[Ze\boldsymbol{A}_0(\boldsymbol{X}) + mV_\parallel\hat{\boldsymbol{b}}_{\boldsymbol{0}}(\boldsymbol{X})\big]\cdot\dot{\boldsymbol{X}} + \frac{1}{2}\frac{m^2v_\perp^2}{ZeB_0(\boldsymbol{X})}\dot{\xi} \\
-\left[\frac{1}{2}mv_\perp^2 + \frac{1}{2}mV_\parallel^2 + Ze\Phi_0(\boldsymbol{X})\right] + \Delta L_0 \\
-[Ze\delta\Phi - ZeV_\parallel\delta A_\parallel - Zev_\perp\hat{\boldsymbol{c}}_0(\boldsymbol{X},\xi)\cdot\delta\boldsymbol{A}_\perp] \\
\{\delta\Phi,\delta A_\parallel,\delta\boldsymbol{A}_\perp\} = \exp(\boldsymbol{\rho}\cdot\nabla_{\boldsymbol{X}})\,\{\delta\Phi(\boldsymbol{X},t),\delta A_\parallel(\boldsymbol{X},t),\delta\boldsymbol{A}_\perp(\boldsymbol{X},t)\} \\
\Delta L_0 = Ze\left[\frac{m^2v_\perp^2}{2Z^2e^2B_0(\boldsymbol{X})}\nabla\hat{\boldsymbol{2}}(\boldsymbol{X})\cdot\hat{\boldsymbol{1}}(\boldsymbol{X}) + \frac{m^2v_\perp V_\parallel}{Z^2e^2B_0(\boldsymbol{X})}\nabla_{\mathrm{X}}\hat{\boldsymbol{a}}(\boldsymbol{X},\xi)\cdot\hat{\boldsymbol{b}}_{\boldsymbol{0}}(\boldsymbol{X}) + \exp_{2..N}(\boldsymbol{\rho}\cdot\nabla_{\boldsymbol{X}})\boldsymbol{A}_{\boldsymbol{0}}(\boldsymbol{X})\right]\cdot\dot{\boldsymbol{X}} + \\
Ze\exp_{2..N}(\boldsymbol{\rho}\cdot\nabla_{\boldsymbol{X}})\boldsymbol{A}_{\boldsymbol{0}}(\boldsymbol{X})\cdot\dot{\boldsymbol{\rho}} - \frac{Ze}{2}\boldsymbol{\rho}\cdot\left(\frac{d}{dt}\nabla_{\boldsymbol{X}}\boldsymbol{A}_0(\boldsymbol{X})\right)\cdot\boldsymbol{\rho} \\
\exp_{2..N}(\boldsymbol{\rho}\cdot\nabla_{\boldsymbol{X}}) \equiv \exp(\boldsymbol{\rho}\cdot\nabla_{\boldsymbol{X}}) - 1 - \boldsymbol{\rho}\cdot\nabla_{\boldsymbol{X}}
\end{gathered}
\tag{9}
$$

The portion independent of gyroangle ξ preceding ΔL_0 is the Littlejohn Lagrangian in a static field, which describes the motion of guiding centres accurate to leading order in Larmor radius normalized to typical equilibrium scale length $\lambda = \left(\frac{mv}{ZeB_0(x)}\right)/\left(\frac{B_0(x)}{|\nabla B_0(x)|}\right)$. The static field quantities in this gyroinvariant portion are all evaluated at the guiding centre position $\boldsymbol{X}$ and represent the approximate guiding centre motion in equilibrium. ΔL_0 contains the $O(\lambda)$ corrections to the equilibrium guiding centre motion and the gyroangle dependent equilibrium motion. The remaining portion contains the influence of the perturbation on the motion. To retain FLR effects in the perturbed fields, we cannot say $\delta\boldsymbol{A}(\boldsymbol{x},t) = \delta\boldsymbol{A}(\boldsymbol{X},t)$ nor $\delta\Phi(\boldsymbol{x},t) = \delta\Phi(\boldsymbol{X},t)$ so we retain these to arbitrary order for the time being by keeping them at the particle position. If we neglect the contribution of ΔL_0 to the Lagrangian, then the resulting equilibrium trajectories of guiding centres will be displaced from the particle locations by an additional amount $\boldsymbol{\Delta X}$ and similarly for the other phase space variables

$$
\begin{gathered}
\{\boldsymbol{X},V_\parallel,v_\perp,\xi\} = \{\check{\boldsymbol{X}},\check{V}_\parallel,\check{v}_\perp,\check{\xi}\} + \{\boldsymbol{\Delta X},\Delta V_\parallel,\Delta v_\perp,\Delta\xi\} \\
L\left(\check{\boldsymbol{X}},\check{V}_\parallel,\check{v}_\perp,\check{\xi},\dot{\check{\boldsymbol{X}}},\dot{\check{V}}_\parallel,\dot{\check{v}}_\perp,\dot{\check{\xi}},t\right) = \big[Ze\boldsymbol{A}_0(\check{\boldsymbol{X}}) + mV_\parallel\hat{\boldsymbol{b}}_{\boldsymbol{0}}(\check{\boldsymbol{X}})\big]\cdot\dot{\check{\boldsymbol{X}}} + \frac{1}{2}\frac{m^2\check{v}_\perp^2}{ZeB_0(\check{\boldsymbol{X}})}\dot{\check{\xi}} \\
-\left[\frac{1}{2}m\check{v}_\perp^2 + \frac{1}{2}m\check{V}_\parallel^2 + Ze\Phi_0(\check{\boldsymbol{X}})\right] \\
-\big[Ze\delta\Phi - ZeV_\parallel\delta A_\parallel - Zev_\perp\hat{\boldsymbol{c}}_0(\check{\boldsymbol{X}},\check{\xi})\cdot\delta\boldsymbol{A}_\perp\big] \\
\{\delta\Phi,\delta A_\parallel,\delta\boldsymbol{A}_\perp\} = \exp(\boldsymbol{\Delta X}\cdot\boldsymbol{\nabla_X})\exp(\boldsymbol{\rho}\cdot\nabla_{\boldsymbol{X}})\,\{\delta\Phi(\boldsymbol{X},t),\delta A_\parallel(\boldsymbol{X},t),\delta\boldsymbol{A}_\perp(\boldsymbol{X},t)\}
\end{gathered}
\tag{10}
$$

The physical interpretation is clear – if we neglect the contribution to the motion from ΔL_0, then we are neglecting the difference between the perturbation as measured at $\check{\boldsymbol{X}},\check{V}_\parallel,\check{v}_\perp,\check{\xi}$ versus the true guiding centre position $\boldsymbol{X},V_\parallel,v_\perp,\xi$. The perturbing fields along the new trajectory require a correction $\exp(\boldsymbol{\Delta X}\cdot\boldsymbol{\nabla_X})$ as well as a correction for the velocity (since for example the magnetic moment is not constant). If we make the estimate $|\boldsymbol{\Delta X}| \sim \lambda|\boldsymbol{\rho}|$ then we conclude that the mode FLR corrections that can be resolved are limited by the ordering in the equilibrium guiding-centre orbits λ. For the remainder of the paper, we ignore these differences taking $\Delta L_0 = 0$ and $\exp(\boldsymbol{\Delta X}\cdot\boldsymbol{\nabla_X}) = 1$, and dropping the accents on variables. Accuracy in computing the particle response to a perturbation is thus limited by the accuracy of the equilibrium orbit tracking.

To obtain a canonical set of variables, we require three angular coordinates and three momenta. To do this, we adopt orthogonal magnetic coordinates where $B^\psi = B_\psi = 0$ and $g_{\psi\theta} = g_{\theta\phi} = g_{\psi\phi} = 0$. This is less convenient for description of the wave phenomena, but a useful simplification for describing the particle motion analytically. Such a coordinate system should be well defined except for regions very close to the magnetic axis in a strongly shaped plasma. Recent work [29] offers an alternative which is both canonical and straight-field line, suitable for numerical calculations.

In orthogonal coordinates we have

$$\begin{gathered}
\widehat{\boldsymbol{b}}_0 = \frac{B_2}{B_0}\nabla\theta + \frac{B_3}{B_0}\nabla\phi \\
\boldsymbol{A}_0 = A_2(\psi)\nabla\theta + A_3(\psi)\nabla\phi \\
\boldsymbol{X} = \psi\widehat{\boldsymbol{e}}_\psi + \theta\widehat{\boldsymbol{e}}_\theta + \phi\widehat{\boldsymbol{e}}_\phi \\
\dot{\boldsymbol{X}} = \dot{\psi}\widehat{\boldsymbol{e}}_\psi + \dot{\theta}\widehat{\boldsymbol{e}}_\theta + \dot{\phi}\widehat{\boldsymbol{e}}_\phi \\
\widehat{\boldsymbol{c}}_{\boldsymbol{0}} = -\sin\xi\,\widehat{\boldsymbol{\psi}} - cos\xi\,\widehat{\boldsymbol{\pi}} \\
\hat{\psi}(\boldsymbol{X}) = \frac{\nabla\psi}{|\nabla\psi|}\,, \hat{\pi}(\boldsymbol{X}) = \widehat{\boldsymbol{b}}(\boldsymbol{X}) \times \hat{\psi}(\boldsymbol{X})
\end{gathered} \tag{11}$$

We now have the Hamiltonian system

$$\begin{gathered}
L = P_\theta\dot{\theta} + P_\phi\dot{\phi} + P_\xi\dot{\xi} - H_0 - \delta H \\
P_\theta = ZeA_2(\psi) + mV_\parallel\frac{B_2(\psi,\theta)}{B_0(\psi,\theta)} \\
P_\phi = ZeA_3(\psi) + mV_\parallel\frac{B_3(\psi,\theta)}{B_0(\psi,\theta)} \\
P_\xi = \frac{m^2v_\perp^2}{2ZeB_0(\psi,\theta)} = \frac{m\mu}{Ze} \\
H_0 = \frac{1}{2}mv_\perp^2 + \frac{1}{2}mV_\parallel^2 + Ze\Phi_0(\psi,\theta) \\
\delta H = -Ze\delta\Phi - ZeV_\parallel\delta A_\parallel - Zev_\perp\widehat{\boldsymbol{c}}_0(\psi,\theta,\xi)\cdot\delta\boldsymbol{A}_\perp
\end{gathered} \tag{12}$$

with the canonical variables $\theta, \phi, \xi, P_\theta, P_\phi, P_\xi$ all being guiding centre variables of the equilibrium motion. In appendix C we show that the choice of orthogonal flux coordinates guarantees canonical guiding centre variables to arbitrary order.

The symmetry in equilibrium with respect to time, gyroangle and toroidal angle mean that H_0, P_ϕ, and P_ξ are conserved quantities. The Liouvile-Arnold theorem says that if there are three independent conserved quantities for three degrees of freedom, then the motion is integrable, meaning that our particles are confined to periodic orbits. These equilibrium orbits give trajectories in $\theta, \phi, \xi, P_\theta$ which are complicated by the field geometry. To solve the linear wave-particle problem, we will ultimately have to integrate a kinetic equation of the form

$$\frac{d}{dt}\delta f(\boldsymbol{x}_0(t), \boldsymbol{v}_0(t), t) = -\frac{\delta\boldsymbol{F}}{m}\cdot\frac{\partial F_0}{\partial\boldsymbol{v}} \tag{13}$$

where the linear perturbation to the particle distribution δf along the equilibrium trajectories $\boldsymbol{x}_0(t)$ $\boldsymbol{v}_0(t)$ varies in time according to the perturbing forces $\delta\boldsymbol{F}$ acting on the equilibrium distribution momentum gradients $\frac{1}{m}\frac{\partial F_0}{\partial v}$. This integration will be greatly simplified provided we express the orbits as steady motion in a new action-angle coordinate system. We therefore split the particle angular position $\theta(t)$ into steady $\bar{\theta}(t)$ and oscillatory $\tilde{\theta}(t)$ components as $\theta = \bar{\theta} + \tilde{\theta}$ for all the coordinates $\theta, \phi, \xi, P_\theta, P_\phi, P_\xi$.

We want a canonical transformation which gives the new equations of motion

$$
\begin{aligned}
\dot{P_{\bar{\theta}}} &= -\frac{\partial \overline{H_0}}{\partial \bar{\theta}} = 0 \\
\dot{P_{\bar{\phi}}} &= -\frac{\partial \overline{H_0}}{\partial \bar{\phi}} = 0 \\
\dot{P_{\bar{\xi}}} &= -\frac{\partial \overline{H_0}}{\partial \bar{\xi}} = 0 \\
\dot{\bar{\theta}} &= \frac{\partial \overline{H_0}}{\partial P_{\bar{\theta}}} = \omega_b \\
\dot{\bar{\phi}} &= \frac{\partial \overline{H_0}}{\partial P_{\bar{\phi}}} = \bar{\omega}_\phi \\
\dot{\bar{\xi}} &= \frac{\partial \overline{H_0}}{\partial P_{\bar{\xi}}} = \bar{\Omega}_c
\end{aligned}
\tag{14}
$$

and new Hamiltonian

$$
\overline{H_0}\left(P_{\bar{\phi}}, P_{\bar{\theta}}, P_{\bar{\xi}}\right) = \bar{\omega}_\phi P_{\bar{\phi}} + \omega_b P_{\bar{\theta}} + \bar{\Omega}_c P_{\bar{\xi}} \tag{15}
$$

where we have introduced the orbit averaged frequencies for poloidal ω_b, toroidal $\bar{\omega}_\phi$, and gyro $\bar{\Omega}_c$ motion.

The canonical transformation to the new action angle proceeds by exploiting the invariance of the Lagrangian under the addition of a total time derivative $\frac{dG_0}{dt}$ for any G_0

$$
P_\theta \dot{\theta} + P_\phi \dot{\phi} + P_\xi \dot{\xi} - H_0(\theta, \phi, \xi, P_\theta, P_\phi, P_\xi) \;\; = P_{\bar{\theta}} \dot{\bar{\theta}} + P_{\bar{\phi}} \dot{\bar{\phi}} + P_{\bar{\xi}} \dot{\bar{\xi}} - \overline{H_0}\left(P_{\bar{\phi}}, P_{\bar{\theta}}, P_{\bar{\xi}}\right) + \frac{dG_0}{dt} \tag{16}
$$

The useful choice of G_0 for our purposes is

$$
G_0 = G\left(\theta, \phi, \xi, P_{\bar{\theta}}, P_{\bar{\phi}}, P_{\bar{\xi}}\right) - \bar{\theta} P_{\bar{\theta}} - \bar{\phi} P_{\bar{\phi}} - \bar{\xi} P_{\bar{\xi}} \tag{17}
$$

this leads to the identities

$$
P_\theta = \frac{\partial G}{\partial \theta} \;\; P_\phi = \frac{\partial G}{\partial \phi} \; P_\xi = \frac{\partial G}{\partial \xi} \;\; \bar{\theta} = \frac{\partial G}{\partial P_{\bar{\theta}}} \;\; \bar{\phi} = \frac{\partial G}{\partial P_{\bar{\phi}}} \;\; \bar{\xi} = \frac{\partial G}{\partial P_{\bar{\xi}}} \tag{18}
$$

which when allowing for the invariance of H_0, P_ϕ, and P_ξ gives the generating function

$$
G\left(\theta, \phi, \xi, P_{\bar{\theta}}, P_{\bar{\phi}}, P_{\bar{\xi}}\right) = P_{\bar{\xi}} \xi + P_{\bar{\phi}} \phi + \int_0^\theta P_\theta [P_{\bar{\phi}}, P_{\bar{\xi}}, \overline{H_0}\left(P_{\bar{\phi}}, P_{\bar{\theta}}, P_{\bar{\xi}}\right), \theta'] d\theta' \tag{19}
$$

This completes the implicit definition of the new action-angle coordinates. In particular, $\bar{\xi} = \frac{\partial G}{\partial P_{\bar{\xi}}}$ is an important definition in the context of compressional modes where gyration is implicated in the power transfer. This gyroangle definition is in direct analogy to the toroidal precession in the drift-kinetic theory [1], where the toroidal angle ϕ can be separated into secular $\bar{\phi}$ and oscillatory $\tilde{\phi}$ parts, so too can the gyro angle $\xi = \bar{\xi} + \tilde{\xi}$. The oscillatory motion $\tilde{\xi}$ is due to the change in the equilibrium magnetic field over a poloidal bounce. This gives a rigorous definition to the intuition that change in the magnetic field over an orbit will change the gyrofrequency, and this change should be accounted for as an average. In appendix B, we show that to first order in λ, $\bar{\Omega}_c$ is the orbit average of the cyclotron frequency evaluated at the guiding centre.

Continuing now with our task of a linear solution to the kinetic equation 13, we must express our perturbed forces in terms of the action-angle coordinates. Looking at the perturbed Hamiltonian again, we expand the perturbed potentials as a Fourier series assuming a discrete eigenmode in an axisymmetric torus

$$\delta H(\theta,\phi,\xi,P_\theta,P_\phi,P_\xi,t) = -Ze\sum_m [\delta\Phi_m + v_\parallel \delta A_{\parallel m} + v_\perp \hat{\boldsymbol{c}}_0\cdot\delta\boldsymbol{A}_{\perp m}]\exp(in\phi + im\theta - i\omega t) + c.c. \tag{20}$$

An additional Fourier transform converts our perturbing Hamiltonian to the right form

$$\delta H(\bar\theta,\bar\phi,\bar\xi,P_{\bar\theta},P_{\bar\phi},P_{\bar\xi},t) = -\sum_{p,l} Ze\Psi_{npl}\left(P_{\bar\theta},P_{\bar\phi},P_{\bar\xi}\right)\exp\left(in\bar\phi + ip\bar\theta + il\bar\xi - i\omega t\right) + c.c.$$

$$\Psi_{npl} \equiv \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi}\sum_m [\delta\Phi_m + V_\parallel\delta A_{\parallel m} + v_\perp\hat{\boldsymbol{c}}_0\cdot\delta\boldsymbol{A}_{\perp m}]\exp(in(\phi-\bar\phi) + im\theta - ip\bar\theta - il\bar\xi)\, d\bar\xi d\bar\theta \tag{21}$$

where the Fourier coefficient Ψ_{npl} can be thought of as an average scalar potential experienced over a bounce orbit labelled by the invariants $(P_{\bar\theta},P_{\bar\phi},P_{\bar\xi})$. Notice also the natural appearance of what we may call a global gyro-average $\frac{1}{2\pi}\int_0^{2\pi} d\bar\xi$. Both the poloidal and gyro-motion change the wave phase experienced by the particle over the course of its orbit, with the aggregate effect captured as the integral of a 'propagator' (an analogy noticed by others as well e.g.: [30]).

We may choose to write in a covariant form

$$\boldsymbol{\Gamma} \equiv (\boldsymbol{x},\boldsymbol{v}) \in \mathbb{R}^6$$
$$\bar{\boldsymbol{Q}} \equiv \bar\theta\frac{\partial\boldsymbol{\Gamma}}{\partial\bar\theta} + \bar\phi\frac{\partial\boldsymbol{\Gamma}}{\partial\bar\phi} + \bar\xi\frac{\partial\boldsymbol{\Gamma}}{\partial\bar\xi}$$
$$\bar{\boldsymbol{P}} \equiv P_{\bar\theta}\frac{\partial\bar\theta}{\partial\boldsymbol{\Gamma}} + P_{\bar\phi}\frac{\partial\bar\phi}{\partial\boldsymbol{\Gamma}} + P_{\bar\xi}\frac{\partial\bar\xi}{\partial\boldsymbol{\Gamma}}$$
$$\bar{\boldsymbol{k}} \equiv n\frac{\partial\bar\phi}{\partial\boldsymbol{\Gamma}} + p\frac{\partial\bar\theta}{\partial\boldsymbol{\Gamma}} + l\frac{\partial\bar\xi}{\partial\boldsymbol{\Gamma}}$$
$$\delta H(\bar{\boldsymbol{Q}},\bar{\boldsymbol{P}},t) = -\sum_{\bar{\boldsymbol{k}}} Ze\Psi_{\bar{\boldsymbol{k}}}(\bar{\boldsymbol{P}})\exp\left(i\bar{\boldsymbol{k}}\cdot\bar{\boldsymbol{Q}} - i\omega t\right) + c.c. \tag{22}$$

with the interpretation of travelling waves in angle $\boldsymbol{Q}$ with wave-vector $\bar{\boldsymbol{k}}$ and amplitude Ψ_{npl} for a given orbit $\boldsymbol{P}$.

The linearized Vlasov equation 13 may instead be written

$$\frac{d}{dt}\delta f = -\{F_0,\delta H\} = \frac{\partial\delta H}{\delta\bar\theta}\frac{\partial F_0}{\delta P_{\bar\theta}} + \frac{\partial\delta H}{\delta\bar\phi}\frac{\partial F_0}{\delta P_{\bar\phi}} + \frac{\partial\delta H}{\delta\bar\xi}\frac{\partial F_0}{\delta P_{\bar\xi}} \tag{23}$$

recalling that equilibrium $\{F_0,H_0\} = 0$ is achieved if and only if $F_0 = F_0(P_{\bar\theta},P_{\bar\phi},P_{\bar\xi})$. Explicitly, we have

$$\frac{d\delta f}{dt} = \left(in\frac{\partial F}{\partial P_{\bar\phi}} + ip\frac{\partial F}{\partial P_{\bar\theta}} + il\frac{\partial F}{\partial P_{\bar\xi}}\right)\sum_{p,l} Ze\Psi_{npl}\left(P_{\bar\theta},P_{\bar\phi},P_{\bar\xi}\right)\exp\left(in\bar\phi + ip\bar\theta + il\bar\xi - i\omega t\right) + c.c. \tag{24}$$

this is the equation we need to integrate in time. We now can take full advantage of the action-angle coordinates, since the time integration is trivial with the only time dependence coming from $\bar{\theta} = \omega_b t$, $\bar{\phi} = \bar{\omega}_\phi t$, and $\bar{\xi} = \overline{\Omega_c} t$. We obtain the full general solution

General solution

$$\delta f = -\sum_{p,l} Ze\Psi_{npl}\left(P_{\bar{\theta}}, P_{\bar{\phi}}, P_{\bar{\xi}}\right) \frac{\left(n\frac{\partial F}{\partial P_{\bar{\phi}}} + p\frac{\partial F}{\partial P_{\bar{\theta}}} + l\frac{\partial F}{\partial P_{\bar{\xi}}}\right)\exp\left(in\bar{\phi} + ip\bar{\theta} + il\bar{\xi} - i\omega t\right)}{n\bar{\omega}_\phi + p\omega_b + l\bar{\Omega}_c - \omega} + c.c.$$

$$\Psi_{npl} \equiv \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} \sum_m [\delta\Phi_m + V_\parallel \delta A_{\parallel m} + v_\perp \hat{\boldsymbol{c}}_0 \cdot \delta\boldsymbol{A}_{\perp m}] \exp\left(in(\phi - \bar{\phi}) + im\theta - ip\bar{\theta} - il\bar{\xi}\right) d\bar{\xi} d\bar{\theta} \quad (25)$$

This result is true to all orders in Larmor radius and for arbitrary mode polarization or frequency provided all perturbed quantities are evaluated at the particle position $\boldsymbol{x}$ and the equilibrium orbit is sufficiently well resolved. An obvious important difference to the standard $\omega \ll \Omega_c$ theories lies in the destruction of μ as an invariant, making $\frac{\partial F}{\partial P_{\bar{\xi}}}$ a source of energy.

Again, drawing out the structure, we may write

$$\frac{\partial}{\partial \bar{\boldsymbol{P}}} \equiv \frac{\partial \boldsymbol{\Gamma}}{\partial \bar{\theta}}\frac{\partial}{\partial P_{\bar{\theta}}} + \frac{\partial \boldsymbol{\Gamma}}{\partial \bar{\phi}}\frac{\partial}{\partial P_{\bar{\phi}}} + \frac{\partial \boldsymbol{\Gamma}}{\partial \bar{\xi}}\frac{\partial}{\partial P_{\bar{\xi}}}$$

$$\dot{\bar{\boldsymbol{Q}}} \equiv \omega_b \frac{\partial \boldsymbol{\Gamma}}{\partial \bar{\theta}} + \bar{\omega}_\phi \frac{\partial \boldsymbol{\Gamma}}{\partial \bar{\phi}} + \bar{\Omega}\frac{\partial \boldsymbol{\Gamma}}{\partial \bar{\xi}}$$

$$\boldsymbol{k} \equiv n\frac{\partial \phi}{\partial \boldsymbol{\Gamma}} + m\frac{\partial \theta}{\partial \boldsymbol{\Gamma}}$$

$$\delta f(t) = -\sum_{\bar{\boldsymbol{k}}} Ze\Psi_{\bar{\boldsymbol{k}}} \frac{\bar{\boldsymbol{k}} \cdot \frac{\partial F_0}{\partial \bar{\boldsymbol{P}}} \exp\left(i\bar{\boldsymbol{k}} \cdot \bar{\boldsymbol{Q}} - i\omega t\right)}{\bar{\boldsymbol{k}} \cdot \dot{\bar{\boldsymbol{Q}}} - \omega} + c.c.$$

$$\Psi_{\bar{\boldsymbol{k}}} \equiv \left(\frac{1}{2\pi}\right)^3 \int \sum_{\boldsymbol{k}} [\delta\Phi_{\boldsymbol{k}} + V_\parallel \delta A_{\parallel \boldsymbol{k}} + v_\perp \hat{\boldsymbol{c}}_0 \cdot \delta\boldsymbol{A}_{\perp \boldsymbol{k}}] \exp\left(i\boldsymbol{k} \cdot \boldsymbol{Q} - i\bar{\boldsymbol{k}} \cdot \bar{\boldsymbol{Q}}\right) d^3\boldsymbol{Q} \quad (26)$$

This result may be compared with taking the linearized Vlasov equation 13 and applying it to the case of free streaming particles interacting with plane-waves in a uniform slab where we can readily show

$$\delta f(t) = -\sum_{\boldsymbol{k}} Ze\Psi_k \frac{\boldsymbol{k} \cdot \frac{1}{m}\frac{\partial F_0}{\partial \boldsymbol{v}} \exp(i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t)}{\boldsymbol{k} \cdot \dot{\boldsymbol{x}} - \omega} + c.c \quad (27)$$

The fundamental differences between the uniform slab and the real tokamak are two-fold. Firstly, the equilibrium orbits are not free-streaming but have a velocity that varies periodically. Secondly, the perturbing fields are non-uniform, so not all orbits will encounter the same perturbing amplitude, and that amplitude will also vary over the course of a single orbit. These differences between cases can be thought of as the difference between the local and global approximations; for sufficiently short times, the local perturbing potential $\Psi_{\boldsymbol{k}}(\boldsymbol{x})$ acts resonantly on particles satisfying $\boldsymbol{k} \cdot \boldsymbol{v} - \omega = 0$ with a local free energy $\frac{\partial F_0}{\partial v}(\boldsymbol{x}, \boldsymbol{v})$. Continuing the local analysis, the particle will move to a different region of the plasma with a new velocity $\boldsymbol{v}'$ moving out of resonance with $\Psi_{\boldsymbol{k}}(\boldsymbol{x}')$ but perhaps moving into resonance instead with $\Psi_{\boldsymbol{k}'}(\boldsymbol{x}')$ through $\boldsymbol{k}'\cdot\boldsymbol{v}'-\omega=0$ and different local gradient $\frac{\partial F_0}{\partial \boldsymbol{v}}(\boldsymbol{x}',\boldsymbol{v}')$. This variation along the orbit is the essence of sideband resonances and why $\boldsymbol{k}\neq\overline{\boldsymbol{k}}$ in the global theory.

In the next sections, we examine applications of equation 25 to various limiting cases, making connection with existing results, then looking at new results for compressional modes $\delta A_\perp \gg \delta A_\parallel$. We will also explore what effects the global variation in gyroangle have on the perturbing potential.

Neglecting FLR effects in mode structure $k_\perp\rho \ll 1$

First, we review the existing derivation [22] for an Alfven wave $\delta A_\parallel \gg \delta A_\perp$ in the drift kinetic approximation where the gyro-variation has no effect on the perturbing forces, meaning that the perturbation experienced by the particle matches exactly the perturbation measured at the guiding centre. Then we have

$$\Psi_{npl}\equiv\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi}\sum_m[\Phi_m+v_\parallel\delta A_{\parallel m}]\exp\big(in(\phi-\bar{\phi})+im\theta-ip\bar{\theta}-il\bar{\xi}\big)\,d\bar{\xi}\,d\bar{\theta} \qquad (28)$$

where all the gyro-variation is contained only within the term $\exp\big(-il\bar{\xi}\big)$. Applying the global gyro-average $\frac{1}{2\pi}\int_0^{2\pi}\exp\big(-il\bar{\xi}\big)\,d\bar{\xi}$ therefore nullifies all contributions to the linear solution except for when $l=0$.

The drift kinetic result for Alfvenic modes is thus

Shear Alfven global solution without mode FLR: $\delta A_\parallel \gg \delta A_\perp \;\; k_\perp\rho \ll 1$

$$\delta f=-\sum_p Ze\Psi_{np0}\left(P_{\bar{\theta}},P_{\bar{\phi}},P_{\bar{\xi}}\right)\frac{\left(n\frac{\partial F}{\partial P_{\bar{\phi}}}+p\frac{\partial F}{\partial P_{\bar{\theta}}}\right)\exp(in\bar{\phi}+ip\bar{\theta}-i\omega t)}{n\bar{\omega}_\phi+p\omega_b-\omega}+c.c.$$

$$\Psi_{np0}\equiv\frac{1}{2\pi}\int_0^{2\pi}\sum_m[\delta\Phi_m+V_\parallel\delta A_{\parallel m}]\exp\big(in\tilde{\phi}+im\theta-ip\bar{\theta}\big)\,d\bar{\theta} \qquad (29)$$

We make connection to the Porcelli theory [1] where $F=F(E,P_{\bar{\xi}},P_{\bar{\phi}})$. Applying the chain rule to $E\equiv\bar{H}_0$ we have

$$\left(\frac{\partial F}{\partial P_{\bar{\phi}}}\right)_{P_{\bar{\theta}}}=\left(\frac{\partial F}{\partial P_{\bar{\phi}}}\right)_E+\frac{\partial F}{\partial E}\bar{\omega}_\phi$$
$$\frac{\partial F}{\partial P_{\bar{\theta}}}=\frac{\partial F}{\partial E}\omega_b$$
$$\left(\frac{\partial F}{\partial P_{\bar{\xi}}}\right)_{P_{\bar{\theta}}}=\left(\frac{\partial F}{\partial P_{\bar{\xi}}}\right)_E+\bar{\Omega}_c\frac{\partial F}{\partial E} \qquad (30)$$

The non-adiabatic response δh is obtained at resonance when

$$n\bar{\omega}_\phi+p\omega_b+l\bar{\Omega}_c-\omega=0 \qquad (31)$$

giving immediately the Porcelli result

$$\omega_* \equiv \frac{\partial F/\partial P_{\bar{\phi}}}{\partial F/\partial E}$$

$$\delta h = -(n\omega_* + \omega)\frac{\partial F}{\partial E}\sum_p Ze\Psi_{np0}\left(E, P_{\bar{\phi}}, P_{\bar{\xi}}\right)\frac{\exp(in\bar{\phi} + ip\bar{\theta} - i\omega t)}{n\bar{\omega}_\phi + p\omega_b - \omega} + c.c. \tag{32}$$

Looking at the compressional limit $\delta A_\parallel \ll \delta A_\perp$, particle gyration appears in the perturbed potential

$$\widehat{\boldsymbol{c}_0} = -\sin\xi\,\widehat{\boldsymbol{\psi}} - cos\xi\,\widehat{\boldsymbol{\pi}} = \left(i\widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\pi}}\right)\frac{1}{2}\exp(i\xi) + c.c. \equiv \frac{1}{2}\hat{C}\exp(i\xi) + c.c.$$

$$\Psi_{npl} \equiv \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi}\sum_m\left[\frac{v_\perp}{2}\hat{C}\cdot\delta\boldsymbol{A}_{\perp m}\exp(i\xi)\right]\exp\left(in(\phi - \bar{\phi}) + im\theta - ip\bar{\theta} - il\bar{\xi}\right)d\bar{\xi}d\bar{\theta} \tag{33}$$

the gyro-angle can be split into secular and oscillatory pieces, with the oscillation coming from poloidal angle $\xi = \bar{\xi} + \tilde{\xi}\left(\bar{\theta}, P_{\bar{\theta}}, P_{\bar{\phi}}, P_{\bar{\xi}}\right)$. The global gyro-average nullifies all terms except for $l = 1$ giving the global result

Compressional Alfven global solution without mode FLR: $\delta A_\parallel \ll \delta A_\perp \;\; k_\perp\rho \ll 1$

$$\delta f = -\sum_p Ze\Psi_{np1}\left(P_{\bar{\theta}}, P_{\bar{\phi}}, P_{\bar{\xi}}\right)\frac{\left(n\frac{\partial F}{\partial P_{\bar{\phi}}} + p\frac{\partial F}{\partial P_{\bar{\theta}}} + \frac{\partial F}{\partial P_{\bar{\xi}}}\right)\exp\left(in\bar{\phi} + ip\bar{\theta} + i\bar{\xi} - i\omega t\right)}{n\bar{\omega}_\phi + p\omega_b + \bar{\Omega}_c - \omega} + c.c.$$

$$\Psi_{np1} \equiv \frac{1}{2\pi}\int_0^{2\pi}\sum_m\left[\frac{v_\perp}{2}\hat{\boldsymbol{C}}\cdot\delta\boldsymbol{A}_{\perp m}\right]\exp\left(in\tilde{\phi} + im\theta - ip\bar{\theta} + i\tilde{\xi}\right)d\bar{\theta} \tag{34}$$

We see that information about the particle gyration is essential for resonant interaction with the compressional mode, in contrast to the shear mode. We also see perfect symmetry in $\tilde{\phi}$ and $\tilde{\xi}$, showing that the global variation in gyroangle is completely captured in the usual way in the poloidal sidebands p, rather than entering through cyclotron sidebands l. We will discover that, as with the local theory, cyclotron sidebands appear only because of a finite Larmor effect. This gives the compact and intuitive result that p sidebands capture the variations experienced by the guiding centre (including the modifications to the gyroaverage as the cyclotron frequency and Larmor radius changes) whereas l sidebands capture the additional variation experienced by the particle. These bounce and gyration timescales need not be well separated, though of course, if they aren't, it would mean more elaborate equations of equilibrium guiding centre motion would be required for faithful representation of the actual orbits. This allows a separation of concerns between accuracy of the orbit versus accuracy of the power transfer, which can be tailored to a given problem.

The key results from this section are that when a mode is large with respect to the gyration of the particles, then Alfvenic modes only resonate with $l = 0$ and compressional modes only resonate with $l = 1$, and both modes are subject to the same global sideband physics. Under these circumstances when $k_\perp\rho \ll 1$, the field experienced by the particle is taken to be the field evaluated at the guiding centre.

Guiding centre solution including mode FLR effects for arbitrary $k_\perp\rho$

In the previous section, we supposed that the modes were sufficiently large in comparison to the Larmor radius at all points along the equilibrium orbit that we could ignore the difference between perturbation measured at the guiding centre versus that experienced by the actual particle. In this section, we see the effects of applying the correction due to circular gyration about the equilibrium guiding centre.

We start by expressing each of the potential terms as a Fourier series such as

$$\delta\Phi(\boldsymbol{x},t)=\sum_{\boldsymbol{m}}\delta\Phi_m(\psi_x)\exp(im\theta_x+in\phi_x-i\omega t) \tag{35}$$

where $\boldsymbol{x}=(\psi_x,\theta_x,\phi_x)$ is the actual particle position. The Fourier series in the guiding centre variables $\boldsymbol{X},\boldsymbol{\rho}$ are of the form

$$\begin{gathered}\delta\Phi(\boldsymbol{X},\boldsymbol{\rho},t)=\sum_{\boldsymbol{m}}\delta\Phi_m\big(\psi_x(\boldsymbol{X},\boldsymbol{\rho})\big)\exp(im\theta_x(\boldsymbol{X},\boldsymbol{\rho})+in\phi_x(\boldsymbol{X},\boldsymbol{\rho})-i\omega t)\\ \psi_x(\boldsymbol{X},\boldsymbol{\rho})=\psi+\boldsymbol{\rho}\cdot\nabla\psi\\ \theta_x(\boldsymbol{X},\boldsymbol{\rho})=\theta+\boldsymbol{\rho}\cdot\nabla\theta\\ \phi_x(\boldsymbol{X},\boldsymbol{\rho})=\phi+\boldsymbol{\rho}\cdot\nabla\phi\\ \boldsymbol{\rho}(\boldsymbol{X},v_\perp,\xi)=\frac{v_\perp}{\Omega_c(\boldsymbol{X})}\Big(\cos\xi\,\widehat{\boldsymbol{\psi}}(\boldsymbol{X})+\sin\xi\,\widehat{\boldsymbol{\pi}}(\boldsymbol{X})\Big)\end{gathered} \tag{36}$$

Giving

$$\delta\Phi(\boldsymbol{X},v_\perp,\xi,t)=\sum_{\boldsymbol{m}}\delta\Phi_m\left(\psi+\frac{v_\perp|\nabla\psi|}{\Omega_c(X)}\cos\xi\right)\exp(i\boldsymbol{k}\cdot\boldsymbol{X}-i\omega t)\;\exp\left(\left[\frac{ik_\pi v_\perp}{\Omega_c(\boldsymbol{X})}\sin\xi\right]\right) \tag{37}$$

Using the Jacobi–Anger expansion gives exponentials of trigonometric functions as an infinite series of Bessel functions of the first kind

$$\begin{gathered}\delta\Phi(\boldsymbol{X},v_\perp,\xi,t)=\sum_{\boldsymbol{m}}\delta\Phi_m\left(\psi+\frac{v_\perp|\nabla\psi|}{\Omega_c(\boldsymbol{X})}\cos\xi\right)\left[\sum_{l_\pi=-\infty}^{\infty}J_{l_\pi}\left(\frac{k_\pi v_\perp}{\Omega_c}\right)\exp(il_\pi\xi)\right]\exp(i\boldsymbol{k}\cdot\boldsymbol{X}-i\omega t)\\ \delta\Phi_m\left(\psi+\frac{|\nabla\psi|\;v_\perp}{\Omega_c(\boldsymbol{X})}\cos\xi\right)=\delta\Phi_m(\psi)+\frac{|\nabla\psi|\;v_\perp}{\Omega_c(\boldsymbol{X})}\cos\xi\,\delta\Phi_m'(\psi)+\cdots\end{gathered} \tag{38}$$

The Taylor expansion can also be written as the exponential differential operator

$$\delta\Phi_m\left(\psi+\frac{|\nabla\psi|\;v_\perp}{\Omega_c(\boldsymbol{X})}\cos\xi\right)=\exp\left(\rho|\nabla\psi|\cos\xi\frac{\partial}{\partial\psi}_{\rho\nabla\psi}\right)\Phi_m(\psi) \tag{39}$$

The Jacobi–Anger expansion can also be applied to the exponential differential operator without ambiguity to create a Bessel-Function differential operator, understood to be an infinite series as with the exponential operator. We then arrive at the desired Fourier series in terms of guiding centre variables

$$\begin{gathered}\delta\Phi(\boldsymbol{X},v_\perp,\xi,t)=\sum_{integers}\delta\Phi\big(\boldsymbol{X},v_\perp;m,l_{\nabla\psi},l_\pi\big)\exp(i\boldsymbol{k}\cdot\boldsymbol{X}-i\omega_k t)\exp\big(il_{\nabla\psi}\xi\big)\exp(il_\pi\xi)\\ \delta\Phi_{\mathrm{X,\rho}}\equiv\Phi\big(\boldsymbol{X},v_\perp;m,l_{\nabla\psi},l_\pi\big)=J_{l_\pi}(k_\pi\rho)J_{l_{\nabla\psi}}\left(\rho|\nabla\psi|\frac{\partial}{\partial\psi}_{\rho\nabla\psi}\right)\Phi_m(\psi)\end{gathered} \tag{40}$$

The Fourier coefficients of the potential at the particle position can then be considered to be a series of contributions oscillating rapidly as $\exp\big(il_{\nabla\psi}\xi\big)\exp(il_\pi\xi)$ with a magnitude less than the potential at the guiding centre by a factor $J_{l_\pi}(k_\pi\rho)J_{l_{\nabla\psi}}\left(\rho|\nabla\psi|\frac{\partial}{\partial\psi}_{\rho\nabla\psi}\right)$.

These Bessel functions capture all of the FLR effects due to the mode and are free of gyrating terms. The total effective potential is then

Potential evaluated at guiding centre for arbitrary $k_\perp\rho$

$$\Psi_{npl} \equiv \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi}\sum_{integers}\left[\delta\Phi_{\mathbf{X},\boldsymbol{\rho}} + v_\parallel \delta A_{\parallel\,\mathbf{X},\boldsymbol{\rho}} + \frac{v_\perp}{2}\widehat{\boldsymbol{C}}_0(\boldsymbol{X})\cdot\delta\boldsymbol{A}_{\perp\mathrm{X},\rho}\exp(i\xi)\right] \exp\left(in(\phi-\overline{\phi}) + im\theta + l_{\nabla\psi}\xi + l_\pi\xi - ip\overline{\theta} - il\overline{\xi}\right)d\overline{\xi}d\overline{\theta} \tag{41}$$

noting that $\rho(\boldsymbol{X})\Omega_c(\boldsymbol{X}) = v_\perp$ by definition and energy is an invariant of the motion making $v_\parallel = v_\parallel(v_\perp)$. All the gyroangle dependent terms are therefore now explicit. Noting that along an orbit $\xi = \overline{\xi} + \tilde{\xi}(\overline{\theta})$, we realize that nonzero contributions may only come when $l_{\nabla\psi} + l_\pi - l = 0$ for $\delta A_\parallel \gg \delta A_\perp$ and $l_{\nabla\psi} + l_\pi - l = 1$ for $\delta A_\parallel \ll \delta A_\perp$. These requirements show that one of the integer summations is redundant because of the Neumann's Addition Theorem for Bessel Functions

$$J_a(x+y) = \sum_b J_{a-b}(x)J_b(y) \tag{42}$$

which becomes useful if we wish to write expressions using $k_\perp$ or $\nabla_\perp$

$$\delta\Phi_{\mathrm{X},\rho} = J_{l_\pi}(k_\pi\rho)J_{l_{\nabla\psi}}\left(\rho|\nabla\psi|\frac{\partial}{\partial\psi}_{\rho\nabla\psi}\right)\delta\Phi_m(\psi) = J_{l_\perp}\left(k_\pi\rho + \rho|\nabla\psi|\frac{\partial}{\partial\psi}_{\rho\nabla\psi}\right)\delta\Phi_m(\psi) \tag{43}$$

Introducing FLR variation due to narrow mode-widths has a clear implication; the global gyro-average becomes non-zero for $l \neq 0$ in the shear Alfven case and $l \neq 1$ in the compressional case. The resonance condition in equation 13 will now admit contributions from l sidebands with a magnitude which depends on $J_l(k_\perp\rho)$. In addition, p poloidal sidebands have a new Larmor radius contribution from variation in $k_\perp\rho$ and gyroangle adjustment $l\tilde{\xi}$ along the bounce orbit.

The path to quantify these effects is intuitively clear but tedious – the exponential terms in gyroangle $\exp(i\xi)$ will need to be converted into secular and oscillatory pieces $\exp(il\xi) = \exp(il\overline{\xi})\exp(il\tilde{\xi})$, the oscillations will occur because of the poloidal variation in the gyrofrequency. This is precisely the same global sideband effect that occurs for the drift resonances where for example $\exp(in\phi) = \exp(in\overline{\phi})\exp(in\tilde{\phi})$. In both cases, poloidal sidebands are generated with relative strength which depends predominantly on aspect ratio through a terms that look like $\exp(\sim il\epsilon sin\theta)$, much in the same way $\exp\,(k_\perp\rho\sin\xi)$ terms appeared in the shift from particle position to guiding centre; "it's Bessel functions all the way down".

The lowest order FLR correction for shear Alfven waves is obtained for $l_\pi = l = 0$ and resembles a linear gyrokinetic equation [3] when we eliminate the redundant sum over $l_{\nabla\psi}$

Shear Alfven global solution with lowest order FLR: $\delta A_\parallel \gg \delta A_\perp \;\; k_\perp\rho\sim 1$

$$\delta f = -\sum_p Ze\Psi_{np0}\left(P_{\overline{\theta}}, P_{\overline{\phi}}, P_{\overline{\xi}}\right)\frac{\left(n\frac{\partial F}{\partial P_{\overline{\phi}}} + p\frac{\partial F}{\partial P_{\overline{\theta}}}\right)\exp(in\overline{\phi} + ip\overline{\theta} - i\omega t)}{n\overline{\omega}_\phi + p\omega_b - \omega} + c.c.$$

$$\Psi_{np0} \equiv \frac{1}{2\pi}\int_0^{2\pi}\sum_m J_0\left(k_\pi\rho + \rho|\nabla\psi|\frac{\partial}{\partial\psi}_{\rho\nabla\psi}\right)[\delta\Phi_m(\psi) + V_\parallel\delta A_{\parallel m}(\psi)]\exp(in\tilde{\phi} + im\theta - ip\overline{\theta})\,d\overline{\theta} \tag{44}$$

The lowest order FLR correction for the compressional modes requires the inclusion of particle gyration and the associated $l = 1$ resonance. Remembering the global gyro-average we obtain

Compressional Alfven global solution with lowest order FLR: $\delta A_\parallel \ll \delta A_\perp \;\; k_\perp \rho \sim 1$

$$\delta f = -\sum_p Ze\Psi_{np1}\left(P_{\bar\theta}, P_{\bar\phi}, P_{\bar\xi}\right)\frac{\left(n\frac{\partial F}{\partial P_{\bar\phi}} + p\frac{\partial F}{\partial P_{\bar\theta}} + \frac{\partial F}{\partial P_{\bar\xi}}\right)\exp(in\bar\phi + ip\bar\theta - i\omega t)}{n\bar\omega_\phi + p\omega_b + \bar\Omega_c - \omega} + c.c.$$

$$\Psi_{np1} \equiv \frac{1}{2\pi}\int_0^{2\pi}\sum_m J_0\left(k_\pi\rho + \rho|\nabla\psi|\frac{\partial}{\partial\psi}_{\rho\nabla\psi}\right)\left[\frac{v_\perp}{2}\widehat{\boldsymbol{C}}_0(\boldsymbol{X})\cdot\delta\boldsymbol{A}_{\perp\boldsymbol{m}}(\psi)\right]\exp(in\tilde\phi + i\tilde\xi + im\theta - ip\bar\theta)\, d\bar\theta \quad (45)$$

Showing again the generation of poloidal sidebands due to the poloidal variation in $\tilde\phi, \tilde\xi$.

For both shear and compressional modes, we can make connection with various local theories by taking poloidal variation out of consideration. We can do this either for free-streaming particles in a cylinder, or for deeply trapped particles in a tokamak. For passing particles in a cylinder, we set $\bar\phi = \phi$, $\bar\xi = \xi$, $\bar\theta = \theta$, then the poloidal sidebands vanish. For the deeply trapped particles in arbitrary aspect ratio we have $\bar\phi = \phi$ $\bar\xi = \xi$ and $\theta \approx 0$ and then $p = 0$ is the only contribution. We obtain the limiting cases

Cylindrical approximation for passing particles

$$\delta f = -\sum_{l,m} Ze\Psi_{nml}\left(P_{\bar\theta}, P_{\bar\phi}, P_{\bar\xi}\right)\frac{\left(n\frac{\partial F}{\partial P_{\bar\phi}} + m\frac{\partial F}{\partial P_{\bar\theta}} + l\frac{\partial F}{\partial P_{\bar\xi}}\right)\exp(in\phi + im\theta + il\xi - i\omega t)}{k_{\parallel m}V_\parallel + l\Omega_c - \omega} + c.c.$$

Deeply trapped approximation in arbitrary aspect ratio

$$\delta f = -\sum_{l,m} Ze\Psi_{nml}\left(P_{\bar\theta}, P_{\bar\phi}, P_{\bar\xi}\right)\frac{\left(n\frac{\partial F}{\partial P_{\bar\phi}} + l\frac{\partial F}{\partial P_{\bar\xi}}\right)\exp(in\phi + il\xi - i\omega t)}{k_\phi v_\phi + l\Omega_c - \omega} + c.c.$$

Shear $\delta A_\parallel \gg \delta A_\perp$

$$\Psi_{nml} \equiv J_l\left(k_\pi\rho + \rho|\nabla\psi|\frac{\partial}{\partial\psi}_{\rho\nabla\psi}\right)\left[\delta\Phi_m(\psi) + V_\parallel\delta A_{\parallel m}(\psi)\right]$$

Compressional $\delta A_\parallel \ll \delta A_\perp$

$$\Psi_{nml} \equiv J_{l-1}\left(k_\pi\rho + \rho|\nabla\psi|\frac{\partial}{\partial\psi}_{\rho\nabla\psi}\right)\frac{v_\perp}{2}\widehat{\boldsymbol{C}}_0(X)\cdot\delta\boldsymbol{A}_{\perp\boldsymbol{m}}(\psi) \quad (46)$$

Large aspect ratio action-angle calculation for passing particles

We would like to now perform the lowest order non-trivial calculation of the conversion to action-angle coordinates, making connection with the existing TAE theory [31][22], but also showing the magnitude of the gyro-angle induced poloidal sidebands. We will proceed with the 1st order in equilibrium drift motion.

For Shafranov-shifted circular flux surfaces in large aspect ratio choosing minor radius r to be the flux surface label, we have the following equilibrium relations in orthogonal coordinates [32][24]

$$A_3 = -B_{mag}\int_0^r \frac{r_1}{q(r_1)}dr_1$$
$$A_2 = \frac{1}{2}B_{mag}r^2 - B_{mag}\alpha(r)\cos\theta$$
$$B_3 = B_{mag}R_{mag}$$
$$B_2 = \frac{B_{mag}r^2}{R_{mag}q(r)}\left(1 - \frac{\alpha'(r)}{r}\cos\theta\right)$$
$$B_0 = B_{mag}\left(1 - \frac{r}{R_{mag}}cos\theta\right)$$
$$\alpha(r) = \int_0^r \left(\frac{r_1^2}{R_{mag}} + r_1\Delta'(r_1) + r_1\int_0^{r_1}\frac{\Delta'(r_2)}{r_2}dr_2\right)dr_1 \tag{47}$$

For Shafranov shift $\Delta(r)$. Expanding the Hamiltonian around $\frac{r}{R_{mag}} \ll 1$ gives the parallel velocity equation

$$V_\parallel = \pm V_{\parallel mag}\left[1 + \frac{\epsilon(r)}{2}\frac{\mu B_{mag}}{H_0 - \mu B_{mag}}cos\theta\right]$$
$$V_{\parallel mag} \equiv \sqrt{\frac{2}{m}}\sqrt{H_0 - \mu B_{mag}}, \epsilon(r) \equiv \frac{r}{R_{mag}} \tag{48}$$

we make the ansatz

$$\bar{r} = r - \Delta^* cos\theta \tag{49}$$

with average radius $\bar{r}$ and orbit width Δ^* as yet unknown. Looking at the invariant

$$P_\phi = -ZeB_{mag}\int_0^r \frac{r_1}{q(r_1)}dr_1 + \frac{1}{B_0}mV_\parallel B_3 \tag{50}$$

we provide the implicit definition of $\bar{r}$ in terms of the invariant P_ϕ (defined by setting $\theta = \pi/2$)

$$P_\phi = -ZeB_{mag}\int_0^{\bar{r}} \frac{r_1}{q(r_1)}dr_1 + mV_{\parallel mag}R_{mag} \tag{51}$$

making $\bar{r}$ an invariant of the motion by definition. $\Delta^*$is obtained from P_ϕ using the extreme poloidal angles $\theta = 0, \pi$ and taking $\frac{\Delta^*}{\bar{r}} \ll 1$

$$\Delta_* = q(\bar{r})\frac{mV_{\parallel mag}}{2ZeB_{mag}}\left[\frac{2H_0 - \mu B_{mag}}{H_0 - \mu B_{mag}}\right] \tag{52}$$

giving that Δ_* is also an invariant of the motion. Turning attention to P_θ, we omit terms in Δ_*^2, $\epsilon\Delta_*$, $\epsilon\alpha'(\mathrm{r})$, and $\Delta_*\alpha'(\mathrm{r})$ to obtain

$$P_\theta = Ze\frac{1}{2}B_{mag}\bar{r}^2 + ZeB_{mag}\bar{r}\Delta_*\cos\theta - ZeB_{mag}\alpha(\bar{r})\cos\theta + \frac{mV_{\parallel mag}\bar{r}^2}{R_{mag}q(r)} \tag{53}$$

Identifying the invariant portion of P_θ as the value taken at $\theta = \frac{\pi}{2}$ and eliminating the parallel velocity we arrive at

$$
\begin{aligned}
P_\phi &= -ZeB_{mag}\int_0^{\bar r}\frac{r_1}{q(r_1)}dr_1 + R_{mag}\sqrt{2m}\sqrt{H_0-\mu B_{mag}} \\
P_\theta &= P_{\bar\theta} + \bar r q(\bar r)\sqrt{\frac{m}{2}}\left[\frac{2H_0-\mu B_{mag}}{\sqrt{H_0-\mu B_{mag}}}\right]\cos\theta - ZeB_{mag}\alpha(\bar r)\cos\theta
\end{aligned} \tag{54}
$$

The generating function for the action-angle transformation is then

$$
\begin{aligned}
G &= P_{\bar\xi}\xi + P_{\bar\phi}\phi + P_{\bar\theta}\theta + \tilde g\left(P_{\bar\xi}, P_{\bar\phi}, P_{\bar\theta}\right)sin\theta \\
\tilde g\left(P_{\bar\xi}, P_{\bar\phi}, P_{\bar\theta}\right) &\equiv \bar r q(\bar r)\sqrt{\frac{m}{2}}\left[\frac{2H_0-\mu B_{mag}}{\sqrt{H_0-\mu B_{mag}}}\right] - ZeB_{mag}\alpha(\bar r)
\end{aligned} \tag{55}
$$

and using the coordinate definitions in equation 18 we can see the form of the secular and oscillatory pieces in the motion

$$
\begin{aligned}
\bar\theta &= \theta + \frac{\partial\tilde g}{\partial P_{\bar\theta}}sin\theta \\
\bar\xi &= \xi + \frac{\partial\tilde g}{\partial P_{\bar\xi}}sin\theta \\
\bar\phi &= \phi + \frac{\partial\tilde g}{\partial P_{\bar\phi}}sin\theta
\end{aligned} \tag{56}
$$

To compute the amplitude of oscillation becomes an exercise in the chain rule, taking account of equations 14 and 15 and using the approximations

$$
\begin{aligned}
\omega_b &\approx \frac{V_{\parallel mag}}{qR_{mag}} \\
\bar\omega_\phi &\approx \frac{V_{\parallel mag}}{R_{mag}} \\
\bar\Omega_c &\approx \frac{ZeB_{mag}}{m}
\end{aligned} \tag{57}
$$

we have

$$
\begin{aligned}
\frac{\partial\,\bar r}{\partial P_{\bar\theta}} &= \frac{1}{\bar r ZeB_{mag}} \\
\frac{\partial\,\bar r}{\partial P_{\bar\phi}} &= 0 \\
\frac{\partial\,\bar r}{\partial P_{\bar\xi}} &= 0
\end{aligned} \tag{58}
$$

Neglecting shear we obtain finally

$$
\boxed{
\begin{aligned}
\bar\theta &= \theta + \frac{\Delta_*}{\bar r}sin\theta \\
\bar\xi &= \xi + \frac{\bar r}{R_{mag}}\frac{\overline{\Omega_c}}{\omega_b}sin\theta \\
\bar\phi &= \phi
\end{aligned}
} \tag{59}
$$

The new result is the identification of the relation $\tilde\xi \approx \frac{\bar r}{R_{mag}}\frac{\overline{\Omega_c}}{\omega_b}sin\theta$ for passing particles in large aspect ratio circular equilibria. The interpretation is that the cumulative error in gyro-angle is a function of how many particle gyrations have time to occur during a bounce orbit $\frac{\overline{\Omega_c}}{\omega_b}$ and the degree to which the magnetic field is changing over the orbit $\frac{\bar{r}}{R_{mag}}$. This new parameter is implicated in the generation of poloidal sidebands in addition to the usual finite-orbit width contribution $\frac{\Delta^*}{\bar{r}}$. Contrary to orbit width for confined particles, $\frac{\bar{r}}{R_{mag}}\frac{\overline{\Omega_c}}{\omega_b}$ is not usually a small number. We will take $\frac{\bar{r}}{R_{mag}}\frac{\overline{\Omega_c}}{\omega_b} \gg \frac{\Delta^*}{\bar{r}}$

Noting also

$$\sin\theta = \sin\left[\bar{\theta} - \frac{\Delta_*}{\bar{r}} sin\theta\right] \approx \sin\bar{\theta} - \frac{\Delta_*}{\bar{r}}\sin 2\bar{\theta} \tag{60}$$

we can compare the generation of poloidal sidebands between compressional modes and shear modes. Allowing for up to $k_\perp\rho \sim 1$ leading order FLR effects, we insert equations 59 into equation 44 to obtain for shear modes

$$\Psi_{np0} \equiv \frac{1}{2\pi}\int_0^{2\pi} \sum_m J_0\left(k_\pi\rho + \rho|\nabla\psi|\frac{\partial}{\partial\psi}_{\rho\nabla\psi}\right)[\delta\Phi_m(\psi) + V_\parallel \delta A_{\parallel m}(\psi)]$$
$$\exp\left(-i\frac{m\Delta_*}{\bar{r}} sin\bar{\theta}\right)\exp(i(m-p)\bar{\theta})\,d\bar{\theta} \tag{61}$$

We haven't yet accounted for the poloidal variation in the mode Fourier coefficient $\psi(\theta)$, Larmor radius or k_π. Replacing the radial variable and Taylor expanding

$$\Psi_{np0} \equiv \frac{1}{2\pi}\int_0^{2\pi} \exp\left(\Delta^* cos\bar{\theta}\frac{\partial}{\partial\bar{r}}\right)\sum_m J_0\left(k_\pi(\bar{r})\rho(\bar{r}) + \rho(\bar{r})\frac{\partial}{\partial r_\rho}\right)[\delta\Phi_m(\bar{r}) + V_\parallel \delta A_{\parallel m}(\bar{r})]$$
$$\exp\left(-i\frac{m\Delta_*}{\bar{r}} sin\bar{\theta}\right)\exp(i(m-p)\bar{\theta})\,d\bar{\theta} \tag{62}$$

making the Jacobi–Anger expansion again, we arrive at

passing particles large aspect ratio, shear modes $\delta A_\parallel \gg \delta A_\perp$ $k_\perp\rho\sim 1$

$$\Psi_{np0} \equiv \sum_{m,p} J_{m-p}\left(\frac{m\Delta_*}{\bar{r}} + \Delta^*\frac{\partial}{\partial\bar{r}}\right) J_0\left(k_\pi(\bar{r})\rho(\bar{r}) + \bar{\rho}\frac{\partial}{\partial\bar{r}_\rho}\right)\left[\delta\Phi_m(\bar{r}) + V_{\parallel mag}\delta A_{\parallel m}(\bar{r})\right] \tag{63}$$

giving the standard intuition that for small orbit width $m \approx p$ dominates, but with sidebands that appear as $\frac{m\Delta_*}{\bar{r}} + \Delta^*\frac{\partial}{\partial\bar{r}}$ becomes finite, with relative importance of sidebands scaling as $J_1\left(\frac{m\Delta_*}{\bar{r}} + \Delta^*\frac{\partial}{\partial\bar{r}}\right)/J_0\left(\frac{m\Delta_*}{\bar{r}} + \Delta^*\frac{\partial}{\partial\bar{r}}\right) \approx \frac{m\Delta_*}{2\,\bar{r}} + \frac{\Delta^*}{2}\frac{\partial}{\partial\bar{r}}$. Simply put - narrow modes produce more poloidal sidebands for drifting orbits.

Turning to compressional modes, the global effect of gyro-angle oscillation is a much stronger generator of poloidal sidebands since normally $\frac{\bar{r}}{R_{mag}}\frac{\overline{\Omega_c}}{\omega_b} \gg \frac{\Delta^*}{\bar{r}}$ we neglect the orbit width terms when inserting equation 59 into equation 45

passing particles large aspect ratio, compressional modes $\delta A_\parallel \ll \delta A_\perp$ $k_\perp\rho\sim 1$

$$\Psi_{np1} \equiv \sum_m J_0\left(k_\pi(\bar{r})\rho(\bar{r}) + \bar{\rho}\frac{\partial}{\partial\bar{r}_\rho}\right) J_{m-p}\left(\frac{\bar{r}}{R_{mag}}\frac{\overline{\Omega_c}}{\omega_b}\right)\left[\sqrt{\frac{\mu B_{mag}}{2m}}\hat{C}_0(\bar{r})\cdot\delta\boldsymbol{A}_{\perp\boldsymbol{m}}(\bar{r})\right] \tag{64}$$

Conclusion

The solution of the linear kinetic equation for global modes has been clarified for arbitrary frequency and Larmor radius. The equilibrium motion when represented using orthogonal flux coordinates and the Littlejohn Lagrangian gives canonical coordinates in poloidal, toroidal and cyclotron motion of the guiding centre. The global resonance condition is constructed from a linear combination of average bounce, precession and cyclotron frequencies, where the average is computed over a poloidal bounce time. The difference between the local and global resonance behaviour is due to the non-uniformity in equilibrium causing the variation in particle velocity , resulting in variation of the instantaneous resonance condition when viewed as a sequence of local solutions within a bounce period.

The resonance condition can now be summarized as

$$n\overline{\omega}_{\phi} + p\omega_b + l\overline{\Omega}_c - \omega = 0 \tag{65}$$

with p standing for "poloidal" and l standing for "Larmor". Where the equilibrium or mode is varying significantly from the point of view of the guiding centre, a single orbit has many poloidal harmonics that can be considered to be similarly important, and the local passing particle theory $m = p$ will not suffice. Similarly, where the perturbing field length scale is sufficiently small, particle gyration is causing the perturbing field experienced by the particle to vary and $l = 0$ will no longer suffice for shear Alfven modes (and $l = 1$ will not suffice for compressional modes).

The complexity of the resonance maps generated for a given experiment, which are implied by the arguments on poloidal and Larmor uniformity, will impact the implied fast ion transport. Allowing for wave-particle nonlinearity, the resonances develop a width which scales as the square-root of the perturbation amplitude Ψ. Increasing the number of resonances provides additional opportunity for those resonances to overlap and produce stochastic motion [33]. We have found in this work that the strength of poloidal sidebands for Alfvenic modes goes as orbit width, with fewer sidebands as we go to a reactor. Interestingly for compressional modes, as particles become better confined in reactors, the separation between cyclotron and bounce motion $\frac{\overline{\Omega}_c}{\omega_b}$ becomes larger and the resonance maps become richer and more prone to transport, particularly in spherical tokamaks. This latter finding motivates studies of resonance and losses in existing spherical tokamaks where compressional modes are readily observed.

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Appendix

Appendix A: Neglect of $\dot{\boldsymbol{x}} \cdot \delta\boldsymbol{A}$ term for resonant particles

We wish to demonstrate that

$$\boldsymbol{v} \to \boldsymbol{v} + \frac{Ze}{m}\delta\boldsymbol{A} \tag{66}$$

is a satisfactory approximation for resonant particles for sufficiently small amplitudes.

We start with the full-orbit Hamiltonian

$$H(\boldsymbol{x},\boldsymbol{p},t) = \frac{(\boldsymbol{p} - Ze\boldsymbol{A}_0(\boldsymbol{x}) - Ze\delta\boldsymbol{A}(\boldsymbol{x},t))^2}{2m} + Ze\Phi_0(\boldsymbol{x}) + Ze\delta\Phi(\boldsymbol{x},t) \tag{67}$$

the relationship between canonical momentum and velocity is

$$\boldsymbol{p} = m\boldsymbol{v} + Ze\boldsymbol{A}_0(\boldsymbol{x}) + Ze\delta\boldsymbol{A}(\boldsymbol{x},t) \tag{68}$$

with flows in $(\boldsymbol{x},\boldsymbol{p})$ phase space

$$\begin{aligned}\dot{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{p},t) &= \frac{\left(\boldsymbol{p} - Ze\boldsymbol{A}_0(\boldsymbol{x}) - Ze\delta\boldsymbol{A}(\boldsymbol{x},t)\right)}{m} \\ \dot{\boldsymbol{p}}(\boldsymbol{x},\boldsymbol{p},t) &= Ze(\boldsymbol{E}_0 + \delta\boldsymbol{E}_0 + \dot{\boldsymbol{x}}\times\boldsymbol{B}_0 + \dot{\boldsymbol{x}}\times\delta\boldsymbol{B}) + \frac{d}{dt}Ze\boldsymbol{A}_0(\boldsymbol{x}) + \frac{d}{dt}Ze\delta\boldsymbol{A}(\boldsymbol{x},t)\end{aligned} \tag{69}$$

Let's consider now a point transformation to new variables

$$\begin{aligned}\boldsymbol{\mathcal{X}}(\boldsymbol{x},\boldsymbol{p},t) &= \boldsymbol{x} \\ \boldsymbol{\mathcal{P}}(\boldsymbol{x},\boldsymbol{p},t) &= \boldsymbol{p} - Ze\delta\boldsymbol{A}(\boldsymbol{x},t) \\ \mathcal{H}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{P}},t) &= \frac{\left(\boldsymbol{\mathcal{P}} - Ze\boldsymbol{A}_0(\boldsymbol{\mathcal{X}})\right)^2}{2m} + Ze\Phi_0(\boldsymbol{\mathcal{X}}) + Ze\delta\Phi(\boldsymbol{\mathcal{X}},t)\end{aligned} \tag{70}$$

which is not a canonical transformation and does not produce the same equations of particle motion. The new erroneous Hamiltonian $\boldsymbol{\mathcal{H}}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{P}},t)$ produces the following flows in $(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{P}})$ space

$$\begin{aligned}\dot{\boldsymbol{\mathcal{X}}}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{P}},t) &= \frac{\boldsymbol{\mathcal{P}} - Ze\boldsymbol{A}_0(\boldsymbol{\mathcal{X}})}{m} \\ \dot{\boldsymbol{\mathcal{P}}}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{P}},t) &= Ze\left(\boldsymbol{E}_0 - \nabla\delta\Phi + \dot{\boldsymbol{\mathcal{X}}}\times\boldsymbol{B}_0\right) + \frac{d}{dt}Ze\boldsymbol{A}_0(\boldsymbol{\mathcal{X}})\end{aligned} \tag{71}$$

Both the erroneous and original Hamiltonians correctly give $\dot{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{v},t) = \boldsymbol{v}$ when we transform back $(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{P}},t) \to (\boldsymbol{x},\boldsymbol{p},t) \to (\boldsymbol{x},\boldsymbol{v},t)$, however the equations for velocity evolution are different

$$\begin{aligned}\text{original } H(\boldsymbol{x},\boldsymbol{p},t)\text{: } m\dot{\boldsymbol{v}}(\boldsymbol{x},\boldsymbol{v},t) &= Ze\left(\boldsymbol{E}_0 - \frac{\partial\delta\boldsymbol{A}}{\partial t} - \nabla\delta\Phi + \boldsymbol{v}\times\boldsymbol{B}_0 + \boldsymbol{v}\times\delta\boldsymbol{B}\right) \\ \text{erroneous } \mathcal{H}(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{P}},t)\text{: } m\dot{\boldsymbol{v}}(\boldsymbol{x},\boldsymbol{v},t) &= Ze(\boldsymbol{E}_0 - \nabla\delta\Phi + \boldsymbol{v}\times\boldsymbol{B}_0)\end{aligned} \tag{72}$$

This suggests that when the perturbing fields are omitted from "the symplectic part of the Lagrangian", only the electrostatic portion of the perturbing forces survive.

We are primarily concerned with the resonant wave-particle problem, so we wish to establish the scaling of perturbed velocity with amplitude. Before attempting this in realistic geometry, it is instructive to consider the simplified situation of a particle confined in a periodic potential

$$\delta\Phi(x,t) = -\delta\Phi_k \exp(ikx - i\omega t) + c.c. \tag{73}$$

giving an equation of motion

$$m\ddot{x} = -iZek\delta\Phi_k \exp(ikx - i\omega t) + c.c. \tag{74}$$

we shift to the wave frame and obtain the large amplitude pendulum problem with pendulum angle Θ

$$r \equiv x - \frac{\omega}{k} t$$
$$\Theta \equiv kr$$
$$\ddot{\Theta} = -\frac{iZek^2}{m} \exp(i\Theta) + c.c.$$
$$\frac{d}{dt}\left(\frac{d\Theta}{dt}\right)^2 = \frac{d}{dt}\frac{2Zek^2}{m}\delta\Phi_k \exp(i\Theta) + c.c.$$
$$\dot{r} = \pm\sqrt{\frac{2Ze(\delta\Phi_k + \delta\Phi_k^*)}{m}(\cos(kr(t)) - \cos(kr_0))} \quad (75)$$

Resonant particles have velocities which oscillate with an amplitude $v \propto \sqrt{\frac{Ze\delta\Phi_k}{m}}$ depending on their initial position and velocity. During this oscillation they do work on the wave. If $v \ll \sqrt{\frac{Ze\delta\Phi_k}{m}}$ then they traverse less of the wave potential doing less work and being less important to the growth rate. For $v \gg \sqrt{\frac{Ze\delta\Phi_k}{m}}$, particles are unbounded by the potential "going over hill and dale" with no average power transfer. The velocity responsible for the power transfer therefore scales as $v \sim \sqrt{\frac{Ze\delta\Phi_k}{m}}$

Now instead of electrostatic motion in the x direction, we look at electromagnetic waves in real toroidal geometry. We wish to show clearly that our erroneous motion behaves like a large amplitude pendulum with the same resonant velocity scaling. This will be easiest to show in our action-angle coordinates. Starting with the generalized force obtained with equation 22

$$\dot{\overline{\boldsymbol{P}}} = -\frac{\partial\delta H}{\partial\overline{\boldsymbol{Q}}}$$
$$\delta H(\overline{\boldsymbol{Q}},\overline{\boldsymbol{P}}) = -\sum_{\overline{\boldsymbol{k}}} Ze\Psi_{\overline{\boldsymbol{k}}}(\overline{\boldsymbol{P}}) \exp\left(i\overline{\boldsymbol{k}}\cdot\overline{\boldsymbol{Q}} - i\omega t\right) + c.c. \quad (76)$$

we find

$$\dot{\overline{\boldsymbol{P}}} = -i\sum_{\overline{\boldsymbol{k}}} \overline{\boldsymbol{k}} Ze\Psi_{\overline{\boldsymbol{k}}} \exp\left(i\overline{\boldsymbol{k}}\cdot\overline{\boldsymbol{Q}} - i\omega t\right) + c.c. \quad (77)$$

For integrable motion, it is well known that only one resonance can impart force on a trajectory [33], so we suppose that we make the field amplitude sufficiently small that each resonance works in isolation; we cannot have multiple pendulums. In the neighbourhood of exactly one resonance $\overline{k}$ we have motion obeying

$$\dot{\overline{\boldsymbol{P}}} = -i\overline{\boldsymbol{k}} Ze\Psi_{\overline{\boldsymbol{k}}} \exp\left(i\overline{\boldsymbol{k}}\cdot\overline{\boldsymbol{Q}} - i\omega t\right) + c.c \quad (78)$$

Changing frames to the resonance frame

$$\overline{\boldsymbol{k}}\cdot\boldsymbol{r} \equiv \overline{\boldsymbol{k}}\cdot\overline{\boldsymbol{Q}} - \omega t$$
$$\dot{\overline{\boldsymbol{P}}} = -i\overline{\boldsymbol{k}} Ze\Psi_{\overline{\boldsymbol{k}}} \exp\left(i\overline{\boldsymbol{k}}\cdot\boldsymbol{r}\right) + c.c \quad (79)$$

Multiplying the generalised force by the generalised velocity in the resonance frame gives

$$\dot{\boldsymbol{r}}\cdot\dot{\overline{\boldsymbol{P}}} = -i\overline{\boldsymbol{k}}\cdot\dot{\boldsymbol{r}} Ze\Psi_{\overline{\boldsymbol{k}}} \exp\left(i\overline{\boldsymbol{k}}\cdot\boldsymbol{r}\right) + c.c \quad (80)$$

For sufficiently small amplitude, we take the resonance condition as fixed and ignore possible nonlinear shifting of the resonance $\frac{d}{dt}\overline{\boldsymbol{k}} = 0$, then the work done in the resonance frame is the rate of change of the particle energy in the same frame

$$\frac{d}{dt}\frac{1}{2}mv^2 = \frac{d}{dt}Ze\Psi_{\bar{k}}\exp\left(i\overline{\boldsymbol{k}}\cdot\boldsymbol{r}\right) + c.c \tag{81}$$

Giving our large amplitude pendulum

$$v = \pm\sqrt{\frac{2Ze\left(\Psi_{\overline{\boldsymbol{k}}} + \Psi_{\overline{\boldsymbol{k}}}^*\right)}{m}\left[\cos(\boldsymbol{k}\cdot\boldsymbol{r}(t)) - \cos(\boldsymbol{k}\cdot\boldsymbol{r}_0)\right]} \tag{82}$$

We are now in a position to label terms in the velocity equation 72 for resonant particles according to their scaling with mode amplitude

$$m\dot{\boldsymbol{v}}\Psi^0 = Ze(\boldsymbol{E}_0\Psi^0 - \frac{\partial\delta\boldsymbol{A}}{\partial t}\Psi^1 - \nabla\delta\Phi\,\Psi^1 + \boldsymbol{v}\times\boldsymbol{B}_0\Psi^{0.5} + \boldsymbol{v}\times\delta\boldsymbol{B}\Psi^{1.5}) \tag{83}$$

(reobtaining the original physical formulas by setting the label $\Psi = 1$). For sufficiently small values of Ψ, the dominant contribution to the perturbed force comes from the term $\boldsymbol{v}\times\boldsymbol{B}_0$ which is retained in the erroneous equations of motion. Since the amplitude of the mode may be put as arbitrarily small in the linear analysis, we have shown that $\dot{\boldsymbol{x}}\cdot\delta\boldsymbol{A}$ may indeed be dropped.

Appendix B: Interpretation of $\overline{\Omega}_c$

We can confirm our intuitions about the quantity $\overline{\Omega_c}$ by integrating our definition of it to obtain $\xi(t) \equiv \overline{\Omega_c}t + \tilde{\xi}(t)$. The equation of 1st order equilibrium motion for the gyroangle is readily obtained from our Lagrangian

$$L_0\left(\boldsymbol{X}, V_\parallel, \mu, \xi, \dot{X}, V_\parallel, \dot{\mu}, \dot{\xi}, t\right) = \left[Ze\boldsymbol{A}_0(\boldsymbol{X}) + mV_\parallel\widehat{\boldsymbol{b}}_0(\boldsymbol{X})\right]\cdot\dot{\boldsymbol{X}} + \frac{m\mu}{Ze}\dot{\xi} - \left[\mu\boldsymbol{B}_0(\boldsymbol{X}) + \frac{1}{2}mV_\parallel^2 + Ze\Phi_0(\boldsymbol{X})\right]$$

$$\frac{d}{dt}\left(\frac{\partial L_0}{\partial\dot{\mu}}\right) = \frac{\partial L_0}{\partial\mu}$$

$$\dot{\xi} = \frac{ZeB_0(\boldsymbol{X})}{m} \tag{84}$$

The magnetic field can then be split into orbit-average and oscillatory pieces giving the clear interpretation

$$B_0\left(X(t)\right) = \overline{B_0} + \tilde{B}_0\left(X(t)\right)$$

$$\xi(t) = \frac{Ze\bar{B}_0}{m}t + \int_0^t\frac{Ze\tilde{B}_0\left(X(t')\right)}{m}dt'$$

$$\overline{\Omega_c} \equiv \frac{Ze\bar{B}}{m} \tag{85}$$

true at least to first order. Whether a similar relation can be obtained at higher order isn't obvious; on the one hand, one can envisage a guiding centre non-inertial reference frame drifting alongside the particle that makes the gyration appear a perfect circle, the perpendicular speed of the particle adjusting itself to keep a constant radius of curvature as it encounters varying field at the particle position. On the other hand, the neglected ΔL contribution to the motion will require a re-definition of the gyroangle $\Xi \neq \xi$ in order to make a higher order accurate Lagrangian[25].

Appendix C: Proof that orthogonal flux coordinates are canonical to arbitrary order

In arbitrary flux coordinates, the structure of the guiding centre Lagrangian is always of the form

$$L = \boldsymbol{A}_0^* \cdot \dot{\boldsymbol{X}} + (\dots)\dot{\Xi} - H$$
$$\Xi = \xi + O(\lambda^2)$$
$$L = (\dots)\dot{\psi} + (\dots)\dot{\theta} + (\dots)\dot{\phi} + (\dots)\dot{\Xi} - H \tag{86}$$

which poses the inconvenience that the number of time derivative terms exceeds the number of spatial dimensions. We would like an explicit coordinate transformation where each of the time derivative terms corresponds to the characteristic periods of the motion: the poloidal θ, toroidal ϕ, and cyclotron oscillations Ξ. The troublesome term is to do with radial oscillation $(\dots)\dot{\psi}$. For lowest order drift motion, we have

$$A_0^* = A_0 + \frac{m}{Ze} V_\parallel \hat{\boldsymbol{b}}$$
$$L = \boldsymbol{A}_0^* \cdot \dot{\boldsymbol{X}} + \frac{m\mu}{Ze}\dot{\xi} - H$$
$$A_0 = A_2(\psi)\nabla\theta + A_3(\psi)\nabla\phi \tag{87}$$

Giving

$$\hat{\boldsymbol{b}} = \frac{B_1}{B}\nabla\psi + \frac{B_2}{\mathrm{B}}\nabla\theta + \frac{B_3}{\mathrm{B}}\nabla\phi$$
$$L = mV_\parallel \frac{B_1}{B}\dot{\psi} + \left(ZeA_2(\psi) + mV_\parallel\frac{B_2}{B}\right)\dot{\theta} + \left(ZeA_3(\psi) + mV_\parallel\frac{B_3}{B}\right)\dot{\phi} + \frac{m\mu}{Ze}\dot{\xi} - H \tag{88}$$

Orthogonal flux coordinates have $B_1 = 0$ which removes the $\dot{\psi}$ term without need for further manipulation or approximation

$$\hat{\boldsymbol{b}} = \frac{B_2}{\mathrm{B}}\nabla\theta + \frac{B_3}{\mathrm{B}}\nabla\phi$$
$$L = \left(ZeA_2(\psi) + mV_\parallel\frac{B_2}{B}\right)\dot{\theta} + \left(ZeA_3(\psi) + mV_\parallel\frac{B_3}{B}\right)\dot{\phi} + \frac{m\mu}{Ze}\dot{\xi} - H \tag{89}$$

For all higher orders in λ (if we correctly interpret Littlejohn [25], but this has unclear correspondence to more recent derivations [34]) one needs to make the modification to A_0^*

$$\boldsymbol{A}_0^*(\boldsymbol{X}, V_\parallel, \mu) = \boldsymbol{A}_0(\boldsymbol{X}) + \frac{m}{Ze}(\dots)\hat{\boldsymbol{b}}(\boldsymbol{X}) - \frac{m}{Z^2e^2}(\dots)\left(\nabla\hat{\boldsymbol{1}}(\boldsymbol{X})\right)\cdot\hat{\boldsymbol{2}}(\boldsymbol{X}) \tag{90}$$

so to ensure that orthogonal flux coordinates are still canonical, we must investigate the effect of the additional term in producing a $(\dots)\dot{\psi}$ contribution. The radial component is obtained from the derivative of the contravariant basis vectors

$$\left(\nabla\hat{\mathbf{1}}(\boldsymbol{X})\right)\cdot\hat{\mathbf{2}}(\boldsymbol{X})=\left(\dot{\psi}\partial_\psi\frac{\boldsymbol{e}^\psi}{|\nabla\psi|}+\dot{\theta}\partial_\theta\frac{\boldsymbol{e}^\psi}{|\nabla\psi|}+\dot{\phi}\partial_\phi\frac{\boldsymbol{e}^\psi}{|\nabla\psi|}\right)\cdot\left(\hat{\boldsymbol{b}}\times\frac{\boldsymbol{e}^\psi}{|\nabla\psi|}\right)$$
$$\hat{\mathbf{1}}(\boldsymbol{X})=\frac{\nabla\psi}{|\nabla\psi|}=\frac{\boldsymbol{e}^\psi}{|\nabla\psi|}$$
$$\hat{\mathbf{2}}(\boldsymbol{X})=\hat{\boldsymbol{b}}\times\frac{\boldsymbol{e}^\psi}{|\nabla\psi|} \tag{91}$$

Paying attention only to the term proportional to $\dot{\psi}$ that isn't perpendicular to $\hat{2}$, we have

$$\dot{\psi}\partial_\psi\boldsymbol{e}^\psi=-\dot{\psi}\left(\Gamma^\psi_{\psi\theta}\nabla\theta+\Gamma^\psi_{\psi\phi}\nabla\phi\right) \tag{92}$$

Using the properties of the Christoffel symbols we have

$$\Gamma^\psi_{\psi\theta}=\frac{1}{2}g^{\psi l}\left(\partial_\psi g_{l\theta}+\partial_\theta g_{\psi l}-\partial_l g_{\psi\theta}\right)$$
$$\Gamma^\psi_{\psi\phi}=\frac{1}{2}g^{\psi l}\left(\partial_\psi g_{l\phi}+\partial_\phi g_{\psi l}-\partial_l g_{\psi\phi}\right) \tag{93}$$

In axisymmetric orthogonal coordinates we have $\partial_\phi g_{ij}=0$ and g_{ij} diagonal, and further choosing normalized flux label $g_{\psi\psi}=1$ makes obvious that

$$\dot{\psi}\partial_\psi\boldsymbol{e}^\psi=-\dot{\psi}\left(\frac{1}{2}g^{\psi\psi}\left(\partial_\theta g_{\psi\psi}\right)\nabla\theta\right)=0 \tag{94}$$

This proves that there are still no terms in $\dot{\psi}$ and that orthogonal flux coordinates are still canonical at higher order in λ.

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