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Abstract

It has been suggested (Kleva and Guzdar, *Phys. Plasmas* **6**, 116 (1999)), that reconnecting ballooning modes in which electron inertia replaces resistivity in a non-ideal magnetohydrodynamic Ohm's law can have substantial growth rates in the low collisionality regime. Numerical calculation, albeit necessarily at unrealistically large values of the collisionless skin depth, showed that strongly growing ballooning modes exist at beta values which are below the ideal beta limit. In order to investigate stability at more realistic values of the skin depth we exploit an analytic approach. As in the case of resistive ballooning modes, we find that inertial ballooning modes are stabilised by favourable average curvature effects at moderate values of Δ'_B, the stability index for resistive ballooning. Instability only becomes possible close to the ideal stability boundary ($\Delta'_B \to \infty$) or at unrealistically large values of the toroidal mode number n (eg $n \gtrsim 10^2$). Another ballooning mode, the collisionless analogue of the Carreras-Diamond mode (Carreras, Diamond, Murakami, Dunlap et al., *Phys Rev Lett* **50**, 503 (1983)) can also be excited at larger values of the collisionless skin depth, but this mode is not valid for realistic parameters in a hot plasma.

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I. Introduction

Major disruptions of tokamak plasmas are regularly observed as $\beta = 2\mu_0 p/B^2$, the ratio of plasma pressure to magnetic field pressure, is increased above a critical value, β_{crit}. Theoretical stability analyses of ideal magnetohydrodynamic (MHD) modes predict linear instability at β values in excess of β_{ideal}. However, it appears that, in practice, the critical value for experimental disruptions is considerably less than that predicted for ideal MHD instability. Typically, $\beta_{crit} \sim \beta_{ideal}/2$.

In a recent paper, Kleva and Guzdar[1] seek to explain the very fast ($\sim 100\mu sec$), thermal collapse of high-β disruptions occurring in large tokamaks in terms of instability of collisionless ballooning modes below the ideal MHD β limit. These modes are analogous to resistive ballooning modes,[2-6] but with electron inertia replacing the collisional resistivity η in a non-ideal Ohm's law. They fall into two separate classes:

(i) those driven by Δ'_B, the ballooning space analogue of the tearing mode instability index Δ', representing the source of instability from the ideal region. These modes are stabilised by favourable average curvature, D, in much the same way as low-n tearing modes are in toroidal equilibria.[7] In the case of resistive ballooning modes, Connor et al.[3] and Drake and Antonsen[4] found that instability was only possible in one of two limits:
(a) very close to the ideal MHD stability ballooning boundary, $\beta \sim \beta_{ideal}$, where $\Delta'_B \to \infty$, or
(b) at very short wavelength: $n \gtrsim 100$, where n is the toroidal mode number.

(ii) an analogue of the Carreras-Diamond resistive ballooning mode.[5,6] The Carreras-Diamond mode is a purely growing, pressure driven, instability, localised within the non-ideal layer where resistivity enters the Ohm's law. However, a necessary condition for its validity[3] is that its growth rate must exceed the sound frequency, ie that $\gamma > C_s/Rq$, with $C_s = \Gamma p/(n_i m_i)$ being the sound speed, R the major radius, q the safety factor, n_i and m_i the ion density and mass respectively and Γ is the adiabatic index.

In reference 1 instability of collisionless ballooning modes was demonstrated numerically, using an initial value stability code, for a range of values of the 'effective Lundquist number', a^2/d_e^2, where $d_e = c/\omega_{pe}$ is the collisionless skin depth, with ω_{pe} the plasma frequency and a a typical equilibrium scale length. However, the values considered were still two orders of magnitude smaller than the values typical of the present generation of tokamaks.

In this note we consider more realistic values of d_e by obtaining and solving the analytic dispersion relations for both resistive and collisionless ballooning modes and then compare with the results of reference.[1]

Considering the class (i) ballooning modes,[2] the twisting parity dispersion relation takes the form[3,4]:

$$\frac{2Z_0}{G^{5/6}\Delta'_B} = \frac{Q^{1/4}}{\tau}\left\{\frac{\Gamma(\sigma_-)}{\Gamma\left(\frac{1}{2}+\sigma_-\right)} - \frac{\Gamma(\sigma_+)}{\Gamma\left(\frac{1}{2}+\sigma_+\right)}\right\} \quad (1)$$

where

$$\begin{aligned} Z_0 &= \left(\frac{\tau_\eta}{n^2q^2\tau_A}\right)^{1/3} \\ Q &= (\gamma\tau_A)Z_0G^{2/3} \\ \sigma_\pm &= (2+Q^{3/2}\pm\tau)/8 \\ \tau &= (Q^3+4GD)^{1/2} \end{aligned}$$

and Δ'_B is the stability index obtained by solving the ideal-MHD high-n ballooning equation, $\tau_\eta = \mu_0 r^2/\eta$ is the resistive diffusion time, and $\tau_A = Rq\sqrt{(1+2q^2)}/(sC_A)$ is the Alfvén tim with $s = (r/q)(dq/dr)$ the magnetic shear and C_A the Alfvén speed $B/\sqrt{(m_i n_i)}$. The quantiti G and D are the expressions defined by Glasser, Greene and Johnson[7] for general toroid equilibria. In the large aspect ratio, low β, circular cross-section limit they take the form:

$$G = B^2/\left[\Gamma p(1+2q^2)\right]$$

$$D = -\frac{2rp'}{B^2s^2}\left[1-q^2-sq^2\frac{R}{r}\Delta'_s\right]$$

with p the plasma pressure and Δ_s the Shafranov shift of equilibrium magnetic surfaces.

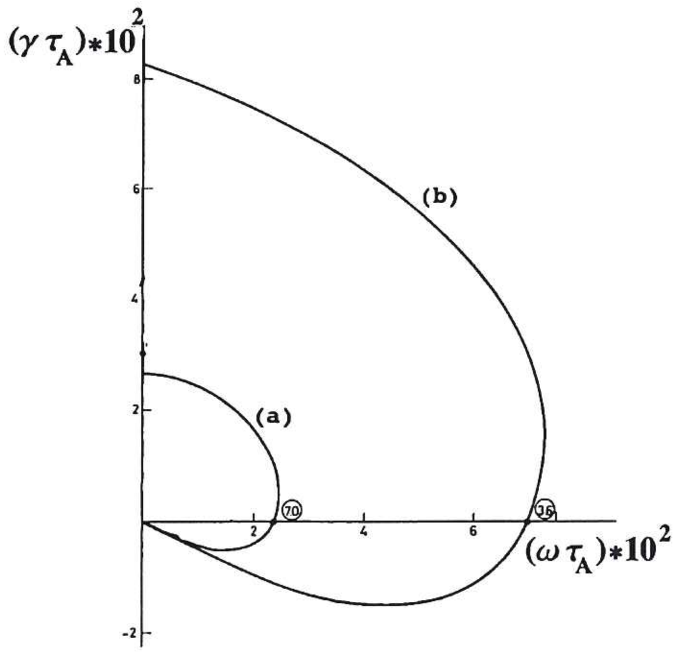

Figure 1: Path of the resistive ballooning eigenvalue in complex frequency space as Z_0/Δ (circled values) varies. (a) $G = 20$, $D = -0.1$; (b) $G = 5, D = +0.1$. From reference 3.

In reference 3 this dispersion relation was derived and solved for a variety of values of G a

D. It was shown that when $GD > 1$, a purely growing resistive instability is always present, but that when $GD < 1$, only damped oscillatory modes exist in the limit of large $Z_0 \rightarrow \infty$ (ie $\eta n^2 \rightarrow 0$). These modes become unstable, as shown in Figure 1, for values of Z_0 below a critical value. In fact, by considering values of GD close to unity

$$GD = 1 - \delta \qquad ; \qquad \delta \ll 1 \tag{2}$$

it is possible to derive (Appendix A) the analytic stability criterion

$$Z_0 \gtrsim 1.3\Delta'_B / \left[\frac{1}{G} - D\right]^{5/6} \tag{3}$$

Although this inequality is only asymptotically correct in the limit $\delta \equiv (1 - GD) \ll 1$, comparison with numerical solutions of dispersion relation (1) shows that it is accurate to within 10% for all values of $\delta < 3$. Figure 2 shows the numerically computed stability boundary in terms of $Z = Z_0/(\Delta'_B G^{5/6})$ (solid curve) as a function of δ, compared to the approximate values (broken curve) obtained from inequality (3). .

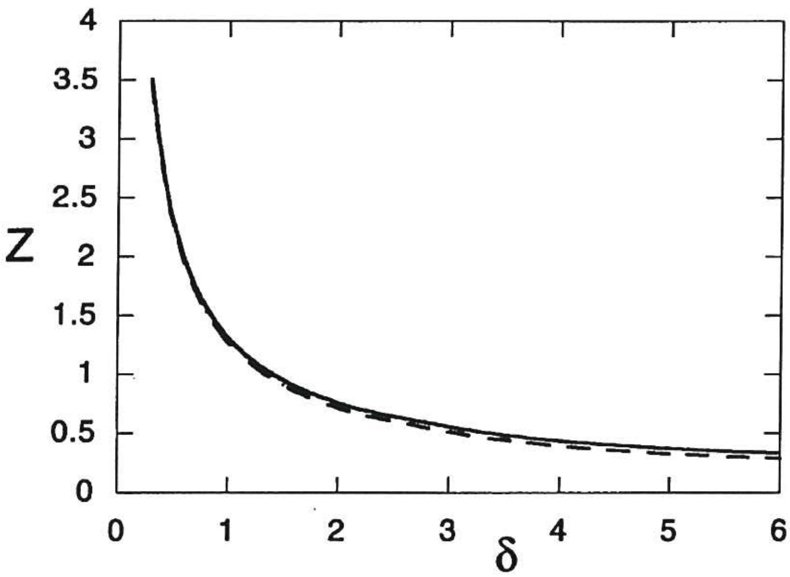

Figure 2: Comparison of the approximate analytic (broken curve) and the exact numerical (solid curve) resistive stability boundaries for $Z_0/(\Delta'_B G^{5/6})$ as a function of $\delta \equiv (1 - GD)$.

We turn now to the class (ii) resistive ballooning modes.[5,6] The Carreras-Diamond ballooning mode is a purely growing instability with growth rate given by

$$\gamma\tau_A = A^{1/3}/Z_0 \tag{4}$$

where, in the large aspect ratio, low β tokamak approximation,

$$A \sim \frac{\alpha^2}{2s^2} \sim (\epsilon\beta_p/s)^2, \tag{5}$$

with $\alpha = -(2Rq^2dp/dr/B^2)$. It was originally proposed theoretically as an explanation for hi values of transport in the edge region of the ISX-B tokamak[5]: at the relatively low temperatu and high q values of the edge plasma in ISX-B the validity condition, $\gamma > C_s/Rq$, could satisfied at moderate values of wavenumber, n. Expressed as a condition on n this validi condition takes the form:

$$n \quad > \quad \left[\frac{2\tau_\eta C_A(1+2q^2)}{R}\right]^{1/2}\frac{(\Gamma\beta/2)^{3/4}}{\alpha q^{3/2}}, \qquad \propto \qquad 1/q^{5/2} \qquad ($$

so that excessively large values of n would be required for validity of this mode in the h core plasma of present generation tokamaks. For example, with $q = 2$, $T = 5KeV$, $R = 3$ $r = 0.5m$, $n_i = 3 \times 10^{19}m^{-3}$, $r/L_p \equiv (r/p)(dp/dr) = 2$, $B = 4$Tesla and Deuterium, inequali (6) requires that $n > 2300$. However the $q^{-5/2}$ scaling of n at large q in inequality (6) mea that this resistive ballooning mode could be a valid, vigorously unstable, mode in the H-mo pedestal of a separatrix plasma boundary. Thus, taking $q = 7$, $T = 1KeV$ and $r/L_P = 1$ modes with $n > 15$ should be unstable.

The relationship between these two classes of ballooning mode is best understood by consi ering the stability of a given equilibrium as ηn^2 is increased from zero, (or, alternatively is decreased from ∞). At very small values of ηn^2 there are two damped, oscillatory class modes: $\omega = \pm\omega_r - i\gamma$. When ηn^2 is increased beyond the critical value given by (3), the oscillatory modes become unstable. At a yet larger value these unstable modes coalesce as purely growing instability. At still larger values of ηn^2, the growth rate of one of these mod increases with ηn^2, while the growth rate of the other decreases: the faster mode growth ra scales as $\gamma \propto (\eta n^2)^{3/5}$. These class (i) modes remain valid solutions of the eigenvalue equatio provided that their frequencies and growth rates are smaller than the sound frequency C_s/R This is normally very easily satisfied for the parameters of real tokamak plasmas. Howeve at large enough values of ηn^2 the more rapidly growing of the class (i) modes becomes co parable to the class (ii), or Carreras-Diamond, mode: the growth rate of this mode scales $\gamma \propto (\eta n^2)^{1/3}$.
In Section 2 we obtain the dispersion relations for collisionless ballooning modes of both class and in Section 3 we discuss the stability of such modes. Section 4 is devoted to a brief discu sion of diamagnetic effects and the validity of single fluid MHD equations for the analysis collisionless ballooning modes, and in Section 5 we summarize the main results of the paper.

II. Inertial Ballooning

The equations for inertially driven ballooning modes are obtained from the resistive ones the replacement:

$$\eta \rightarrow m_e\gamma/n_ee^2 \qquad ,$$

where m_e and e are the electron mass and charge respectively and n_e is the electron densit Thus, to derive the dispersion relation for collisionless ballooning modes, we make the followi substitutions in equation (1);

$$\tau_\eta = \frac{\mu_0 r^2}{\eta} \rightarrow \frac{\omega_{pe}^2}{c^2}\frac{r^2}{\gamma} = \frac{r^2}{d_e^2}\frac{1}{\gamma} \quad ,$$

where $d_e = c/\omega_{pe}$ is the collisionless skin depth,

$$Q = G^{2/3}\gamma\tau_A\left[\frac{\tau_\eta}{n^2q^2\tau_A}\right]^{1/3} \rightarrow \left[G\gamma\tau_A\frac{r}{d_e nq}\right]^{2/3} \equiv Q_i^{2/3}.$$

and

$$Z_0 = \left[\frac{\tau_\eta}{n^2q^2\tau_A}\right]^{1/3} \rightarrow \frac{r}{nqd_e}\frac{G^{1/3}}{Q_i^{1/3}}.$$

Thus for the class (i) mode, defining $Z_{0i} \equiv (r/nqd_e) \gg 1$ as the new expansion parameter, the twisting parity dispersion relation becomes:

$$\frac{2Z_{0i}}{\sqrt{G}\Delta'_B} = \frac{Q_i^{1/2}}{\tau}\left[\frac{\Gamma(\sigma_-)}{\Gamma\left(\sigma_- + \frac{1}{2}\right)} - \frac{\Gamma(\sigma_+)}{\Gamma\left(\sigma_+ + \frac{1}{2}\right)}\right]$$

with

$$\tau = (Q_i^2 + 4GD)^{1/2} \qquad \text{and} \qquad \sigma_\pm = (2 + Q_i \pm \tau)/8$$

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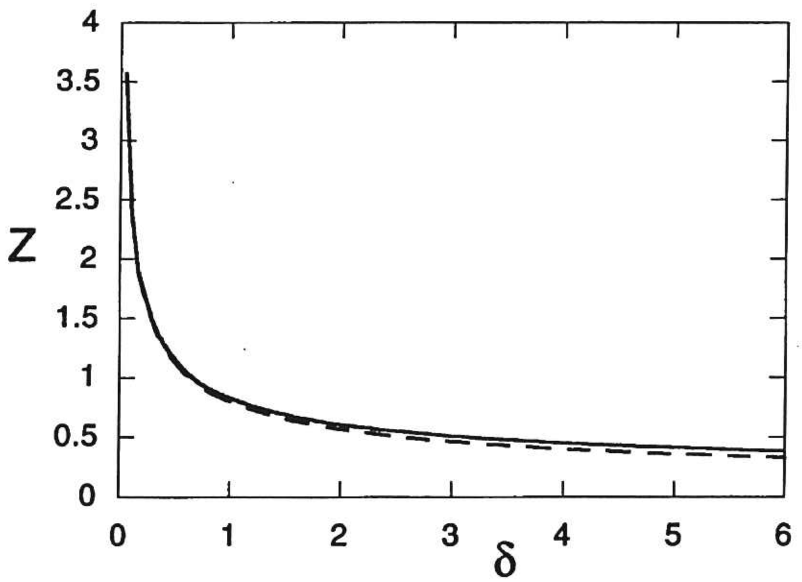

Figure 3: Comparison of the approximate analytic (broken curve) and the exact numerical (solid curve) stability boundaries for the inertial ballooning mode for $Z \equiv Z_{0i}/(\Delta'_B G^{1/2})$ as a function of $\delta \equiv (1 - GD)$.

As for the resistive ballooning case, a purely growing instability always exists where $GD > 1$

and favourable average curvature stabilises the inertial ballooning mode when Z_{0i} is larg
By again considering the limit $\delta \ll 1$, a useful stability criterion can be derived analytica
(Appendix A). The stability criterion is:

$$Z_{0i} > \Delta'_B \left(\frac{2G}{\pi(1-GD)} \right)^{1/2} \qquad (\ $$

This expression turns out to be accurate to $\approx 7\%$ throughout the range $-1 < GD < 1$. T
numerical (solid curve) and analytic (broken curve) stability boundaries are shown in Figure
in this figure $Z = Z_{0i}/(\sqrt{G}\Delta'_B)$.
For completeness, an analogous expression for tearing parity modes is given in Appendix B.

III. Discussion

The resistive stability criterion (3), and the inertial stability criterion (7) for class (i) mod
may both be expressed in the form:

$$n < n_{c1}, \qquad (\ $$

predicting that, for a given equilibrium, only modes of sufficiently short wavelength (high-
can be unstable. The relevant expressions for n_{c1} in the two cases are:

$$\begin{aligned} n_{c1}(\eta) &= 0.7\frac{(\tau_\eta/\tau_A)^{1/2}}{q(\Delta'_B)^{3/2}}\left[\frac{1-GD}{G}\right]^{5/4}, & (\ \\ n_{c1}(inertial) &= \frac{r}{qd_e}\frac{1}{\Delta'_B}\left[\frac{\pi}{2}\left(\frac{1-GD}{G}\right)\right]^{1/2}. & (1 \end{aligned}$$

It is of interest to compare these values for typical, high temperature, tokamak plasma p
rameters. Taking, $\Delta'_B \simeq 1$, $s \simeq 1$ and the same plasma parameters as in section (1), so th
$\beta \simeq 0.75 \times 10^{-2}$, $G \simeq 18$ and $D \simeq -0.06$, we find:

$$n_{c1}(\eta) \simeq 210$$

$$n_{c1}(inertial) \simeq 100$$

These results show that the favourable average curvature stabilization is almost as strong
effect for collisionless ballooning modes as it is for resistive ballooning modes. It stabilises su
modes for values of n in the range $n < 100$, unless Δ'_B is large: ie unless β is close to the ide
MHD stability limit, β_{ideal}.
For the class (ii) ballooning mode, employing the same transformation from Q and Z_0 to
and Z_{0i} as before, the collisionless version of the dispersion relation is:

$$\gamma\tau_A = A^{1/2}/Z_{0i}. \qquad (1$$

The condition for the validity of the resistive and inertial forms of this mode, $\gamma > C_s/Rq$, can also be expressed in the form:

$$n > n_{c2} \; , \tag{12}$$

where

$$\begin{aligned} n_{c2}(\eta) &= \left[\frac{2\tau_\eta C_A(1+2q^2)}{R}\right]^{1/2} \frac{(\Gamma\beta/2)^{3/4}}{\alpha q^{3/2}} && \propto && 1/q^{5/2}, && (13) \\ n_{c2}(inertial) &= \frac{r}{d_e}\frac{[\Gamma\beta(1+2q^2)]^{1/2}}{\alpha q} && \propto && 1/q^2. && (14) \end{aligned}$$

and the $\propto 1/q^x$ entries on the right hand sides of equations (13) and (14) refer to the scaling at large q. For the plasma parameters used above, these expressions yield:

$$\begin{aligned} n_{c2}(\eta) &\simeq 2300 \; , && (15) \\ n_{c2}(inertial) &\simeq 250 \; . && (16) \end{aligned}$$

As discussed in the next section, these values are far beyond the limit of validity of the fluid equations.

Taking equilibrium data to resemble that used in [1] ($\Gamma = 1$, $r \simeq a/2$, $q \simeq 2$, $\beta \simeq 0.65 \times 10^{-2}$, $r/d_e \simeq 15$ and (from Fig.1 of [1]) $r/L_p \sim 2$) the validity condition becomes $n > 6$. Thus the instabilities found in reference[1] appear to be of the vigorously unstable class (ii), or Carreras-Diamond, modes rather than the, more weakly growing, class (i) ballooning modes. In addition the growth rates computed in reference [1] and shown in Figs. 3,4 and 6 all exceed the sound frequency, which has a value $\sim 0.9 \times 10^{-2}$ at $\beta_p = 1$ in the normalised variables of [1]. Furthermore, the strong stabilisation apparent in Fig 8 of [1], as β_p is reduced, occurs just as γ drops below the sound frequency C_s/Rq, suggesting that this may be the point of transition from the strongly growing (class (ii)) mode to the more weakly growing class (i) mode. As we have seen, this latter mode does not merely have a reduced growth rate as (r/d_e) increases, in fact it becomes absolutely stable at realistic values of d_e.

IV. Validity of the Single Fluid Equations

The theoretical model underlying the equations of reference 1 and the dispersion relations of 3,4 and the present paper, is that of single fluid MHD with an Ohm's law containing electron inertia or resistivity. Neglect of Hall terms and thermoelectric terms in Ohm's law, the ion gyro-viscous stress in the momentum equation, and diamagnetic heat flux terms in the energy equations, are all equivalent to neglecting terms of order (ω_*/ω) in the dispersion relation. Here

$$\omega_* \simeq (nq/r)(\rho_i V_{T_i}/L_n)$$

is the diamagnetic frequency, with ρ_i the ion Larmor radius, V_{T_i} the ion thermal speed and L_n the characteristic scale length for density (or alternatively temperature) variation. At marginal stability, the electron inertial class (i) ballooning modes have finite frequency with

$$\omega\tau_A = \frac{(1-GD)}{G}\frac{1}{Z_{0i}} \quad (1$$

Evaluating the ratio ω_*/ω, we note that it is independent of the toroidal mode number n, a that

$$\frac{\omega_*}{\omega} \simeq \left(\frac{\rho_i}{d_e}\right)\left(\frac{R}{L_n}\right)\frac{1}{s\sqrt{\beta}} \quad (1$$

which typically exceeds unity. It is also worth noting that for these plasma parameters the Fini Larmor Radius parameter $k_\perp \rho_i = nq\rho_i/r \simeq (n/80)$, so that fluid equations become invalid smaller values of the toroidal mode number, n, than the predicted instability threshold.
For the class (ii) collisionless ballooning mode, neglect of diamagnetic effects is also inval since,

$$\frac{\omega_*}{\gamma} \simeq \left(\frac{\rho_i}{d_e}\right)\frac{1}{\sqrt{\beta}}\,. \quad (1$$

V. Summary and Conclusions

Resistive ballooning modes, in which electron inertia replaces resistivity in Ohm's law, ha been investigated. Both class (i) modes,[2–4] driven by Δ'_B, the energy source from the ide region, and class (ii), Carreras-Diamond-type, modes,[5,3,6] driven from within the non-ide region, have been investigated. With plasma parameters which are characteristic of prese large tokamaks, such as JET, the class (i) modes are found to be stable for $n < 100$, exce very close to the ideal ballooning stability boundary ($\Delta'_B \gg 1$), where lower n values can l unstable. Unstable Carreras-Diamond modes are predicted only for $n > 250$, far beyond tl validity range of fluid equations. These results suggest that the numerical results of referenc for modes with n = 10, 20 and 30 would have indicated stability had computation at mo realistic values of $(a/d_e)^2 \sim 10^6$ been possible.

It is also noted that, for realistic plasma parameters, the neglect of diamagnetic effects is nev justified. Thus a two fluid treatment retaining Hall terms and the gyro-viscous stress, or a ful kinetic treatment, is required for a consistent treatment of the layer physics.

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Appendix A

An analytic stability criterion for resistive ballooning modes can be derived for the case where $GD = 1 - \delta$ with $\delta \ll 1$. Writing $GD = 1 - \delta$ and $Q = -i\omega$, with $\omega^{3/2} \simeq O(\delta)$, one finds:

$\tau \simeq 2 - \delta, \qquad \sigma_+ \simeq \frac{1}{2}, \qquad \sigma_- \simeq \left[\delta/8 - 2(1+i)(\omega/8)^{3/2}\right],$

so that, at marginal stability, the dispersion relation takes the form:

$$\frac{Z_0}{G^{5/6}\Delta'_B} = \frac{2}{\sqrt{\pi}} \frac{\omega^{1/4} e^{-i\pi/8}}{[\delta - 2(1+i)(\omega/2)^{3/2}]}, \tag{A1}$$

which has the solution

$$\begin{aligned} \omega^{3/2} &= \delta\,[\tan(\pi/8)], && \text{(A2)} \\ Z_0 &= 2\sqrt{2/\pi}\Delta'_B \left[\frac{G}{1-GD}\right]^{5/6} \cos(\pi/8)\,[\tan(\pi/8)]^{1/6}, && \text{(A3)} \\ &= 1.273\Delta'_B \left[\frac{G}{1-GD}\right]^{5/6}. \end{aligned}$$

An analogous marginal stability treatment for the case with electron inertia instead of resistivi
yields, with $Q_i = -i\omega_i$:

$$\omega_i = \delta, \qquad \text{(A}$$

$$Z_{0i} = \Delta'_B \left[\frac{2G}{\pi(1-GD)}\right]^{1/2}. \qquad \text{(A}$$

In both cases instability occurs at smaller values of Z_0, Z_{0i}, ie at larger values of the toroid mode number n.

Appendix B

The dispersion relation for tearing modes (equation (8) of reference [2]) can also be modified f the case where electron inertia dominates over resistivity in the reconnecting layer. Solution the dispersion relation numerically shows that the favourable average curvature stabilisation resistive tearing modes, discovered by Glasser et al,[7] persists in the inertial limit ($m_e\gamma/n_e e^2$: η). At marginal stability the tearing mode has a finite frequency (provided $D < 0$) and a usef analytic stability criterion, and estimate of the frequency, can be derived by considering tl limit

$$-GD = \delta_1 \ll 1$$

The mode frequency is found to be

$$\omega\tau_A = -\frac{\pi}{4}\frac{D}{Z_{0i}} \qquad \text{(B}$$

and the stability criterion is:

$$Z_{0i} > \Delta'_T \frac{\Gamma(1/4)}{\Gamma(3/4)} \frac{1}{(-2\pi D)^{1/2}}. \qquad \text{(B}$$

where Δ'_T is the usual tearing stability index.