UKAEA FUS 422

EURATOM/UKAEA Fusion

On neoclassical transport near the magnetic axis

P Helander

January 2000

EURATOM/UKAEA Fusion Association

Culham Science Centre, Abingdon
Oxfordshire, OX14 3DB
United Kingdom
Telephone +44 1235 463546
Facsimile +44 1235 463647

On neoclassical transport near the magnetic axis

P. Helander

Euratom/UKAEA Fusion Association, Culham Science Centre
Abingdon, OX14 3DB, United Kingdom

Abstract

The theory of neoclassical transport near the magnetic axis in a tokamak is discussed. It is shown that the ordinary treatment of transport in the plateau regime holds close to the magnetic axis, and is not modified by "potato" orbits. It is also demonstrated that transport at low collisionality cannot be described independently of the sources of particles and heat in the region near the magnetic axis, in contradiction to several recently published theories. A variational principle is formulated for the near-axis transport problem.

PACS Numbers: 52.25.Fi, 52.25.Dg, 52.55.Fa, 52.20.Dq

1 Introduction

In the last few years there has been a revival of interest in the old problem of neoclassical transport near the magnetic axis of a tokamak. Conventional neoclassical theory [1, 2, 3] assumes that the ion orbit width is much smaller than the distance r from the magnetic axis. Far from the axis, where this assumption is valid, the widest orbits are shaped like bananas and have a width of the order

$$r_b = 2\epsilon^{1/2} v/\Omega_\theta, \tag{1}$$

where $\Omega_\theta = eB_\theta/m$ is the poloidal gyrofrequency, v the velocity, and $\epsilon = r/R$ the inverse aspect ratio. The banana width r_b increases towards the magnetic axis, and when $r \lesssim r_b$ the shapes of trapped orbits change noticeably and resemble potatoes rather than bananas. These orbits are the widest ones that exist in a tokamak; they have a width that can be estimated by equating the banana width (1) to r. This gives the "potato width" [4]

$$r_p = (2q^2\rho^2 R)^{1/3}, \tag{2}$$

where $q = rB/RB_\theta$ is the safety factor and $\rho = mv/eB$ the gyroradius.

Recently there have been a number of attempts to improve on conventional neoclassical theory, which is valid for $r \gg r_p$, so as to be able to treat the near-axis region $r = O(r_p)$, see Refs [5] - [11]. There are two reasons for the interest in this topic. First, the ion potato width can be a considerable fraction of the minor radius in a tokamak if the current density in the center is small, which is common in discharges with negative magnetic shear. In such plasmas the neoclassical ion confinement in the core should be governed by "potato" transport rather than by the usual theory. Second, the bootstrap current is proportional to the fraction of trapped particles, $f_t \sim \epsilon^{1/2}$, in the banana regime and therefore vanishes on the magnetic axis [12]. It is impossible to find a simultaneous solution to the Grad-Shafranov equation and the transport equation for particles or energy if the current is exclusively given by the bootstrap current [12, 13]. Therefore it is difficult to attain a steady-state tokamak without driving a seed current in the center. It has been suggested that it may be possible to create a plasma with a current "hole", so that the current density vanishes completely in a region near the plasma center and there is no magnetic axis [14]. However, it is uncertain whether such a configuration would be stable or attainable in practice. On the other hand, the presence of (electron) potato orbits might give rise to a neoclassical bootstrap current near the axis after all. Such a current was calculated in Ref [8], and, based on this, a stable, completely bootstrapped equilibrium was found numerically in Ref [10].

It is the purpose of the present paper to comment on the recent literature and to point out a number of mathematical shortcomings and physical misconceptions. In fact, we have found that all of the recently published papers [5]-[11] contain errors, some of which are of a fundamental nature. Our paper is organized as follows. In Sec II, we analyze the guiding-center orbits, which play a basic role in the kinetic transport theory developed in the next two sections. In Sec III, we treat the potato-plateau regime introduced by Shaing and Hazeltine in Ref [9]. In contradiction to their work, we find that in this collisionality regime the transport is similar to that found in conventional neoclassical theory: potato orbits do not affect the plateau kinetics. In the next section we discuss the collisionless, "potato regime", which is analogous to the usual banana regime. This has been the subject of some controversy recently as Ref [5] and Refs [6]-[8] arrive at opposite results: the former finds that the ion heat diffusivity vanishes at the magnetic axis, the latter that it becomes infinite. We show that the transport is nonlocal in nature in the region $r \lesssim r_p$ and cannot be treated as in Refs [5]-[8] and [11] without taking the source term or time evolution into account. We formulate a variational principle for the resulting kinetic problem. In the final section our conclusions are summarized.

2 Orbits

In this section we briefly review the theory of guiding center trajectories in a tokamak, including the near-axis region. A general axisymmetric magnetic field can be written as

$$\mathbf{B} = I(\psi)\nabla\varphi + \nabla\varphi \times \nabla\psi,$$

where ψ is the poloidal flux and φ the toroidal angle. For simplicity, we shall assume that the flux surfaces are elliptical, so that

$$r^2 = z^2/k^2 + x^2$$

is a flux function, that the toroidal field dominates over the poloidal field, $B_\theta \ll B_\varphi$, where $B_\theta = |\nabla\psi|/R$ and $B_\varphi = I/R$, and that the tokamak safety factor

$$q = \frac{1}{2\pi}\oint \frac{\mathbf{B}\cdot\nabla\varphi}{\mathbf{B}\cdot\nabla\theta} d\theta = krB/\psi'(r)$$

is approximately constant over the region we are considering. Here x and z are the horizontal and vertical distances from the magnetic axis, and k is the flux-surface elongation. These assumptions are usually well satisfied near the center of a tokamak.

In general, the shape of the orbits can be deduced from the three constants of motion (E, μ, p_φ), where $E = mv^2/2 + e\Phi$ is the energy, $\mu = mv_\perp^2/2B$ the magnetic moment, and $p_\varphi = e(Iv_{||}/\Omega - \psi) = -e\psi_*$ the toroidal canonical momentum. To simplify the analysis we assume that there is no strong electric field, so that the electrostatic potential Φ varies by much less than does E/e over a guiding center orbit. We can then use the particle speed v as a constant of motion rather than E, and we can use $\lambda = v_\perp^2 B_0/v^2 B \simeq (1-\xi^2)R/R_0$ instead of μ, where $B_0 = I/R_0$ denotes the magnetic field strength at the magnetic axis, $R = R_0$, and $\xi = v_{||}/v$ the cosine of the pitch angle.

By eliminating ξ from these relations one obtains

$$\left(\frac{R}{R_0}\right)^2 - \frac{\lambda R}{R_0} = \left[\frac{e\,(\psi - \psi_*)}{mR_0 v}\right]^2,$$

which determines the shape of the orbits in the coordinates (R, ψ). It is now straightforward to calculate where the orbits intersect the midplane $z = 0$. Since $\psi = kB_0 r^2/2q$ the points of intersection $\epsilon = x/R_0$ are given by the quartic equation

$$(1+\epsilon)^2 - \lambda\,(1+\epsilon) = \left[\frac{kR_0}{2q\rho_0}\left(\epsilon^2 - \epsilon_*^2\right)\right]^2, \qquad (3)$$

where $\rho_0 = mv/eB_0$ and $\epsilon_*^2 = 2q^2\psi_*/kR_0^2 B_0$. This equation has either two or four real roots, corresponding to whether there are one or two orbits with a given set of invariants (v, λ, ψ_*). (Each orbit intersects the midplane twice.) Far from the magnetic axis, these cases correspond to whether the orbits are trapped or circulating. For given (v, λ, ψ_*) there is either one trapped orbit, or two circulating ones with different signs of $v_{||}$.

Let us first analyse the orbits that pass through the magnetic axis. For such orbits the constant term in the quartic equation (3) must vanish, $1 - \lambda - (kR_0\epsilon_*^2/2q\rho_0)^2 = 0$, so that $\epsilon = 0$ is a solution. By finding the other solutions we can determine the other point where an orbit intersects the midplane and thus obtain the orbit width. If we denote the value of ξ at the magnetic axis by ξ_0, then we have $\epsilon_*^2 = -2q\rho_0\xi_0/kR_0$ and $\lambda = 1 - \xi_0^2$, and we can write Eq (3) as

$$\epsilon^3 + \frac{4q\rho_0}{kR_0}\xi_0\epsilon - \left(\frac{2q\rho_0}{kR_0}\right)^2 \left(1 + \xi_0^2 - \epsilon\right) = 0.$$

There are two types of solutions to this equation. For most orbits $\xi_0 = O(1)$ and the term ϵ^3 is small, so that the solution becomes

$$\epsilon \simeq \frac{1+\xi_0^2}{\xi_0}\frac{q\rho_0}{kR_0}.$$

Thus, these orbits stay within about a gyroradius of the magnetic axis. If ξ_0 is very small, $\xi_0 \sim (2q\rho_0/R_0)^{1/3}$, then the cubic term is no longer negligible and

$$\epsilon \sim \left(\frac{2q\rho_0}{kR_0}\right)^{2/3},$$

which corresponds to the potato width estimate (2).

More generally, it is instructive to write Eq (3) as

$$\epsilon^2 = \epsilon_*^2 \left(1 \pm \frac{2q\rho_0}{k\epsilon_*^2 R_0}\sqrt{(1+\epsilon)(1+\epsilon-\lambda)}\right), \tag{4}$$

Since $\rho_0/R_0 \ll 1$ it is apparent that for most solutions we must have $\epsilon^2 \simeq \epsilon_*^2$, so that the orbit stays close to a particular magnetic surface. The maximum excursion for these orbits is $\Delta r = O(q\rho_0\epsilon^{-1/2})$, which occurs for $1-\lambda = O(\epsilon)$. The only orbits which have large excursions, $\Delta r/r = O(1)$, are those for which $\epsilon \lesssim (q\rho/kR_0)^{2/3}$ and $1-\lambda \ll 1$ simultaneously, i.e., the potato orbits analyzed above.

In summary, most orbits stay within a distance of the order of the Larmor radius of a particular flux surface. This is true *everywhere* in the plasma, even in the potato region $r \lesssim r_p$. A small proportion, $f_t \ll 1$, of the particles are trapped (or barely untrapped) and have wider orbits. These particles are characterized by small parallel velocity, $|\xi| \lesssim f_t$. Far from the magnetic axis, the trapped fraction is $f_t \sim \epsilon^{1/2}$. In the region close to the center, as defined by (2), the fraction is $f_t \sim (2q\rho_0/kR_0)^{1/3}$. Figure 1 shows a few particle trajectories close to the magnetic axis. Most orbits in this region are of the type C, i.e., identical to passing orbits farther away from the magnetic axis.

3 Plateau regime

The plateau regime of collisionality is defined by

$$1 \ll \nu_* \ll f_t^{-3}, \tag{5}$$

where $\nu_* = \nu/f_t^3\omega_t$ is the collisionality, with ν the collision frequency and $\omega_t = v_T/qR$ the transit frequency of thermal, well circulating particles. In this regime, the effective collision frequency for scattering across the trapped-passing boundary, $\nu_{\text{eff}} = \nu/f_t^2$, exceeds the poloidal (bounce) frequency $\omega_\theta \sim f_t\omega_t$ for trapped and barely circulating particles. As a result, the orbits of these particles are interrupted by collisions, while the well circulating ones are collisionless.

Because of the smallness of f_t near the magnetic axis, the plateau regime is of fairly wide applicability. Indeed, if the conventional expression, $f_t \sim \epsilon^{1/2}$, were valid all the way to the magnetic axis, one would conclude that the near-axis region is always in the plateau regime if $\nu < \omega_t$. Because of the correction from potato orbits, f_t does not quite vanish at the axis. However, since $f_t \sim (2q\rho_0/kR_0)^{1/3} = (r_p/R)^{1/2}$ is usually a very small number, even quite hot plasmas can be in the plateau regime near the axis. Shaing and Hazeltine have therefore developed a theory for plateau transport in the near-axis region [9]. Their theory takes potato orbits into account, and these orbits play a central role in the transport.

Before discussing this theory, we review the physics of conventional plateau-regime transport. The latter is dominated by a class of resonant particles for which the effective collision frequency $\nu_{\text{eff}} = \nu/\xi^2$ is equal to the poloidal frequency $\omega_\theta = d\theta/dt = \xi\omega_t$, so that $\xi_{\text{res}} \sim (\nu/\omega_t)^{1/3}$. These particles are well circulating, $\xi_{\text{res}} \gg f_t$, and therefore follow ordinary untrapped orbits. Their excursions from the flux surface on which they are centered are small (of the order $q\rho$). In first order (in the smallness of the Larmor radius) the drift kinetic equation is

$$\dot{r}\frac{\partial f_1}{\partial r} + \dot{\xi}\frac{\partial f_1}{\partial \xi} + \dot{\theta}\frac{\partial f_1}{\partial \theta} - C(f_1) = -\mathbf{v}_d \cdot \nabla f_0, \tag{6}$$

where $f_0(r)$ is the zeroth-order, Maxwellian distribution function, $\mathbf{v}_d$ the drift velocity, and C the linearized collision operator. It is convenient to split off a part of the distribution function by writing

$$f_1 = g - \frac{Iv_\parallel}{\Omega}\left(\frac{d\ln p}{d\psi} + \frac{e}{T}\frac{d\Phi}{d\psi} + y\frac{d\ln T}{d\psi}\right) f_0,$$

where $p = nT$ is the pressure and y is a constant, whose value is later chosen suitably (see Ref [2]). The first two terms on the left of Eq (6) are small since r and ξ are nearly constant over the orbits of resonant particles. Moreover, because of the narrowness (in ξ) of the resonant region, the collision operator can be approximated by its pitch-angle scattering part, and the kinetic equation becomes

$$\frac{v\xi}{qR}\frac{\partial g}{\partial \theta} - \frac{\nu}{2}\frac{\partial^2 g}{\partial \xi^2} = Q \sin\theta, \tag{7}$$

where $Q = (v_d r B/kq)(E/T - 5/2 - y)(dT/d\psi) f_0$. (Although the resonant layer is narrow in this sense, it is still wide enough that the mirror force may be ignored.) The solution is easily found by elementary means [2], and becomes

$$g = \pi Q \delta(v\xi/qR) \sin\theta \tag{8}$$

in the limit $\nu \to 0$. The collision operator is small in most of velocity space but is important in the resonant region $\xi \sim (\nu/\omega_t)^{1/3}$, where it resolves a singularity which otherwise arises. As a result, the transport is insensitive to both the collision frequency and the structure of the collision operator. The latter can, for instance, be replaced by a simple Krook operator, $C(g) \to -\nu g$, without affecting the result.

Note that the only properties about orbits which were used were that r and ξ are constant for resonant particles. Both these assertions hold also in most of the potato region $r \lesssim r_p$. Well circulating orbits (including the resonant ones) are no different in this region than farther away from the magnetic axis. The trapped orbits are different, but they play no role in plateau transport! In fact, they do not exist as they are interrupted by collisions. Thus, the transport close to the magnetic axis is similar to that found in the conventional theory. Of course, this theory breaks down at very small distances (of the order of $q\rho$) from the magnetic axis, where the orbits do have relatively large excursions. On the other hand, transport theory is not meaningful on such short length scales, as will be discussed further in the Conclusions. In addition, there is a further constraint associated with the disparity between kinetic and transport time scales, which limits the applicability of the plateau regime close to the magnetic axis, as will be discussed in the next section.

We now turn to the recently published theory of Ref [9]. This paper focuses on particles whose orbits pass through the magnetic axis, and the drift kinetic equation is written as

$$\omega_\theta \left(\frac{\partial g}{\partial \theta}\right)_{E,\mu,\psi_*} - C(g) = Q \sin\theta.$$

The poloidal frequency $\omega_\theta = d\theta/dt$ varies strongly along potato orbits. By changing the independent variables from (E, μ, ψ_*, θ) to $(E, \omega_\theta, \psi_*, \theta)$, the equation is transformed into

$$\omega_\theta \left(\frac{\partial g}{\partial \theta} + \frac{\partial \omega_\theta}{\partial \theta}\frac{\partial g}{\partial \omega_\theta}\right) - C(g) = Q \sin\theta. \tag{9}$$

It is then argued that the second term on the left is small in the resonant region and can be dropped. The problem is thus transformed into a form which is mathematically equivalent to the usual plateau problem, and the solution becomes

$$g = \pi Q \delta\left(\omega_\theta\right) \sin\theta$$

by analogy with Eq (8). The transport can thus be calculated along familiar lines.

However, the neglect of the second term in Eq (9) does not appear to be justified. The poloidal speed varies significantly along orbits passing through the magnetic axis, $\partial\omega_\theta/\partial\theta \sim \omega_\theta$, and in the resonant region $\partial g/\partial\omega_\theta = O(g/\omega_\theta)$, so that

$$\frac{\partial \omega_\theta}{\partial \theta}\frac{\partial g}{\partial \omega_\theta} = O(g)$$

for the relevant class of particles. The terms on the left of Eq (9) are thus comparable, and the kinetic problem is two-dimensional in nature when written in the variables (ω_θ, ψ_*).

In summary, plateau transport is not affected by potato orbits since these orbits are interrupted by collisions and the transport fluxes are carried by well circulating ones. Thus, conventional transport laws [2] apply in the potato region $q\rho < r \lesssim r_p$ rather than the (rather different) ones derived in Ref [9]. In particular, the bootstrap current scales as $j \propto r^2/\nu_*$ and is thus very weak near the axis.

4 Potato regime

We now turn to the regime of very low collisionality, $\nu_* \ll 1$, where not only circulating orbits, but also trapped ones, are collisionless. Transport in this regime has been treated in several recent papers [5]-[8], [10], [11]. The starting point of these papers is the drift kinetic equation

$$v_{\|} \nabla f_1 + \mathbf{v}_d \cdot \nabla \left(f_0 + f_1\right) = C(f_1), \tag{10}$$

where the gradient is taken at fixed (E, μ). The new feature as compared with conventional neoclassical theory is the retention of the term $\mathbf{v}_d \cdot \nabla f_1$, which is as large as the usual term $\mathbf{v}_d \cdot \nabla f_0$ since f_1 varies on the scale of the potato width while the equilibrium scale length associated with f_0 is much longer.

In Ref [5] the kinetic equation (10) for ions was solved numerically, and the results were compared with a random-walk estimate of the ion heat flux

$$q_r \simeq -\alpha \int f_t \frac{(\Delta r)^2}{\tau_{\text{eff}}} \frac{\partial f_0}{\partial r} \frac{mv^2}{2} d^3v = -\frac{4\alpha n}{\pi^{1/2}\tau} \frac{dT_i}{dr} \int_0^\infty \frac{(\Delta r)^2}{f_t} \left(u - \frac{3}{2}\right) e^{-u} du,$$

where the effective collision time was taken as $\tau_{\text{eff}} = \tau u^{3/2} f_t^2$, with τ the ordinary ion collision time and $u = mv^2/2T_i$. The constant α was chosen so as to match the usual neoclassical result far from the magnetic axis, and only the temperature gradient term was included in $\partial f_0/\partial r$. The trapped particle fraction f_t and the step length Δr were both regarded as functions of the velocity v and the radius r. For velocities so small that the banana width is smaller than the radius, $r_b < r$, the banana expressions $\Delta r = r_b$, $f_t = \epsilon^{1/2}$ were used, while for v such that $r_b > r$ the potato results $\Delta r = r_p$, $f_t = (r_p/R)^{1/2}$ were employed. At the magnetic axis the latter apply for all velocities, and the heat flux vanishes

$$q_r \simeq -\frac{4\alpha n R^{1/2}}{\pi^{1/2}\tau} \frac{dT_i}{dr} \int_0^\infty r_p^{3/2} \left(u - \frac{3}{2}\right) e^{-u} du = 0,$$

since $r_p \propto u^{1/3}$, and in the potato region $r/r_p \to 0$ the heat diffusivity approaches zero by a formula derived in Ref [5]. This result appears questionable as it results from the accidental cancellation of two terms, which have only been estimated in an approximate way. For instance, if the density gradient were included or if the collision time had a different dependence on velocity than $u^{-3/2}$ (for instance involving errror functions), a rather different result (more like that of Shaing et al [6]-[8]) is obtained.

In Refs [6] - [8] a more ambitious solution of the kinetic equation (10) was attempted, by making two approximations. Only orbits passing through the magnetic axis were considered, and the collision operator was simplified by including only pitch-angle scattering across the trapped-passing boundary, $\omega_\theta \simeq 0$. Both these approximations are difficult to justify. When an orbit passing through the magnetic axis undergoes a collision, it is normally scattered onto an orbit which does not go through the axis. Recall that although there are trapped, wide orbits in the potato region $r \lesssim r_p$, most particle trajectories in this region are circulating and approximately follow magnetic field lines.

In Fig 1, if a particle on potato orbit A undergoes a collision, it typically ends up on a passing orbit of type C. The kinetics of these different types of orbits cannot be considered independently of each other. This issue is also related to the replacement of the collision operator by scattering across the boundary $\omega_\theta \simeq 0$, which is made in Refs [6] - [8]. This approximation was inspired by earlier work by Hazeltine and Catto on transport in a bumpy torus [15], but does not appear to be justified in the present context. The pitch-angle derivatives in the scattering operator transform as

$$\frac{\partial f_1}{\partial \xi} = \frac{\partial \omega_\theta}{\partial \xi}\frac{\partial f_1}{\partial \omega_\theta} + \frac{\partial \psi_*}{\partial \xi}\frac{\partial f_1}{\partial \psi_*},$$

where both terms must be retained in general. Indeed, in the potato region, scattering across the boundary $\omega_\theta \simeq 0$,

$$\frac{\partial \omega_\theta}{\partial \xi}\frac{\partial f_1}{\partial \omega_\theta} \sim \frac{f_1}{f_t},$$

is comparable to the radial derivative

$$\frac{\partial \psi_*}{\partial \xi}\frac{\partial f_1}{\partial \psi_*} \sim \frac{Iv}{\Omega}\frac{f_1}{\psi_p},$$

since the latter involves the short scale length $\psi_p = kBr_p^2/2q$, the scale length in ψ corresponding to the potato width (2). The point is that orbits change noticeably on the scale length of the orbit width.

However, there is a more fundamental difficulty with the formulation of the transport problem itself as in Eq (10). Normally the time derivative $\partial f/\partial t$ and any source terms can be neglected in Eq (10), and Refs [5]-[8], [10], [11] follow this practice. The usual reason for this neglect is that there is a separation of time scales between the establishment of a local equilibrium within each flux surface (which is fast) and the cross-field transport (which is slow). However, such a situation does not prevail in the region $r \lesssim r_p$ near the magnetic axis.

For instance, consider the ion energy transport equation

$$\frac{3}{2}\left\langle \frac{\partial nT_i}{\partial t} \right\rangle = -\frac{1}{V'}\frac{\partial}{\partial \psi}V' \langle \mathbf{q} \cdot \nabla\psi \rangle + S_E, \tag{11}$$

and suppose that the energy source S_E vanishes. Here $\langle \cdots \rangle$ is the flux-surface average and $V(\psi)$ the volume enclosed by the flux surface ψ. In the literature cited the heat transport was found to be diffusive

$$\langle \mathbf{q} \cdot \nabla\psi \rangle = -n\chi_\psi \frac{dT_i}{d\psi},$$

where the heat diffusivity χ_ψ scales as

$$\chi_\psi \sim \nu\psi_p^2/f_t$$

in the papers by Shaing et al [6, 7, 8]. This can be understood from a random-walk argument. The transport is mainly due to the trapped and barely passing particles, which constitute a fraction f_t of the total population and have an effective collision frequency $\nu_{\text{eff}} = \nu/f_t^2$. The random-walk step size is of the order of the potato width ψ_p, and the heat diffusivity therefore scales as $\chi_\psi \sim f_t\nu_{\text{eff}}\psi_p^2$. The theory assumes that $dT_i/d\psi$ is nearly constant on the scale length of the potato width. This is questionable and will be discussed below, but suppose it holds at some initial instant, $t = 0$. The energy equation (11), which can now be written as

$$\frac{3}{2}\left\langle\frac{\partial nT_i}{\partial t}\right\rangle = \frac{\partial}{\partial\psi}n\chi_\psi\frac{\partial T_i}{\partial\psi},$$

governs the subsequent evolution of the temperature gradient. Since there is no heat source at the magnetic axis, $dT_i/d\psi$ is immediately flattened there, and after the time

$$\tau \simeq \frac{\psi_p^2}{\chi_\psi} \sim \frac{f_t}{\nu}$$

$dT_i/d\psi \simeq 0$ in the entire potato region $r \lesssim r_p$. However, this is shorter than the time ν^{-1} required to establish an equilibrium for the distribution function in velocity space.[1] It is thus clear that the time derivative $\partial f_i/\partial t$ cannot be neglected in the kinetic equation (10).

Alternatively, the source term must be retained. The energy equation (11) then implies

$$\frac{dT}{d\psi} = -\frac{1}{n\chi_\psi}\int_0^\psi S_E(\psi')d\psi',$$

in steady state. Realistically, the source S_E must be taken to be constant over the near-axis region, but then $dT/d\psi$ is not constant as required by the theory. For instance, if χ_ψ is taken from Shaing et al [6] - [8], then $dT/d\psi$ vanishes on axis, while it becomes infinite if the heat diffusivity of Lin, Tang and Lee [5] is used. (The latter actually

[1] It may perhaps be argued that it should be sufficient that the transport time scale $\tau \sim f_t/\nu$ exceeds the time scale for establishing a collisional equilibrium within the trapped region, which is relatively short, of the order $O(f_t^2/\nu)$. While this may be true for energy transport, it certainly cannot hold when considering the bootstrap current, which is mostly carried by passing particles.

predicts infinite temperature on axis!) At any rate, retaining the source term is likely to influence the kinetics of the transport problem.

We are thus led to consider the full, steady-state drift kinetic equation

$$\left(\mathbf{v}_{\|} + \mathbf{v}_d\right) \cdot \nabla f = C(f) + S, \tag{12}$$

where S is a source term, upon which the transport will depend in general. The density and temperature gradients must be determined from this equation; they cannot be prescribed independently. This issue has recently been explained eloquently by Hazeltine [16] in the context of transport along a magnetic field: spatial gradients are usually considered to be the driving force for transport, but this point of view is only fruitful on length scales much longer than the step length in the collisional random walk. On shorter length scales, in the present case $r \lesssim r_p$, the transport is non-local in nature and cannot be related uniquely to local gradients. Instead, the source term acts as the driving force for the transport, which determines the gradients.

In the potato regime, where the source term and the collision term are smaller than the left-hand side, it is appropriate to expand the distribution function, $f = f_0 + f_1 + \ldots$, so that

$$\left(\mathbf{v}_{\|} + \mathbf{v}_d\right) \cdot \nabla f_0 = 0,$$

which implies that f_0 depends only on constants of motion, $f_0 = f_0(v, \lambda, \psi_*, \sigma)$, where $\sigma = v_{\|}/|v_{\|}|$. This function is determined by the constraint equation that is obtained by taking the orbit average of Eq (12),

$$\oint \left[C\left(f_0\right) + S\right] dt = 0. \tag{13}$$

We shall make no attempt to solve Eq (13), which is a formidable problem, but we would like to point out that it is not difficult to formulate a variational principle for Eq (13). The variational form is the entropy functional

$$\Lambda = -\int \frac{f_0}{f_M}\left[C(f_0) + 2S\right] d^3r d^3v,$$

where the volume integral is taken over a radial region which is much larger than the orbit width but much smaller than the length scale associated with the density and temperature in f_0. The function f_M is Maxwellian, with constant density and temperature in this region. It is assumed that f_0 is nearly Maxwellian, which is the case if the source term is weak enough, and that f_0 is close to f_M. If Λ is varied subject to the constraint $f_0 = f_0(v, \lambda, \psi_*, \sigma)$, then $\delta\Lambda = 0$ is equivalent to Eq (13). To see this, we recall that the phase-space volume element can be expressed as

$$d^3 r d^3 v = d^3 J d^3 \vartheta,$$

where $(\mathbf{J}, \boldsymbol{\vartheta})$ are action-angle variables [17, 18, 19]. The action variables $\mathbf{J} = (J_1, J_2, J_3)$ are constants of motion and thus functions of $(v, \lambda, \psi_*, \sigma)$, while the angle variables $\boldsymbol{\vartheta}$ are phases which evolve linearly in time along an orbit, $d\vartheta_i/dt = \text{const.}$, and run from 0 to 1. The first angle corresponds to the gyrophase, and the third one to the toroidal angle φ. Thus, in drift kinetics of an axisymmetric plasma, only the second angle, ϑ_2, is of importance; it measures the phase along the guiding-center trajectory. Varying the functional Λ subject to the constraint that f_0 should depend only on the action variables now gives

$$\delta\Lambda = -\int \frac{2\delta f_0}{f_0} d^3 J \oint [C(f_0) + S] \, d\vartheta_2,$$

and it follows that $\delta\Lambda = 0$ is equivalent to Eq (13). Here we have used the self-adjointness of the linearized collision operator and assumed that the source term is known.

In principle, the transport problem (12) can thus be solved by using trial functions to minimize the functional Λ. When minimized, Λ becomes equal to the total entropy production in the region,

$$\Lambda[f_0] = \int \frac{f_0}{f_M} C(f_0) d^3 r d^3 v.$$

5 Conclusions

The topic of neoclassical transport in the region near the magnetic axis has been the subject of several recent papers [5] - [11]. These works have convincingly shown that the transport properties of a nearly collisionless plasma depend sensitively on the particle orbits near the axis. However, the properties of the orbits themselves depend strongly on the radius, so that the transport changes on a length scale that is comparable to the step length in the collisional random-walk process. Under such circumstances the conventional picture of diffusive transport is inappropriate, as it considers length scales larger than the step length. It is no longer meaningful to write transport equations such as Eq (11) since the heat flux is not determined by the local temperature gradient alone. Moreover, the temperature gradient is neither known nor constant in the region close to the magnetic axis; it must be determined as part of the transport problem. To some extent, these issues are recognized in Refs [6] - [9], where a radial average is taken in the calculation of the heat flux. However, unlike the situation in a bumpy torus [15], there is no intermediate length scale to average over. (A potato orbit which is displaced radially by as little as a potato width changes shape noticeably.)

Although we have argued that the literature cited contains several errors, both mathematical and conceptual, we believe that the basic scalings derived in some of these works are correct and valuable. It is certainly true that the fraction of trapped particles does not vanish on the magnetic axis, but is $f_t \sim (2q\rho_0/kR_0)^{1/3}$. This implies that the particle and heat fluxes should scale approximately as derived in Refs [6] - [8] (rather than as in Ref [5]), although the numerical coefficients (and indeed the form of the diffusive transport laws) are incorrect. Furthermore, there should be a non-vanishing bootstrap current and a trapped-particle correction to the resistivity on the magnetic axis, as pointed out in Refs [10] and [11]. Unfortunately, the bootstrap current tends to be very small – far lower than what is required for a conventional current profile. It should be noted that the electron potato region (which is where potato-orbit effects "fill in" the hole left by the conventional bootstrap current) is very small. It can easily fall in the plateau regime, $\nu_* > 1$, where the current is suppressed by collisions and scales like $j \propto r^2/\nu_*$. This is, for instance, the case for the plasma parameters used in Ref [10].

Finally, we have shown that the plateau regime, as defined in Eq (5), is no different in the potato region $r \lesssim r_p$ than farther away from the magnetic axis. The physical reason for this is that the transport is dominated by well-passing particles, which approximately follow magnetic field lines. Only in a region very close to the magnetic axis, $r \sim q\rho$, do well-passing orbits deviate significantly from flux surfaces. [2]

[2]However, there is also the condition that the transport time scale $\tau \sim r^2/\chi$, where $\chi \sim \nu(q\rho)^2$ in the plateau regime, should be longer than the collision time ν^{-1}. This limits the applicability of the plateau regime to $r > q\rho\sqrt{\omega_t/\nu}$.

6 Acknowledgments

I would like to acknowledge very useful and detailed discussions with Peter Catto, Jack Connor, Richard Hazeltine and Dieter Sigmar, and the hospitality of the Plasma Science and Fusion Center at Massachusetts Institute of Technology, where some of this work was carried out. It was funded jointly by the UK Department of Trade and Industry, and Euratom.

References

[1] A.A. Galeev and R.Z. Sagdeev, Sov. Phys. JETP **26**, 233 (1968).

[2] F.L. Hinton and R.D. Hazeltine, Rev. Mod. Phys **48**, 239 (1976).

[3] S.P. Hirshman and D.J. Sigmar, Nucl. Fusion **21**, 1079 (1981).

[4] T.E. Stringer, Plasma Phys. **16**, 651 (1974).

[5] Z. Lin, W.M. Tang, and W.W. Lee, Phys. Rev. Lett. **78**, 456 (1997); Phys. Plasmas **4**, 1707 (1997).

[6] K.C. Shaing, R.D. Hazeltine, and M.C. Zarnstorff, Phys. Plasmas **4**, 771 (1997).

[7] K.C. Shaing, R.D. Hazeltine, and M.C. Zarnstorff, Phys. Plasmas **4**, 1371 (1997).

[8] K.C. Shaing, R.D. Hazeltine, and M.C. Zarnstorff, Phys. Plasmas **4**, 1375 (1997).

[9] K.C. Shaing and R.D. Hazeltine, Phys. Plasmas **4**, 4331 (1997).

[10] K.C. Shaing, A.Y. Aydemir, Y.R. Lin-Liu, and R.L. Miller, Phys. Rev. Lett. **4**, 1375 (1997).

[11] S. Wang, Phys. Plasmas **5**, 3319 (1998).

[12] R.J. Bickerton, J.W. Connor, and J.B. Taylor, Nature Phys. Sci. **229**, 110 (1971).

[13] D.J. Sigmar, Nucl. Fusion **13**, 17 (1973).

[14] T.H. Jensen, R.L. Miller, and Y.R. Lin-Liu, Phys. Plasmas **3**, 1656 (1996).

[15] R.D. Hazeltine and P.J. Catto, Phys. Plasmas **24**, 290 (1981).

[16] R.D. Hazeltine, Phys. Plasmas **5**, 3282 (1998).

[17] A. Kaufman, Phys. Fluids **15**, 1063 (1972).

[18] R.D. Hazeltine, S.M. Mahajan, and D.A. Hitchcock, Phys. Fluids **24**, 1164 (1981).

[19] I.B. Bernstein and K. Molvig, Phys. Fluids **26**, 1491 (1983).

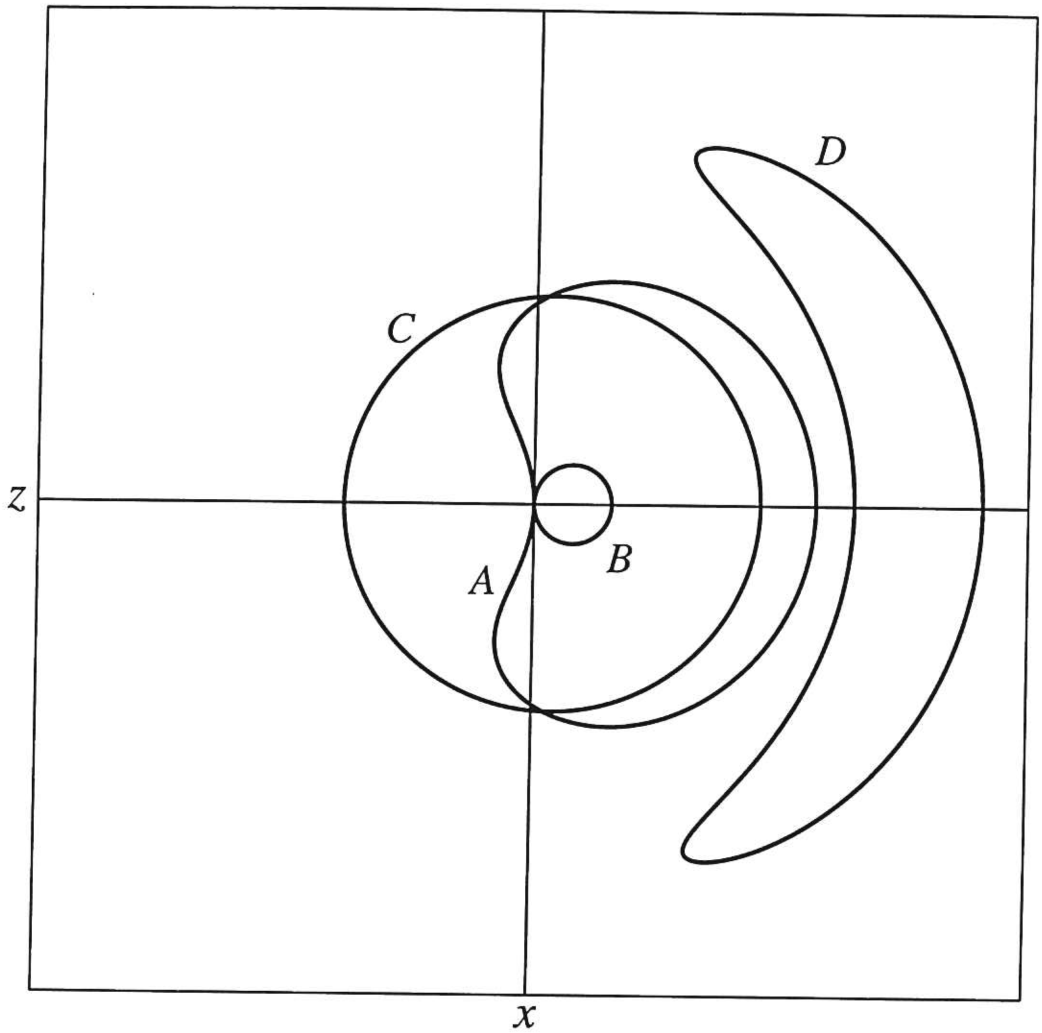

Figure 1. Orbits close to the magnetic axis.Trapped orbits that pass through the axis (A), also known as potato orbits, have relatively large widths. Circulating orbits passing through the axis (B) stay within a few Larmor radii of the axis. Most orbits in the near-axis region are circulating (C) and stay close to a particular flux surface. At a distance of only one potato width from the axis, the trapped orbits resemble bananas (D).