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Azimuthally symmetric MHD and two-fluid equilibria with arbitrary flows

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Abstract

Magnetohydrodynamic (MHD) and two-fluid quasi-neutral equilibria with azimuthal symmetry, gravity and arbitrary ratios of (nonrelativistic) flow speed to acoustic and Alfvén speeds are investigated. In the two-fluid case, the mass ratio of the two species is arbitrary, and the analysis is therefore applicable to electron-positron plasmas. The methods of derivation can be extended in an obvious manner to several charged species. Generalized Grad-Shafranov equations, describing the equilibrium magnetic field, are derived. Flux function equations and Bernoulli relations for each species, and Poisson's equation for the gravitational potential, complete the set of equations required to determine the equilibrium. These are straightforward to solve numerically. The two-fluid system, unlike the MHD system, is shown to be free of singularities. It is demonstrated analytically that there exists a class of incompressible MHD equilibria with magnetic field-aligned flow. A special sub-class first identified by S. Chandrasekhar, in which the flow speed is everywhere equal to the local Alfvén speed, is compatible with virtually any azimuthally symmetric magnetic configuration. Potential applications of this analysis include extragalactic and stellar jets, and accretion disks.

Key words: plasmas – MHD – Galaxies: jets

1 Introduction

Observations across a range of wavelengths dating back over several decades have established the existence of jets associated with stars (Camenzind 1998; Henriksen 1998) and active galactic nuclei (Zensus 1997; Ferrari 1998). These structures, and also accretion disks, have in common the following features: they contain flows of bulk matter; they are embedded in magnetic fields; and they can be treated, to a first approximation, as having an axis of symmetry. All three features are also found in most of the laboratory magnetic confinement systems which have been investigated experimentally, such as tokamaks (Wesson 1997). There is increasing evidence generally that magnetic fields play an important role in governing the physics of a wide variety of astrophysical processes. These include, for example, the formation and early evolution of galaxies, and interstellar gas dynamics (Zweibel & Heiles 1997). In the case of accretion disks, it is believed that purely hydrodynamic models cannot provide the effective viscosity required to account for observationally-inferred accretion rates, and that magnetohydrodynamic (MHD) effects must be taken into account (Hawley & Stone 1998). Magnetic fields may also provide a means of connecting accretion disks with jets, via the generation of MHD waves (Tagger & Pellat 1999). Finally, evidence has emerged recently that plasma confinement by dipole-like magnetic fields can account for X-ray and radio emission from active late-type stars (Kellet, Bingham & Tsikoudi 2000).

The first step in the construction of a theoretical model of a quasi-steady magnetized plasma structure (whether astrophysical or in the laboratory) is a determination of its equilibrium state. In general, this requires all time derivatives to be set equal to zero in the combined system of Maxwell and magnetized fluid equations, and solutions determined for the magnetic field, density, and, if applicable, flow velocity in three dimensions. Having determined the equilibrium configuration, observed time variations can then often be interpreted theoretically as perturbations of the equilibrium state.

MHD equilibrium studies of axisymmetric systems with flow have been carried out by many authors, including Chandrasekhar (1956), Woltjer (1959), Morozov and Solov'ev (1963), Zehrfeld & Green (1972), Maschke & Perrin (1980) and Throumoulopoulos & Pantis (1989). Early work in this field was restricted to the case of incompressible flow (Chandrasekhar 1956; Woltjer 1959). The problem of MHD equilibrium in toroidal systems with compressible flow was studied in the ideal limit by Morozov & Solov'ev (1963), and in the resistive case by Zehrfeld & Green (1972). The case of purely toroidal flow has been studied by Maschke & Perrin (1980) and Throumoulopoulos & Pantis (1989). Lovelace et al. (1986) developed a general theory of axisymmetric MHD equilibria with relativistic flows, which they applied to disks associated with rotating magnetized stars and black holes. Rosso & Pelletier (1994) used a variational method to resolve mathematical problems arising from MHD flow singularities. Bogoyavlenskij (2000) recently demonstrated the existence of exact axisymmetric MHD equilibria which do not include flow, but may nevertheless have applications to astrophysical jets. Krasheninnikov et al. (2000) included flow effects in a study of magnetic dipole equilibria.

These analyses of axisymmetric MHD equilibria are generally based on the Grad–Shafranov equation, derived originally to describe magnetic field equilibria in nuclear fusion experiments (Shafranov 1958; Grad & Rubin 1958): it is essentially an expression of momentum balance for one or more magnetized fluids. The analysis in this paper is also based on various forms of the Grad–Shafranov equation: we investigate ideal axisymmetric MHD (Section 2) and two–fluid (Section 3) equilibria with flows which are nonrelativistic but otherwise arbitrary. Whereas MHD equilibria have been studied in considerable detail by previous authors (in particular, Lovelace et al. 1986), little attention has been paid to two–fluid effects. We use a similar formalism for the MHD and two–fluid models, thus making it straightforward (and instructive) to compare and contrast them. The two–fluid model provides the basis for a more comprehensive description of jets and accretion disks than the MHD model, and is actually more

tractable numerically. In the MHD case, special classes of solutions can be identified analytically (Section 4), which provide useful benchmarks for more realistic numerical solutions.

2 General equilibrium analysis: MHD

We present an alternative derivation of the “generalised Grad–Shafranov” equation of ideal MHD with arbitrary flows in azimuthally symmetric systems. This equation was first obtained by Lovelace et al. (1986): our alternative derivation is simpler than that of Lovelace et al., and can be readily generalized to the two–fluid case. We consider non–relativistic MHD equilibria of a quasi–neutral plasma. The crucial simplifications are due to the assumed azimuthal symmetry (about the z –axis) and steady conditions. We adopt a cylindrical coordinate system, denoting the azimuthal angle by ϕ , and distance from the symmetry axis by r . It is useful to consider an “external” source of gravitation, creating an azimuthally symmetric gravitational potential $V(r, z)$, although it will be seen that Newtonian self–gravitation of the plasma can be easily incorporated into the analysis.

The Maxwell equation $\nabla \cdot \mathbf{B} = 0$ ensures that the magnetic field \mathbf{B} has potential representation

$$\mathbf{B} = \left[-\frac{1}{r} \frac{\partial \Psi}{\partial z} \mathbf{e}_r + B_\phi \mathbf{e}_\phi + \frac{1}{r} \frac{\partial \Psi}{\partial r} \mathbf{e}_z \right]. \quad (1)$$

In equation (1) $\Psi(r, z)$ is the poloidal magnetic flux function and $B_\phi(r, z)$ is the toroidal field. It is easily shown that Ψ/r is the toroidal (azimuthal) component of the magnetic vector potential \mathbf{A} . Since $\mathbf{B} \cdot \nabla \Psi = 0$, the function Ψ is constant along magnetic field lines. In an analogous manner, mass continuity ensures that the plasma mass flux vector, defined to be the product of mass density ρ and flow velocity \mathbf{v} , can be represented in the form

$$\rho \mathbf{v} = \left[-\frac{1}{r} \frac{\partial \chi}{\partial z} \mathbf{e}_r + \rho v_\phi \mathbf{e}_\phi + \frac{1}{r} \frac{\partial \chi}{\partial r} \mathbf{e}_z \right]. \quad (2)$$

where χ is the mass flow function. Because of the assumed azimuthal symmetry, this is simply related to the poloidal magnetic flux function Ψ as follows. The ideal MHD Ohm's law in Gaussian cgs units is

$$\nabla\Phi = \frac{\mathbf{v}}{c} \times \mathbf{B}, \quad (3)$$

where c is the speed of light and Φ has gradient equal to minus the electric field \mathbf{E} : the existence of such a potential follows from the steady-state assumption. Since Φ cannot depend on ϕ , the azimuthal component of equation (3) yields

$$v_r B_z = v_z B_r.$$

Expressing the velocity and magnetic field components in terms of Ψ and χ using equations (1) and (2), we obtain

$$\frac{\partial(\chi, \Psi)}{\partial(r, z)} = 0,$$

from which it follows that

$$\chi = F(\Psi), \quad (4)$$

where F is an arbitrary function. This, and other arbitrary functions appearing in the MHD and two-fluid systems of equations, are determined ultimately by plasma transport processes (Freidberg 1982). However, valuable physical insights can often be gained by adopting simple forms for these functions (see, e.g., Bogoyavlenskij 2000).

The r and z components of equation (3) can be written in the form

$$-c \frac{\partial\Phi}{\partial r} = \frac{(\frac{B_\phi F'}{\rho} - v_\phi)}{r} \frac{\partial\Psi}{\partial r}, \quad (5)$$

and

$$-c \frac{\partial\Phi}{\partial z} = \frac{(\frac{B_\phi F'}{\rho} - v_\phi)}{r} \frac{\partial\Psi}{\partial z}, \quad (6)$$

where $F' \equiv dF/d\Psi$. Introducing the quantity $\Omega \equiv (v_\phi - B_\phi F'/\rho)/r$, and eliminating Φ in the two equations above by cross-differentiation and subtraction, we find that

$$\frac{\partial(\Omega, \Psi)}{\partial(r, z)} = 0.$$

This indicates that Ω is a function of Ψ . From the definition of Ω , we thus have

$$rv_\phi - \frac{rB_\phi F'}{\rho} = r^2\Omega(\Psi). \quad (7)$$

In a similar fashion, by eliminating Ω from equations (5) and (6), it is straightforward to show that Φ is also a function of Ψ and that $c\Phi' = \Omega$ (the prime again denoting differentiation with respect to Ψ). These results have several interesting physical interpretations. First, azimuthally symmetric, steady MHD equilibria with flows have electrostatic potentials which are functions of the poloidal magnetic flux, regardless of centrifugal and Coriolis effects at arbitrary flow Mach number. Second, in any poloidal plane (i.e. any (r, z) plane with ϕ fixed), although not necessarily in three dimensional space, the flow and magnetic field components are parallel. Third, the poloidal mass flow function depends only on the poloidal magnetic flux. However, in general the density is not a flux function.

To proceed further, it is necessary to calculate the components of the vorticity $\mathbf{K} \equiv \nabla \times \mathbf{v}$, the current density $\mathbf{j} = (c/4\pi)\nabla \times \mathbf{B}$, and the vector products $(\nabla \times \mathbf{B}) \times \mathbf{B}$, $\mathbf{K} \times \mathbf{v}$. The following relations are easily derived from the definitions:

$$\nabla \times \mathbf{B} = \left[-\frac{\partial B_\phi}{\partial z} \mathbf{e}_r + j_\phi^* \mathbf{e}_\phi + \frac{1}{r} \frac{\partial}{\partial r} (rB_\phi) \mathbf{e}_z \right], \quad (8)$$

$$j_\phi^* = -\frac{1}{r} \left[\frac{\partial^2 \Psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) \right], \quad (9)$$

$$\mathbf{K} = \left[-\frac{\partial v_\phi}{\partial z} \mathbf{e}_r + K_\phi^* \mathbf{e}_\phi + \frac{1}{r} \frac{\partial}{\partial r} (rv_\phi) \mathbf{e}_z \right], \quad (10)$$

$$K_\phi^* = -\frac{1}{r} \left[\frac{\partial}{\partial z} \left(\frac{F'}{\rho} \frac{\partial \Psi}{\partial z} \right) + r \frac{\partial}{\partial r} \left(\frac{F'}{r\rho} \frac{\partial \Psi}{\partial r} \right) \right], \quad (11)$$

$$\begin{aligned} (\nabla \times \mathbf{B}) \times \mathbf{B} = & \left[\left(j_\phi^* \frac{1}{r} \frac{\partial \Psi}{\partial r} - \frac{rB_\phi}{r^2} \frac{\partial}{\partial r} (rB_\phi) \right) \mathbf{e}_r + \frac{1}{r^2} \frac{\partial (\Psi, rB_\phi)}{\partial (r, z)} \mathbf{e}_\phi \right. \\ & \left. + \left(j_\phi^* \frac{1}{r} \frac{\partial \Psi}{\partial z} - \frac{rB_\phi}{r^2} \frac{\partial}{\partial z} (rB_\phi) \right) \mathbf{e}_z \right], \end{aligned} \quad (12)$$

$$\mathbf{K} \times \mathbf{v} = \left[\left(K_\phi^* \frac{F'}{\rho r} \frac{\partial \Psi}{\partial r} - \frac{rv_\phi}{r^2} \frac{\partial}{\partial r} (rv_\phi) \right) \mathbf{e}_r + \frac{F'}{\rho r^2} \frac{\partial (\Psi, rv_\phi)}{\partial (r, z)} \mathbf{e}_\phi \right]$$

$$+\left(K_\phi^* \frac{F'}{r} \frac{\partial \Psi}{\partial z} - \frac{rv_\phi}{r^2} \frac{\partial}{\partial z}(rv_\phi)\right) \mathbf{e}_z \Big]. \quad (13)$$

The quantity $j_\phi^* = 4\pi j_\phi/c$, where j_ϕ is the azimuthal current density.

The system of MHD equations is completed by the ideal isentropic equation and the equation of motion. For a perfect gas with ratio of specific heats γ the specific entropy s is a function of $p/\rho^\gamma \equiv \sigma$. In the absence of dissipation, σ must be conserved in the fluid frame:

$$\mathbf{v} \cdot \nabla \sigma = 0.$$

It is clear from equations (2) and (4) that this condition is equivalent to

$$\frac{\partial(\sigma, \Psi)}{\partial(r, z)} = 0. \quad (14)$$

The steady-state equation of motion is the usual Eulerian one, with Lorentz, pressure and gravitational forces included. In terms of vorticity \mathbf{K} it can be written in the form

$$\mathbf{K} \times \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \frac{\mathbf{v}^2}{2} - \nabla V + \frac{1}{4\pi\rho} (\nabla \times \mathbf{B}) \times \mathbf{B}. \quad (15)$$

To obtain equation (15) we have used the vector identity

$$\mathbf{v} \cdot \nabla \mathbf{v} = (\nabla \times \mathbf{v}) \times \mathbf{v} + \nabla \frac{\mathbf{v}^2}{2}. \quad (16)$$

Equation (14) has the immediate consequence that σ (a measure of the specific entropy) must be an arbitrarily specifiable flux function.

We next consider the azimuthal component of the equation of motion. Noting the fact that all quantities are independent of ϕ , due to the assumed azimuthal symmetry, we infer from equations (12), (13) and (15) that

$$\frac{F'}{r^2} \frac{\partial(\Psi, rv_\phi)}{\partial(r, z)} = \frac{1}{4\pi r^2} \frac{\partial(\Psi, rB_\phi)}{\partial(r, z)}.$$

Cancelling r^2 in the denominator on both sides and observing that F' depends only upon Ψ , we see that this relation is equivalent to

$$\frac{\partial(F'rv_\phi - \frac{1}{4\pi}rB_\phi, \Psi)}{\partial(r, z)} = 0. \quad (17)$$

This relation, an expression of the law of conservation of canonical angular momentum of the fluid about the symmetry axis, connects the toroidal flow and magnetic field components. Defining

$$\Lambda = F'rv_\phi - \frac{1}{4\pi}rB_\phi, \quad (18)$$

it is clear from equation (17) that Λ is a flux function.

It will be seen that the quantities defined by equations (7) and (18) play a key role in the reduction of the equations of motion. Making use of equations (12) and (13), we find that equation (15) in the (r, z) plane can be written in the form

$$\begin{aligned} \left[K_\phi^* F' - \frac{1}{4\pi} j_\phi^* \right] \frac{\nabla \Psi}{\rho r} &= \frac{1}{2r^2} \nabla [(rv_\phi)^2] - \frac{1}{8\pi \rho r^2} [\nabla (rB_\phi)^2] - \frac{\gamma}{\gamma - 1} \nabla (\sigma \rho^{\gamma-1}) \\ &\quad - \nabla \frac{\mathbf{v}^2}{2} - \nabla V. \end{aligned} \quad (19)$$

If the gravitational potential V and the “structure functions”, $F(\Psi)$, $\Omega(\Psi)$, $\Lambda(\Psi)$ and $\sigma(\Psi)$ are prescribed, equation (19) represents two partial differential equations for the determination of Ψ and ρ as functions of r, z , subject to suitable boundary conditions. The explicit forms of these equations will now be derived in full generality.

We note that the left hand side of equation (19) is annihilated by taking the scalar product with the vector $\nabla \Psi \times \mathbf{e}_\phi$, which is always in a poloidal plane and tangential to the flux curves $\Psi = \text{constant}$. The right hand side is simply related to the tangential derivatives of various quantities, by virtue of a geometrical relation valid for any function $f(r, z)$:

$$(\nabla \Psi \times \mathbf{e}_\phi) \cdot \nabla f = \frac{\partial(\Psi, f)}{\partial(r, z)} = |\nabla \Psi| \frac{\partial f}{\partial l}, \quad (20)$$

where l denotes arc length along the flux line. Using this, we annihilate the left hand side of equation (19) and derive the relation

$$\frac{1}{2r^2} \frac{\partial [(rv_\phi)^2]}{\partial l} - \frac{1}{8\pi \rho r^2} \frac{\partial [\nabla (rB_\phi)^2]}{\partial l} = \frac{\partial}{\partial l} \left[\frac{\gamma}{\gamma - 1} (\sigma \rho^{\gamma-1}) + \frac{\mathbf{v}^2}{2} + V \right]. \quad (21)$$

This equation represents a far-reaching generalization of the well-known Bernoulli equation of gas dynamics. A formal integral of it will now be obtained.

Taking account of equation (18), we introduce the following representations of rv_ϕ and rB_ϕ :

$$rv_\phi = J(\Psi) + \Theta(r, z), \quad (22)$$

$$rB_\phi = I(\Psi) + 4\pi F'(\Psi)\Theta(r, z). \quad (23)$$

Equation (17) can then be written in the form

$$\Lambda(\Psi) = JF' - \frac{I}{4\pi}. \quad (24)$$

Here, Θ is a function to be determined and I, J are arbitrary functions of Ψ related to Λ and Ω . Substitution in equation (7) yields the relation

$$(J + \Theta) - \frac{F'}{\rho}(I + 4\pi F'\Theta) = r^2\Omega(\Psi). \quad (25)$$

This equation expresses Θ in terms of the flux functions Ω, I and J , and in terms of ρ and r^2 , which are not necessarily flux functions. The quantity Θ measures the variation of rv_ϕ and rB_ϕ on flux surfaces. In terms of J and Θ , the two terms on the the left hand side of equation (21) are given by

$$\begin{aligned} \frac{1}{2r^2} \frac{\partial [(rv_\phi)^2]}{\partial l} &= \frac{1}{2r^2} \frac{\partial [(J^2 + 2J\Theta + \Theta^2)]}{\partial l} = \frac{1}{r^2} [(J + \Theta)] \frac{\partial \Theta}{\partial l}, \\ \frac{1}{8\pi\rho r^2} \frac{\partial [(rB_\phi)^2]}{\partial l} &= \frac{1}{8\pi\rho r^2} \frac{\partial [(I^2 + 2I4\pi F'\Theta + 16\pi^2(F')^2\Theta^2)]}{\partial l} \\ &= \frac{1}{\rho r^2} [IF' + 4\pi(F')^2\Theta] \frac{\partial \Theta}{\partial l}. \end{aligned}$$

Thus we have

$$\frac{1}{2r^2} \frac{\partial [(rv_\phi)^2]}{\partial l} - \frac{1}{8\pi\rho r^2} \frac{\partial [\nabla(rB_\phi)^2]}{\partial l} = \frac{1}{r^2} \left[(J + \Theta) - \frac{F'}{\rho}(I + 4\pi F'\Theta) \right] \frac{\partial \Theta}{\partial l} = \frac{\partial}{\partial l}(\Omega\Theta), \quad (26)$$

where we have made use of equation (25) and the fact that Ω is a flux function (i.e. $\partial\Omega/\partial l = 0$). Equation (21) can now be exactly integrated by writing it in the form

$$\frac{\partial}{\partial l} \left[\Omega\Theta - \frac{\gamma}{\gamma-1}(\sigma\rho^{\gamma-1}) - \frac{v^2}{2} - V \right] = 0, \quad (27)$$

where Θ is given by

$$\Theta = \frac{r^2\Omega + \left(\frac{F'}{\rho}I - J\right)}{1 - \frac{4\pi(F')^2}{\rho}}. \quad (28)$$

It is clear from equation (27) that there exists a generalized Bernoulli integral

$$\Omega\Theta - \frac{\gamma}{\gamma - 1}(\sigma\rho^{\gamma-1}) - \frac{\mathbf{v}^2}{2} - V \equiv -H(\Psi), \quad (29)$$

where the arbitrary function $H(\Psi)$ may be regarded as an MHD generalization of the “stagnation enthalpy” (total pressure) of gas dynamics (Meyer 1971).

We return to equation (19) and use the representations given by equations (22) and (23) for rv_ϕ and rB_ϕ . From the above analysis, we deduce from equation (19) that

$$\left[K_\phi^* F' - \frac{1}{4\pi} j_\phi^* - \frac{\rho}{r} J J' + \frac{1}{4\pi r} I I' + \rho r H' \right] \nabla \Psi = 0. \quad (30)$$

Nontrivial solutions of this equation have $\nabla \Psi \neq 0$: setting the bracketed quantity in equation (30) equal to zero, and using our expressions for K_ϕ^* , j_ϕ^* [equations (9) and (11)], we infer that Ψ and ρ satisfy

$$\frac{\partial^2 \Psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) - 4\pi F' \left[\frac{\partial}{\partial z} \left(\frac{F'}{\rho} \frac{\partial \Psi}{\partial z} \right) + r \frac{\partial}{\partial r} \left(\frac{F'}{r\rho} \frac{\partial \Psi}{\partial r} \right) \right] - 4\pi \rho J J' + I I' + 4\pi \rho r^2 H' = 0.$$

This is a generalized form of the Grad-Shafranov equation. It is convenient to rearrange it in the form

$$\frac{\partial}{\partial z} \left(\Delta \frac{\partial \Psi}{\partial z} \right) + r \frac{\partial}{\partial r} \left(\Delta \frac{1}{r} \frac{\partial \Psi}{\partial r} \right) + 4\pi F' F'' \frac{(\Psi_r^2 + \Psi_z^2)}{\rho} - 4\pi \rho J J' + I I' + 4\pi \rho r^2 H' = 0, \quad (31)$$

where

$$\Delta(\rho, F') = 1 - \frac{4\pi(F')^2}{\rho}. \quad (32)$$

Given the arbitrary flux functions I , J , H and the potential V , equations (28), (29), (31) and (32) determine Ψ , Θ and ρ . Equation (31) becomes singular when $\Delta = 0$: physically, this corresponds to the projection of the flow velocity onto any (r, z) surface being equal to an Alfvén speed defined in terms of the (r, z) components of \mathbf{B} . Singularities of this type were identified by Blandford & Payne (1982) and Lovelace et al. (1986), and aggravate considerably the difficulties involved in finding solutions (see

e.g. Rosso & Pelletier 1994). In principle, they can be removed by invoking dissipation or electron inertia, or by special, compatible choices of the arbitrary functions involved. It should be noted that equation (31), unlike the Grad–Shafranov equation without flows (e.g. Bogoyavlenskiy 2000), is fundamentally nonlinear, regardless of the choice of arbitrary functions, due to the presence of the density ρ , which must be obtained by solving equation (29) in terms of Ψ and its first derivatives.

We have so far assumed that the gravitational potential V is due to a distribution of masses which is external to the plasma. It is straightforward to include Newtonian self-gravitation of the plasma by adding the gravitational Poisson equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial z^2} = 4\pi G(\rho + \rho_{\text{ext}}), \quad (33)$$

where ρ_{ext} represents the density distribution of external, uncharged bodies (e.g. a neutron star or black hole). The Poisson integral solution of this equation can be used to eliminate V from the generalized Bernoulli integral [equation (29)]: this would have the effect of making the governing equations non-local.

3 General equilibrium analysis: two-fluid theory

It is possible to generalize the above model to include two-fluid effects. There are both physical and mathematical reasons for seeking such a generalization. Ferrari (1998) has noted that single-fluid models are unlikely to describe adequately the microphysics of extragalactic jets, and that a two-fluid theory would provide a useful intermediate step towards the development of a fully kinetic model. Moreover, the two-fluid equations of motion include inertial terms which, as noted above, remove troublesome singularities appearing in the MHD theory. It is convenient to have a symmetrical notation for both ions (mass m_i , charge $e_i = e$) and electrons (mass m_e , charge $e_e = -e$). In principle, the ions can be any charged species, including positrons. Indeed, the methods are also applicable, *mutatis mutandis* to quasi-neutral plasmas with several charged species.

The equations derived below are non-relativistic, assume quasi-neutrality, azimuthal symmetry and steady conditions. Electron inertia and temperature will be included, but all dissipative and irreversible terms are neglected.

As before, we take the magnetic field in the form given by equation (1). In place of the mass density ρ we introduce the common number density, $n = n_e = n_i$. We replace equation (2) by equations for the particle flux functions, the existence of which follows from the continuity equations for the two species ($\chi_j \rightarrow \Theta_j m_j$, $j = i, e$)

$$n\mathbf{v}_j = \left[-\frac{1}{r} \frac{\partial \Theta_j}{\partial z} \mathbf{e}_r + n v_{\phi j} \mathbf{e}_\phi + \frac{1}{r} \frac{\partial \Theta_j}{\partial r} \mathbf{e}_z \right], \quad (34)$$

where Θ_j , $v_{\phi j}$ are, respectively, the particle flux function and toroidal flow speed of species j . Introducing the vorticities \mathbf{K}_j of the two species, we infer a set of relations analogous to equations (10), (11) and (13):

$$\mathbf{K}_j = \left[-\frac{\partial v_{\phi j}}{\partial z} \mathbf{e}_r + K_{\phi j}^* \mathbf{e}_\phi + \frac{1}{r} \frac{\partial}{\partial r} (r v_{\phi j}) \mathbf{e}_z \right], \quad (35)$$

$$K_{\phi j}^* = -\frac{1}{r} \left[\frac{\partial}{\partial z} \left(\frac{1}{n} \frac{\partial \Theta_j}{\partial z} \right) + r \frac{\partial}{\partial r} \left(\frac{1}{rn} \frac{\partial \Theta_j}{\partial r} \right) \right], \quad (36)$$

$$\begin{aligned} \mathbf{K}_j \times n\mathbf{v}_j &= \left(K_{\phi j}^* \frac{1}{r} \frac{\partial \Theta_j}{\partial r} - \frac{rn v_{\phi j}}{r^2} \frac{\partial}{\partial r} (r v_{\phi j}) \right) \mathbf{e}_r + \frac{1}{r^2} \frac{\partial (\Theta_j, r v_{\phi j})}{\partial (r, z)} \mathbf{e}_\phi \\ &\quad + \left(K_{\phi j}^* \frac{1}{r} \frac{\partial \Theta_j}{\partial z} - \frac{rn v_{\phi j}}{r^2} \frac{\partial}{\partial z} (r v_{\phi j}) \right) \mathbf{e}_z. \end{aligned} \quad (37)$$

The two equations of motion governing momentum balance now take the form

$$m_j \mathbf{K}_j \times n\mathbf{v}_j = -\nabla p_j - m_j n \nabla \mathbf{v}_j^2 / 2 - m_j n \nabla V - e_j n \nabla \Phi + e_j n \mathbf{v}_j \times \mathbf{B} / c, \quad (38)$$

where $j = i, e$, p_j is the pressure of species j , $e_i = e = -e_e$ and, as before, Φ , V are, respectively, electrostatic potential and gravitational potential.

We begin by considering the azimuthal component of equation (38). Substituting from equation (37) and evaluating the azimuthal component of $\mathbf{v}_j \times \mathbf{B}$, we obtain

$$\frac{\partial (\Pi_j, \Theta_j)}{\partial (r, z)} = 0, \quad (39)$$

where Π_j are canonical momenta of the two fluids [cf. equation (18)]:

$$\Pi_j = m_j r v_{\phi_j} + e_j \Psi / c, \quad (40)$$

whence it follows that

$$\Pi_j = F_j(\Theta_j), \quad (41)$$

where F_j are arbitrary functions of the respective particle flux functions. We now apply Ampère's law,

$$\nabla \times \mathbf{B} = \frac{4\pi e}{c} (n \mathbf{v}_i - n \mathbf{v}_e). \quad (42)$$

Only two of the three components of this are independent. The r and z components integrate to give

$$r B_\phi = \frac{4\pi e}{c} (\Theta_i - \Theta_e), \quad (43)$$

while equation (8) indicates that the ϕ component can be written as

$$j_\phi^* = \frac{4\pi e n}{c} (v_{\phi_i} - v_{\phi_e}). \quad (44)$$

Using equations (9), (40) and (41) we rewrite equation (44) in the form

$$\left[\frac{\partial^2 \Psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) \right] = - \frac{4\pi e n}{c} \left[\frac{F_i}{m_i} - \frac{F_e}{m_e} - \frac{e(m_i + m_e)\Psi}{m_i m_e c} \right]. \quad (45)$$

This is the Grad–Shafranov equation for the two–fluid system.

We assume, in the absence of dissipation, that the entropies of the two species are constant following the flow:

$$\frac{\partial \sigma_j}{\partial l} = 0, \quad (46)$$

where $\sigma_j = p_j / n^\gamma$. Taking dot products of \mathbf{v}_j with equation (38), and using equation (45), we derive two Bernoulli relations

$$\frac{\gamma}{\gamma - 1} \frac{p_j}{n} + \frac{m_j \mathbf{v}_j^2}{2} + m_j V + e_j \Phi = H_j(\Theta_j), \quad (47)$$

where, as in the MHD case, the stagnation enthalpies $H_j(\Theta_j)$ are arbitrarily prescribable functions of the respective particle flux functions. We note that these two equations effectively determine the number density n and the electrostatic potential Φ ,

given $\sigma_j(\Theta_j)$, $F_j(\Theta_j)$, Θ_j , Ψ and V . Since Φ appears nowhere else explicitly, it may be eliminated by simply adding the two equations to obtain the following equation for n :

$$\frac{\gamma}{\gamma-1}(\sigma_i + \sigma_e)n^{\gamma-1} + \left(\frac{m_i \mathbf{v}_i^2}{2} + \frac{m_e \mathbf{v}_e^2}{2}\right) + (m_i + m_e)V = H_i(\Theta_i) + H_e(\Theta_e). \quad (48)$$

Using equation (47) we may rewrite equation (38) in the form

$$(m_j \mathbf{K}_j + \frac{e_j}{c} \mathbf{B}) \times \mathbf{v}_j = -H'_j \nabla \Theta_j. \quad (49)$$

From the radial or z components of this we immediately obtain

$$m_j K_{\phi j}^* + \frac{e_j}{c} B_\phi - \frac{n}{m_j r} F'_j (F_j - \frac{e_j}{c} \Psi) = -nr H'_j,$$

i.e.

$$\left[\frac{\partial}{\partial z} \left(\frac{1}{n} \frac{\partial \Theta_j}{\partial z} \right) + r \frac{\partial}{\partial r} \left(\frac{1}{rn} \frac{\partial \Theta_j}{\partial r} \right) \right] - \frac{e_j}{m_j c} r B_\phi + \frac{n}{m_j^2} F'_j (F_j - \frac{e_j}{c} \Psi) = nr^2 H'_j / m_j.$$

Eliminating rB_ϕ using equation (43), we obtain the following closed system for Θ_j , Ψ , n and V :

$$\left[\frac{\partial}{\partial z} \left(\frac{1}{n} \frac{\partial \Theta_i}{\partial z} \right) + r \frac{\partial}{\partial r} \left(\frac{1}{rn} \frac{\partial \Theta_i}{\partial r} \right) \right] - \frac{e}{m_i c} \frac{4\pi e}{c} (\Theta_i - \Theta_e) + \frac{n}{m_i^2} F'_i (F_i - \frac{e}{c} \Psi) = nr^2 H'_i / m_i, \quad (50)$$

$$\left[\frac{\partial}{\partial z} \left(\frac{1}{n} \frac{\partial \Theta_e}{\partial z} \right) + r \frac{\partial}{\partial r} \left(\frac{1}{rn} \frac{\partial \Theta_e}{\partial r} \right) \right] + \frac{e}{m_e c} \frac{4\pi e}{c} (\Theta_i - \Theta_e) + \frac{n}{m_e^2} F'_e (F_e + \frac{e}{c} \Psi) = nr^2 H'_e / m_e, \quad (51)$$

$$\left[\frac{\partial^2 \Psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) \right] = -\frac{4\pi en}{c} \left[\frac{F_i}{m_i} - \frac{F_e}{m_e} - \frac{e(m_i + m_e)\Psi}{m_i m_e c} \right], \quad (52)$$

$$\frac{\gamma}{\gamma-1}(\sigma_i + \sigma_e)n^{\gamma-1} + \left(\frac{m_i \mathbf{v}_i^2}{2} + \frac{m_e \mathbf{v}_e^2}{2}\right) + (m_i + m_e)V = H_i(\Theta_i) + H_e(\Theta_e), \quad (53)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial z^2} = 4\pi G [(m_i + m_e)n + \rho_{\text{ext}}]. \quad (54)$$

The first two equations are the equations of motion, the third represents Ampère's law, equation (53) is the Bernoulli relation for the total pressure, and equation (54) is the gravitational Poisson equation.

Given the six arbitrary functions F_j , σ_j , H_j and the "external" mass density ρ_{ext} , these equations have to be solved subject to suitable boundary conditions on Ψ , Θ_j and

the gravitational potential V . They fully determine the structure of both flows and magnetic fields. The velocities, currents and so on can be obtained from the various auxiliary relations. This system represents an exact generalization of the equations derived by Lovelace et al. (1986) to two-fluid, dissipationless, azimuthally symmetric, quasi-neutral, nonrelativistic, gravitating equilibria with arbitrary flows. It is important to note that the leading order operators in the partial differential equations are entirely nonsingular and elliptic, provided the number density (n) remains bounded. For this reason, the system is much easier to deal with numerically than the equations of MHD equilibrium. We note also that equation (53) is somewhat simpler to deal with than the corresponding equations of ideal MHD [equations (28) and (29)].

4 Special solutions: field-aligned flows

In this section a particular class of MHD equilibria is considered. In Section 2 it was shown quite generally that the flow function χ is a function of the magnetic flux, Ψ : we now assume that mass density ρ is also a flux function. In this case, it is easily shown from equations (2) and (4) that the flow velocity is divergence-free, i.e. the plasma is incompressible: in general, this requires that the flow speed v be less than the sound speed c_s (e.g. Landau & Lifshitz 1987). Axisymmetric MHD equilibrium equations were obtained for general incompressible flow by Woltjer (1959). Here we consider incompressible flows with $\Omega \equiv 0$ [cf. equation (7)]: in such cases rv_ϕ is proportional to rB_ϕ . Denoting F'/ρ by G' [cf. equations (2) and (4)], we obtain

$$\mathbf{v} = G'\mathbf{B}, \tag{55}$$

Where $G(\Psi)$ is a new flux function. This equation indicates that the flow and the field are aligned in three dimensions [in general, the alignment is in (r, z) planes only]. The ideal Ohm's law [equation (3)] indicates that $\Phi \equiv 0$ for such flows.

From equation (54) we infer an expression for the vorticity $\mathbf{K} = \nabla \times \mathbf{v}$ in terms of G

and \mathbf{B} :

$$\mathbf{K} = G''\nabla\Psi \times \mathbf{B} + G'\nabla \times \mathbf{B}.$$

Substituting this in the equation of motion [equation (15)], we get

$$\rho G' [G''\nabla\Psi \times \mathbf{B} + G'\nabla \times \mathbf{B}] \times \mathbf{B} = -\nabla p - \rho\nabla\frac{\mathbf{v}^2}{2} - \rho\nabla V + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}.$$

Using $\mathbf{B} \cdot \nabla\Psi = 0$, we find that the above equation may be written in the form

$$-\rho G' G'' \mathbf{B}^2 \nabla\Psi + (\rho G'^2 - \frac{1}{4\pi})(\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla p - \rho\nabla\frac{\mathbf{v}^2}{2} - \rho\nabla V. \quad (56)$$

If we take the dot product of this equation with \mathbf{B} , consistency requires that the right hand side must vanish. Using the fact that ρ is a flux function, we infer from this that

$$\mathbf{B} \cdot \nabla(p + \rho\frac{\mathbf{v}^2}{2} + \rho V) = 0.$$

We thus obtain the Bernoulli relation

$$p + \rho\frac{\mathbf{v}^2}{2} + \rho V = H^*(\Psi), \quad (57)$$

where H^* is an arbitrary function of Ψ . Using this to eliminate p from equation (56), we obtain

$$-\rho G' G'' \mathbf{B}^2 \nabla\Psi + (\rho G'^2 - \frac{1}{4\pi})(\nabla \times \mathbf{B}) \times \mathbf{B} = -H^{*\prime} \nabla\Psi + \rho' \nabla\Psi(\frac{\mathbf{v}^2}{2} + V).$$

Putting $\mathbf{v} = G'\mathbf{B}$ [equation (55)], this reduces to

$$-(\rho G' G'' + \rho' \frac{G'^2}{2}) \mathbf{B}^2 \nabla\Psi + (\rho G'^2 - \frac{1}{4\pi})(\nabla \times \mathbf{B}) \times \mathbf{B} = -(H^{*\prime} - \rho' V) \nabla\Psi. \quad (58)$$

This is our fundamental equation. We first note that since Ψ depends only on r and z , the azimuthal component is satisfied if $\mathbf{j} \times \mathbf{B}$ vanishes in the ϕ direction. It is clear from equation (12) that this requires rB_ϕ to be a flux function, which we identify with $I(\Psi)$ defined by equation (23) [$\Theta \equiv 0$ in this case]. It follows from our field alignment assumption that $rv_\phi = J(\Psi) = G'I$. We observe next that if $\rho G'^2$ is a constant, the first term on the left hand side of equation (58) vanishes. In this case, making use of the r or z components of equation (12), we find that the equation reduces to

$$\lambda(j_\phi^* \frac{1}{r} - \frac{II'}{r^2}) = -(H^{*\prime} - \rho' V), \quad (59)$$

where

$$\lambda = (\rho G'^2 - \frac{1}{4\pi}).$$

Equation (59) can be simplified, using equation (9), to give a Grad–Shafranov equation for field–aligned MHD flows:

$$\lambda \left[\frac{\partial^2 \Psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) + II' \right] + r^2 \rho' V = r^2 H^*. \quad (60)$$

This is algebraically similar to the Grad–Shafranov equation used by Bogoyavlenskij (2000), but represents a generalization of it to include field–aligned flow in a spatially–varying gravitational potential. From the definition of G' [equation (55)], it is clear that $4\pi\rho G'^2 = v^2/c_A^2$, where $c_A = B/\sqrt{4\pi\rho}$ is the Alfvén speed. Although we assumed that $\rho G'^2$ was a constant in order to obtain equation (60), this does not, of course, preclude the possibility of v , B and ρ individually varying in space.

There are several limiting cases of equation (60) which are of physical interest. If V is a prescribed gravitational potential, ρ and H^* are quadratic in Ψ , and II' is linear in Ψ , we obtain a linear eigenvalue problem for Ψ and λ , which can be solved by a variety of methods (e.g. Bogoyavlenskij 2000). Even when equation (60) is nonlinear, it can be solved numerically in a straightforward way.

A special case of great interest is that of constant density and force–free (Beltrami) magnetic fields, defined by the property $\nabla \times \mathbf{B} = \mu\mathbf{B}$ for some scalar function μ , so that $(\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{0}$ (Lundquist 1951; Mahajan & Yoshida 1998). Consistency with equation (58), with constant $\rho G'^2 \neq 1/4\pi$, requires that H^* also be a constant. More generally, the force–free condition is satisfied for constant $\rho G'^2$ whenever $H^* = \rho'V$. From equation (60), it is clear that λ is then an arbitrary finite constant, and that Ψ satisfies the equation

$$\frac{\partial^2 \Psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) + II' = 0. \quad (61)$$

Having solved equation (61) for ψ , one can obtain the velocity field $\mathbf{v}(r, z)$ from equation (55): the pressure is then determined by the Bernoulli relation [equation (57)], as befits an incompressible flow.

A second class of solutions has $\lambda = 0$. The equilibrium is then compatible with a completely arbitrary azimuthally symmetric field! It is clear from equation (58) that it is not necessary for the azimuthal component of $\mathbf{j} \times \mathbf{B}$ to vanish in this case: the only consistency requirements are that $\rho G'^2$ is a constant and that $H^{*'} = \rho'V$. Since $\lambda = \rho G'^2 - 1/4\pi = (v^2/c_A^2 - 1)/4\pi$, the condition $\lambda = 0$ means that the flow is everywhere “trans-Alfvénic” ($v = c_A$), as well as being field-aligned. This solution was first obtained by Chandrasekhar (1956), who proved moreover that it is stable. Consistency with the assumption of incompressibility ($\nabla \cdot \mathbf{v} = 0$) requires in this case that $c_s > c_A$, i.e. that the plasma beta $\beta \sim c_s^2/c_A^2$ be greater than unity. It is not clear how restrictive this condition is, since β in most astrophysical plasmas is not accurately known.

The most general field-aligned flow governed by equation (57) is one in which λ , and hence $\rho G'^2$, are flux functions, the equation then taking the form

$$\lambda(\Psi) \left[\frac{\partial^2 \Psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) + II' \right] + \frac{1}{2} \lambda' ((\nabla \Psi)^2 + I^2) = r^2 (H^{*'} - \rho'V), \quad (62)$$

where

$$\lambda(\Psi) = \rho G'^2 - \frac{1}{4\pi}. \quad (63)$$

It is plain that we can extend the same ideas to the two-fluid case by taking n to be a flux function and $\mathbf{v}_j = G'_j(\Psi)\mathbf{B}$. Specific applications of these solutions are left to future work.

5 Conclusions and Discussion

We have derived equilibrium equations for azimuthally symmetric MHD and two-fluid systems with arbitrary nonrelativistic flows. Our method of derivation in the MHD case is distinct from that used by Lovelace et al. (1986), and provides the basis for our two-fluid calculation. We have identified some special exact solutions of the MHD system of equations which are of independent interest. Since there are no restrictions on

the geometry (other than azimuthal symmetry), both systems of equations are equally applicable to astrophysical and laboratory systems with azimuthal symmetry. The two-fluid model is more suitable than the MHD model for astrophysical applications, not only because it contains more physics, but also from the computational point of view: the two-fluid equations do not involve singularities which are associated with ideal MHD (Blandford & Payne 1982; Lovelace et al. 1986). The physical reason for this is that the two-fluid model, unlike the MHD model, takes into account electron inertia, which introduces a new fundamental length, the collisionless skin depth c/ω_{pe} (in the MHD limit, $c/\omega_{pe} \rightarrow 0$). In future work we intend to solve numerically both the MHD and two-fluid systems of equations for specific astrophysical scenarios, including extragalactic jets.

The equations we have derived apply to nonrelativistic flows. However, if the assumptions of stationarity and azimuthal symmetry are retained, it is reasonable to expect that similar but more complicated equations will exist for relativistic conditions (this was demonstrated by Lovelace et al. for the case of ideal MHD). Although we considered plasmas with only one or two separate species, the methods clearly extend to any number of charged species, provided that overall quasi-neutrality and other equilibrium assumptions apply. It is relatively straightforward to take into account higher order collisional or collisionless effects (using, for example, the reciprocals of the Reynolds or Lundquist numbers as expansion parameters): one would expect the inclusion of such effects to provide transport equations for the various arbitrary functions appearing in the MHD and two-fluid systems of equations.

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