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The Resistive Wall Instability and Critical Flow Velocity

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Abstract

A cylindrical model with an equilibrium surface current and a uniform equilibrium plasma flow velocity parallel to the axis of the cylinder is used to investigate resistive wall instability. This system can be unstable to the ideal, external kink mode, which can be stabilised by the presence of a perfectly conducting wall. This is the classic condition for the resistive wall instability and the effect of a plasma flow velocity on this mode is explored. It is noted that a stable kink mode, Doppler shifted by the flow velocity, can pass through zero frequency for a velocity which depends on the marginal condition for the external kink instability. The passage through zero frequency is the condition for the kink mode to carry negative energy. It is shown how this mode implies a critical flow speed at which the resistive wall mode is further destabilized, with a growth rate inversely proportional to the square root of the wall time. Under these circumstances, the resistive wall mode behaves more like an ideal instability. All flow velocities are shown to be potentially destabilizing and the flow velocity can produce a resistive wall instability even when the plasma is stable to the external kink mode in the absence of a wall. At velocities well above the critical flow speed, the resistive wall growth rate is much reduced (inversely proportional to the wall time and to the flow speed).

I INTRODUCTION

With the advent of advanced tokamaks with improved confinement there has been renewed interest in resistive wall instability[1] since this would be a threat to the success of these devices. The recent observation[2] of an extension of the wall stabilised lifetime of DIII-D[3] to more than 30 times the resistive wall time in the presence of toroidal rotation has also focussed attention on the effect of rotation on resistive wall instability. An early treatment of the effect of rotation on resistive wall modes was given by Gimblett[4] who later considered the effect of a secondary wall[5] rotating relative to the first wall. Since this early work several authors[6–12] have analyzed the effect of rotation with the aim of identifying a stabilizing effect due to rotation.

Recently, Wesson[13] discussed a very simple model in order to clarify the role of a flow velocity on the resistive wall instability. For a uniform incompressible slab of fluid in the presence of a uniform flow velocity along a uniform magnetic field it was shown that the flow velocity resulted in a resistive wall instability if $v_0 > \sqrt{2}c_A$ where v_0 is the flow speed and c_A the Alfvén speed. An extension of this model to a compressible plasma[14,15] showed that, in addition to this instability, a second resistive wall instability occurred when $v_0 > c_S$ where c_S is the sound speed, and for low beta conditions, $c_S \ll c_A$. However, these very simple models are not relevant to a tokamak. The most important feature missing from these models is free magnetic energy. The purpose of this paper is to analyse the effect of a flow velocity along the equilibrium magnetic field on resistive wall instability for a model which contains free magnetic energy. The characteristic feature of the resistive wall instability of a magnetically confined plasma is that in the absence of a wall the plasma is unstable to an ideal kink mode which is stabilized by the presence of a perfectly conducting wall close to the plasma-vacuum boundary. When the perfectly conducting wall is replaced by one with finite resistivity the plasma becomes unstable to the resistive wall mode.

In order to keep the analysis as simple as possible a cylindrical model with a surface current at the plasma-vacuum boundary is used. A uniform plasma flow velocity along the axis of the cylinder is assumed which is parallel to the equilibrium magnetic field in the plasma. The motivation for this study is to elucidate the effect of a flow velocity on the resistive wall instability of the type defined at the end of the previous paragraph. It is also of interest to discover whether there are any critical values of the flow velocity for the instability. The outline of the paper is the following. In Section II the cylindrical model is defined and the equations of ideal magnetohydrodynamics (MHD) are used to obtain a second order equation for perturbations to the equilibrium state. The boundary conditions at the plasma-vacuum interface at $r = a$ and at the wall at $r = b$ are given in Section III and used to obtain the dispersion relation. Solutions of the dispersion relation are given in Section IV and a summary and conclusions are given in Section V.

II THE CYLINDRICAL MODEL

The starting point for the analysis in this paper is Ref 16 in which an infinitely long cylinder of plasma, of radius a, with uniform density, pressure, and magnetic field is considered. The confining magnetic field is produced by a surface current J_0 flowing parallel to the axis of the cylinder at the plasma-vacuum boundary. Thus

$$\boldsymbol{J}_0 = \hat{z} J_0 \delta(r-a) \tag{1}$$

where $\delta(x)$ is the Dirac δ-function. The magnetic field resulting from this current is

$$\begin{aligned} B^p_{0\theta}(r) &= 0 \quad , \quad 0 \le r < a \\ B^v_{0\theta}(r) &= \frac{\mu_0 a}{r} J_0 \quad , \quad a \le r \end{aligned}$$

There is also a constant axial magnetic field in both plasma, B^p_{0Z}, and vacuum B^v_{0Z}. Since there is no poloidal surface current it is assumed that $B^p_{0Z} = B^v_{0Z}$. The equilibrium pressure balance at $r = a$ gives

$$p_0 = \frac{(B^v_{0\theta}(a))^2}{2\mu_0} \tag{2}$$

In Ref 16 the plasma is assumed to be stationary in the equilibrium state and to be surrounded by a vacuum which extends to infinity. In this paper, the plasma is assumed to have a uniform flow velocity v_0 parallel to the axis. In addition, there is a thin wall, of thickness Δ, having finite resistivity and positioned at $r = b$, concentric with the plasma cylinder. In the regions between the plasma and the wall and beyond the wall ($r > b$) there is a vacuum.

Perturbations about this equilibrium are described by the linearized equations of ideal MHD. In the presence of a uniform axial flow of the plasma, the linearized equations are

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \rho_0 (\boldsymbol{v}_0 \cdot \boldsymbol{\nabla}) \boldsymbol{v}_1 = -\boldsymbol{\nabla} p_1 - \boldsymbol{\nabla} \frac{(\boldsymbol{B}_0 \cdot \boldsymbol{B}_1)}{\mu_0} + \frac{1}{\mu_0} (\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}) \boldsymbol{B}_1 \tag{3}$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0) + \boldsymbol{\nabla} \times (\boldsymbol{v}_0 \times \boldsymbol{B}_1) \tag{4}$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 + (\boldsymbol{v}_0 \cdot \boldsymbol{\nabla}) \rho_1 = 0 \tag{5}$$

Assuming an isothermal equation of state, $p_1 = c_S^2 \rho_1$, where c_S is the sound speed $(p_0/\rho_0)^{1/2}$, and that all perturbations vary as $f(r) \exp i(kz + m\theta - \omega t)$, the perturbed variables can be expressed in terms of v_{1Z}. Carrying out this elimination, the following second order equation for v_{1Z} is obtained

$$\frac{d^2 v_{1Z}}{dr^2} + \frac{1}{r} \frac{d v_{1Z}}{dr} - \alpha^2 v_{1Z} - \frac{m^2}{r^2} v_{1Z} = 0 \tag{6}$$

where

$$\alpha^2 = \frac{\left(k^2 - \frac{\bar{\omega}^2}{c_A^2}\right)\left(k^2 - \frac{\bar{\omega}^2}{c_S^2}\right)}{\left(k^2 - \frac{\bar{\omega}^2}{c_A^2} - \frac{\bar{\omega}^2}{c_S^2}\right)} \tag{7}$$

and $\bar{\omega} = \omega - kv_0$. The solution of Eq (6) which is finite at $r = 0$ is

$$v_{1Z} = CI_m(\alpha r) \tag{8}$$

where I_m is a modified Bessel function of the first kind. In addition, p_1, v_{1r} and B_{1Z} are also required. These variables can be expressed in terms of v_{1Z} and are

$$p_1 = \frac{\bar{\omega}\rho_0}{k} v_{1Z} \tag{9}$$

$$v_{1r} = -\frac{i}{k}\frac{\left(k^2 - \frac{\bar{\omega}^2}{c_S^2}\right)}{\alpha^2}\frac{dv_{1Z}}{dr} \tag{10}$$

$$B_{1Z} = \frac{kB_{0Z}}{\bar{\omega}}\frac{\left(\bar{\omega}^2 - k^2c_S^2\right)}{k^2c_S^2} v_{1Z} \tag{11}$$

III THE BOUNDARY CONDITIONS

The perturbations in the plasma must be matched to the corresponding perturbations in the vacuum region. The perturbed magnetic field in vacuum is given by $\boldsymbol{B}_1^v = \nabla\psi$. In the region $a < r < b$, ψ is given by

$$\psi(r) = DK_m(kr) + FI_m(kr) \tag{12}$$

where $I_m(kr), K_m(kr)$ are modified Bessel functions of the first and second kinds respectively. For $r > b$,

$$\psi(r) = EK_m(kr) \tag{13}$$

which satisfies the condition that $\psi \to 0$ as $r \to \infty$. The boundary conditions at the thin resistive wall[4] at $r = b$ are:

B_{1r}^v is continuous, and

$$\left.\frac{dB_{1r}^v}{dr}\right|_b^{b+\Delta} = -\frac{i\omega}{c_W}B_{1r}^v(b) \tag{14}$$

where $c_W = (\mu_0\sigma\Delta)^{-1}$, with Δ the thickness of the resistive wall and σ its conductivity.

Substituting Eqs (12) and (13) into these two boundary conditions, and eliminating the constants E and F in favour of D, ψ, given in Eq (12), can be written as

$$\psi(r) = DK_m(kr) + \frac{i\omega(K_m'(kb))^2 DI_m(kr)}{\{kc_W\left[I_m''(kb)K_m'(kb) - K_m''(kb)I_m'(kb)\right] - i\omega K_m'(kb)I_m'(kb)\}} \tag{15}$$

where a single prime denotes the first radial derivative of the corresponding Bessel function and a double prime the second radial derivative.

In order to eliminate the two remaining constants, two further boundary conditions are required. These are both obtained at the plasma vacuum interface at $r = a$. The first condition is obtained from the continuity of the total pressure which can be obtained by integrating the radial component of Eq (3) across the plasma vacuum boundary, giving

$$p_1(a) + \left.\frac{B^p_{0Z}B^p_{1Z}}{\mu_0}\right|_a = \left.\frac{B^v_{0Z}B^v_{1Z}}{\mu_0}\right|_a + \left.\frac{B^v_{0\theta}B^v_{1\theta}}{\mu_0}\right|_a + \left.\frac{iB^v_{0\theta}}{\mu_0}\frac{dB^v_{0\theta}}{dr}\frac{v_{1r}}{\bar{\omega}}\right|_a \tag{16}$$

where the third term on the right-hand-side of Eq (16) results from evaluating the equilibrium magnetic pressure at the perturbed surface (see, for example Ref 18). The second boundary condition at the plasma vacuum boundary can be written[18]

$$B^v_{1r}(a) = i\left(\frac{m}{a}B^v_{0\theta}(a) + kB^v_{0Z}\right)\frac{iv_{1r}}{\bar{\omega}} \tag{17}$$

Substituting Eqs (8) - (11) and Eq (15) into Eqs (16) and (17) the dispersion relation for the modes of oscillation of a cylindrical plasma with uniform flow, a surface current and a thin resistive shell at $r = b$ is

$$\frac{\bar{\omega}^2 c_S^2}{(\bar{\omega}^2 - k^2c_S^2)} + c_A^2 = \frac{I'_m(\alpha a)}{\rho_0\mu_0\alpha I_m(\alpha a)}\left\{\frac{(B^v_{0\theta}(a))^2}{a} + \left(kB^v_{0Z} + \frac{m}{a}B^v_{0\theta}\right)^2 \times\right. \tag{18}$$

$$\times\frac{\left\{[K_m(ka)I'_m(kb) - I_m(ka)K'_m(kb)]\,K'_m(kb) + \frac{ikc_W}{\omega}\,[I''_m(kb)K'_m(kb) - K''_m(kb)I'_m(kb)]\,K_m(ka)\right.}{\left\{[K'_m(ka)I'_m(kb) - I'_m(ka)K'_m(kb)]\,kK'_m(kb) + \frac{ikc_W}{\omega}\,[I''_m(kb)K'_m(kb) - K''_m(kb)I'_m(kb)]\,kK'_m(ka\right.}$$

Within the limitations of the present sharp boundary, surface current model, Eq (18) is the most general dispersion relation. It describes all the modes of the system, namely, shear Alfvén waves, fast and slow magnetosonic waves and kink modes. Solutions of Eq (18) will now be obtained for various limiting cases.

IV SOLUTIONS OF THE DISPERSION RELATION

In order to make contact with earlier work consider the case when the wall at $r = b$ is a perfect conductor, ie $c_W \to 0$. In the limit $b \to \infty$, Eq (18) is then identical to Eq (11) of Shafranov[16] for the equilibrium assumed here and taking $v_0 = 0$.

Since the main aim of the present paper is to analyse the effect of a flow velocity on resistive wall modes attention will be concentrated on the kink modes. For this purpose it is sufficient to consider the incompressible approximation to Eq (18), obtained by letting $c_S \to \infty$, giving

$$\bar{\omega}^2 = k^2c_A^2 - \frac{kI'_m(ka)}{\rho_0\mu_0 I_m(ka)}\left\{\frac{(B^v_{0\theta}(a))^2}{a} + \left(kB^v_{0Z} + \frac{m}{a}B^v_{0\theta}\right)^2\frac{K_m(ka)\left[1 - \frac{I_m(ka)K'_m(kb)}{K_m(ka)I'_m(kb)}\right]}{kK'_m(ka)\left[1 - \frac{I'_m(ka)K'_m(kb)}{K'_m(ka)I'_m(kb)}\right]}\right\} \tag{19}$$

where, for the moment, it is still assumed that $c_W = 0$ and $\alpha \to k$ when $c_S \to \infty$. In the long wavelength limit, $ka \ll 1$, $kb \ll 1$, Eq (19) reduces to

$$\bar{\omega}^2 = k^2 c_A^2 + \frac{\left(kB_{0Z}^v + \frac{m}{a}B_{0Z}^v\right)^2}{\rho_0 \mu_0} \frac{\left[1 + \left(\frac{a}{b}\right)^{2m}\right]}{\left[1 - \left(\frac{a}{b}\right)^{2m}\right]} - \frac{(B_{0\theta}^v(a))^2}{\rho_0 \mu_0} \frac{m}{a^2} \tag{20}$$

Again, in the limit $b \to \infty$, and taking $v_0 = 0$, Eq (20) gives the well known dispersion relation for external kink modes (see for example, Ref 18). The kink mode is unstable for $m = 1$ when $-mB_{0\theta}^v/(aB_{0Z}^v) < k < 0$. For $m = 0, 2$ the external kink is marginally stable and for all higher m it is stable.

For the later discussion of resistive wall modes it is helpful to review the corresponding properties of kink modes when the perfectly conducting wall is at a finite distance from the plasma. The $m = 1$ mode is still unstable but the band of unstable wave numbers is reduced to $-B_{0\theta}^v/aB_{0Z}^v < k < -B_{0\theta}^v(G-1)/(aB_{0Z}^v(G+1))$ where $G \equiv [1+(a/b)^{2m}]/[1-(a/b)^{2m}]$. For this case the $m = 2$ mode is stable and does not reach the marginal condition. However, it is instructive to obtain the frequency of the $m = 2$ mode. This is done for the wave number k which minimises the first two terms on the right-hand-side of Eq (20). The resulting value of k is $-2B_{0\theta}^v G/aB_{0Z}^v(1 + G)$. Using this value (for k) in Eq (20) the solution for $\bar{\omega}$ is given by

$$\bar{\omega}^2 = \frac{4}{a^2} \frac{(B_{0\theta}^v)^2}{\rho_0 \mu_0} \frac{(G-1)}{2(G+1)} \tag{21}$$

For $v_0 = 0$, the frequency is progressively down-shifted as the conducting wall is moved further from the plasma. As the distance to the wall tends to infinity, $G \to 1$ and $\omega \to 0$, the marginal condition. For intermediate positions of the conducting wall, the kink mode will have a phase velocity along the magnetic field which is significantly smaller than the Alfvén speed. For example, when $a/b = 0.25$, and using Eq (21), the phase velocity is $\omega / \mid k \mid \simeq 0.09 c_A$. Clearly, the phase velocity can be reduced to zero at the marginal condition. This property has already been noted in another context[19].

Returning to Eq (21) with the plasma flow velocity v_0 included, the kink mode solutions can be written

$$\omega = kv_0 \pm \frac{2}{a} \frac{B_{0\theta}^v}{(\rho_0 \mu_0)^{1/2}} \left[\frac{(G-1)}{2(G+1)}\right]^{1/2} \tag{22}$$

where the solutions of Eq (23) describe stable kink modes propagating parallel or anti-parallel to the magnetic field. However, when the flow speed exceeds the phase velocity of the kink mode one of the kink modes becomes a negative energy wave, namely the one whose frequency passes through zero. It will be found that this will have important consequences for resistive wall modes, especially as rather low velocities can cause the change in sign of the wave energy when the kink mode is close to the marginal condition.

The effect of a plasma flow velocity on resistive wall modes can now be discussed by returning to the general dispersion relation given in Eq (18). The same approximations are made as in the previous case, namely, the incompressible limit, $c_S \to \infty$, and the long wavelength conditions, $ka \ll 1$, $kb \ll 1$. However, a resistive wall is now assumed instead of a perfect conductor. Hence, finite values of c_W are now included. Under these conditions, Eq (18) can be reduced to

$$\bar{\omega}^2 = k^2 c_A^2 + \frac{(kB_{0Z}^v + \frac{m}{a}B_{0\theta}^v)^2}{\rho_0\mu_0}\frac{G\,(1 + 2imkc_W/(\omega kb(1+(a/b)^{2m})))}{(1 + 2imkc_W/(\omega kb(1-(a/b)^{2m})))} - \frac{(B_{0\theta}^v(a))^2\, m}{\rho_0\mu_0 a^2} \tag{23}$$

The dispersion relation given in Eq (23) is identical to the one obtained recently by Veeresha et al[15] although these authors wrote the equation in a different form. It is helpful to re-write Eq (23) as follows

$$\omega\left\{\bar{\omega}^2 - k^2c_A^2 - \frac{\left(kB_{0Z}^v + \frac{m}{a}B_{0\theta}^v\right)^2 G}{\rho_0\mu_0} + \frac{(B_{0\theta}^v(a))^2}{\rho_0\mu_0}\frac{m}{a^2}\right\}$$
$$= -\frac{i2mc_W}{b\left[1-\left(\frac{a}{b}\right)^{2m}\right]}\left\{\bar{\omega}^2 - k^2c_A^2 - \frac{\left(kB_{0Z}^v + \frac{m}{a}B_{0\theta}^v\right)^2}{\rho_0\mu_0} + \frac{(B_{0\theta}^v(a))^2 m}{\rho_0\mu_0 a^2}\right\} \tag{24}$$

Introducing the notation

$$\omega_0^2 \equiv k^2c_A^2 + \frac{\left(kB_{0Z}^v + \frac{m}{a}B_{0\theta}^v\right)^2 G}{\rho_0\mu_0} - \frac{(B_{0\theta}^v(a))^2 m}{\rho_0\mu_0 a^2} \quad , \tag{25}$$

$$\omega_1^2 \equiv k^2c_A^2 + \frac{\left(kB_{0Z}^v + \frac{m}{a}B_{0\theta}^v\right)^2}{\rho_0\mu_0} - \frac{(B_{0\theta}^v(a))^2\, m}{\rho_0\mu_0 a^2} \quad , \tag{26}$$

the dispersion relation given in Eq (24) can now be written in the compact form

$$\omega(\bar{\omega}^2 - \omega_0^2) = -\frac{i2mc_W}{b\left[1-\left(\frac{a}{b}\right)^{2m}\right]}(\bar{\omega}^2 - \omega_1^2) \quad . \tag{27}$$

The meaning of the quantities ω_0 and ω_1 is that $\bar{\omega}^2 = \omega_0^2$ is the dispersion relation for kink modes with a perfectly conducting wall at $r = b$ and $\bar{\omega}^2 = \omega_1^2$ is the corresponding dispersion relation in the absence of a wall. The dispersion relation, Eq (27), is in exactly the same form as the one given by Eq (3a) of Finn and Gerwin[10] for a different equilibrium. It is also worth noting that Eq (27) has the same structure as the dispersion relation derived by Wesson[13].

The dispersion relation given in Eq (27) is the basis of the discussion which follows on resistive wall instabilities and their dependence on a plasma flow velocity. Since the kink mode dispersion relation with a perfectly conducting wall at $r = b$ contains the three cases of interest, unstable, marginally stable and stable, Eq (27) is used to consider the various cases which might arise. Before continuing with the discussion of resistive wall modes, it should be emphasised that this discussion is of a heuristic nature. Although the dispersion relation given in Eq (27) is treated as being representative of the resistive wall instability, there is always a band of k-values for which the $m = 1$ ideal kink is unstable. The justification for the model is that it allows a comprehensive treatment of the problem to be given and enables further understanding of the underlying mechanisms to be gained. It is hoped that the information obtained from this simple model will serve as a guide for the analysis of more realistic situations and be of some qualitative help to experiment.

It is useful to begin with the case without a flow velocity. The classical resistive wall mode is readily obtained from Eq (27). For this case the plasma is unstable to the ideal kink mode in the absence of a wall, ie $\omega_{\infty}^2 < 0$ or $\omega_{\infty}^2 = -\gamma_{\infty}^2$. In the presence of a perfectly conducting wall at $r = b$, the ideal kink mode is stable, so that $\omega_0^2 > 0$. The dispersion relation, Eq (27) is now written as

$$\omega = -\frac{i2mc_W(\omega^2 + \gamma_{\infty}^2)}{b\left[1 - \left(\frac{a}{b}\right)^{2m}\right](\omega^2 - \omega_0^2)} \tag{28}$$

Treating c_W as a perturbation, Eq (28) is solved for the wall mode which is approximated as a zero frequency mode. The correction, $\delta\omega$, to the wall mode frequency, due to a weakly resistive wall, can be obtained perturbatively from Eq (28) by substituting $\omega = 0$ on the right hand side, giving

$$\delta\omega \simeq \frac{i\Gamma\gamma_{\infty}^2}{\omega_0^2} \tag{29}$$

where

$$\Gamma \equiv \frac{2mc_W}{b\left[1 - \left(\frac{a}{b}\right)^{2m}\right]} \tag{30}$$

Hence, the stabilised ideal kink mode is destabilized due to the inclusion of finite resistivity of the wall. The growth rate is inversely proportional to the wall time. This is the definition of the MHD resistive wall instability. If $\omega_0^2 < 0$ then the plasma is unstable to an ideal kink with a perfectly conducting wall and the addition of a resistive wall is not significant. It will therefore be assumed that $\omega_0^2 > 0$. Note, also, that if the kink mode is stable without a wall, so that $\omega_{\infty}^2 > 0$, then the wall mode solution is

$$\delta\omega \simeq \frac{-i\Gamma\omega_{\infty}^2}{\omega_0^2} \tag{31}$$

In this case, the wall mode is damped.

Next, the effect of a plasma flow velocity is considered. Again choosing the condition that $\omega_{\infty}^2 = -\gamma_{\infty}^2$ it cannot be assumed that the wall mode will still be a zero frequency mode.

Therefore, substituting $\omega = \omega_r + i\gamma$ into Eq (27) and assuming small ω_r

$$\omega_r \simeq \frac{2\gamma k v_0(\gamma + \Gamma)}{(\omega_0^2 - k^2 v_0^2)} \tag{32}$$

$$\gamma \simeq \frac{\Gamma(k^2 v_0^2 + \gamma_1^2)}{(\omega_0^2 - k^2 v_0^2)} \tag{33}$$

Assuming that Γ is small, $\omega_r \ll \gamma$. As before, the wall mode is unstable due to a weakly resistive wall. Since $\omega_r \ll \gamma$, the growth rate can be obtained by perturbing the wall mode about zero frequency as in the case without a flow velocity. Thus, in the presence of a flow velocity, Eq (28) becomes

$$\omega = -\frac{i\Gamma(\bar{\omega}^2 + \gamma_1^2)}{(\bar{\omega}^2 - \omega_0^2)} \tag{34}$$

Perturbing about zero frequency, the correction, $\delta\omega$, to the wall mode frequency is

$$\delta\omega \simeq \frac{i\Gamma(k^2 v_0^2 + \gamma_1^2)}{(\omega_0^2 - k^2 v_0^2)} \tag{35}$$

which is in agreement with Eq (33).

It will be noticed that Eqs (33) and (35) have a pole $\omega_0 = kv_0$. The meaning of this is as follows. When $kv_0 = \omega_0$ one of the kink modes passes through zero frequency and its energy changes sign. When the wall mode and kink mode both have frequencies close to zero they are able to couple. This effect has also been discussed in Refs 10 and 14. The dispersion relation can still be solved perturbatively, as follows.

Assuming the condition

$$kv_0 = \omega_0 \tag{36}$$

the dispersion relation can be written

$$\omega(\omega - kv_0 - \omega_0)(\omega - kv_0 + \omega_0) = -i\Gamma(\bar{\omega}^2 + \gamma_1^2) \tag{37}$$

Perturbing about zero frequency, Eq (37) becomes

$$\delta\omega \cdot (-2\omega_0) \cdot \delta\omega \simeq -i\Gamma(k^2 v_0^2 + \gamma_1^2) \tag{38}$$

Hence

$$(\delta\omega)^2 \simeq i\Gamma\frac{(k^2 v_0^2 + \gamma_1^2)}{2\omega_0} \tag{39}$$

The perturbed frequency is given by

$$\delta\omega \simeq \pm\left[\frac{\Gamma(k^2 v_0^2 + \gamma_1^2)}{4\omega_0}\right]^{\frac{1}{2}}(1 + i) \tag{40}$$

The growth rate of the resistive wall mode is therefore enhanced by the coupling to the negative energy kink mode. The growth rate now varies inversely as the square root of the wall time, ie $(c_W/b)^{\frac{1}{2}}$. It is also worth noting that the frequency of the wall mode is comparable to the growth rate under these conditions.

For still higher flow velocities, $kv_0 > \omega_0$, it is the slow kink mode (negative energy) which is destabilized by the resistive wall in a manner analogous to the resistive wall amplifier of Birdsall et al[20]. In this case, the perturbation solution of Eq (37) is obtained by assuming

$$\omega = kv_0 - \omega_0 + \delta\omega \tag{41}$$

Substituting Eq (41) into Eq (37) gives

$$\delta\omega \simeq \frac{i\Gamma(\omega_0^2 + \gamma_1^2)}{2\omega_0(kv_0 - \omega_0)} \tag{42}$$

In this case the frequency of the resistive wall instability is $kv_0 - \omega_0$ and the growth rate is again inversely proportional to the resistive wall time.

The final case to consider is when the plasma is stable to the ideal kink mode in the absence of a wall, ie $\omega_1^2 > 0$. Without a plasma flow velocity, the wall mode is damped by a resistive wall. Now consider the effect of a plasma flow velocity. It can again be shown that $\omega_r \ll \gamma$ for the wall mode so that wall mode stability can still be analysed by perturbing about zero frequency. In this case, the correction to the wall mode frequency can be obtained from Eq (27) and is given by

$$\delta\omega \simeq \frac{i\Gamma(kv_0 - \omega_1)(kv_0 + \omega_1)}{(\omega_0^2 - k^2v_0^2)} \tag{43}$$

It can be seen, that even for this case, the wall mode can become unstable when $v_0 > (\omega_1/k)$. At the threshold $v_0 = (\omega_1/k)$, the real part of the frequency is zero.

As the velocity increases, the growth rate increases. As v_0 approaches (ω_0/k) the wall mode will couple to the negative energy kink mode, again producing a strong enhancement of the growth rate. The resistive wall mode behaves rather like an ideal mode in this case since the higher growth rate $\sim (c_W/b)^{\frac{1}{2}}$ results from a coupling of two modes. A similar interpretation has been given by Finn and Gerwin[9], although these authors refer to the coupling between the wall mode and a backward MHD mode. In fact, as demonstrated above, the coupling occurs when the backward kink mode changes to a forward wave as the frequency passes through zero and the wave energy changes sign. A similar quadratic perturbation analysis for $\delta\omega$ of the dispersion relation, Eq (27), yields

$$\delta\omega \simeq \pm\left[\frac{\Gamma(k^2v_0^2 - \omega_1^2)}{4\omega_0}\right]^{\frac{1}{2}}(1+i) \tag{44}$$

For still higher flow speeds, $v_0 > (\omega_0/k)$, the growth rate falls with increasing v_0 and the corresponding result to Eqs (41) and (42) is

$$\delta\omega \simeq \frac{i\Gamma(\omega_0^2 - \omega_1^2)}{2\omega_0(kv_0 - \omega_0)} \tag{45}$$

The instability again corresponds to the slow kink mode with a frequency $\omega = kv_0 - \omega_0$.

V SUMMARY AND CONCLUSIONS

Experimental results[3] from the DIII-D tokamak have suggested that, in the presence of toroidal rotation the lifetime of the discharge can be significantly extended. This has provided the motivation for the present attempt to gain some insight into the role of a plasma flow velocity on the resistive wall instability and whether there are any critical flow velocities. For this purpose a simple, sharp boundary cylindrical model with a skin current, axial flow and a thin resistive wall has been studied.

The classic resistive wall instability results from an external kink mode which is unstable without a wall but is stabilized by the presence of a perfectly conducting wall. When the finite resistivity of the conducting wall is included the system is unstable to the resistive wall mode, growing on the slower time scale of the resistive wall. These are the essential features of the resistive wall instability and are all contained in the present model. However, it should be remembered that the results obtained from this model can only be used as a qualitative guide to the behaviour of resistive wall modes. This is because the sharp boundary, surface current model is always unstable to an $m = 1$, ideal kink mode.

Although this model is oversimplified, it allows the physical mechanisms to be more easily identified. These mechanisms would be expected to play a role in more realistic models. This expectation is supported by the work of Finn and Gerwin[10] who have also examined a cylindrical model but one in which an equilibrium current flows in the plasma. The dispersion relation obtained in Ref 10 has the same form as the one derived in this paper. The analytic results given in the present paper are complementary to the numerical results obtained in Ref 10.

The effect of a flow velocity was first considered for the ideal kink mode with a perfectly conducting wall. In particular, the properties of wall stabilized kink modes have been discussed. Far from marginal stability the phase velocity of kink modes along the magnetic field is of the order of the Alfvén speed. However, these modes are down-shifted in frequency as the condition of marginal stability for the ideal kink mode is approached with the result that the parallel phase speed can become much smaller than the Alfvén speed, tending to zero at the marginal condition.

In the presence of a parallel flow velocity, the kink modes are Doppler shifted in frequency so that the waves propagating parallel and anti-parallel to the magnetic field have frequencies $\omega = kv_0 \pm \omega_0$, where ω_0 is the frequency of the mode without flow. It is noted that the mode

$\omega = kv_0 - \omega_0$ passes through zero frequency when $v_0 = \omega_0/k$ and that the change in sign of the frequency corresponds to a change in sign of the wave energy. This is an MHD example of a wave carrying negative energy when the flow speed exceeds the phase speed of the wave in the medium[21]. Under these conditions the two waves, $\omega = kv_0 \pm \omega_0$ are often referred to as fast and slow waves.

Resistive wall modes are discussed making use of this point of view. These modes are only of significance when the ideal kink modes can be stabilised by a perfectly conducting wall. It is clear that any rotation can destabilize a resistive wall mode depending on the proximity of the marginal condition for a perfectly conducting wall. If a plasma without a parallel flow velocity is stable in the absence of a wall then it remains so if a resistive wall is introduced. However, in the presence of a flow velocity, a resistive wall instability can occur when $v_0 > \omega_1/k$ where ω_1 is the kink mode frequency without a wall. When the plasma is unstable without a wall, $\omega_1^2 < 0$, but stabilized by a perfectly conducting wall, the resistive wall instability is further destabilized at a critical flow speed, $v_0 = \omega_0/k$, when the zero frequency, negative energy kink mode couples to the zero frequency wall mode. Under these conditions the resistive wall instability behaves more like an ideal instability and the growth rate is proportional to $(c_W/b)^{\frac{1}{2}}$, ie to the inverse of the square root of the wall time.

For still larger values of $v_0 > (\omega_0/k)$ it is the negative energy kink mode, rather than the wall mode, with a frequency $kv_0 - \omega_0$, which is unstable in the presence of a resistive wall, in a manner reminiscent of the resistive wall amplifer[20]. In this case, the growth rate is again inversely proportional to the wall time (c_W/b) and varies inversely with the flow speed v_0. Hence, although all flow speeds are evidently destabilizing, a system driven at a higher flow velocity, ie $v_0 > (\omega_0/k)$, will be subject to a weaker instability than for the smaller flow speed, $v_0 \simeq (\omega_0/k)$.

The final conclusion is that there are two critical velocities. The first, $v_0 = (\omega_0/k)$, is the more threatening since at this velocity, the resistive wall instability would have a growth rate closer to an ideal instability. The other critical velocity is $v_0 = (\omega_1/k)$, corresponding to the case when the plasma is stable in the absence of a wall. When this velocity is exceeded, the plasma is again unstable to a resistive wall instability. Since $\omega_1 < \omega_0$, this instability is also enhanced when $v_0 = (\omega_0/k)$ and becomes weaker as v_0 increases further as discussed for the case where $\omega_1^2 < 0$. If (ω_0/k) is low, ie the plasma is close to marginal stability, a velocity $v_0 \gg (\omega_0/k)$ would be preferred whereas if the plasma is not near the marginal condition, $v_0 < (\omega_0/k)$ might be preferable. In either case, the plasma would only be subjected to a weak instability rather than the strong instability associated with the critical velocity, $v_0 = (\omega_0/k)$.

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