UKAEA FUS 446

EURATOM/UKAEA Fusion

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January 2001

EURATOM/UKAEA Fusion Association

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Abstract

The Grad–Shafranov equation for a plasma with axial symmetry has been generalised by Iacono and co-workers [Phys. Fluids B **2**, 1794 (1990)] to include arbitrary flow, pressure anisotropy, and an arbitrary equation of state. In this report numerically–tractable forms of the equation are derived for two different scenarios: one is the double adiabatic model of Chew, Goldberger and Low [Proc. R. Soc. London Ser. A **236**, 112 (1956)]; the other ("single adiabatic") model is based on the assumption of constant temperature parallel to the magnetic field on magnetic flux surfaces. The equations derived are found to be equivalent, in the appropriate limits, to anisotropic Grad–Shafranov equations with and without flow obtained by previous authors. Physical constraints on magnetic flux functions appearing in the analysis are discussed. In the limit of low flow speed and low plasma beta, for a given set of flux functions and boundary conditions, it is shown that there exist two self–consistent solutions for the plasma density, in both the double and single adiabatic cases. In a tokamak, the lower density solution is applicable if the poloidal flow speed v_θ exceeds a value of the order of $c_s(r/R_0q)$, where c_s is the sound speed, R_0/r is the local aspect ratio, and q is the local safety factor.

1 Introduction

Equilibrium analyses of axisymmetric plasma configurations are generally based on the assumption that momentum balance in the plasma fluid can be described by the equation

$$\nabla p = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B},$$

where $\mathbf{B}$ is the magnetic field, p is a scalar plasma pressure, and μ_0 is free space permeability. Equation (1) is valid if the pressure is approximately isotropic, and if flow velocities $\mathbf{v}$ are of sufficiently small magnitude that the $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term in the fluid equation of motion can be neglected. There are grounds for believing that these assumptions may not be valid in certain tokamak experiments. For example, waves in the ion cyclotron range of frequencies (ICRF) preferentially accelerate ions perpendicular to the magnetic field, leading to a higher plasma pressure in that direction. Conversely, neutral beam injection (NBI) tangential to the magnetic axis of a tokamak can cause the parallel pressure $p_\|$ to exceed the perpendicular pressure $p_\perp$. NBI can also result in the injection of significant momentum to the plasma. However, although several authors have derived modified forms of the Grad–Shafranov equation which can, in principle, take such effects into account, there have been few if any attempts to actually solve these equations. The main purpose of this report is to derive from first principles a form of the generalised Grad–Shafranov equation which is tractable numerically and, in certain limits, analytically (Sect. 2). Having derived such an equation, we take the limit in which there is no flow, but retain pressure anisotropy, in order to compare our analysis with the work of previous authors (Sect. 3). In general, the incorporation of flow and pressure anisotropy makes it necessary to specify six functions of poloidal magnetic flux. These functions are all formally arbitrary, but in practice they are severely constrained by the requirement that the values obtained for the variables describing the equilibrium are physically meaningful. We discuss this issue in Sect. 4. In principle, the flux functions are governed by appropriate transport equations. The problems connected with self-consistent evolution of the latter, subject to appropriate sources, will be discussed in future work.

2 General Analysis

Our starting point is the steady–state single fluid equation of motion with anisotropic plasma pressure, which can be written in the form

$$\rho \mathbf{K} \times \mathbf{v} = -\nabla \cdot \mathsf{P} - \rho \nabla \left(\frac{v^2}{2} \right) + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{1}$$

where ρ is plasma density, P is the pressure tensor and $\mathbf{K} = \nabla \times \mathbf{v}$. Equation (1) is equivalent to Eq. (15) in Ref. 1, except that external gravity has been neglected and the possibility of anisotropic pressure has been taken into account. Following Ref. 2, we write

$$\mathsf{P} = \mathsf{I} p_\perp + \frac{(p_\parallel - p_\perp)}{B^2} \mathbf{B}\mathbf{B},$$

where I is the unit tensor. Substituting this expression in $\nabla \cdot \mathsf{P}$, introducing $\tau \equiv (p_\parallel - p_\perp)/B^2$, and using $\nabla \cdot \mathbf{B} = 0$, we obtain

$$\nabla \cdot \mathsf{P} = \nabla p_\perp + \nabla \cdot (\tau \mathbf{B}\mathbf{B}), \tag{2}$$

$$\nabla \cdot \mathsf{P} = \nabla p_\perp + \mathbf{B}(\mathbf{B} \cdot \nabla \tau) + \tau \mathbf{B} \cdot \nabla \mathbf{B}. \tag{3}$$

Using the vector identity

$$\mathbf{B} \cdot \nabla \mathbf{B} \equiv (\nabla \times \mathbf{B}) \times \mathbf{B} + \nabla \left(\frac{B^2}{2} \right), \tag{4}$$

we infer that

$$\nabla \cdot \mathsf{P} = \nabla p_\perp + \tau (\nabla \times \mathbf{B}) \times \mathbf{B} + \tau \nabla \left(\frac{B^2}{2} \right) + \mathbf{B}(\mathbf{B} \cdot \nabla \tau). \tag{5}$$

We now adopt a cylindrical coordinate system (R, ϕ, Z) and, as in Ref. 1, use the following potential representation of $\mathbf{B}$

$$\mathbf{B} = -\frac{1}{R} \frac{\partial \Psi}{\partial Z} \mathbf{e}_R + B_\phi \mathbf{e}_\phi + \frac{1}{R} \frac{\partial \Psi}{\partial R} \mathbf{e}_Z, \tag{6}$$

where $\mathbf{e}_R$, $\mathbf{e}_\phi$, $\mathbf{e}_Z$ denote basis vectors in the R, ϕ, Z directions. We can then write

$$\mathbf{B}\cdot\nabla\tau \equiv \frac{1}{R}\frac{\partial(\Psi,\tau)}{\partial(R,Z)} = \frac{|\nabla\Psi|}{R}\frac{\partial\tau}{\partial l}, \tag{7}$$

where l denotes arc length along a curve of constant poloidal magnetic flux, $2\pi\Psi$, at fixed ϕ.

Using Eq. (13) in Ref. 1 and Eqs. (5) and (7) above, we infer that the azimuthal (toroidal) component of the equation of motion, Eq. (1), can be written as follows

$$\frac{F'}{R^2}\frac{\partial(\Psi,Rv_\phi)}{\partial(R,Z)} = -\frac{RB_\phi}{R^2}\frac{\partial(\Psi,\tau)}{\partial(R,Z)} + \left(\frac{1}{\mu_0}-\tau\right)\frac{1}{R^2}\frac{\partial(\Psi,RB_\phi)}{\partial(R,Z)}. \tag{8}$$

where F' is a flux function relating the poloidal components of $\mathbf{v}$ and $\mathbf{B}$:

$$\rho\mathbf{v} = -\frac{F'}{R}\frac{\partial\Psi}{\partial Z}\mathbf{e}_R + \rho v_\phi\mathbf{e}_\phi + \frac{F'}{R}\frac{\partial\Psi}{\partial R}\mathbf{e}_Z. \tag{9}$$

As shown in Ref. 1, the existence in toroidal symmetry of a function F' satisfying Eq. (9) follows from the steady–state mass continuity equation and the toroidal component of the ideal MHD Ohm's law:

$$\mathbf{v}\times\mathbf{B} = \nabla\Phi, \tag{10}$$

where Φ is the electrostatic potential. Equation (8) can be expressed concisely in the form

$$\frac{\partial\left(F'Rv_\phi - \left(\frac{1}{\mu_0}-\tau\right)RB_\phi, \Psi\right)}{\partial(R,Z)} = 0. \tag{11}$$

This implies the existence of a flux function $f(\Psi)$ given by

$$f(\Psi) = \left(\frac{1}{\mu_0}-\tau\right)RB_\phi - F'Rv_\phi. \tag{12}$$

We now consider Eq. (1) in full, using the expression for $\nabla\cdot\mathsf{P}$ given by Eq. (5):

$$\rho\mathbf{K}\times\mathbf{v} = -\nabla p_\perp - \mathbf{B}(\mathbf{B}\cdot\nabla\tau) - \tau\nabla\left(\frac{B^2}{2}\right) - \rho\nabla\left(\frac{v^2}{2}\right) + \left(\frac{1}{\mu_0}-\tau\right)(\nabla\times\mathbf{B})\times\mathbf{B}. \tag{13}$$

Introducing

$$\lambda = \frac{1}{\mu_0} - \tau, \tag{14}$$

and noting that

$$[\nabla\times(\lambda\mathbf{B})]\times\mathbf{B} = (\nabla\lambda\times\mathbf{B})\times\mathbf{B} + \lambda(\nabla\times\mathbf{B})\times\mathbf{B},$$

so that

$$\lambda(\nabla\times\mathbf{B})\times\mathbf{B} = [\nabla\times(\lambda\mathbf{B})]\times\mathbf{B} + B^2\nabla\lambda - \mathbf{B}(\mathbf{B}\cdot\nabla\lambda), \tag{15}$$

we find that Eq. (13) can be re–written in the form

$$\frac{1}{\rho}[\nabla\times(\lambda\mathbf{B})]\times\mathbf{B}=\mathbf{K}\times\mathbf{v}+\nabla\left(\frac{v^2}{2}\right)+\frac{1}{\rho}\left[\nabla p_{\parallel}-\left(\frac{p_{\parallel}-p_{\perp}}{B}\right)\nabla B\right]. \quad (16)$$

Iacono and co–workers[2] obtained a Grad–Shafranov equation for a plasma with arbitrary equations of state. However, for reasons of clarity it is appropriate to specify an equation of state at this stage of the analysis. We consider two separate cases.

2.1 Double Adiabatic Model

The steady–state "double adiabatic" equations are the following:[3]

$$\mathbf{v}\cdot\nabla\left(\frac{p_{\perp}}{\rho B}\right)=0, \quad (17)$$

$$\mathbf{v}\cdot\nabla\left(\frac{p_{\parallel}B^2}{\rho^3}\right)=0. \quad (18)$$

Because of the assumed toroidal symmetry, these equations imply the existence of flux functions $\sigma_{\perp}$, $\sigma_{\parallel}$ such that

$$p_{\perp}=\sigma_{\perp}(\Psi)\rho B, \quad (19a)$$

$$p_{\parallel}=\sigma_{\parallel}(\Psi)\rho^3/B^2. \quad (19b)$$

Expressing the quantity in square brackets in Eq. (16) in terms of these new flux functions, one obtains

$$\frac{1}{\rho}\left[\nabla p_{\parallel}-\left(\frac{p_{\parallel}-p_{\perp}}{B}\right)\nabla B\right]=\nabla\left[\frac{3}{2}\sigma_{\parallel}\left(\frac{\rho}{B}\right)^2+\sigma_{\perp}B\right]-\left[\frac{1}{2}\sigma_{\parallel}'\left(\frac{\rho}{B}\right)^2+B\sigma_{\perp}'\right]\nabla\Psi. \quad (20)$$

The equation of motion [Eq. (16)] thus takes the form

$$\frac{1}{\rho}[\nabla\times(\lambda\mathbf{B})]\times\mathbf{B}=\mathbf{K}\times\mathbf{v}+\nabla\left[\frac{v^2}{2}+\frac{3}{2}\sigma_{\parallel}\left(\frac{\rho}{B}\right)^2+\sigma_{\perp}B\right]-\left[\frac{1}{2}\sigma_{\parallel}'\left(\frac{\rho}{B}\right)^2+B\sigma_{\perp}'\right]\nabla\Psi. \quad (21)$$

We now evaluate the components of the left hand side of this equation, noting first that

$$\nabla\times(\lambda\mathbf{B})=-\frac{\partial}{\partial Z}(\lambda B_{\phi})\mathbf{e}_R+\hat{J}_{\phi}\mathbf{e}_{\phi}+\frac{1}{R}\frac{\partial}{\partial R}(R\lambda B_{\phi})\mathbf{e}_Z, \quad (22)$$

where

$$\hat{J}_{\phi}=-\frac{1}{R}\left[\frac{\partial}{\partial Z}\left(\lambda\frac{\partial\Psi}{\partial Z}\right)+R\frac{\partial}{\partial R}\left(\frac{\lambda}{R}\frac{\partial\Psi}{\partial R}\right)\right]. \quad (23)$$

Hence

$$[\nabla\times(\lambda\mathbf{B})]\times\mathbf{B}=\left(\hat{J}_{\phi}\frac{1}{R}\frac{\partial\Psi}{\partial R}-\frac{B_{\phi}}{R}\frac{\partial}{\partial R}(R\lambda B_{\phi})\right)\mathbf{e}_R+\frac{1}{R^2}\frac{\partial(\Psi,\lambda RB_{\phi})}{\partial(R,Z)}\mathbf{e}_{\phi}$$

$$+\left(\hat{J}_\phi \frac{1}{R}\frac{\partial\Psi}{\partial Z} - \frac{B_\phi}{R}\frac{\partial}{\partial Z}(R\lambda B_\phi)\right)\mathbf{e}_Z. \tag{24}$$

Substituting this result in the equation of motion, taking the scalar product with $\mathbf{e}_\phi$, and noting that the assumed toroidal symmetry implies $\mathbf{e}_\phi \cdot \nabla \equiv 0$, one recovers Eq. (11). Taking the scalar product with $\mathbf{B}$, we obtain

$$0 = \mathbf{B}\cdot\mathbf{K}\times\mathbf{v} + \frac{|\nabla\Psi|}{R}\frac{\partial}{\partial l}\left[\frac{v^2}{2} + \frac{3}{2}\sigma_\parallel\left(\frac{\rho}{B}\right)^2 + \sigma_\perp B\right], \tag{25}$$

i.e.

$$0 = \mathbf{K}\cdot\mathbf{v}\times\mathbf{B} + \frac{|\nabla\Psi|}{R}\frac{\partial}{\partial l}\left[\frac{v^2}{2} + \frac{3}{2}\sigma_\parallel\left(\frac{\rho}{B}\right)^2 + \sigma_\perp B\right]. \tag{26}$$

Using the R and Z components of the ideal MHD Ohm's law [Eq. (10)], it can be shown[1] that the electrostatic potential Φ is a flux function whose derivative $\Omega = \Phi'$ is given by

$$Rv_\phi - \frac{RB_\phi F'}{\rho} = R^2\Omega(\Psi), \tag{27}$$

Since Φ is a flux function it follows that $\mathbf{v}\times\mathbf{B} = \Omega\nabla\Psi$, and hence

$$\mathbf{K}\cdot\mathbf{v}\times\mathbf{B} = \Omega\mathbf{K}\cdot\nabla\Psi = -\frac{\Omega}{R}\frac{\partial(\Psi, Rv_\phi)}{\partial(R,Z)} = -\frac{|\nabla\Psi|}{R}\frac{\partial}{\partial l}(\Omega R v_\phi), \tag{28}$$

where we have used the fact that Ω, being a flux function, commutes with the operator $\partial/\partial l$. Combining this result with Eq. (26), we obtain the Bernoulli relation

$$\frac{v^2}{2} - \Omega R v_\phi + \frac{3}{2}\sigma_\parallel\left(\frac{\rho}{B}\right)^2 + \sigma_\perp B = H^*(\Psi), \tag{29}$$

the quantity $H^*(\Psi)$ being an arbitrary flux function. The full equation of motion can now be written in the form

$$\frac{1}{\rho}[\nabla\times(\lambda\mathbf{B})]\times\mathbf{B} = \mathbf{K}\times\mathbf{v} + H^{*\prime}\nabla\Psi + \nabla(\Omega R v_\phi) - \left[\frac{1}{2}\sigma_\parallel'\left(\frac{\rho}{B}\right)^2 + B\sigma_\perp'\right]\nabla\Psi. \tag{30}$$

The components of this equation in the (R, Z) plane are

$$\frac{1}{\rho}\left[\frac{\hat{J}_\phi}{R}\nabla\Psi - \frac{B_\phi}{R}\nabla(RB_\phi\lambda)\right] = K_\phi^*\left(\frac{F'}{\rho R}\right)\nabla\Psi - \frac{1}{2R^2}\nabla(Rv_\phi)^2 + \Omega\nabla(Rv_\phi)$$
$$+\left[H^{*\prime} + Rv_\phi\Omega' - \frac{1}{2}\sigma_\parallel'\left(\frac{\rho}{B}\right)^2 - B\sigma_\perp'\right]\nabla\Psi. \tag{31}$$

where K_ϕ^* is defined as in Ref. 1:

$$K_\phi^* = -\frac{1}{R}\left[\frac{\partial}{\partial Z}\left(\frac{F'}{\rho}\frac{\partial\Psi}{\partial Z}\right) + R\frac{\partial}{\partial R}\left(\frac{F'}{R\rho}\frac{\partial\Psi}{\partial R}\right)\right]. \tag{32}$$

Using the definition of f [Eq. (12)], Eq. (31) can be rearranged to give

$$\frac{1}{\rho}\left[\frac{\hat{J}_\phi}{R}\nabla\Psi\right] - K_\phi^*\left(\frac{F'}{\rho R}\right)\nabla\Psi = \left(\frac{RB_\phi}{\rho R^2}\right)\nabla\left[F'Rv_\phi + f\right] - \left(\frac{F'}{\rho R}B_\phi\right)\nabla(Rv_\phi)$$
$$+\left[H^{*\prime} + Rv_\phi\Omega' - \frac{1}{2}\sigma_\parallel'\left(\frac{\rho}{B}\right)^2 - B\sigma_\perp'\right]\nabla\Psi, \tag{33}$$

i.e.

$$\frac{1}{\rho R}\left[\hat{J}_\phi - K_\phi^* F'\right]\nabla\Psi = \left(\frac{RB_\phi}{\rho R^2}\right)(Rv_\phi)F''\nabla\Psi + f'\left(\frac{RB_\phi}{\rho R^2}\right)\nabla\Psi$$
$$+\left[H^{*\prime} + Rv_\phi\Omega' - \frac{1}{2}\sigma_\parallel'\left(\frac{\rho}{B}\right)^2 - B\sigma_\perp'\right]\nabla\Psi. \tag{34}$$

Equating the coefficients of $\nabla\Psi$, we obtain finally a generalised Grad–Shafranov equation for a toroidally–symmetric plasma with flow and anisotropic pressure

$$\frac{1}{\rho R}\left[\hat{J}_\phi - K_\phi^* F'\right] = H^{*\prime} - \frac{1}{2}\sigma_\parallel'\left(\frac{\rho}{B}\right)^2 - B\sigma_\perp' + Rv_\phi\Omega' + f'\left(\frac{RB_\phi}{\rho R^2}\right) + \left(\frac{RB_\phi}{\rho R^2}\right)(Rv_\phi)F''. \tag{35}$$

Writing this explicitly in terms of Ψ, we obtain

$$R\frac{\partial}{\partial R}\left(\frac{\Delta}{R}\frac{\partial\Psi}{\partial R}\right) + \frac{\partial}{\partial Z}\left(\Delta\frac{\partial\Psi}{\partial Z}\right) + \frac{F'F''\mu_0}{\rho}\left[\left(\frac{\partial\Psi}{\partial R}\right)^2 + \left(\frac{\partial\Psi}{\partial Z}\right)^2\right]$$
$$= \mu_0\rho R^2\left[\frac{1}{2}\sigma_\parallel'\left(\frac{\rho}{B}\right)^2 + B\sigma_\perp' - Rv_\phi\Omega' - H^{*\prime}\right] - RB_\phi\mu_0(f' + Rv_\phi F''), \tag{36}$$

where

$$\Delta \equiv \mu_0\left(\lambda - \frac{F'^2}{\rho}\right) = 1 - \mu_0\frac{p_\parallel - p_\perp}{B^2} - \mu_0\frac{F'^2}{\rho}. \tag{37}$$

The equilibrium is completely determined by the partial differential equation Eq. (36) and the following set of algebraic relations:

$$B^2 = \frac{1}{R^2}|\nabla\Psi|^2 + B_\phi^2, \tag{38}$$

$$v^2 = \frac{F'^2}{R^2\rho^2}|\nabla\Psi|^2 + v_\phi^2, \tag{39}$$

$$f(\Psi) = \lambda RB_\phi - F'Rv_\phi, \tag{40}$$

$$R^2\Omega(\Psi) = Rv_\phi - \frac{F'RB_\phi}{\rho}, \tag{41}$$

$$H^*(\Psi) = \frac{v^2}{2} - \Omega Rv_\phi + \frac{3}{2}\sigma_\parallel\left(\frac{\rho}{B}\right)^2 + \sigma_\perp B. \tag{42}$$

We thus have a system of six equations, six unknowns (Ψ, ρ, B_ϕ, v_ϕ, B, v), and six arbitrary flux functions (F', Ω, f, $\sigma_\parallel$, $\sigma_\perp$, H^*).

2.2 Single Adiabatic Model

We now consider the case of an "isothermal" model for $p_\parallel$: that is to say, we take $T_\parallel = mp_\parallel/\rho$ rather than $p_\parallel B^2/\rho^3$ to be a flux function (m being the mean particle mass). This is justified physically by the fact that parallel transport in tokamaks is generally extremely fast. Equation (17) remains valid in this case, while Eq. (18) is replaced with

$$\mathbf{v} \cdot \nabla \left(\frac{p_\parallel}{\rho}\right) = 0.$$

We now reconsider Eq. (16), noting that

$$\frac{1}{\rho}\left[\nabla p_\parallel - \frac{p_\parallel}{B}\nabla B\right] + \frac{1}{\rho}\frac{p_\perp}{B}\nabla B = \frac{1}{\rho}\nabla\left(\frac{\rho}{m}T_\parallel(\Psi)\right) - \frac{T_\parallel(\Psi)}{m}\nabla \ln\left(\frac{B}{B_0}\right) + \sigma_\perp(\Psi)\nabla B$$

$$= -\left[\frac{T_\parallel'}{m}\ln\left(\frac{\rho B_0}{\rho_0 B}\right) + B\sigma_\perp'\right]\nabla\Psi + \nabla\left[\frac{T_\parallel}{m}\left\{\ln\left(\frac{\rho B_0}{\rho_0 B}\right) + 1\right\} + \sigma_\perp B\right]. \qquad (43)$$

Here, B_0 and ρ_0 are arbitrary constants with the dimensions of magnetic field and density respectively, introduced in order to make the logarithm arguments dimensionless. Taking the scalar product of Eq. (16) with $\mathbf{B}$, as before, we infer a Bernoulli relation of the form

$$\frac{v^2}{2} - \Omega R v_\phi + \frac{T_\parallel}{m}\left\{\ln\left(\frac{\rho B_0}{\rho_0 B}\right) + 1\right\} + \sigma_\perp B = H^*(\Psi), \qquad (44)$$

and a Grad–Shafranov equation

$$R\frac{\partial}{\partial R}\left(\frac{\Delta}{R}\frac{\partial\Psi}{\partial R}\right) + \frac{\partial}{\partial Z}\left(\Delta\frac{\partial\Psi}{\partial Z}\right) + \frac{F'F''\mu_0}{\rho}\left[\left(\frac{\partial\Psi}{\partial R}\right)^2 + \left(\frac{\partial\Psi}{\partial Z}\right)^2\right]$$

$$= \mu_0\rho R^2\left[\frac{T_\parallel'}{m}\ln\left(\frac{\rho B_0}{\rho_0 B}\right) + B\sigma_\perp' - Rv_\phi\Omega' - H^{*\prime}\right] - RB_\phi\mu_0(f' + Rv_\phi F''). \qquad (45)$$

3 Anisotropic Pressure without Flow

In the isotropic limit without flow, $\Delta = 1$, $F' = \Omega = 0$, and $f = RB_\phi/\mu_0$. Equations (36) and (45) then reduce to equations which are similar to the standard Grad–Shafranov form.[4] They are not quite identical, however, because of the use of the double adiabatic and single adiabatic equations of state. It can be shown, however, that Eqs. (36) and (45) are particular forms of a Grad–Shafranov equation obtained independently in Ref. 2: we can be confident, therefore, that they are correct.

Considerable simplification occurs when flow is neglected but anisotropy is retained. We then have $\Delta = \mu_0\lambda$, and in the double adiabatic case the Grad–Shafranov equation

[Eq. (36)] reduces to

$$R\frac{\partial}{\partial R}\left(\frac{\Delta}{R}\frac{\partial\Psi}{\partial R}\right)+\frac{\partial}{\partial Z}\left(\Delta\frac{\partial\Psi}{\partial Z}\right)=\mu_0\rho R^2\left[\frac{1}{2}\sigma_{\parallel}'\left(\frac{\rho}{B}\right)^2+B\sigma_{\perp}'-H^{*\prime}\right]-\frac{\mu_0^2 ff'}{\Delta}, \quad (46)$$

where the Bernoulli flux function H^* is now given by

$$H^*(\Psi)=\frac{3}{2}\sigma_{\parallel}\left(\frac{\rho}{B}\right)^2+\sigma_{\perp}B. \quad (47)$$

It is possible to show that Eq. (46) corresponds exactly to Eq. (12) of Ref. 5 when the double–adiabatic equations of state are used. It is also identical to the Grad–Shafranov equation in Ref. 2 in the double adiabatic case without flow. In order to establish this equivalence, it is essential to recognise that the quantity $\partial p_{\parallel}/\partial\psi$ in Ref. 2 differs from $(\partial p_{\parallel}/\partial\psi)_B$ in Ref. 5: in the former case, $p_{\parallel}$ is treated as a function of Ψ, B and ρ; in the latter, $p_{\parallel}$ is a function only of Ψ and B. To see what this means, one can write Eq. (19b) in the form

$$p_{\parallel}(\Psi)=\sigma_{\parallel}(\Psi)\rho(B,\Psi)^3/B^2, \quad (48)$$

where the function $\rho(B,\Psi)$ can be obtained for given H^*, $\sigma_{\parallel}$ and $\sigma_{\perp}$ from Eq. (47). From Eq. (48), we obtain

$$\left(\frac{\partial p_{\parallel}}{\partial\Psi}\right)_B=\frac{\sigma_{\parallel}'\rho^3}{B^2}+\frac{3\sigma_{\parallel}\rho^2}{B^2}\left(\frac{\partial\rho}{\partial\Psi}\right)_B. \quad (49)$$

Differentiating Eq. (47) with respect to Ψ at fixed B, we find that the expression in square brackets in Eq. (46) is proportional to $(\partial p_{\parallel}/\partial\psi)_B$: it corresponds to the first term on the right hand side of the Grad–Shafranov equation in Ref. 5.

4 Solution Procedure

We have seen that when flow and anisotropy are taken into account the number of flux functions which must be specified increases from two to six. These functions cannot be chosen arbitrarily, however: they are constrained, for example, by the requirement that solutions for the plasma density ρ be real and positive for all R and Z. One can ensure that this requirement is satisfied as follows. From the definition of f [Eq. (12) or Eq. (40)] we have that

$$\left(\lambda-\frac{F'^2}{\rho}\right)RB_{\phi}=R^2\Omega F'+f(\Psi). \quad (50)$$

Introducing

$$S\equiv\frac{1}{R}\left(R^2\Omega F'+f(\Psi)\right), \quad (51)$$

and denoting the poloidal component of the magnetic field by $B_\theta \equiv |\nabla\Psi|/R$, we find from Eq. (50) that the total field B can be expressed in the form

$$B^2 = B_\theta^2 + \frac{\mu_0^2 S^2}{\left\{1 - \frac{\mu_0(p_\parallel - p_\perp)}{B^2} - \frac{\mu_0 F'^2}{\rho}\right\}^2}. \tag{52}$$

We now consider the limit

$$\frac{\mu_0(p_\parallel - p_\perp)}{B^2} \ll 1, \quad \frac{\mu_0 F'^2}{\rho} \ll 1, \tag{53}$$

i.e. we assume that the difference between the parallel and perpendicular plasma betas and the square of the poloidal Alfvénic Mach number (the ratio of poloidal flow speed to Alfvén speed defined in terms of the poloidal magnetic field) are everywhere much less than unity. Both of these approximations are justified in the case of tokamaks. If Eq. (53) holds, it is clear from Eq. (52) that we can write

$$B_\phi \simeq \mu_0 S = \frac{\mu_0}{R}\left(R^2\Omega F' + f(\Psi)\right). \tag{54}$$

Equation (54) gives the toroidal field in terms of prescribed functions of Ψ: if $\Psi(R,Z)$ is known we can then use calculate the total field B using $B^2 = |\nabla\Psi|^2/R^2 + B_\phi^2$. In this limit, B is independent of density ρ. To calculate the latter, we can proceed as follows. It follows from Eqs. (9) and (27) that

$$v^2 = v_\phi^2 + F'^2\left(\frac{|\nabla\Psi|}{R\rho}\right)^2 = R^2\Omega^2 + \frac{2R\Omega B_\phi F'}{\rho} + F'^2\left(\frac{B}{\rho}\right)^2. \tag{55}$$

Hence the double adiabatic Bernoulli relation [Eq. (29)] can be written in the form

$$\frac{F'^2}{2}\left(\frac{B}{\rho}\right)^2 + \frac{3}{2}\sigma_\parallel\left(\frac{\rho}{B}\right)^2 = H^*(\Psi) + \frac{\Omega^2 R^2}{2} - \sigma_\perp B. \tag{56}$$

Putting $(\rho/B)^2 = X$,

$$b = \frac{2}{3\sigma_\parallel}\left(H^*(\Psi) + \frac{\Omega^2 R^2}{2} - \sigma_\perp B\right), \tag{57}$$

and

$$c = \frac{F'^2}{3\sigma_\parallel}, \tag{58}$$

one can write Eq. (56) as a quadratic equation for X:

$$X^2 - bX + c = 0. \tag{59}$$

From its definition [Eq. (19b)], it is obvious that the flux function $\sigma_\parallel$ must be positive definite: the quantity c is then also positive definite. Having specified $\Omega(\Psi)$ and $\sigma_\perp(\Psi)$, it is necessary to choose $H^*(\Psi)$ to be sufficiently large that $b \geq 0 \;\forall\; R, Z$, since otherwise Eq. (56) cannot be satisfied for real ρ/B. We emphasize here that the absolute values of H^* are completely arbitrary: since the Grad–Shafranov equation [Eq. (36)] depends only on $H^{*\prime}$ rather than H^* itself, one can always add a constant to the latter without changing the physics of the problem [the *solutions* of Eq. (36) would, however, be affected by such a transformation, since Eq. (56) indicates that the density would also change]. Solutions of Eq. (59)

$$X_{\pm} = \frac{1}{2}\left[b \pm (b^2 - 4c)^{1/2}\right], \tag{60}$$

are represented schematically by the curve in Fig. 1a. The condition for these solutions to be real is that c be less than or equal to $b^2/4$, i.e. that

$$\left(H^*(\Psi) + \frac{\Omega^2 R^2}{2} - \sigma_\perp B\right)^2 \geq 3\sigma_\parallel F'^2. \tag{61}$$

If this condition is satisfied, the two roots will both be positive if $b > 0$ and $c > 0$. An intriguing result is that there will then be two solutions for the density:

$$\rho_{\pm} = B X_{\pm}^{1/2}. \tag{62}$$

It should be stressed that these "solutions" are purely formal, since in order to evaluate $\rho(R, Z)$ it is still necessary to solve the Grad–Shafranov equation [Eq. (36)]. Moreover, the absolute values of the density are indeterminate, since the addition of an arbitrary flux function to H^* necessarily changes $\rho(R, Z)$. In practice, however, ρ is constrained by the total mass of plasma and the size of the system. Finally, Eq. (62) only represents an explicit solution for the density if we assume that B is independent of ρ. This assumption, which is valid if Eq. (53) holds, needs to be verified *a posteriori*. Low density implies a relatively high poloidal Alfvén Mach number $\mu_0 F'^2/\rho$ (for a given F'), and so it is particularly important to check that Eq. (53) remains valid in the case of the low density solution. To quantify this remark, we note that the first term on the left hand side of Eq. (56) is of the order of $v_\theta^2 (B/B_\theta)^2$, where v_θ is the poloidal flow speed. The second term in Eq. (56) is of the order of c_s^2, where $c_s \sim (p_\parallel/\rho)^{1/2}$ is the sound speed. The low density solution is applicable if the first term is dominant, which requires that

$$v_\theta > \frac{B_\theta}{B} c_s. \tag{63}$$

The second of the two inequalities in Eq. (53), on the other hand, requires that

$$v_\theta \ll \frac{B_\theta}{B} c_A, \tag{64}$$

where $c_A = B/(\mu_0 \rho)^{1/2}$ is the Alfvén speed. Equations (63) and (64) can be consistent if $c_A \gg c_s$, i.e. if the total plasma beta $\beta \sim c_s^2/c_A^2$ is small. When this condition is satisfied, the first of the two inequalities in Eq. (53) is also satisfied, and the low density solution will be self-consistent. In a large aspect ratio tokamak $B_\theta/B \simeq r/(R_0 q) \ll 1$, where R_0 is major radius, r is minor radial distance from the magnetic axis, and $q \gtrsim 1$ is the safety factor. The low density solution could thus be realized even if the poloidal flow speed were subsonic.

For cases in which $\mu_0(p_\parallel - p_\perp)/B^2$ or $\mu_0 F'^2/\rho$ are not negligibly small, one could obtain an iterative solution for ρ and B, with Eq. (54) as the initial estimate of B_ϕ. In general, Eqs. (52) and (56) constitute a pair of algebraic equations which must be solved self-consistently for B and ρ. An analogous pair of equations was obtained for the case of scalar pressure and an isothermal equation of state in Ref. 6. The above discussion indicates that there are circumstances in which at least two physically-acceptable density solutions exist, for a given set of flux functions and boundary conditions.

In the case of the single adiabatic model, the form of the Bernoulli relation analogous to Eq. (56) is

$$\frac{F'^2}{2}\left(\frac{B}{\rho}\right)^2 + \frac{T_\parallel}{m}\left\{\ln\left(\frac{\rho B_0}{\rho_0 B}\right) + 1\right\} = H^*(\Psi) + \frac{\Omega^2 R^2}{2} - \sigma_\perp B. \tag{65}$$

As in the double adiabatic case, there are two possible real roots. This is shown graphically in Fig. 1b, where X, again defined as $(\rho/B)^2$, is plotted versus $c' \equiv mF'^2/T_\parallel$ for a particular value of $b' \equiv 2m(H^* + \Omega^2 R^2/2 - \sigma_\perp B)/T_\parallel - 2 + \ln(\rho_0^2/B_0^2)$. To examine the nature of the roots, it is useful to consider, as in the double adiabatic model, limiting cases in which one or other of the density-dependent terms on the left hand side of Eq. (65) is dominant. Considering first the limit of large ρ/B, we obtain from Eq. (65)

$$\rho = \rho_0 \frac{B}{B_0 e} \exp\left\{\frac{m(H^* + \frac{\Omega^2 R^2}{2} - \sigma_\perp B)}{T_\parallel}\right\}. \tag{66}$$

This corresponds to the upper branch of the curve in Fig. 1b, and provides a suitable initial estimate of the density in an iteration solution when the poloidal flow speed is known to be much lower than $(B_\theta/B)c_s$ [cf. Eq. (63)]. At sufficiently low ρ/B the first term on the left hand side of Eq. (65) is dominant and we can write

$$\rho = \frac{BF'}{\left[2\left(H^* + \frac{\Omega^2 R^2}{2} - \sigma_\perp B\right)\right]^{1/2}}. \tag{67}$$

Equation (67) represents a solution for the density if Eq. (53) holds, i.e. if B_ϕ does not depend explicitly on ρ. As in the double adiabatic model, this must be verified *a posteriori*. The coupled equations for B and ρ [Eqs. (52) and (65)] are now transcendental rather than algebraic, and so the problem of determining all the physically-acceptable solutions is more complex. However, it is clear from Eq. (65) and Fig. 1b that the existence of a self-consistent low density solution of the form given by Eq. (67) requires the same criteria as those which apply in the double adiabatic case, namely $(r/R_0 q)c_A \gg v_\theta > (r/R_0 q)c_s$ and $\beta \ll 1$.

For both equations of state, there is a critical poloidal flow speed separating the high and low density solutions: this corresponds to the points in Fig. 1 at which the curves have infinite gradient. In the double adiabatic case, it is straightforward to show from Eq. (60) that the critical v_θ is $v_{\theta c} \equiv (B_\theta/B)(3p_{\|}/\rho)^{1/2}$. Taking the low beta limit, one can demonstrate that the determinant corresponding to the second order derivatives in the Grad–Shafranov equation [Eq. (36)] vanishes when v_θ is equal to the critical value.[2] Corresponding singularities exist in the equations describing isothermal[6,7] and adiabatic[8] axisymmetric plasmas with isotropic pressure. For poloidal flow speeds below $v_{\theta c}$, the Grad–Shafranov equation is elliptic; above the critical value, there is a range of flow speeds in which the equation is hyperbolic. It is possible that both high and low density solutions could be realised in different regions of a single tokamak plasma equilibrium. In that case, there must exist points in the plasma where the Grad–Shafranov equation is singular. In general, the presence of such singularities is likely to aggrevate the problem of determining numerical solutions.

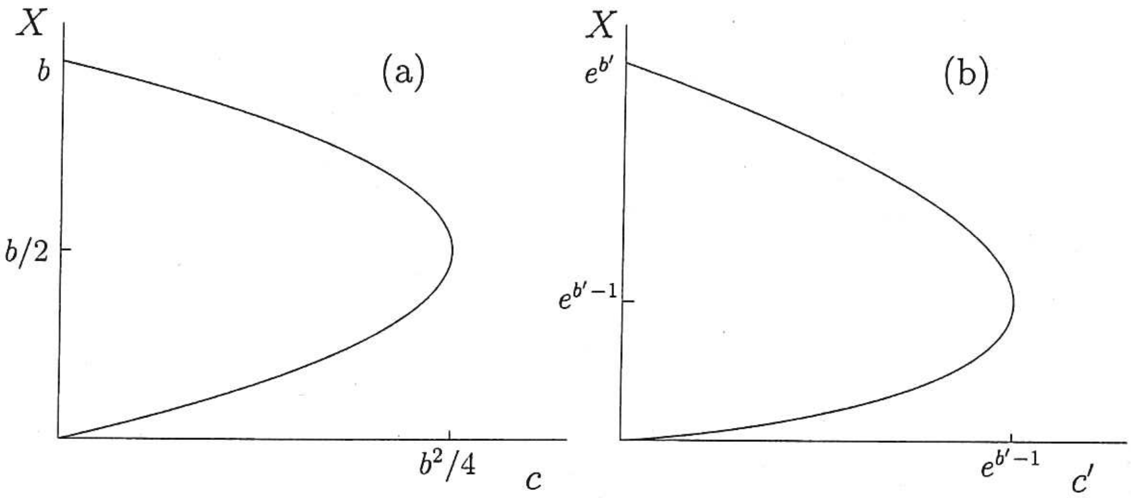

Fig. 1. Schematic plots of normalised density variable $X \equiv (\rho/B)^2$ versus (a) $c \equiv F'^2/(3\sigma_{\|})$ (double adiabatic model) and (b) $c' \equiv mF'^2/T_{\|}$ (single adiabatic model). The point of infinite gradient on each curve defines the critical flow speed separating the high and low density solutions.

An example of a plasma configuration in which both high and low density solutions might occur is a tokamak in high confinement (H) mode. Such plasmas are characterised by an edge transport barrier, indicated by a steep pressure gradient. Transport barriers generally are believed to be associated with strong poloidal flows (see, e.g., Ref. 9). On the other hand, the poloidal field can be very small close to the plasma edge (in the case of a divertor plasma, $B_\theta \to 0$ at the separatrix). The critical flow speed is thus relatively low, and it is possible that the inequality $v_\theta > v_{\theta c}$ could be satisfied in the edge plasma. Indeed, it has been proposed[10] that the transition from low confinement to H–mode is facilitated by poloidal flows approaching the critical value.

It is important to note that when the Grad–Shafranov equation is solved for a region of space in which it can change from being elliptic to hyperbolic, the solution procedure and the boundary data must take full and consistent account of this transition (similar considerations apply to transonic aerodynamics). To the best of our knowledge, such solutions have not yet been obtained for tokamak plasmas.

5 Conclusions

We have derived from first principles a generalised Grad–Shafranov equation for an axisymmetric plasma with flow and pressure anisotropy. In particular, we have obtained numerically tractable forms of the equation for two specific models of the parallel and perpendicular pressures: the double adiabatic theory of Chew and co-workers;[3] and a "single adiabatic" model, based on the assumption of constant temperature on magnetic flux surfaces. Anisotropic Grad–Shafranov equations with[2] and without[5] flow, and with arbitrary equations of state, have been obtained by previous authors: these reduce, in the appropriate limits and with appropriate equations of state, to the equations derived in the present analysis. We have discussed physical constraints on otherwise arbitrary magnetic flux functions appearing in the analysis. In particular, we have identified conditions which must be satisfied for the plasma density to be real and positive. In the limit of low flow speed and low plasma beta, for a given set of flux functions and boundary conditions, we have found that two self-consistent solutions exist for the plasma density, in both the double and single adiabatic models. The low density solution is applicable when the poloidal flow speed exceeds a value $v_{\theta c}$ which, in a tokamak, is of the order of the sound speed divided by the product of the local aspect ratio and the local safety factor. It is possible that the low density solution could be applicable to existing tokamaks. In this connection we note that Betti and Freidberg[11] have recently obtained shock-free solutions of the steady MHD equations with anisotropic pressure, viscosity and poloidal flows of the order of $v_{\theta c}$. It should be stressed that the arbitrary flux functions which appear in the equilibrium MHD solutions cannot in general be prescribed *a priori*, but must be obtained using appropriate transport equations. We will consider the problem of self-consistent determination of these flux functions in a future report.

Appendix: Notation and Units

Comparisons between the various equilibrium analyses in the literature are complicated by differences in notation. For this reason, we present a summary of the relationships between the variables used in Ref. 1, the present treatment, Ref. 2, and Ref. 5. In some cases the comparison between variables in Ref. 5 and corresponding variables in the other papers are only meaningful if the limit of zero flow is taken in the latter. Note that Gaussian units were used in Ref. 1, while SI units are used in the present paper. The final sets of equations obtained in these two papers can be compared by replacing factors of 4π in Ref. 1 with μ_0. Refs. 2 and 5 employ units in which $\mu_0 \equiv 1$.

Table 1. Equivalent variables used in treatments of Grad–Shafranov equation with flow and/or pressure anisotropy.

McClements & Thyagaraja[1]	Thyagaraja et al. (this paper)	Iacono et al.[2]	Salberta et al.[5]
Ψ	Ψ	$-\psi$	$-\psi$
r	R	r	X
z	Z	z	Z
–	Δ	τ	σ
$\Lambda(\Psi)$	$-f(\Psi)$	$-I_M$	$-(2G/\mu_0)^{1/2}$
rB_ϕ	RB_ϕ	I	g
F'	F'	ψ'_M	–
Ω	Ω	$-\phi'_E$	–
H	H^*	H_M	–

Acknowledgements

This work was supported in part by the UK Department of Trade and Industry and EURATOM.

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