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Finite Larmor-radius theory of magnetic island evolution

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Gyro-kinetic theory is used to investigate the effect of the polarization drift on magnetic island evolution. Three regimes are found. For island phase velocities between the ion and electric drift-velocities the polarization current is shown to be stabilizing. For phase velocities between the electric and electron drift-velocities, the island emits drift waves. This results in a radiative drag force. For all other phase velocities the polarization current is destabilizing, in agreement with the fluid limit.

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The polarization current has an important influence on the stability of thin magnetic islands[1–9]. In particular, it has been invoked as a drive for magnetic turbulence[1–5], as an agent in the process of mode locking in tokamaks[6, 7], and as the source of the observed stability threshold against the growth of neoclassical tearing modes[8, 9]. Questions have been raised, however, concerning the role of a narrow layer surrounding the island separatrix where the polarisation current is large[7]. It has been shown recently, using fluid models, that accounting for this layer reverses the effect of the polarization current on stability[10–12].

The occurrence of a current-layer on the separatrix is a consequence of the assumption, motivated by experimental observation, that magnetic islands propagate at different velocities than the surrounding plasma. For islands wide enough to flatten the density profile, such a difference in velocity requires the presence at the magnetic separatrix of a pedestal in the electric field. The polarization drift of the ions in the pedestal gives rise to a density perturbation that is neutralized by an electron current flowing along the field lines. We will follow the practice of referring to this neutralizing electron current as the polarization current.

In this paper, we present the first kinetic analysis of nonlinear magnetic island evolution that includes the effect of the excitation of drift waves and extends over the complete range of frequencies comparable to the drift frequency. Our analysis brings to light three new phenomena that have important implications for our understanding of island dynamics. We find that first, the emission of drift waves leads to a radiative torque on the island. Second, the island interacts resonantly with the drift wave it excites when its width is a multiple of the radial wavelength of this drift wave. Third, there exist two bands of frequency where the polarization drift is stabilizing: the first extending from the ion diamagnetic frequency almost to the electric drift frequency, and the second extending from the electron drift frequency towards the electric drift frequency by an amount that varies inversely with the ratio of the island width to the ion gyroradius.

We consider a periodic sheared slab geometry which serves as a model for many physical plasma systems. The magnetic field takes the form $\mathbf{B} = B_0\hat{\mathbf{z}} - \nabla\psi \times \hat{\mathbf{z}}$, where B_0 is a constant magnetic field pointing in the symmetry direction $\hat{\mathbf{z}} = \nabla z$. In the reference state $\mathbf{B} = B_0(\hat{\mathbf{z}} - (w/L_s)x\hat{\mathbf{y}})$, where x is the distance from the resonant magnetic surface normalized to the island width w and L_s is the magnetic shear length. We consider a perturbed azimuthal flux of the form $\psi = \psi_0 + \tilde{\psi}\cos\xi$, where $\xi = k_y y - \int^t \omega(t')\,dt'$ is the azimuthal angle in the frame such that the island is at rest. In this frame, the electric field far from the island approaches a spatially constant value such that $k_y v_E = -\omega$, where v_E is the electric drift velocity and ω is the rotation frequency of the island in the (x, y, z) frame where the unperturbed electric field vanishes. The perturbed flux describes a magnetic island of half-width $w = \sqrt{4L_s\tilde{\psi}/B_0}$. We use the normalized flux-surface label Ω defined by $\Omega = -\psi/\tilde{\psi} = 2x^2 - \cos\xi$, so that $\Omega = -1$ is the island 'O'-point, $\Omega = 1$ is the island separatrix and $\Omega > 1$ is the region outside the island.

Following Rutherford[13], we obtain an equation describing the evolution of the island's width by integrating Ampère's law radially across the island region, making use of the constant-$\tilde{\psi}$ approximation ($\tilde{\psi} \gg \partial_x\tilde{\psi}$). We write the condition that the $\cos\xi$ and $\sin\xi$ components of the current match the jump in the perturbed magnetic field across the island region as

$$\oint d\xi \int_{-\infty}^{\infty} dx\, J_{\|} e^{-i\xi} = \frac{c\Delta'}{4w}\tilde{\psi}, \tag{1}$$

where $\Delta' = [(\partial\psi/\partial x)_{x=0^+} - (\partial\psi/\partial x)_{x=0^-}]/\tilde{\psi}$ and $0^{\pm}$ indicates the asymptotic limits as x approaches the resonant surface from either side. Here $\Re(\Delta')$ describes the part of the external perturbation that is in phase-quadrature with the perturbed field B_x: it is proportional to the free energy available in the equilibrium current distribution. $\Im(\Delta')$, by contrast, describes the part of the external perturbation that is in phase with B_x. This is proportional to the external electromagnetic torque that may arise, for example, in the presence of a resistive boundary or of a synchronous external perturbation.

From the asymptotic matching Eq. (1), Ohm's law, and the vorticity equation, it can be shown[14] that the island's evolution in the fluid limit is governed by

$$\frac{2}{D_\eta}\frac{dw}{dt} - \frac{i\lambda}{k_{\parallel}'^2 v_A^2 w^3}\frac{d\omega}{dt} = \Delta' + \Delta_{\rm pol}, \tag{2}$$

where $D_\eta = \eta c^2/4\pi$ is the resistive diffusion coefficient, $\lambda \approx 32\sqrt{2}$ is a numerical coefficient, $k_\parallel = \mathbf{k}\cdot\mathbf{B}/B_0$ is the wavevector, $k_\parallel' = dk_\parallel/dx = k_y/L_s$, v_A is the Alfvén velocity and

$$\Delta_{\rm pol} = -\frac{16L_s}{cB_0 w}\oint d\xi \int_{-\infty}^{\infty} dx\, J_{\rm pol} e^{-i\xi}, \tag{3}$$

measures the effect of the polarization current $J_{\rm pol}$ on the island growth. Our aim is to calculate $J_{\rm pol}$, and to use Eq. (3) to evaluate its effects on the island evolution.

For $\omega \ll k_\parallel v_{te}$, where $v_{te} = \sqrt{T_e/m_e}$ and T_e are respectively the electron thermal velocity and temperature, we may use a fluid model to describe the electron response. In the vicinity of the island, $e\phi/T_e \ll 1$, so that the density is adequately represented by an expansion of the Boltzmann response,

$$n_e(\Omega,\xi) = n_0[1 + e\phi(\Omega,\xi)/T_e(\Omega) + H(\Omega)] \tag{4}$$

The quantity $H(\Omega)$ is a stream function for the transverse component of the electron fluid velocity. We adopt the following model, similar to the one introduced in Ref. [3]:

$$H(\Omega) = \sigma_x\left(1 - \frac{\omega}{\omega_{*e}}\right)\frac{w}{L_n}\left(\frac{\sqrt{\Omega}-1}{\sqrt{2}}\right)\Theta(\Omega - 1), \tag{5}$$

where $\sigma_x = \mathrm{sign}(x)$ and Θ is the Heaviside step function. This model, representing a quantity with a transverse gradient that has been flattened inside the separatrix, was shown in previous work[12] to be a good approximation for the exact solution of the transport equation in the fluid limit. The gradient of H, proportional to the electron transverse velocity, is shown in Fig. 1.

The electric current is obtained from the continuity equation, $\nabla_\parallel J_\parallel = e\mathbf{v}_E \cdot \boldsymbol{\nabla} n_e$, where parallel ion velocity is neglected on the grounds that $k_\parallel c_s \ll \omega$. Here, $c_s = \sqrt{T_e/m_i}$ is the sound speed. Substituting the electron density found in Eq. (4) into the continuity equation and integrating, we find

$$J_\parallel = I(\Omega) + \frac{n_0 ec}{\tilde{\psi}}\frac{dH}{d\Omega}\sigma_x(\phi - \langle\phi\rangle_\xi). \tag{6}$$

Here the angle brackets represent the flux-surface average. The first term of Eq. (6) is the inductive part of the current, while the second term represents the polarization current.

We consider next the gyro-kinetic equation for the ions. Using the total particle energy as an independent variable, the gyro-kinetic equation takes the form

$$v_\parallel \nabla_\parallel f_i + \langle \mathbf{v}_E\rangle_\alpha \cdot \boldsymbol{\nabla} f_i = 0, \tag{7}$$

where $\langle \mathbf{v}_\mathbf{E}\rangle_\alpha$ is the average of the electric drift velocity over the gyration phase. For $k_\parallel v_{ti} \ll \omega$, where $v_{ti} = \sqrt{T_i/m_i}$ and T_i are respectively the ion thermal velocity and temperature, the solution is

$$f_i(X, \Xi, \mu, \mathcal{E}_i) = g(\langle\phi\rangle_\alpha, \mu, \mathcal{E}_i), \tag{8}$$

where X, Ξ are the radial and azimuthal coordinates of the guiding center, and g is an arbitrary function. The solution given by Eq. (8) expresses the fact that the distribution function depends on position only through the gyro-averaged electrostatic potential.

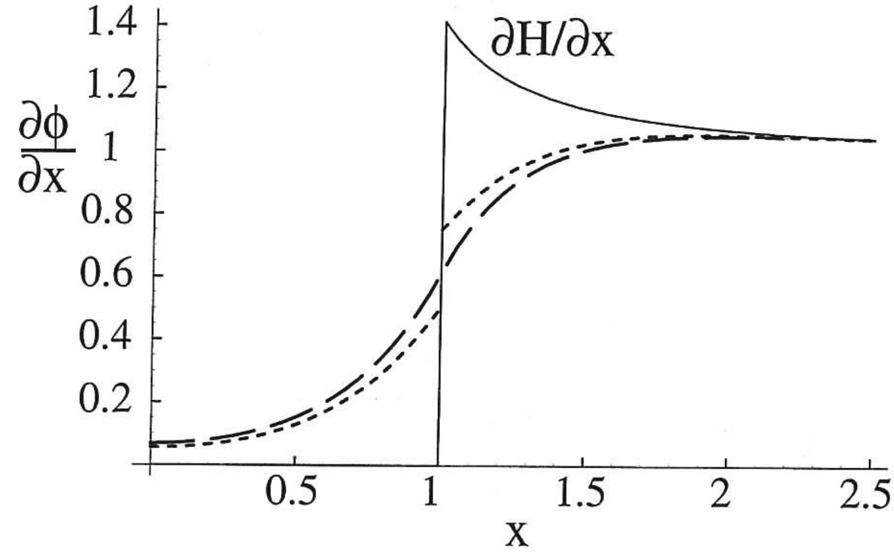

FIG. 1: Comparison of the electric drift velocity profile across the island's O-point for $\rho_i = .2w$ and $\tau = 1$ (dotted line) with that found with cold-ion fluid theory for $\rho_s = .2w$ (dashed line). The solid line represents the derivative of the electron stream-function H in Eq. (5), normalised to $1 - \omega/\omega_{*e}$. This is equivalent to the electric drift velocity in the limit $\rho_s/w \to 0$ corresponding to Magnetohydrodynamics

In principle, the form of the function g is determined at higher order by transport processes. Here, for simplicity, we will use a linearized model for g, expanding with respect to the equilibrium Maxwellian distribution f_{Mi}. There follows $f_i = (1 - e\phi(x,\xi)/T_i)n_0 f_{Mi}(v) + \delta f_i$, where

$$\delta f_i(X, \Xi, v) = \frac{e\langle\phi\rangle_\alpha}{T_i}\left(1 - \frac{\omega_{*i}^T}{\omega}\right) n_0 f_{Mi}(v) \tag{9}$$

is the non-adiabatic part of the ion response. Here $\omega_{*i}^T = [1 + (v^2/v_{ti}^2 - 3/2)\eta_i]\omega_{*i}$, $\eta_i = d\log T_i/d\log n$, and $\omega_{*i} = k_y cT_i/eBL_n$. The above solution was obtained previously by Connor and Wilson[5]. Note that the adiabatic part of the ion response depends on the local (unaveraged) electrostatic potential at the position of the particle. The ion distribution function and ion density will thus have a discontinuous gradient if the electric field is discontinuous.

To obtain the ion density it is necessary to express the ion distribution function in the particle coordinates before integrating over the velocity. This is most easily done by using Fourier transformations in the transverse direction. Evaluating the spatial and velocity integrals, and using the quasi-neutrality relation, we obtain the governing equation for $\hat{\phi}(k_x, \xi)$, the Fourier transform of ϕ:

$$G(k_x^2\rho_i^2/2)\, e\hat{\phi}(k_x, \xi)/T_e = -\omega \hat{H}(k_x, \xi), \tag{10}$$

where $\rho_i = v_{ti}/\omega_{ci}$, $\omega_{ci} = eB/mc$, and $G(b) = \omega(1+\tau) - [(\omega\tau + \omega_{*e})I_0(b) - \eta_i\omega_{*e}b(I_0(b) - I_1(b))]e^{-b}$. Here I_0 and I_1 are the modified Bessel functions of the first kind, $\tau = T_e/T_i$, and $\hat{H}$ is the Fourier transform of H. The response function G is clearly proportional to the dielectric permittivity. An important feature of Eq. (10) is that the dispersion relation $G(k_x^2\rho_i^2/2) = 0$ has one or more pairs of real roots for k_x whenever $0 < \omega/\omega_{*e} < 1$ (corresponding to islands with a phase velocity intermediate between the electric drift velocity and the drift velocity of the electrons). When G has real roots, $\hat{\phi}$ has a pole on the real axis, indicating the excitation of a drift wave by the island. It is then necessary to apply an outgoing-wave boundary condition to determine the form of $\phi(x, \xi)$[15]. We have verified the appropriateness of the outgoing wave boundary condition by matching the solution at large x to the WKB solutions of the linear electrostatic eigenmode equations including the effect of parallel ion streaming and ion Landau damping. The WKB analysis shows that the outgoing wave decays, as expected, for $\omega/\omega_{*e} > 0$. For $1 < \eta_i \lesssim 3$ however, we find that there appears a narrow range of frequencies in the ion direction for which the outgoing drift wave couples to the ion temperature gradient instability. In this range of frequencies the outgoing waves will amplify as they propagate away from the island.

The application of the outgoing-wave boundary condition is complicated here by the nonlinear and nonlocal nature of the solution, as well as by the secular divergence of H at large $|x|$. In order to treat the secular divergence, we seek the response $\phi_+(x,\xi)$ for a one-sided electron stream-function $H_+(x,\xi) = H(\Omega(x,\xi))\Theta(x)$. The Fourier transform of $H_+(x,\xi)$ is well-defined for $\Im(k_x) < 0$. A particular solution ϕ_{+p} follows by performing the Fourier inversion integral for $e\hat{\phi}_+/T_e = -\omega\hat{H}_+/G$ on a contour lying slightly below the real k_x-axis. The general solution ϕ_+ is the sum of this particular solution with the forward and backward propagating plane waves satisfying the dispersion relation. We may evaluate the asymptotic form of $\phi_{+p}(x,\xi)$ for $x \to +\infty$ by deforming the contour upwards. In the most common case where the dispersion relation has a single pair of real roots $\pm k_{x0}$, we find

$$\phi_{+p}(x,\xi) \sim \bar{\phi}(x) + \tilde{\phi}(x,\xi); \quad x \to \infty, \tag{11}$$

where $\bar{\phi}(x) = (x - \sigma_x/\sqrt{2})w\omega/L_n\omega_{*e}$ and $e\tilde{\phi}(x,\xi)/T_e = \omega k_{x0}\Im[\hat{H}(k_{x0},\xi)e^{ik_{x0}x}]/b_0G'(b_0)$ are the residues of the poles at $k_x = 0$ and $k_x = \pm k_{x0}$, respectively. Here $b_0 = k_{x0}^2\rho_i^2/2$ and $G' = dG/db$. For $x \to -\infty$, by contrast, deforming the integration contour downwards shows that $\phi_{+p}(x,\xi)$ decays exponentially.

We next construct the general solution of Eq. (10), $\phi(x,\xi)$, by antisymmetrizing the solution ϕ_+ described above: $\phi(x,\xi) = \phi_+(x,\xi) - \phi_+(-x,\xi)$. The function $\phi(x,\xi)$ has the asymptotic behavior $\phi \sim \bar{\phi}(x) + \sigma_x c(\xi)\cos k_{x0}x + s(\xi)\sin k_{x0}x$, where $c(\xi) = \tilde{\phi}(0,\xi)$ and $s(\xi)$ is a free function resulting from the antisymmetric homogeneous solution. We determine this free function by matching $\tilde{\phi}$ to a Fourier superposition of outgoing waves,

$$\tilde{\phi}(x,\xi) \sim \sigma_x \sum_{m=0}^{\infty} d_m \cos[k_{x0}|x| - m\sigma_\omega\xi], \tag{12}$$

where the d_m are free parameters and the coefficient $\sigma_\omega = \mathrm{sign}(\omega)$ ensures that (12) represents waves with incoming phase velocity. This corresponds to an outgoing group velocity, since the drift wave is a backward-wave[15]. We complete the asymptotic matching by observing that $c(\xi)$ is an even function of ξ and thus has the Fourier expansion $c(\xi) = \sum_{m=0}^{\infty} c_m \cos m\xi$. It follows that $d_m = c_m$ and

$$s(\xi) = \sigma_\omega \sum_{m=1}^{\infty} c_m \sin m\xi. \tag{13}$$

This completes the description of the solution of Eq. (10) in the regime $0 < \omega/\omega_{*e} < 1$. A salient feature of the solution is the presence of odd terms in ξ. These terms give rise to polarization currents in phase with B_x which, when crossed into the magnetic field, exert a tangential force on the island.

The results of the numerical solution of Eq. (10) are shown in Figs. 1-3. Fig. 1 shows the velocity profile along a chord crossing the island through the O-point for $\rho_i = .2w$ and $\tau = 1$ (dotted line). Note the jump in velocity stemming from the adiabatic part of the ion response. The fluid limit (dashed line), by contrast, is continuous. Fig. 2 shows the effect of the polarization current on stability, given by $\Re[\Delta_{\mathrm{pol}}(\omega)]$ (a), and on the drag force $\Im[\Delta_{\mathrm{pol}}(\omega)]$ resulting from electron drift-wave radiation (b). Parametric studies show that the width of the stable band lying to the left of the electron drift frequency increases rapidly with ρ_i/w, but that the depth of the stabilizing region is approximately independent of ρ_i/w.

The oscillatory behavior of the radiative drag and Δ_{pol} as a function of frequency in the region of wave emission (Fig. 2) is caused by the resonances that occur when the island width is a multiple of the radial wavelength of the drift wave. This is illustrated in Fig. 3, where the equipotentials are shown for an island of width equal to one transverse wavelength of the drift wave. It should be noted that by introducing convection cells or eddies within the separatrix, standing drift waves help to resolve the discontinuity in the electric field on the separatrix. That is, the convection cells act as ball-bearings, reducing the friction associated with the slippage of the island through the plasma. This suggests that the excitation of standing drift waves *inside* the separatrix could be favored by collisional transport processes, a possibility that needs to be explored using a nonlinear ion response. Another noteworthy feature in Fig. 3 is the bow wave emanating from the island as it propagates through the plasma.

The emission of drift waves raises interesting questions regarding the interaction of islands with background electrostatic turbulence, and may be an important element for understanding turbulence in finite β plasmas. Our results indicate that the primary mode of interaction is the exchange of momentum between the turbulence and the island. An improved understanding of this process could lead to the use of magnetic islands to modify zonal flow dynamics, and thus to instigate and control transport barriers in magnetic fusion confinement devices.

In summary, we have shown that there exists two bands of frequency where the polarization drift is stabilizing: the first extending from the ion diamagnetic frequency to very near the electric drift frequency, and the second extending

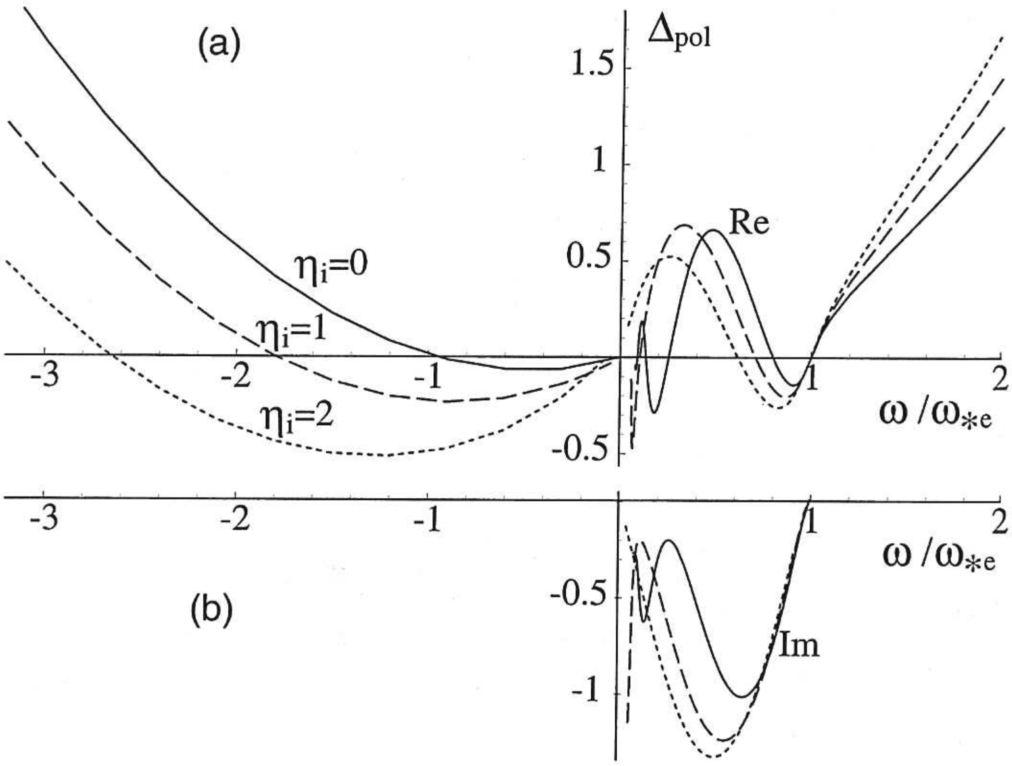

FIG. 2: Real (a) and imaginary (b) parts of the stability parameter Δ_{pol} as a function of frequency. The solid, dashed and dotted lines are for $\eta_i = 0$, 1, and 2 respectively. The real and imaginary parts indicate the effect of the polarization drift on stability ($\Re[\Delta_{pol}]$) and on the radiative torque ($\Im[\Delta_{pol}]$).

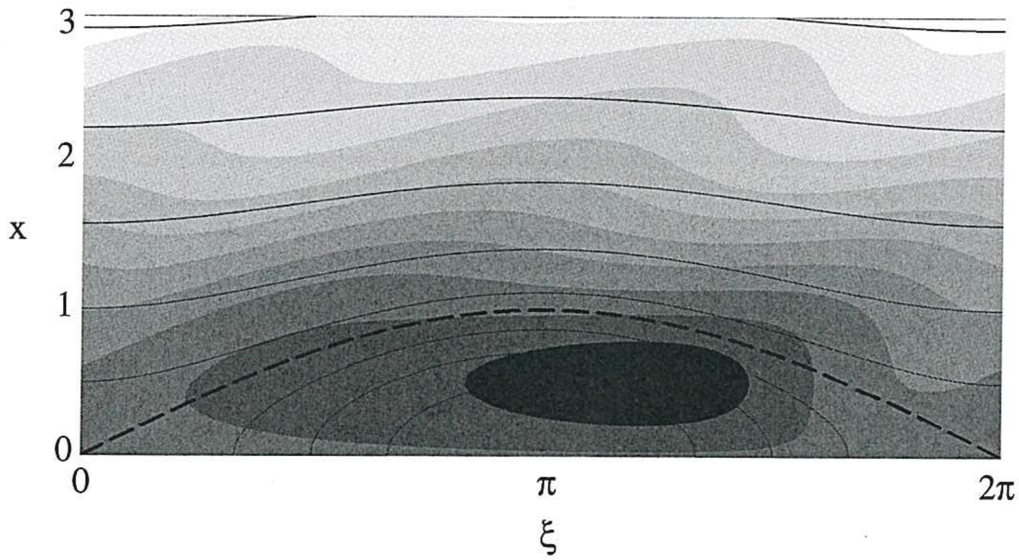

FIG. 3: Contour lines of the electric potential (approximate stream function for the ion flow) in the fluid limit ($\tau \to \infty$) for $w\sqrt{1 - \omega_{*e}/\omega} = \pi\rho_s$. The solid lines represent flux surfaces, and the thick dashed line is the separatrix.

to the left of the electron drift frequency and of variable width. We have further shown that for $0 < \omega/\omega_{*e} < 1$, the island emits drift waves, resulting in a radiative drag. These drift waves propagate to the ion-Landau damping region $k_{\|}v_{ti} \sim \omega$ where they are reabsorbed. The wave-emission is clearly an important ingredient in determining the propagation frequency of the island, and thus its stability in hot plasmas, particularly when collisional dissipation effects are small.

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