UKAEA FUS 481

EURATOM/UKAEA Fusion

Lectures on Plasma Physics: Lectures 1 - 4

A. Thyagaraja

December 2002

EURATOM/UKAEA Fusion Association

Culham Science Centre
Abingdon
Oxfordshire
OX14 3DB
United Kingdom

Telephone: +44 1235 464181
Facsimile: +44 1235 463435

UKAEA Fusion Working with Europe

Cosmic Plasmas, Physics 418

Lecture 1: Elements

Summary

Definition of a plasma; distinction from neutral gas. Debye length, plasma parameter; concept of shielding and quasi neutrality. Two types of description: particle kinetic vs continuum. Self-consistent field concept. Continuum equations for a neutral (ideal) gas. Continuum equations of motion for an ideal, quasi neutral plasma.

N.B. These Lecture Notes are intended to provide a self-contained account of the material. Some topics are included for completeness and may be omitted on a first reading. They are so indicated, where appropriate.

I.1 Introduction

Matter exists in the Universe in roughly four common forms: solids, liquids, neutral gases and ionized plasmas. The word "plasma" was coined by **Irving Langmuir**, who noticed that ionized gases (as opposed to neutral ones) have very peculiar properties which are reminiscent of blood plasma (a very sticky and curious non-Newtonian fluid). It is an interesting fact, that from the point of view of sheer abundance, most of the matter in the Universe exists in the plasma state. Some very common forms of matter are in fact plasmas. For example, the conduction electrons in a metal form a "quantum plasma". Dilute electrolytes, liquid metals, gas discharges, and lightning are other familiar plasma phenomena. In astrophysics, plasmas occur extensively and are of fundamental importance. All modern approaches to controlled thermonuclear fusion power involve a deep and systematic study of fully ionized plasmas at very high temperatures (of the order of 10^7K) confined by magnetic fields or their inertia (as in a hydrogen bomb). In stars and in the cosmos, plasmas are ubiquitous and their interactions in gravitational and magnetic fields is the subject of intense study.

Unlike neutral gases, plasmas are strongly influenced by electromagnetic fields. Indeed, plasma physics (especially in relation to very high temperature, gaseous plasmas) is fundamentally a study of the electrodynamics of a classical, many-body system of charged particles and their interactions. "Exotic" plasmas also exist. An example is provided by the "quark-gluon plasma" found in nucleii and in the early Universe and possibly also in collapsed stars. Recently, pure electron plasmas (no ions!) have also been the subject of beautiful theoretical and experimental studies. "Dusty plasmas" are currently of intense experimental and theoretical study, as are many technologically important, low-temperature plasmas. Plasma chemistry is also of interest and may have applications in the future in environmentally friendly industrial technologies. Quantum plasma behaviour is of great importance in condensed matter physics and may also be of relevance in astrophysics in relation to the equation of state of stellar material. This very partial list should give you some idea of the depth and width of plasma physics which straddles the whole body of classical and quantum many-body phenomena with deep connections to classical mechanics, fluid turbulence, wave propagation theory, astrophysics and many other areas.

I.2 Collective plasma behaviour and quasi neutrality: what is a plasma?

In this first Lecture, I will introduce the concept of a (classical) plasma and some basic properties which characterize it. Let us begin with a thought experiment. Consider a box (say unit cube) which contains hydrogen gas at room temperature and atmospheric pressure. The H_2 molecules are weakly bonded hydrogen atoms. We assume, as with all idealizations, that the walls of the box are "perfect" in that the particles collide with them and are reflected perfectly elastically but do not chemically interact with them. This neutral gas is well understood and approximately obeys the perfect gas equation of state. Maxwell-Boltzmann statistics and kinetics apply to it, and we may even describe the fluid dynamics of it using the standard Navier-Stokes equations of neutral gas dynamics. If we now heat the gas, at constant pressure (by increasing the volume of the box, as necessary), at a well-defined temperature, the bonds of H_2 are broken and we obtain a gas of hydrogen atoms. This is still a neutral gas and is largely uninfluenced by magnetic fields and continues to obey the perfect gas law even better than before, as the temperature rises due to heating. Remember that temperature is directly proportional in this case to the mean translational kinetic energy of the particles making up the gas. If we continue heating at constant pressure, the number of atoms per cubic metre falls whilst the atoms move about with much greater random velocities. At a high enough temperature (of the order of 10^6 K), the atoms are all "ionized". Thus the electrons are literally knocked out from their "shells" in the atoms and it is energetically favourable for them and the protons to exist as two separate, electrically charged fluids. We have arrived at the pure, hydrogen plasma state. For every proton there is an electron (after all we started from the neutral gas!). What is much less trivial is that in every small volume of the box, the *number densities*, n_e and n_i of the two charged species will be found to be very nearly equal. This state is called a "quasi neutral, fully ionized electron-ion plasma".

Let us try to understand why this should be so and the caveats attached to this statement. Firstly, it is clear that in this state the electrons are not "bound" to the protons as in the neutral gas, but are "free". This because their kinetic energy is much larger than the electrostatic binding energy of 13eV$\simeq 2 \times 10^{-18}$joules. Bear in mind the relation, 1 eV, or electron volt is the work done in moving an electron through a potential difference of 1 volt. It is a unit of energy much used in atomic and plasma physics and corresponds to a temperature of 1.16×10^4K and is equal to 1.6×10^{-19}joules.

Secondly, if the difference, $\Delta n = n_i - n_e$ is comparable to n_e, the Poisson equation of electrostatics :

$$-\nabla^2 \phi = e(n_i - n_e)/\epsilon_0 \tag{1}$$

can be used to show that an enormous electric field would be created. Here e is the elementary charge on the proton ($= 1.6 \times 10^{-19}$ coulombs), ϵ_0 is a constant appearing in Maxwell's equations($= 8.85 \times 10^{-12}$coulombs/metre/volt) and ϕ is the electrostatic potential in volts, with $\mathbf{E} = -\nabla\phi$. Electrons would be accelerated by such a field and and would rush in to "equalize" the charge imbalance (bearing in mind that they are 2000 times lighter than the protons which would hardly move!). We see therefore that a plasma "does not like" large electric fields and tends to keep $n_i \simeq n_e$ to reduce this field to "reasonable" values. But, one may ask, what is "reasonable"? The answer turns out to depend upon the temperature, T, of the system.

Following Debye, we ask the following question: suppose we bring in to our box containing a proton-electron plasma, a "test charge" of magnitude e at the centre of the box. We assume that the electrons and protons satisfy Maxwell-Boltzmann equilibrium statistics at a temperature T which we measure in joules.

If the potential due to the test charge is denoted by ϕ, it is clear that $n_e = \bar{n}\exp(e\phi/T), n_i = \bar{n}\exp(-e\phi/T)$, where, $\bar{n}$ is the common number density of the electrons and ions very far from the test charge. If we assume that the temperature is high enough, Eq.(1) can be linearized (by expanding the exponentials in power series). After substituting for $n_{i,e}$, we obtain the Debye shielding equation,

$$\nabla^2\phi = (\frac{2\bar{n}e^2}{\epsilon_0 T})\phi \tag{2}$$

Several things can be learned from this equation. Since the constant, $\lambda_{\text{Debye}} = (\frac{\epsilon_0 T}{2\bar{n}e^2})^{1/2}$ clearly has the dimension of a length (show this!), it is clear that the scale over which the potential ϕ of the test charge varies must be determined by λ_{Debye}. Note also its variation with temperature and density. The equation must be solved with the condition that the potential due to the test charge vanishes at infinity. For this purpose, we assume that the walls of the box are "at infinity" (why is this a reasonable assumption?) The solution is remarkably simple: thus, $\phi = \frac{e}{4\pi\epsilon_0 r}\exp(-r/\lambda_{\text{Debye}})$, where r is the radial distance from the test charge (verify, or prove otherwise!).

It is clear that close to the test charge, within a fraction of a Debye length, we have the usual Coulomb potential, but further out, the charge is "shielded" by the free electrons and ions of the plasma and dies out exponentially like the Yukawa potential of nuclear physics. Thus, in a quasi neutral plasma, the photon, which carries the Coulomb force *appears* to have acquired a finite mass, proportional to the Debye length! This will also be seen when we consider the propagation of light in a plasma.

We can now formulate a criterion for true, "plasma" behaviour and quasi neutrality. Observe that the "plasma parameter", $N_{\text{Debye}} = n_e\lambda^3_{\text{Debye}}$ represents the number of electrons within a Debye cube. This must be large for the system to behave like a truly many-body system characterized by collective dynamics (as opposed to a swarm of nearly noninteracting, individual charged particles). To see this, recall that the average distance d between electrons is measured by $1/n_e^{1/3}$. Therefore, the plasma dynamics will be governed largely by "collective effects" if $\lambda_{\text{Debye}} \gg d$. This implies that $N_{\text{Debye}} \gg 1$. Furthermore, for length-scales much larger than λ_{Debye}, $n_i \simeq n_e$. We shall also encounter a time-scale criterion associated with quasi neutrality a little later.

A very important source of confusion in students (and many text-books!) is this: if it is true that $n_i \simeq n_e$ in a plasma, is it true to say that $\mathbf{E} = 0$? If not, how can we reconcile the Maxwell equation (also called "Gauss' Law"), $\epsilon_0\nabla.\mathbf{E} = e(n_i - n_e)$ with quasi neutrality? The *short* answer is, in a quasi neutral plasma, $\mathbf{E}$ (at least on spatial scales large compared with λ_{Debye}) *is not* determined by Gauss' Law, and is definitely not strictly zero in general. *Instead*, it is obtained from the equations of motion of the charged particles making up the plasma and the other Maxwell equations! After we have found $\mathbf{E}$, we can, if we so wish, *use* Gauss' Law to determine the *charge density* $\rho = e(n_i - n_e)$, by differentiation. It will turn out that typical electrostatic fields in a quasi neutral plasma on length scales (L) large compared with λ_{Debye} are of order $\leq T/eL$, and consequently, the failure of quasi neutrality on these scales is measured by, $(\lambda_{\text{Debye}}/L)^2$, namely, we have the ordering, $|n_i - n_e|/n_i \simeq (\lambda_{\text{Debye}}/L)^2 \ll 1$.

As the development of the subject proceeds you will see how the key concept of quasi neutrality enters the correct physical description of plasmas. For the present, regard quasi neutrality as an important simplification in the treatment of collective plasma effects. When it applies, bear in mind that *other* Maxwell equations must also be *consistently* approximated. We will find that for quasi neutral plasmas, we can safely neglect the famous "displacement current term" in Maxwell's equations (remember that Maxwell introduced this term to achieve consistency between charge conservation and Ampère's Law relating currents and magnetic fields). We shall learn to understand those physical processes for which it is correct to invoke quasi neutrality and when it is essentially violated and the full set of Maxwell equations are required to give a consistent dynamical description of the electromagnetic field in a plasma. When quasi neutrality applies, we may use the "pre-Maxwell" form of Ampère's Law,

$$c^2\epsilon_0 \nabla \times \mathbf{B} = \mathbf{j} \tag{3}$$

where $\mathbf{j}$ is the (quasi neutral) electric current due to the motion of ions and electrons and $\mathbf{B}$ is the magnetic field. Charge conservation requires that $\nabla.\mathbf{j} = 0$. This implies that the quasi neutrality approximation is a quasi static one, and is unlikely to be valid for "fast" processes like electromagnetic waves.

I.3 Modes of description of fluids and plasmas: kinetic and continuum representations

A fluid or a plasma (ie., an *ionized gas*) is an assembly of a very large number (10^{20} or more particles per cubic metre) of molecules, ions or atoms. In principle, there exists (classically) an "exact" description of such systems in terms of the Newtonian (or Einsteinian, if relativistic) equations of motion of all the particles supplemented by Maxwell's equations for the electromagnetic fields. This description of the exact classical dynamics is called "Liouville" or Γ-space description. Unfortunately it is largely useless for two reasons: firstly, it involves an unimaginably complicated system of nonlinear equations, and secondly, it is "too detailed" to be of practical use. Indeed, to solve the equations of motion in this description with the most powerful computers (even for relatively short times like 1 second) would take about 10^{11} years, assuming a time-step of 10^{-10} seconds and a computer which performs about 10^{10} operations per second. Furthermore, even if we could do such calculations, it would require gargantuan amounts of data processing to obtain experimentally measurable parameters. Recall however that despite our manifest inability to perform such calculations, some general "macroscopic laws" like the perfect gas equation of state *do* apply! In equilibrium statistical mechanics we simply cut through this mass of indigestible dynamical equations and use the Boltzman-Maxwell- Gibbs distribution to obtain the thermodynamic (ie., *average*) properties of gases. In the same spirit, two methods have been devised to "reduce" the many-body problem to manageable proportions in nonequilibrium situations. It should, however be borne in mind, that the Liouville description is often useful in proving general theorems about the systems under study and has a definite theoretical (though not practical) role in plasma physics.

You will encounter in these lectures both types of "reduced" approaches. The conceptually simpler approach is the "phase-space" or "kinetic" model. We consider a collection of structureless particles $i = 1, ..., N$ (N is very large) each of which has, at any instant t, a position vector $\mathbf{r_i}$ and a velocity vector, $\mathbf{v}_i$. We consider a 6-dimensional "phase space" called μ-space (the μ stands for "molecular")with coordinates $\mathbf{r}, \mathbf{v}$ (this is *not* to be confused with Liouville's phase space of $6N$ dimensions called Γ-space!). We then introduce a *distribution function*, $F(\mathbf{r}, \mathbf{v}, t)$ in this μ-space, analogous to the distibution functions of equilibrium statistical mechanics like the Maxwell-Boltzmann function. The significance of F is this: if $d\Omega = d\mathbf{r}d\mathbf{v} = dx.dy.dz.dv_x dv_y.dv_z$ represents a reasonably small "volume element" in μ-space, $Fd\Omega$ gives the *probability* of finding a particle in the region surrounding the phase point, $(\mathbf{r}, \mathbf{v})$ of (phase) volume $d\Omega$. Thus F is a *probability density function*. By its very definition, it is a "statistical" object. As such, $NF(\mathbf{r}, \mathbf{v}, t)d\Omega$ represents the expected number of particles in the small region surrounding the phase point. A *kinetic* description of the system is said to be complete, if we can prescribe an equation of evolution for F and specify initial and boundary conditions for it.

Now consider our perfect gas of noninteracting particles in a box with perfectly reflecting walls. For practical purposes, neutrinos with a small mass are excellent for thinking; they can be assumed to be "confined" in a finite volume of space by a gravitational well, instead of a box, if necessary. The Newtonian equations of motion in this instance are extremely simple: the velocity of a particle remains unchanged except when it hits a wall which simply reverses it. Forgetting the walls for the moment, we may write,

$$\frac{d\mathbf{v}_i}{dt} = 0$$
$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i$$

for $i = 1, .., N$. If we consider a group of particles in a small phase volume centred at $(\mathbf{r}, \mathbf{v})$, after a small time t they would occupy the same volume around the point, $(\mathbf{r} + \mathbf{v}dt, \mathbf{v})$. It then follows that $F(\mathbf{r}, \mathbf{v}, t) = F(\mathbf{r} + \mathbf{v}dt, \mathbf{v}, t + dt)$. From this we deduce (show this!) that F must satisfy the "kinetic" equation,

$$\frac{\partial F}{\partial t} + \mathbf{v}.\frac{\partial F}{\partial \mathbf{r}} = 0 \qquad (4)$$

More "formal" and careful proofs of this exist and can be found in the literature.

This is the simplest of all kinetic equations and says that the distribution function of the system stays constant if one "moves" with the flow of particles in μ-space. You will encounter many such equations with more complicated terms. For example, the particles may move in given external force fields (eg. neutrinos in the gravitational field of a black hole, say), or they may interact amongst themselves via "long range" forces of electromagnetic/gravitational origin and/or experience "short range" collisions. The case of external forces $\mathbf{f}$ in the absence of collisions is particularly simple: we now have the rule, $F(\mathbf{r}, \mathbf{v}, t) = F(\mathbf{r} + \mathbf{v}dt, \mathbf{v} + \mathbf{f}dt/m, t + dt)$, modifying Eq.(4) to,

$$\frac{\partial F}{\partial t} + \mathbf{v}.\frac{\partial F}{\partial \mathbf{r}} + \frac{\mathbf{f}}{m}.\frac{\partial F}{\partial \mathbf{v}} = 0 \tag{5}$$

In the particular case of charged particles with charge q and mass m moving in electric and magnetic fields, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, respectively, we have, $\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. This important equation is called "Vlasov's equation". We shall shortly see how $\mathbf{E}, \mathbf{B}$ themselves can be related to the particle distribution functions of the charged species present in the system.

All these cases can, and have been, approximately described by kinetic equations like the above. Collectively, such equations are called by various names like "Master equation", Boltzmann's kinetic equation, Landau-Fokker-Planck equation etc. In this course you will encounter several simple cases of such equations and learn some properties of their solutions in physically interesting situations.

If F can be obtained by solving kinetic equations such as the one above, many familiar properties of the sytem can be calculated. In principle, *all* experimentally measurable quantities relating to our system are expressible as suitable averages over the distribution function. For example, the number density $n(\mathbf{r}, t)$ of the particles is clearly given by the integral (show this from the definitions!),

$$n(\mathbf{r}, t) = N \int_{\mathbf{v}} F(\mathbf{r}, \mathbf{v}, t) d\mathbf{v} \tag{6}$$

where the integral is over the three dimensional *velocity space* (the μ space is obviously the Cartesian product of ordinary position $(\mathbf{r})$, and velocity $(\mathbf{v})$ spaces). Since the combination, $NF(\mathbf{r}, \mathbf{v}, t)$ occurs very frequently, it is convenient to give a special name for it: we thus define the *particle distribution function*, $f(\mathbf{r}, \mathbf{v}, t) \equiv NF(\mathbf{r}, \mathbf{v}, t)$. Evidently, integrating f over all of velocity space gives the particle number density at the location $\mathbf{r}$ at time t. It should be carefully noted that *strictly speaking* the kinetic equation for free particles does not describe the motion in a box, since the velocities of the particles change discontinuously at the walls. However, for many purposes, it is sufficient to imagine that f is simply a periodic function of the spatial coordinate $\mathbf{r}$ with some macroscopic "periodicity length", L (analogous to the box size).

We can define the macroscopic or average **velocity** of the gas/plasma, denoted by $\mathbf{u}(\mathbf{r}, t)$, in the following obvious manner:

$$\mathbf{u}(\mathbf{r}, t) = \frac{1}{n} \int_{\mathbf{v}} \mathbf{v} f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v} \tag{7}$$

This vector is also called, for obvious reasons, the *fluid* velocity at $\mathbf{r}, t$. It must not be confused with $\mathbf{v}$ which represents the *random* velocity of the individual particles making up the fluid. Sometimes you will find in the literature, the term *peculiar velocity* for the difference, $\mathbf{c} = \mathbf{v} - \mathbf{u}$. Plainly, a knowledge of f allows us to calculate the *velocity moments*, $n, \mathbf{u}$ at any $(\mathbf{r}, t)$.

We need not stop here! Bearing in mind the fundamental fact that $(1/2)m\mathbf{v}^2$ represents the kinetic energy of a particle (here, m is the mass of the particle), we may *define* the *mean total kinetic energy per unit volume* at $(\mathbf{r}, t)$ by the obvious formula:

$$\bar{E}(\mathbf{r}, t) = \int_{\mathbf{v}} (1/2) m\mathbf{v}.\mathbf{v} f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v} \tag{8}$$

It is a simple exercise to show from the definitions that,

$$\bar{E}(\mathbf{r}, t) = (1/2) mn\mathbf{u}^2 + \int_{\mathbf{v}} (1/2) m\mathbf{c}.\mathbf{c} f(\mathbf{r}, \mathbf{c} + \mathbf{u}, t) d\mathbf{c} \tag{9}$$

We clearly recognize the quantity, $mn(\mathbf{r}, t) = \rho_{\mathrm{m}}(\mathbf{r}, t)$ as the *mass density* of the gas. Then the first term on the RHS of the above equation is nothing but the local *kinetic energy per unit volume* of the fluid flow, $(1/2)\rho_{\mathrm{m}}\mathbf{u}^2$. The second term may be familiar from statistical mechanics as the *internal energy per unit volume* of the gas, denoted by $U(\mathbf{r}, t)$. It is the kinetic energy of *random motions* of the particles, as opposed to the first term which represents the kinetic energy of the *average* or bulk motions. Clearly, since this is what we mean by "thermal" or heat energy in kinetic theory, it is conventional to *define* the *temperature*, $T(\mathbf{r}, t)$ and the pressure $p(\mathbf{r}, t)$ of the gas by the formulae,

$$\begin{aligned} U(\mathbf{r}, t) &= \int_{\mathbf{v}} (1/2) m\mathbf{c}.\mathbf{c} f(\mathbf{r}, \mathbf{c} + \mathbf{u}, t) d\mathbf{c} \\ &= (3/2) n(\mathbf{r}, t) T(\mathbf{r}, t) \\ p &= nT \end{aligned} \tag{10}$$

These definitions embody the perfect gas equation, which applies whenever the translational kinetic energy of random motions is much larger than any other form of energy of the particles. Note that T as defined here has the units of energy (joules) and represents the *translational kinetic energy per particle per degree of freedom* of particles. Each particle carries on the average, $T/2$ joules for each of its three degrees of freedom. Pressure is usually expressed in $\mathrm{N.m^{-2}}$, but can also be thought of as $\mathrm{Jm^{-3}}$. The conversion factor which expresses the temperature T in kelvins is a universal constant named Boltzmann's constant, and is conventionally denoted by k. It has the value, $k = 1.3807 \times 10^{-23}\mathrm{JK^{-1}}$. Thus, 1K corresponds to k joules. From the formula for pressure, we see that if μ is the mean molar weight (ie., mass of 1 mole of gas having $N_A = 6.0221 \times 10^{23}$ molecules, so that $m = \mu/N_A, \rho_\mathrm{m} = mn$), we may write the equation of state for the pressure in the form,

$$\begin{aligned} p &= nkT(K) \\ \frac{p}{\rho_\mathrm{m}} &= \frac{\mathbf{R}}{\mu}T(K) \\ \mathbf{R} &= kN_A \end{aligned}$$

where $\mathbf{R}$ is the "universal gas constant" ($= 8.3145\mathrm{JK^{-1}mol^{-1}}$). Here, $T(K)$ is the temperature in kelvins, whilst, of course, $kT(K)$ is the temperature in joules.

We have thus far defined two scalar moments (these can be taken to be n, T without loss of generality) and a vector moment ($\mathbf{u}$) of the distribution function. These are the most important observable properties of the gas, though a few higher moments also occur frequently. Note that when we have several *species* of particles which need to be considered (eg. ions and electrons in a fully ionized plasma) we can analogously define distribution functions for each separate species. In general, if the different species interact with each other, we shall have to derive kinetic equations for each distribution function. These equations will contain nonlinear "interaction" terms in general, and must all be simultaneously solved subject to suitable initial and boundary conditions in both position and velocity space. This has only been done in some very special cases, of which the familiar equilibrium statistical mechanics is one.

We conclude this brief introduction to the key ideas underlying the classical, non-equilibrium kinetic description of many-body problems by making an important remark about electron-ion plasmas, where $Z_i e$ is the charge on the ions. The "charge number", Z_i accommodates the fact that ions can be nuclei of atoms, and can consequently have higher charge states than a proton; in these cases, the mass of the ion, $m_i = A_i m_p$, where m_p is the proton mass and A_i is the "isotopic mass number" of the ion in question. As stated above, we introduce $f_e(\mathbf{r}, \mathbf{v}, t), f_i(\mathbf{r}, \mathbf{v}, t)$. It follows from the preceding discussion, that the local number densities, $n_{e,i}(\mathbf{r},, t)$ can be obtained by integrating these functions over the velocities. We immediately note that the *local average charge density,* $\rho = e(Z_i n_i - n_e)$ is obtained as the moment,

$$\rho(\mathbf{r}, t) = e \int_{\mathbf{v}} [Z_i f_i - f_e] \, d\mathbf{v} \tag{11}$$

It also follows that the local *average current density,* $\mathbf{j}$ is given by,

$$\begin{aligned} \mathbf{j}(\mathbf{r}, t) &= eZ_i n_i \mathbf{u}_i - en_e \mathbf{u}_e \\ &= e \int_{\mathbf{v}} \mathbf{v} \, [Z_i f_i - f_e] \, d\mathbf{v} \end{aligned} \tag{12}$$

It can be shown without difficulty that in all cases, the evolution of $f_{e,i}$ is *constrained* by the Law of Conservation of Charge,

$$\frac{\partial \rho}{\partial t} + \nabla.\mathbf{j} = 0 \tag{13}$$

This means that we *can,* if we wish, self-consistently calculate electromagnetic fields, $\mathbf{E}, \mathbf{B}$ due to these charges and currents, according to Maxwell's equations, in all their glory:

$$\epsilon_0 \nabla.\mathbf{E} = \rho \tag{14}$$

$$\nabla.\mathbf{B} = 0 \tag{15}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{16}$$

$$\epsilon_0 c^2 \nabla \times \mathbf{B} = \mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{17}$$

where $c = 2.99 \times 10^8 \mathrm{m.s}^{-1}$, is the speed of light in vacuo, and $\epsilon_0 = 8.85 \times 10^{-12}$(SI unit: coulombs/volt/metre). Sometimes another "fundamental" vacuum constant, $\mu_0 = 1/(\epsilon_0 c^2)$ is introduced. I prefer to stick with c and ϵ_0 (for more essential information on units, see the Supplements to Lecture 1).

You can readily check that the Law of Conservation of Charge is necessary and sufficient for these equations to be consistent. Note also the interesting fact that if, at $t = 0$ we *impose* Eq.(15) as an *initial condition* on $\mathbf{B}$, Eq.(16) ("Faraday's Law of electromagnetic induction") implies the validity of Eq.(15) for *all time*! This remark will have implications later.

The fields $\mathbf{E}, \mathbf{B}$ determined in this manner from Maxwell's equations using the *averaged* charge and current densities, $\rho, \mathbf{j}$ obtained from the plasma distribution functions, $f_{e,i}$

are called **self-consistent** or "smoothed out" fields. The point is that whilst they are determined by the plasma, they, in turn influence the plasma dynamics via the kinetic equations satisfied by the distribution functions (eg. Eq.(5)). We shall leave the kinetic description of fluids and plasmas and consider the "continuum" or fluid description next.

I.4 Continuum description of neutral gases and ideal plasmas

Before we go on to considering plasmas, let us look briefly at the mathematical description of *neutral gases.* We consider a perfect gas with mass density $\rho_{\rm m}(\mathbf{r},t)$ (m is the mass of a gas molecule) at every point in space, with position vector, $\mathbf{r}$ at a time t. At each such point, we may associate a pressure field $p(\mathbf{r},t)$ and a temperature, $T(\mathbf{r},t)$ (for convenience we measure T in joules). We know that the equation of state is,

$$p = \rho_{\rm m} T/m \tag{18}$$

The flow of the gas is described by specifying the instantaneous velocity field at every point $\mathbf{r}$ at time t. We denote this vector field by $\mathbf{u}(\mathbf{r},t)$.

We now wish to set up the equations of motion which govern the variables, $\rho_{\rm m}, T, \mathbf{u}$. Obviously, if $\rho_{\rm m}, T$ are known, the equation of state, Eq.(18) gives p. Consider an arbitrary (but fixed!) closed surface S bounding a volume V within the body of the gas. Clearly the total mass of gas enclosed by V is given by the integral,

$$M_V(t) = \int_V \rho_{\rm m} dV \tag{19}$$

It is easy to see that the net mass flowing out of the volume V per second is given by a surface integral of the mass "flux" (in the absence of sources or sinks within the volume -show this!),

$$-dM_V/dt = \int_S \rho_{\rm m} \mathbf{u}.\mathbf{n} dS \tag{20}$$

Here, $\mathbf{n}$ denotes the unit outward drawn normal to S at the element dS. Applying Gauss' divergence theorem to the volume V, we obtain the *integral equation of continuity,* also called the Law of Conservation of Mass for the volume V:

$$\begin{aligned} dM_V/dt &= \int_V \frac{\partial \rho_{\rm m}}{\partial t} dV \\ &= -\int_V \nabla.(\rho_{\rm m}\mathbf{u}) dV \end{aligned} \tag{21}$$

Now we make use of the fact that we set up this integral equation for an *arbitrary volume* V. Assuming that various functions involved are smooth, the satisfaction of the integral equation requires the following *partial differential equation* at each point within the gas:

$$\frac{\partial \rho_{\rm m}}{\partial t} + \nabla.(\rho_{\rm m}\mathbf{u}) \;=\; 0 \qquad (22)$$

Note that this differential equation is in fact less general than the integral equation, which may hold even when $\rho_{\rm m}$, $\mathbf{u}$ are discontinuous, as for example, at shocks and vortex sheets or free surfaces. In effect, if $\mathbf{u}$ is known (say we measure it), the equation can be used to compute $\rho_{\rm m}$ given suitable initial and boundary conditions. A very important special case of this equation occurs when the gas flows rather slowly compared to the speed of sound in it and the density $\rho_{\rm m}$ can be taken to be nearly constant. We see that $\mathbf{u}$ must then be divergence-free to conserve mass. Fields satisfying $\nabla.\mathbf{u} = 0$ are called "incompressible flows". The flow of many ordinary liquids, including water at speeds much below the sound speed are incompressible to a very good approximation.

We might guess that the Law of Conservation of Momentum (or, more generally Newton's Second Law of force balance) would lead to the equation governing the evolution of the velocity vector. Clearly, from the definition of $\rho_{\rm m}$ and $\mathbf{u}$, the momentum of the gas per unit volume is $\rho_{\rm m}\mathbf{u}$. Hence the total momentum of the fluid contained in an arbitrary volume V is given by,

$$\mathbf{P}_V \;=\; \int_V \rho_{\rm m}\mathbf{u}\,dV$$

Now, let $\mathbf{f}(\mathbf{r},t)$ be any external force per unit volume acting on the fluid within V(this could, for example, be gravity). The time rate of change of $\mathbf{P}$, by Newton's Second Law must be equal to the net flow of momentum into V, plus the force exerted by the pressure of the surrounding gas on the fluid within V, plus the increase in momentum due to the accelerations caused by the body force $\mathbf{f}$. This is true if the flow is "frictionless" and the only internal fluid forces are due to pressure. This can be expressed mathematically as an integral equation:

$$\begin{aligned} d\mathbf{P}_V/dt &= \int_V \frac{\partial \rho_{\rm m}\mathbf{u}}{\partial t} dV \\ &= -\int_S \rho_{\rm m}\mathbf{u}\mathbf{u}.\mathbf{n}dS - \int_S p\mathbf{n}dS + \int_V \mathbf{f}dV \end{aligned}$$

Transforming the surface integrals using Gauss' divergence theorem in the usual manner, we obtain the *integral fluid momentum balance equation* for the volume V:

$$\int_V \left[\frac{\partial \rho_{\rm m}\mathbf{u}}{\partial t} + \nabla.(\rho_{\rm m}\mathbf{u}\mathbf{u}) + \nabla p - \mathbf{f}\right] dV \;=\; 0 \qquad (23)$$

If all the fields are assumed sufficiently smooth, we may derive from this integral equation the vector partial differential equation of motion (or Momentum balance relation),

$$\frac{\partial \rho_{\mathrm{m}} \mathbf{u}}{\partial t} + \nabla.(\rho_{\mathrm{m}} \mathbf{u}\mathbf{u}) \quad = \quad -\nabla p + \mathbf{f} \tag{24}$$

This is Euler's celebrated equation of motion for a compressible, ideal, frictionless gas. Note that it can also be written in the equivalent form,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} \quad = \quad (-\nabla p + \mathbf{f})/\rho_{\mathrm{m}} \tag{25}$$

upon making use of the equation of continuity. Note that we have derived effectively *four* independent evolution equations for the *five* unknowns, $\rho_{\mathrm{m}}, p, \mathbf{v}$ (T can be obtained in principle from ρ_{m}, p using the equation of state). The system is still not *closed* in general. Gases in which it is reasonable to assume that the pressure is a known function of the density *alone* are called *barotropic*. In such cases, we have in addition to the four equations derived, a barotropic equation of state, of the form, $p = F(\rho_{\mathrm{m}})$. For barotropic fluid motions, the above Eulerian system is indeed closed and may be solved in many important special cases. A physically important class of barotropic motions arises when the gas moves in an *isentropic* manner. In this case, for a perfect gas, we have, $p/p^* = (\rho_{\mathrm{m}}/\rho^*_{\mathrm{m}})^\gamma$, where, p^*, ρ^*_{m} are "equilibrium values" and γ is the ratio of the specific heats at constant pressure and constant volume obtained from kinetic considerations ($\gamma = 5/3 = 1.67$ for monatomic gases with only three translational degrees of freedom). It is also called the "adiabatic exponent" and the relation itself is called the adiabatic equation. In many cases, ρ^*_{m}, p^* are constants, but in principle, they could depend upon position, especially in the presence of conservative external fields like gravity. Non adiabatic flows also occur frequently in practice. To treat them, it is necessary to take the First and Second Laws of Thermodynamics into account and formulate an *internal energy balance equation*. This will be done at a later stage.

Finally, I indicate an informal, nonrigorous derivation of the equations of motion satisfied by an "ideal", quasi neutral plasma. We consider a perfect, fully ionized gas in which all dissipative effects like thermal conduction, viscosity and electrical resistivity are negligible (it is a "perfect" conductor, *not* a "super conductor" which is a *quantum system*!). We assume that it is quasi neutral and described by the mass density ρ_{m}, local velocity, $\mathbf{u}$, which are functions of position and time. Since it is ideal in this sense, it makes sense to assume that the pressure p is isentropically related to the mass density ρ_{m}. The *only* additional information we require of the plasma relates to its behaviour under electromagnetic fields. If we consider good conductors of electricity (say liquid metals), it is known from experiment that there is a definite relation between the electric field and the current, called Ohm's Law. In a simple, isotropic conductor at rest, with resistivity η, this takes the general form,

$$\mathbf{E} = \eta \mathbf{j} \tag{26}$$

This is what is called a "constitutive relation", rather like the laws of Snell and Hooke. It describes a property (here electrical resistance) of matter. The quantity, η need not be a constant, but can vary with position and time. Note that in a "perfect" conductor, the resistivity is negligibly small, and we can state that $\mathbf{E} = 0$. Ohm's Law has been generalized in many ways. If there is a magnetic field, $\mathbf{B}$ present, for instance, and the conductor moves with a velocity $\mathbf{u}$, the effective electric field "felt" by the current carriers is, approximately (from Special Relativity), $\mathbf{E} + \mathbf{u} \times \mathbf{B}$. Ohm's law then becomes, for a conducting fluid,

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} \tag{27}$$

Our "ideal" fluid (in which $\eta = 0$) must therefore obey the "ideal" Ohm's Law,

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \tag{28}$$

We need one more piece of information from Special Relativity: when a current carrying conductor moves in a magnetic field, the electromagnetic force, $\mathbf{f}_{\text{em}}$ per unit volume is given by Lorentz' well-known formula, $\mathbf{f}_{\text{em}} = \mathbf{j} \times \mathbf{B}$. In addition to this "Lorentz force", there may be other "body forces" or sources of momentum within the system. The most important of these for an ideal fluid which flows frictionlessly, is of course, the gravity force, $\mathbf{g} = \rho_{\text{m}} \nabla K$, where $K(\mathbf{r}, t)$ is the total gravitational potential due to external masses and the fluid's own self-gravity. This can be assumed known, or calculated by solving an appropriate Poisson equation if all the sources are given.

We can now state the equations of motion for a, quasi neutral, compressible, isentropic, perfectly conducting (ie., "ideal") plasma:

$$\begin{aligned}
\frac{\partial \rho_{\text{m}}}{\partial t} + \nabla.(\rho_{\text{m}} \mathbf{u}) &= 0 && (29)\\
\frac{\partial \rho_{\text{m}} \mathbf{u}}{\partial t} + \nabla.(\rho_{\text{m}} \mathbf{u}\mathbf{u}) &= -\nabla p + \mathbf{j} \times \mathbf{B} + \rho_{\text{m}} \nabla K && (30)\\
p/p^* &= (\rho_{\text{m}}/\rho_{\text{m}}^*)^\gamma && (31)\\
c^2 \epsilon_0 \nabla \times \mathbf{B} &= \mathbf{j} && (32)\\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} && (33)\\
\mathbf{E} &= -\mathbf{u} \times \mathbf{B} && (34)
\end{aligned}$$

If you are wondering what has happened to the *other* two Maxwell equations, $\epsilon_0 \nabla.\mathbf{E} = \rho, \nabla.\mathbf{B} = 0$, use the following two hints to answer your own query. Fact 1: we have stated that our ideal plasma is quasi neutral. Fact 2: if we assume the *initial condition*, $\nabla.\mathbf{B} = 0$, what do the above equations, in particular, Faraday's Law, say about $\nabla.\mathbf{B}$ for later times?

Note carefully that the last three equations may be used to *eliminate* the vector fields $\mathbf{j}, \mathbf{E}$ from the equations of motion which involve *only* $\mathbf{u}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)$ and the single, positive scalar field, $\rho_{\mathrm{m}}(\mathbf{r}, t)$ (pressure having been eliminated using the adiabatic equation of state). Although in many respects, this is a highly simplified reduced description of a perfectly conducting , dissipationless, quasi neutral plasma, this **ideal magnetohydrodynamic** (or "Ideal MHD" for short) approach is surprisingly powerful and extremely widely used as the simplest of all plasma theories. It provides a key bench mark against which all other models of plasma dynamics must be compared and contrasted. Certainly, it is the most systematically explored set of equations in plasma physics and possesses many remarkable and nontrivial properties, some of which will be the subject of later lectures and many applications.

Supplements to Lecture 1

NOTE ON UNITS

We have used SI units. You should be aware that many textbooks and much literature uses another, closely related set of units called Gaussian CGS units. In order to be able to read the literature and use these valuable study materials, it is important to be familiar with Gaussian units, especially as regards the forms which Maxwell's equations take in the two systems.

Here is a summary of the essential information regarding the two widely used systems of units. :

Length: 1 metre (SI)= 100 centimetres (Gaussian)

Time: 1 second(SI)=1 second (Gaussian)

Mass: 1 kilogram(SI)=1000 grams(Gaussian)

Force: 1 newton(SI)=10^5 dynes(Gaussian)

Energy: 1 joule(SI)= 10^7 ergs (Gaussian)

Power: 1 watt(SI)=10^7 erg.s^{-1} (Gaussian)

Charge: 1 coulomb(SI)=3×10^9 statcoulombs(Gaussian)

Current:1 ampere(SI)=3×10^9 statamperes(Gaussian)

Emf: 1 volt(SI)=$(1/3) \times 10^{-2}$ statvolts(Gaussian)

Magnetic field: 1 tesla(SI)=10^4 gauss(Gaussian)

Electric field: 1 volt/metre(SI)=$(1/3) \times 10^{-4}$ statvolts/cm(Gaussian)

Number density:particles.m^{-3}(SI)=10^{-6} particles.cm^{-3}(Gaussian)

We consider the forms of Maxwell equations in the two systems:

They are written as follows in SI units:

$$\begin{aligned}\nabla.\mathbf{B} &= 0\\ \epsilon_0\nabla.\mathbf{E} &= \rho\\ \frac{\partial\mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E}\\ \epsilon_0 c^2\nabla\times\mathbf{B} &= \mathbf{j}+\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\end{aligned}$$

In Gaussian (ecgsu) units they take the form:

$$\begin{aligned}\nabla.\mathbf{B} &= 0\\ \nabla.\mathbf{E} &= 4\pi\rho\\ \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E}\\ \nabla\times\mathbf{B} &= \frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\end{aligned}$$

The particle equation of motion in SI:

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

whilst in Gaussian units, we have,

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{E}+\mathbf{v}\times\mathbf{B}/c)$$

By the same token, in SI units, E/B has the dimensions of a velocity, whereas in Gaussian units, E/B is dimensionless. The Lorentz force (newtons/cu.metre) in SI is, $\mathbf{j}\times\mathbf{B}$, where as in Gaussian units (dynes/cu.cm) it is, $\mathbf{j}\times\mathbf{B}/c$, where $c = 3\times 10^{10}$ cm.s^{-1}.

The translation from SI to Gaussian units is very easy: replace ϵ_0 in the SI formulae by $1/4\pi$ and $\mathbf{B}$ by $\mathbf{B}/c$; you will obtain the Gaussian formulae! For instance, $\epsilon_0 c^2 B^2/2$ is the magnetic field energy density in SI. It is $B^2/8\pi$ in Gaussian.

The Larmor frequency, $\Omega_c = eB/M$ in SI and eB/Mc in Gaussian.

For a list of useful formulae and values of the fundamental constants, it is recommended that you consult: *The NRL Plasma Formulary* by David. L. Book and published by the US Office of Naval Research. Alternatively, Francis Chen's text book, "*Introduction to Plasma Physics and Controlled Fusion*", Volume I, Second Edition (1983), Plenum Press, has, in Appendix A, an excellent and useful set of data relating to units and plasma parameters. This text also has an excellent and accessible coverage of many topics dealt with in the present course, and as such, is a very useful background reference.

Plasma Physics

Lecture 2: Particle orbit theory

Summary

Particle motions, orbit equations. Simple exact solutions in electromagnetic fields with known symmetries. Drifts and Larmor gyrations in homogeneous and inhomogeneous fields. Exact and adiabatic invariants. Trapping of particles. Examples

II.1 Newtonian and Lagrangian equations of motion of charged particles

We now take a step back from the many body problem discussed in the previous Lecture and look in detail at how charged particles move under the action of electric and magnetic fields. Let us consider a single, charged particle, with mass M and charge e. Here e does not necessarily represent the charge on a proton, but can, in principle be any value, as can M. Within Newtonian mechanics and classical electrodynamics, the equations of motion of this particle are:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \tag{1}$$

$$M\frac{d\mathbf{v}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{f} \tag{2}$$

Here, $\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)$ are assumed to be *known* electromagnetic fields and $\mathbf{f}(\mathbf{r}, t)$ represents all forces not involving electromagnetism. For example, if the gravitational potential is $K(\mathbf{r}, t)$, $\mathbf{f} = M\nabla K$. Sometimes we may consider other forces like friction or radiation reaction and the like. If velocities comparable to the speed of light, c, are involved, we must modify the equation. We then replace Eq.(2) with,

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{f} \tag{3}$$

$$\mathbf{p} = m\mathbf{v}(1 - \frac{v^2}{c^2})^{-1/2} \tag{4}$$

where m now represents the *rest mass* of the particle and we take into account the Einsteinian mass variation with the speed of the particle in accordance with Special Relativity Theory. These equations are deceptively simple-looking! Although they describe completely the classical motions of individual charged particles in prescribed electromagnetic fields, we must approach them by systematically building up our knowledge of their properties.

Although the equations of motion given above are exact, they can be put in a more general form which is helpful in many circumstances. As you know, a Newtonian particle with "generalized coordinates" q_i $(i = 1, 2, 3)$, moving in a force-field with a potential energy function, $V(q_i)$ can be associated with a "kinetic energy" $T(\dot{q}, q_i)$ and a Lagrangian function, $T - V$. Here, we use Newton's notation, $\dot{q}_k \equiv \frac{dq_k}{dt}$ for brevity. Lagrange showed that the particle satisfies the equations of motion (entirely equivalent to Newton's Laws and equations),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \;=\; 0 \qquad (5)$$

The beauty of these equations lies in the fact that they take exactly the *same form* whatever transformations we subject the generalized coordinates to! In Cartesian coordinates, we obviously have, $T \equiv (1/2)M\mathbf{v}^2$. It is easy to verify that Lagrange's equations reduce to Newton's in this standard case. It was further shown by Hamilton that Lagrange's equations are the result of extremizing the "action integral", $\int Ldt$, by simultaneous variations of the q_i and $\dot{q}_i$. After this brief recapitulation of Lagrangian mechanics, it is of interest to note that we can write a Lagrangian down for charged particles which will yield the equations, Eq.(1,2).

To construct such a Lagrangian, we first recall some basic facts from electrodynamics. Observe that $\nabla.\mathbf{B} = 0$ is exactly satisfied if we introduce a new vector field, $\mathbf{A}$ such that, $\mathbf{B} = \nabla \times \mathbf{A}$. Such a field, you may remember, is called the *vector potential.* Now Faraday's Law can be completely "solved" by writing, $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi$, where the new scalar function Φ is called the *scalar electrostatic potential.* By introducing $\Phi, \mathbf{A}$ we have managed to "solve" two of Maxwell's equations for $\mathbf{E}, \mathbf{B}$ exactly! However, instead of six independent functions, we now have four independent functions to determine. Bearing in mind the Law of Conservation of Charge, we only have three independent Maxwell equations left to do this. This means that we may, without loss of generality, impose one constraint on the four functions (ie., three components of $\mathbf{A}$ and Φ) to provide as many equations as there are unknowns. This additional equation is called a "gauge condition". It is common to take this condition to be $\nabla.\mathbf{A} = 0$, although other choices which respect relativistic invariance better are also often used. This particular gauge is called the "Coulomb gauge". Let us remember though that *within classical physics*, only $\mathbf{E}$ and $\mathbf{B}$ are "physical" fields which can be measured. The potentials are merely mathematical tricks and all physical properties must be "gauge invariant" in the sense that they must be the same whatever gauge one chooses. Quantum mechanics does give a much more fundamental role to the potentials, but we do not need these finer points here.

With this preamble, we can state the nonrelativistic Lagrangian for charged particle motion. Let us choose Cartesian coordinates, $\mathbf{r}$ for the q_i. Then, $\mathbf{v} = \dot{\mathbf{r}}$, and the nonrelativistic Lagrangian takes the remarkably simple form:

$$L_{\text{nonrelativistic}} = (1/2)M\mathbf{v}^2 - e\Phi(\mathbf{r},t) + e\mathbf{A}(\mathbf{r},t).\mathbf{v} \tag{6}$$

It is definitely *not* obvious that if we now write down Lagrange's equations, we will necessarily get Eqs.(1,2)! However, it is an interesting exercise in vector analysis to show this, using the relations, $\mathbf{B} = \nabla \times \mathbf{A}, \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi$.

Incidentally, no one has succeeded in writing down a Lagrangian for charged particles which is explicitly expressible *only* in terms of $\mathbf{E}, \mathbf{B}$ and does not involve the potentials $\mathbf{A}, \Phi$ in a hidden way. This is a very deep fact about electrodynamics, and may suggest that even within classical Lagrangian dynamics, the potentials are "more fundamental" fields than $\mathbf{E}, \mathbf{B}$!

It is also equally easy to write down the *relativistic* Lagrangian for a charged particle. It is,

$$L_{\text{relativistic}} = -mc^2(1 - \mathbf{v}^2/c^2)^{1/2} - e\Phi(\mathbf{r},t) + e\mathbf{A}(\mathbf{r},t).\mathbf{v} \tag{7}$$

where, as usual, m denotes the *rest mass* of the particle. Again it is a simple exercise to show that Eqs(3,4) follow, with $\mathbf{f} \equiv 0$. It is also elementary to verify that $L_{\text{relativistic}} \rightarrow L_{\text{nonrelativistic}}$ when $v^2/c^2 \rightarrow 0$, ie., in the nonrelativistic limit (apart, of course from an "inessential" constant rest energy, mc^2). I use the notation m for rest mass and M for the relativistic mass parameter. In general, $M = m/(1 - v^2/c^2)^{1/2}$ (and we have, $M \rightarrow m$ in the nonrelativistic limit). While m is a relativistic invariant, M is a measure of the "mass energy" in a particular frame.

Lagrange's equations reveal some very important facts about the system. Suppose we have a Lagrangian which is *independent* of one of the generalized coordinates, q_k. Such a coordinate is called a *cyclic* coordinate. Clearly if we transform $q_k \rightarrow q_k + a$ where a is an arbitrary constant, L does not change. This "translation" of the cyclic coordinate is an example of a "symmetry transformation" of the Lagrangian which is left invariant by it. Now let us look at Lagrange's equation involving derivatives with respect to q_k, our cyclic coordinate. We have, $\frac{\partial L}{\partial q_k} = 0$ by definition of cyclicity or symmetry. But Lagrange's equation says that we must have,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) &= \frac{\partial L}{\partial q_k} \\ &= 0 \end{aligned} \tag{8}$$

We see that cyclicity of q_k *implies* that the function, $\frac{\partial L}{\partial \dot{q}_k}$ is a *constant of the motion*! Thus a symmetry of the Lagrangian is related to the existence of a constant of motion for the system. This profound connection and its considerable ramifications goes under the name, *Noether's theorem* and is of fundamental importance in modern physics. We have but seen a very tiny illustration of it. The function, $\frac{\partial L}{\partial \dot{q}_i}$ is given a special name: it is denoted by p_i and is called the *generalized momentum conjugate to the coordinate* q_i. Evidently, it is a function of all the coordinates q and their time derivatives, $\dot{q}$.

We have just derived an important theorem of Lagrangian dynamics.

Theorem: If a system has a Lagrangian $L(q_i, \dot{q}_i, t)$ and a particular q_k is *cyclic*, ie., $\frac{\partial L}{\partial q_k} = 0$, its conjugate momentum, $p_k \equiv \frac{\partial L}{\partial \dot{q}_k}$ is a *constant of the motion*, ie, $\frac{dp_k}{dt} = 0$.

Suppose now that $L(q_i, \dot{q}_i, t)$ itself has the property that it is independent of t. In cartesian coordinates, for instance, this can happen if $m, e, \Phi, \mathbf{A}$ do not depend explicitly upon the time. Then, we have, $\frac{\partial L}{\partial t} = 0$. Can we find the "generalized momentum" conjugate to the time? The answer is "Yes!"

Firstly note that with $L_{\text{relativistic}}$,

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{v}} &= m\mathbf{v}(1 - v^2/c^2)^{-1/2} + e\mathbf{A} \\ &= M\mathbf{v} + e\mathbf{A} \\ &= \mathbf{P} \end{aligned} \qquad (9)$$

defines the so-called *canonical momentum*, $\mathbf{P}$ conjugate to $\mathbf{r}$. If the Lagrangian does not depend upon x, say, P_x will be conserved!

Consider the new function, H defined by,

$$\begin{aligned} H &= \mathbf{v}.\mathbf{P} - L \\ &= mc^2(1 - v^2/c^2)^{-1/2} + e\Phi \end{aligned} \qquad (10)$$

We now show that,

$$\begin{aligned} \frac{dH}{dt} &= \dot{\mathbf{v}}.\mathbf{P} + \mathbf{v}.\dot{\mathbf{P}} - \frac{\partial L}{\partial \mathbf{r}}.\mathbf{v} - \frac{\partial L}{\partial \mathbf{v}}.\dot{\mathbf{v}} - (\frac{\partial L}{\partial t} = 0) \\ &= 0 \end{aligned}$$

Thus, the function H (this is called the "Hamiltonian function" of the system, in honour of **Sir William Rowan Hamilton** who first introduced it, and, as you may know played a key role in the development of classical and quantum mechanics) is a constant of the motion when the fields are steady and consequently the Lagrangian does not depend explicitly upon the time, t. In this sense, H, which clearly represents the sum of the mechanical and electrostatic energies, is conjugate to t and its constancy is a consequence of *time-translational invariance* of the Lagrangian (hence the general principle, "if the Lagrangian of a system is invariant under time translation, the energy/Hamiltonian of the system is a constant of the motion"). You may remember that in special relativity, $(E, \mathbf{p})$ is a four-vector "conjugate to" the event four-vector, $(t, \mathbf{r})$. Hence these relationships are much deeper than seems at first sight.

Armed with these powerful general principles we are ready to tackle the equations of motion for charged particles under physically interesting conditions and enunciate physically useful sets of approximations.

II.2 Simple cases of charged particle motions

We begin with the simplest of all cases: that of uniform electric fields. Consider a uniform, steady electric field along the x-axis. We then have for force balance,

$$\frac{d\mathbf{p}}{dt} = eE_x\mathbf{e}_x \tag{11}$$

It follows immediately that p_y, p_z are constants of the motion. In the x-direction, the integral is, $p_x = p_x(0) + eE_x t$. I leave it as an interesting exercise for you to obtain x, y, z as explicit functions of t in both relativistic and nonrelativistic cases.

Consider next a steady but nonuniform electric field. Evidently, such a field must be electrostatic, since we have from Faraday's Law under steady contions, $\mathbf{E} = -\nabla\Phi$, where the electrostatic potential can depend on position but not on time. Newton's equations of motion are, in this case,

$$\frac{d\mathbf{p}}{dt} = -e\nabla\Phi(\mathbf{r}) \tag{12}$$

$$\mathbf{p} = m\frac{d\mathbf{r}}{dt}(1 - v^2/c^2)^{-1/2} \tag{13}$$

As already noticed, the *energy*, $H = mc^2(1-v^2/c^2)^{-1/2} + e\Phi$ is a constant of the motion. In general, there are no others, and the equations can only be integrated numerically. In special cases when Φ has a particularly simple structure or symmetries, the solution can be found in terms of known functions. Some examples will be given in the exercises.

Next let us consider the interesting case of motion in steady magnetic fields. First, let us look at a uniform, constant field, $\mathbf{B}$. The equation of motion under relativistic conditions is Eqs(3,4),with $\mathbf{E}, \mathbf{f} = 0$:

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{v} \times \mathbf{B}) \tag{14}$$

$$\mathbf{p} = m\mathbf{v}(1 - \frac{v^2}{c^2})^{-1/2} \tag{15}$$

Note that $\mathbf{B}.\frac{d\mathbf{p}}{dt} = 0$. Since the field is uniform and steady, this says that the momentum component *parallel* to the field direction is a constant of the motion. By taking the dot product of Eq.(14) with $\mathbf{p}$ and noting that $\mathbf{p}$ is always parallel to $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, we find that $\mathbf{p}.\mathbf{p}$ is a constant of the motion. Consequently, v^2 and the kinetic energy, $E = mc^2/(1 - v^2/c^2)^{1/2}$ are also constants of the motion. This fact has already been established for *arbitrary* (not necessarily uniform) steady magnetic fields. To represent the motion more clearly, we can, without loss of generality, choose the magnetic field along the z-axis; $\mathbf{B} = B\mathbf{e}_z$. It follows from the preceding that p_z and v_z are constants of the motion. The particle therefore moves steadily along the z-axis with the velocity, v_z. The mass of the particle in the "laboratory frame" is $M = m/(1 - v^2/c^2)^{1/2}$. We have just seen that this is a constant of the motion. This enables us to write the remaining two components of the equation of motion in the form:

$$\frac{d\mathbf{v}_\perp}{dt} = (eB/M)\mathbf{v}_\perp \times \mathbf{e}_z \tag{16}$$

Here, $\mathbf{v}_\perp = (\frac{dx}{dt}, \frac{dy}{dt}, 0)$, the component of velocity perpendicular to the magnetic field. Note that the combination, eB/M has the dimension of a frequency. As we have already noticed, $\mathbf{v}_\perp^2 = v^2 - v_z^2$ must be a constant of the motion. It is now easy to solve the equation for $\mathbf{v}_\perp$, by separating it into the two components, for example. The solution is, $\mathbf{v}_\perp = c_\perp\left[\mathbf{e}_x \cos(\zeta) + \mathbf{e}_y \sin(\zeta)\right], \zeta(t) = -\Omega_c t + \zeta_0$, where $\Omega_c = eB/M$, and ζ_0 is an initial phase angle determined by the initial values of v_x, v_y. This solution demonstrates that the vector, $\mathbf{v}_\perp$ rotates uniformly round the z-axis in the $x - y$ plane at a constant angular frequency of Ω_c(rads/s) and maintains its length, $c_\perp$. Consequently, the motion of the particle in the plane perpendicular to the field is circular with radius, $r_c = c_\perp/\Omega_c$ and angular frequency, Ω_c. This motion is called **gyromotion** of a charged particle around a uniform magnetic field, or **Larmor precession**. The frequency $\Omega_c = eB/M$ is often called the **gyro frequency** or **Larmor frequency** and the radius r_c of the particle is called the **gyro radius** or **Larmor radius**. The angle, $\zeta(t) = (-\Omega_c t + \zeta_0)$, is called the **gyro-phase** for obvious reasons. Note that the mass involved here is M and is a measure of the particle's *total* energy. The overall motion of the particle is, of course, a helix winding around the field (for positive charges, ζ rotates clockwise for increasing time, looking down the field).

It is obviously the case that if we have in addition to a uniform magnetic field $\mathbf{B}$, an electrostatic field, $\mathbf{E}$, uniform and parallel to $\mathbf{b} = \mathbf{B}/B$, $\mathbf{p}_\perp = m\mathbf{v}_\perp/(1 - v^2/c^2)^{1/2}$ and $\mathbf{p}_{||}$ satisfy,

$$\frac{d\mathbf{p}_\perp}{dt} = (eB/M)(\mathbf{p}_\perp \times \mathbf{b}) \tag{17}$$

$$\frac{d\mathbf{p}_{||}}{dt} = eE_{||}\mathbf{b} \tag{18}$$

Note that $M = m/(1 - v^2/c^2)^{1/2}$ is not a constant in this case since $\mathbf{p}_{||}$ is not. However, we see that, $\mathbf{p}_\perp^2$ is indeed a constant of the motion. We can solve Eq.(18) and get, $p_{||}(t) = p_{||}(0) + eE_{||}t$. Since, $c^2\mathbf{p}^2/(\mathbf{p}^2 + (mc)^2) = v^2$, and $\mathbf{p}^2 = p_{||}^2 + \mathbf{p}_\perp^2$, it is a simple matter to determine the exact relativistic trajectory in this case.

A new and interesting case arises when the $\mathbf{E}$ and $\mathbf{B}$ fields are perpendicular to each other, $\mathbf{E}.\mathbf{B} = 0$. Again we start with uniform fields. We see immediately that $p_{||}$ is a constant of the motion. Note that we must then have (with $M = (m^2 + \mathbf{p}^2/c^2)^{1/2})$),

$$\frac{d\mathbf{p}_\perp}{dt} = e(\mathbf{E} + \mathbf{p}_\perp \times \mathbf{B}/M) \tag{19}$$

We can solve this equation as follows: consider the vector, $\mathbf{u} = \mathbf{E} \times \mathbf{B}/B^2$. By virtue of the fact that $\mathbf{E}.\mathbf{B} = 0$, by construction, we see that $\mathbf{u}$ satisfies the equation, $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$. It follows that $\mathbf{p}_\perp = M\mathbf{u}$ is a particular exact solution of Eq.(19). In fact, to complete this solution, we must substitute $\mathbf{p}_{||}$ (obtained from initial data) and the expression for $\mathbf{p}_\perp$ in the definition of M:

$$\begin{aligned} M^2 &= m^2 + (p_{||}^2 + M^2\mathbf{u}^2)/c^2 \\ &= (m^2 + p_{||}^2/c^2)/(1 - \mathbf{u}^2/c^2) \end{aligned} \tag{20}$$

Further discussion of this is taken up in the problems.

Let us consider now what happens when an "external" force field $\mathbf{f}_\perp$ acts on the particle, transverse to the magnetic field. This external field is assumed to be of "nonelectrodynamic" nature, like the gravitational field. For a change, let us restrict ourselves to the nonrelativistic case, in this instance. The equation of motion reads,

$$M\frac{d\mathbf{v}}{dt} = [e\mathbf{v} \times \mathbf{B} + \mathbf{f}_\perp] \tag{21}$$

Plainly, $\mathbf{v}_{||}$ is a constant. As before, we can find a steady *particular* solution, $\mathbf{v}_\perp^f = \frac{\mathbf{f}_\perp \times \mathbf{B}}{eB^2}$. We again have a drift perpendicular to both the magnetic field and $\mathbf{f}_\perp$. However, *unlike* the $\mathbf{E} \times \mathbf{B}$ drift, this one *does* depend on the charge. Indeed, if $\mathbf{f}_\perp$ is of "gravitational" origin, it can be written in the form, $\mathbf{f}_\perp = M\mathbf{g}_\perp$, since this is after all, Galileo's famous observation that all "masses fall with the same acceleration". In *this* case, the drift,$\mathbf{v}_\perp^g = \frac{M\mathbf{g}_\perp \times \mathbf{B}}{eB^2}$, depends upon the mass-to-charge ratio, M/e of the particle.

Note also that the charge dependence distinguishes the *sign* as well as the magnitude. Thus, in a gravitational field perpendicular to a magnetic field, not only do electrons and protons move differently, but so do electrons and positrons, whereas a crossed electric field cannot detect any changes! This has many important implications, particularly when we consider the behaviour of, not individual charges, as here, but of lots of them, as we did in Lecture 1.

Let us summarise briefly the non relativistic behaviour of a charged particle (mass M, charge e) under the combined action of *uniform and steady* $\mathbf{B} = B\mathbf{b}, \mathbf{E}$, and $\mathbf{f}$ fields, as governed by the equation:

$$M\frac{d\mathbf{v}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{f}$$

We may decompose the vector, $\mathbf{v}$ as follows:

$$\mathbf{v} = v_{||}\mathbf{b} + (\mathbf{E} \times \mathbf{b}/B) + (\mathbf{f} \times \mathbf{b}/eB) + \mathbf{c}_\perp(\zeta(t)) \tag{22}$$

where the quantities, $v_{||}, \mathbf{c}_\perp$ satisfy,

$$v_{||} = v_{||}(0) + t\,[e\mathbf{E} + \mathbf{f}]\,.\mathbf{b}/M \tag{23}$$

$$\mathbf{c}_\perp = c_\perp\,[\mathbf{n}\cos(\zeta) + \mathbf{d}\sin(\zeta)] \tag{24}$$

$$\zeta(t) = -\Omega_c t + \zeta(0) \tag{25}$$

where $\Omega_c = eB/M$, $c_\perp$ is the intial "gyro speed", $\zeta(0)$, the initial gyro phase and $\mathbf{n}, \mathbf{d}$ are unit vectors orthogonal to $\mathbf{b} = \mathbf{B}/B = \mathbf{n} \times \mathbf{d}$. It is left as an (important!) exercise to verify that this is indeed the solution, obtain the spatial coordinates $\mathbf{r}$ by integration, interpret the motion geometrically and generalize to the case of relativistic particles (the last part is somewhat "advanced" and may be attempted with the additional assumption that $\mathbf{E}, \mathbf{f}$ have no components parallel to $\mathbf{B}$).

A key observation based on this exact solution is that in the expression for the velocity components only the gyro phase, $\zeta(t)$ varies with time, and it does so at the gyro frequency, $\Omega_c = eB/M$. The particle position vector, $\mathbf{r}(\mathbf{t})$ of course depends on t via the drifts (uniform motions) and via the Larmor precession of $\mathbf{c}_\perp$ at the gyro frequency.

II.3 Charged particle motions in inhomogeneous fields: drift orbit theory for homogeneous magnetic fields

For arbitrary fields, the equations of motion cannot be integrated in "closed" form (but can be, numerically, in every case. Thus we have **Richard Feynman's** famous remark, "the only really *general* methods for integrating equations in Physics are *numerical* methods!").

In practice, in most applications of charged particle orbit theory we will encounter, a very happy circumstance occurs which enables us to go a long way towards solving the equations of motion analytically. If the force fields concerned vary *slowly* in space and time, in a sense to be clarified shortly, it turns out that the integration of the orbit equations can be greatly simplified.

We have identified a typical time scale associated with the "ambient" $\mathbf{B}$, viz, the gyro period, $2\pi/\Omega_c$. Let us take the gyration speed, $|\mathbf{c}_\perp(0)|$ to represent a typical velocity scale for the particle. It leads to the gyro radius, $r_{\mathrm{L}} = |\mathbf{c}_\perp(0)|/\Omega_c$ as a typical spatial scale of the gyromotion. If the time rates of change of $\mathbf{B}, \mathbf{E}, \mathbf{f}$ are all *small* relative to the gyro frequency, it seems natural to say the fields are *slowly varying*.

This can be formally expressed by the *ordering* relations:

$$\begin{aligned} \rho_t^* &= (\frac{2\pi}{\Omega_c}) Max \left[|\frac{1}{|\mathbf{E}|}|\frac{\partial \mathbf{E}}{\partial t}|, |\frac{1}{|\mathbf{B}|}|\frac{\partial \mathbf{B}}{\partial t}|, |\frac{1}{|\mathbf{f}|}|\frac{\partial \mathbf{f}}{\partial t}| \right] \\ &\ll 1 \end{aligned} \tag{26}$$

Evidently, the same idea can be extended to *spatial* variations. Thus the fields vary slowly in space if the ordering relations,

$$\begin{aligned} \rho_s^* &= r_L Max \left[|\frac{1}{|\mathbf{E}|}|\nabla(\mathbf{E})|, |\frac{1}{|\mathbf{B}|}|\nabla(\mathbf{B})|, |\frac{1}{|\mathbf{f}|}|\nabla(\mathbf{f})| \right] \\ &\ll 1 \end{aligned} \tag{27}$$

This simply means that the typical scale lengths of variation of these fields are long compared to the Larmor/gyro radius, r_L of the particle.

If we now assume that the fields are slowly varying in both senses, it is reasonable to define the parameter, $\rho^* = Max(\rho_s^*, \rho_t^*)$ and require the **drift ordering**,

$$\rho^* \ll 1 \tag{28}$$

to hold uniformly in the region of interest.

Now that we can quite precisely define what we mean by "slowly varying" fields, let us tackle the problem of integrating orbits in the presence of such fields. We use the oldest trick in applied mathematics called "perturbation theory". This simply says, "if you know a solution to an equation in some case, it should be possible to calculate the solution to a *nearby* case."

In addition to the small parameter ρ^* introduced above, we also assume that the electric and external fields are such that, $Max(|\mathbf{E}|/B, |\mathbf{f}|/eB)/|\mathbf{c}_\perp| \leq \rho^*$. This says that the "drift speeds" are small in comparison with the gyro velocity of the particle. In practical cases, the gyro motions proceed at the "thermal speed", $c_s \simeq (T/M)^{1/2}$, whereas the drift speeds are at the more sedate pace of, $E/B \simeq T/(eLB) \simeq \rho^* c_s$.

Now consider the equation of motion for the following special case first. We take the $\mathbf{B}$ field to be *uniform and constant* but allow $\mathbf{E}, \mathbf{f}$ to vary "slowly" in the above sense:

$$M\frac{d\mathbf{v}}{dt} = e\mathbf{E}(\mathbf{r}, t) + eB\mathbf{v} \times \mathbf{b} + \mathbf{f}(\mathbf{r},t) \tag{29}$$

Dividing this by eB, we see that every term on the right except the $\mathbf{v} \times \mathbf{b}$ term is of order $\rho^* \ll 1$, by assumption. Suppose we wish to write the motion of the particle in terms of a *guiding centre* which experiences only the slow variations and superpose on it a fast Larmor gyration, we may set, $\mathbf{v} = \mathbf{v}_{gc} + \mathbf{c}_\perp, \mathbf{r} = \mathbf{r}_{gc} + \mathbf{r}_L$. Ultimately, we are interested in the guiding centre motion, since the Larmor gyration is a simple locally circular motion about the magnetic field line. Evidently, we have, in highest order ("zeroth order in a power series in ρ^*"),

$$\frac{d\mathbf{c}_\perp}{dt} = (eB/M)\mathbf{c}_\perp \times \mathbf{b} \tag{30}$$

Here, $\mathbf{r}_{gc}$ is a constant on the "fast" times scale set by eB/M. The equation is easily solved and gives, $\mathbf{c}_\perp = c_\perp [\mathbf{n}\cos(\zeta) + \mathbf{d}\sin(\zeta)]$, with, $\frac{d\zeta}{dt} = -\Omega_c = -eB/M$. This can be integrated with respect to t once more to get, $\mathbf{r}_L = r_L [-\mathbf{n}\sin(\zeta) + \mathbf{d}\cos(\zeta)]$; $r_L = c_\perp/|\Omega_c|$.

We have determined the Larmor motion on the field line. What of $\mathbf{v}_{gc}$? In highest relevant order, this must satisfy Eq.(29). Substituting, we see that we must solve,

$$\frac{1}{\Omega_c}\frac{d\mathbf{v}_{gc}}{dt} = (\mathbf{E}/B) + \mathbf{v}_{gc} \times \mathbf{b} + (\mathbf{f}/eB) \tag{31}$$

Since the dominant term is clearly $\mathbf{v}_{gc} \times \mathbf{b}$, the equation is re-written as,

$$\mathbf{v}_{gc} \times \mathbf{b} \;=\; -(\mathbf{E}/B) - (\mathbf{f}/eB) + \frac{1}{\Omega_c}\frac{d\mathbf{v}_{gc}}{dt} \tag{32}$$

We want to solve this equation for $\mathbf{v}_{gc}$, which, if you remember is a slowly varying, "guiding centre" velocity. Note that the left side vanishes identially upon taking a dot product with $\mathbf{b}$. Hence, for a consistent solution to exist, the RHS must also vanish when dotted with $\mathbf{b}$. This consistency condition now reads,

$$M\frac{dv_{\|}}{dt} \;=\; eE_{\|} + f_{\|} \tag{33}$$

where use has been made of the fact that, $v_{\|} = \mathbf{v}_{gc}.\mathbf{b}$ and similar expressions for the forces. Returning to Eq.(32) we see that it is solved in highest order by,

$$\mathbf{v}_{gc} \;=\; v_{\|}\mathbf{b} + (\mathbf{E} \times \mathbf{b}/B) + (\mathbf{f} \times \mathbf{b}/eB) \tag{34}$$

The "undetermined" function, $v_{\|}$ is of course to be obtained from Eq.(33). The last two terms are the now familiar perpendicular drifts of the guiding centre (ie., $\mathbf{v}_{gc}^{\perp} = \mathbf{E} \times \mathbf{b}/B + \mathbf{f} \times \mathbf{b}/(eB)$). Using this expression on the RHS of Eq.(31) gives the next order correction to $\mathbf{v}_{gc}$. We therefore have the expression,

$$\begin{aligned} \frac{d\mathbf{r}_{gc}}{dt} &= \mathbf{v}_{gc} \\ &= v_{\|}\mathbf{b} + (\mathbf{E} \times \mathbf{b}/B) + (\mathbf{f} \times \mathbf{b}/eB) + \frac{1}{\Omega_c}(\frac{d}{dt})\,[\mathbf{E}_{\perp} + \mathbf{f}_{\perp}/e]\,/B \end{aligned} \tag{35}$$

Every term in this is slowly varying. The last term is smaller than the first three terms by ρ^*. It arises from the inertia of the charged particles and is often called the "inertial drift" of the charged particles. Since ion masses are much larger than electron masses, for given electric fields, this drift is much more important for the former.

Is this then the whole story? As it happens, we did not carry out the "perturbation expansion" of the basic equation completely consistently. What we have done so far applies strictly only to variations in $\mathbf{E}, \mathbf{f}$ and not in $\mathbf{B}$. In the next section we consider the new drifts which arise when the magnetic field itself is inhomogeneous. This also naturally leads to the concept of **adiabatic invariants**.

II.4 Charged particle motions in inhomogeneous fields: drift orbit theory for general magnetic fields

It can be anticipated that inhomogeneous magnetic fields might cause new drifts by just considering the terms that arise in higher order (in ρ^*) from the $\mathbf{v} \times \mathbf{B}$ term in the general equations of motion. To consider this effect, we are going to adopt a very elegant and efficient technique called the "method of averaged Lagrangians" pioneered by **G.B. Whitham** and used specifically in the present context by **Per Helander** who will be one of the lecturers in this course. Of course, all the results derived can be obtained by more traditional, and sometimes more intuitively understandable perturbation methods. However, they are algebraically messy and sometimes unedifying. You can find several such treatments in the literature and in the problems.

For clarity, I shall only consider nonrelativistic motions, and omit all fields except the magnetic. Recall that the Lagrangian is,

$$L_{\text{nonrelativistic}} = (1/2)M\mathbf{v}^2 + e\mathbf{A}(\mathbf{r}, t).\mathbf{v} \tag{36}$$

Now suppose we have determined an *approximate periodic solution* to the Lagrange's equation derived from this equation. This will involve a *fast* time-scale. However, we may be interested in constructing a solution on a much longer "slower" time-scale over which the terms in the Lagrangian might change. Whitham's idea is to substitute the *form* of the solution and average over the fast time-scale, deriving a new "averaged Lagrangian" with respect to the slowly changing variables. Let us see how this works in our case.

We have seen that the equation of motion can be solved for gyro motions:

$$M\frac{d\mathbf{v}_\perp}{dt} = e\mathbf{v}_\perp \times \mathbf{B} \tag{37}$$

$$\mathbf{v}_\perp = \mathbf{c}_\perp(\zeta(t)) \tag{38}$$

where, $\mathbf{c}_\perp$ is given by,

$$\mathbf{c}_\perp(\zeta) = c_\perp[\mathbf{n}\cos(\zeta) + \mathbf{d}\sin(\zeta)] \tag{39}$$

$$\frac{d\zeta}{dt} = -\Omega_c \tag{40}$$

$$\mathbf{r}_L(\zeta) = r_L[-\mathbf{n}\sin(\zeta) + \mathbf{d}\cos(\zeta)] \tag{41}$$

and $c_\perp$ denotes the Larmor gyration speed, ζ, the "gyro phase" and $\Omega_c = eB/M$, with the previously stated notation. The velocity can also be written in the form,

$$\mathbf{c}_\perp(\zeta) = -r_L\dot{\zeta}[\mathbf{n}\cos(\zeta) + \mathbf{d}\sin(\zeta)] \tag{42}$$

Now suppose we seek a solution in the slow variables and write, $\mathbf{v} = \frac{d\mathbf{r}_{gc}}{dt} + \mathbf{c}_\perp(\zeta)$, $\mathbf{r} = \mathbf{r}_{gc} + \mathbf{r}_L(\zeta)$. Note that in the velocity, $|\frac{d\mathbf{r}_{gc}}{dt}| \simeq \rho^*|\mathbf{c}_\perp(\zeta)|$, whilst, in the position vectors, $|\mathbf{r}_L(\zeta)| \simeq \rho^*|\mathbf{r}_{gc}|$. Furthermore, we know that the gyro phase varies on the "fast" time with a characteristic frequency, Ω_c. The latter may now depend on the *local* B, itself a function of the guiding centre position, $\mathbf{r}_{gc}$. Let us substitute this form into the Lagrangian. Consider the kinetic energy first:

$$\begin{aligned} L^T_{\rm nr} &= (M/2)\left[\frac{d\mathbf{r}_{gc}}{dt} + \mathbf{c}_\perp(\zeta)\right]^2 \\ &= (M/2)\left[\frac{d\mathbf{r}_{gc}}{dt} - r_L\dot{\zeta}(\mathbf{n}\cos(\zeta) + \mathbf{d}\sin(\zeta))\right]^2 \end{aligned}$$

We average this over a complete gyro period by applying the averaging operator, $<>\equiv \frac{1}{2\pi}\int_0^{2\pi} d\zeta$. The average over the first term is trivial since it is a constant, as far as the gyro phase is concerned. In the second term, r_L is similarly independent, as are the vectors $\mathbf{n}, \mathbf{d}$. Remember that $\mathbf{B}$ itself only varies on the slow scale! It then follows from the properties of trigonometric functions that,

$$\begin{aligned} < L^T_{\rm nr} > (\mathbf{r}_{gc}, \mathbf{v}_{gc}, \dot{\zeta}, t) &= \frac{1}{2\pi}\int_0^{2\pi} L^T_{\rm nr} d\zeta \\ &= \frac{M}{2}\left[(\frac{d\mathbf{r}_{gc}}{dt})^2 + (r_L\dot{\zeta})^2\right] \end{aligned}$$

The following derivation of Eq.(43) is not required material for this course and may be omitted on a first reading, although it only involves quite simple ideas and elementary algebra.

[So far so good! Now we must tackle the term, $L^B_{\rm nr} \equiv e\mathbf{A}.\mathbf{v}$ which involves the magnetic field. Observe that we can express this as, $e\mathbf{A}(\mathbf{r}_{gc} + \mathbf{r}_L(\zeta), t).(\frac{d\mathbf{r}_{gc}}{dt} + \mathbf{c}_\perp(\zeta))$. It is clear that we can expand the vector potential in a Taylor series about the guiding centre position. Thus we have,

$$e\mathbf{A}.\mathbf{v} = e\left[\mathbf{A}(\mathbf{r}_{gc}, t) + \mathbf{r}_L(\zeta).\nabla\mathbf{A}(\mathbf{r}_{gc}, t)\right].(\frac{d\mathbf{r}_{gc}}{dt} - r_L\dot{\zeta}\left[\mathbf{n}\cos(\zeta) + \mathbf{d}\sin(\zeta)\right])$$

Averaging the above expression over a gyro period gives,

$$\begin{aligned} < L^B_{\rm nr} > (\mathbf{r}_{gc}, \mathbf{v}_{gc}, \dot{\zeta}, t) &= e\mathbf{A}(\mathbf{r}_{gc}, t).\mathbf{v}_{gc} + \\ &\quad e r_L^2\dot{\zeta} < \left[\mathbf{n}\sin(\zeta) - \mathbf{d}\cos(\zeta)\right].\left[\mathbf{n}\cos(\zeta) + \mathbf{d}\sin(\zeta)\right] > .\nabla\mathbf{A}(\mathbf{r}_{gc}, t)) \\ &= e\mathbf{A}(\mathbf{r}_{gc}, t).\mathbf{v}_{gc} + (e r_L^2/2)\dot{\zeta}\left[\mathbf{nd} - \mathbf{dn}\right].\nabla\mathbf{A}(\mathbf{r}_{gc}, t)) \end{aligned}$$

Now choosing a locally cartesian coordinate system with $\mathbf{n} = \mathbf{x}, \mathbf{d} = \mathbf{y}, \mathbf{b} = \mathbf{z}$, and observing that $\mathbf{B} = \nabla \times \mathbf{A} = B\mathbf{b}$, it is easily seen that $\left[\mathbf{nd} - \mathbf{dn}\right].\nabla\mathbf{A}(\mathbf{r}_{gc}, t)) = B$.]

This identity enables us to write,

$$< L_{\rm nr}^B > (\mathbf{r}_{gc}, \mathbf{v}_{gc}, \dot{\zeta}, t) \quad = \quad e\mathbf{A}(\mathbf{r}_{gc}, t).\mathbf{v}_{gc} + (\frac{eBr_L^2}{2})\dot{\zeta}$$

Consequently, the "averaged" Lagrangian becomes,

$$< L_{\rm nr}^T + L_{\rm nr}^B > \quad = \quad \frac{M}{2}\left[(\frac{d\mathbf{r}_{gc}}{dt})^2 + (r_L\dot{\zeta})^2\right] + e\mathbf{A}(\mathbf{r}_{gc}, t).\mathbf{v}_{gc} + (\frac{eBr_L^2}{2})\dot{\zeta} \tag{43}$$

Note that it depends upon the "fast" variable, $\dot{\zeta}$ but *not* on ζ! Correspondingly, it is a function of r_L but not explicitly with respect to its slow rate of change. If we "vary" this Lagrangian with respect to r_L, we obtain,

$$\begin{aligned} \frac{\partial < L_{\rm nr} >}{\partial r_L} &= Mr_L\dot{\zeta}^2 + eBr_L\dot{\zeta} \\ &= 0 \end{aligned} \tag{44}$$

The solution of this, is of course, $\dot{\zeta} = -eB/M = -\Omega_c(\mathbf{r}_{gc}, t)$. Thus, for the averaged Lagrangian to be truly variational, the gyro phase must be an *integral* with respect to time of the slowly varying gyro frequency!

This is a characteristic result of Whitham theory (ie, the dependence of the fast periodic variable on the slowly varying frequency). Note also that the averaged Lagrangian is independent of the gyro phase (this should not be too surprising as, after all, we actually averaged it out!). This means that we can employ the theorem that the generalized "canonical momentum" conjugate to ζ must be a constant of the motion.

Thus, we have,

$$\begin{aligned} \frac{d}{dt}(\frac{\partial < L_{\rm nr} >}{\partial \dot{\zeta}}) &= \frac{d}{dt}(Mr_L^2\dot{\zeta} + \frac{eBr_L^2}{2}) \\ &= \frac{\partial < L_{\rm nr} >}{\partial \zeta} \\ &= 0 \end{aligned} \tag{45}$$

Now substituting for $\dot{\zeta}$, we see that, $\frac{d}{dt}(-eBr_L^2/2) = 0$. We have thus proved that the quantity defined by, $\mu = (1/2)Mc_\perp^2/B$ is a constant of the motion. This has several simple interpretations. The quantity, $Mr_L^2\Omega_c$ is the angular momentum of the gyromotion about the field line. This is clearly an invariant as the averaged Lagrangian is independent of the gyro phase and thus has rotational symmetry. The gyro period is $T_c = 2\pi/\Omega_c$. As the charge revolves, it creates a current equal to, $I_c = e/T = e\Omega_c/2\pi$. From electrodynamics, this current threading an area, πr_L^2 is equivalent to a magnetic moment, $I_c \pi r_L^2 = (1/2)(Mc_\perp^2/B) = \mu$. This also yields the relation, $\mu B = (1/2)(Mc_\perp^2) = (1/2)r_L^2 eB\Omega_c$. This result is of fundamental importance.

The derivation given here of Eqs.(48-50) is advanced material and is given here for completeness. However, the equations themselves and the terms appearing in them are very important and the student should be able to apply them.

[We shall next proceed to derive the magnetic drifts. Using Eq.(45), we see that, the term, $eBr_L^2/2 = p_\zeta - Mr_L^2\dot{\zeta}$, where p_ζ is a constant. It follows that we can eliminate $(eBr_L^2/2)\dot{\zeta}$ from the averaged Lagrangian (any total time derivative can be dropped from a Lagrangian!) to obtain the full (ie., including the electrostatic term) averaged Lagrangian in the form:

$$
\begin{aligned}
< L'_{\rm nr} > &= (1/2)M\mathbf{v}_{gc}^2 + e\mathbf{A}(\mathbf{r}_{gc}, t).\mathbf{v}_{gc} - \mu B(\mathbf{r}_{gc}, t) - e\Phi(\mathbf{r}_{gc}, t) && (46) \\
\frac{d\mathbf{r}_{gc}}{dt} &= \mathbf{v}_{gc} && (47)
\end{aligned}
$$

Varying this Lagrangian with respect to $\mathbf{r}_{gc}, \dot{\mathbf{r}}_{gc}$ leads to,

$$
\frac{d}{dt}\left[\frac{\partial < L'_{\rm nr} >}{\partial \mathbf{v}_{gc}}\right] = \left[\frac{\partial < L'_{\rm nr} >}{\partial \mathbf{r}_{gc}}\right]
$$

It is convenient to write, $\mathbf{v}_{gc} = v_\parallel(t)\mathbf{b} + \mathbf{v}_{gc}^\perp$ and note the identity, $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}_{gc}.\nabla$. The equation becomes],

$$
M\frac{dv_\parallel}{dt}\mathbf{b} = -Mv_\parallel(\frac{\partial \mathbf{b}}{\partial t} + \mathbf{v}_{gc}.\nabla\mathbf{b}) - e\nabla\Phi - \mu\nabla B - e(\frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}_{gc}.\nabla\mathbf{A}) + e\nabla(\mathbf{A}.\mathbf{v}_{gc}) - M\frac{d\mathbf{v}_{gc}^\perp}{dt}
$$

Since $\mathbf{r}_{gc}, \mathbf{v}_{gc}$ are to be independently varied, we have,

$$
\begin{aligned}
e\nabla(\mathbf{v}_{gc}.\mathbf{A}) &= e\mathbf{v}_{gc}.\nabla\mathbf{A} + e\mathbf{v}_{gc} \times (\nabla \times \mathbf{A}) \\
&= e\mathbf{v}_{gc}.\nabla\mathbf{A} + e\mathbf{v}_{gc} \times \mathbf{B}
\end{aligned}
$$

Using the facts that $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi$ and $\mathbf{b}.\mathbf{b} = 1$ so that, $\mathbf{b}.\frac{d\mathbf{b}}{dt} = 0$, we may take the parallel component and obtain,

$$
M\frac{dv_\parallel}{dt} = eE_\parallel - \mu\nabla_\parallel B \qquad (48)
$$

[The perpendicular component becomes, upon re-arranging slightly,

$$\begin{aligned} e\left[\mathbf{E} + \mathbf{v}_{gc}^{\perp} \times \mathbf{B}\right] &= \mu \nabla_{\perp} B + M \frac{d\mathbf{v}_{gc}^{\perp}}{dt} + M v_{||} \left(\frac{\partial \mathbf{b}}{\partial t} + \mathbf{v}_{gc} . \nabla \mathbf{b}\right) \\ &= \mu \nabla_{\perp} B + M v_{||}^2 \mathbf{b} . \nabla \mathbf{b} \end{aligned}$$

The remaining terms are one order smaller in ρ^* (it is a good exercise to show this explicitly!) with respect to the terms retained.]

It is useful to introduce the geometrical "curvature vector" of the field line. It is defined by,

$$\begin{aligned} \mathbf{k} &= \mathbf{b}.\nabla \mathbf{b} \\ &= (\nabla \times \mathbf{b}) \times \mathbf{b} \end{aligned} \tag{49}$$

We are now able to express $\mathbf{v}_{gc}^{\perp}$:

$$\mathbf{v}_{gc}^{\perp} = \left(\frac{\mathbf{E}}{B} \times \mathbf{b}\right) + \left(\frac{\mu}{e}\right)(\mathbf{b} \times \nabla \ln B) + \left(\frac{v_{||}^2}{\Omega_c}\right)(\mathbf{b} \times \mathbf{k}) \tag{50}$$

This completes the derivation of the "drifts" experienced by the charged particle.

In summary, we find that, $\mu = (1/2)Mc_{\perp}^2/B$ is an adiabatic invariant; the parallel speed, $v_{||}(t)$ varies according to Eq.(48) and $\mathbf{v}_{gc}^{\perp}$ is given by Eq.(50). The first term on the RHS is the "electric drift", the second is called the "grad-B" drift (by convention) and the last term is the so-called, "curvature drift", sometimes also known as the "centrifugal drift". There are smaller, "inertial" drifts which provide next order corrections. In general, the drift equations must be integrated numerically, but have many fascinating properties.

The most important deduction we will make is the concept of particle "trapping" contained in Eq.(48). If Φ does not vary with time, we obtain the Energy Conservation Law, $E = (1/2)Mv_{||}^2 + \mu B + e\Phi$ is a constant along each field line. Now suppose that $\Phi \equiv 0$ and we have an inhomogeneous field. As the particle moves from a region of low to high field, since μ is a constant fixed by its initial conditions, the μB-perpendicular energy term rises. This must mean that the parallel energy and $v_{||}$ must decrease. The term responsible for this effect is an averaged form of the Lorentz force and is called the "mirror force", since a suitably shaped, inhomogeneous magnetic field can act like a magnetic mirror and confine particles. This happens for instance to charged particles in the Earth's dipole field which increases strongly towards the magnetic poles. If the to a potential force field K are:

$$
\begin{aligned}
\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla.(\rho_{\mathrm{m}}\mathbf{u}) &= 0 &(1)\\
\frac{\partial \rho_{\mathrm{m}}\mathbf{u}}{\partial t} + \nabla.(\rho_{\mathrm{m}}\mathbf{u}\mathbf{u}) &= -\nabla p + \rho_{\mathrm{m}}\nabla K &(2)\\
p/p^{*} &= (\rho_{\mathrm{m}}/\rho_{\mathrm{m}}^{*})^{\gamma} &(3)
\end{aligned}
$$

There are many possibilities for solving such equations, although in most cases, a numerical approach is required.

Let us first consider a neutral gas filling a region Δ. We assume that the walls of the domain are "impermeable" and apply the condition $\mathbf{u}.\mathbf{n} = 0$, where $\mathbf{n}(S)$ is the unit outer normal to the wall (S) at each point of it. By integrating Eq.(1) over Δ, we see that the total mass of the gas, $M_{\Delta} = \int_{\Delta} \rho_{\mathrm{m}} dV$ is a constant of the motion. This should not be too surprising, as we derived this differential equation from the Law of Conservation of Mass! Let us now take the dot product of the momentum balance equation with $\mathbf{u}$ and integrate the resulting scalar equation over Δ:

$$
\int_{\Delta} dV\mathbf{u}.\left[\frac{\partial \rho_{\mathrm{m}}\mathbf{u}}{\partial t} + \nabla.(\rho_{\mathrm{m}}\mathbf{u}\mathbf{u})\right] = \int_{\Delta} dV\mathbf{u}.\left[-\nabla p + \rho_{\mathrm{m}}\nabla K\right]
$$

Making use of the equation of continuity (and Gauss' divergence theorem), we see that,

$$
\begin{aligned}
\int_{\Delta} dV\mathbf{u}.\left[\frac{\partial \rho_{\mathrm{m}}\mathbf{u}}{\partial t} + \nabla.(\rho_{\mathrm{m}}\mathbf{u}\mathbf{u})\right] &= \int_{\Delta} dV(1/2)\left[\rho_{m}\frac{\partial \mathbf{u}^2}{\partial t} + (\rho_{\mathrm{m}}\mathbf{u}.\nabla(\mathbf{u}^2)\right]\\
&= \int_{\Delta} dV(1/2)\left[\frac{\partial \rho_{\mathrm{m}}\mathbf{u}^2}{\partial t} + \nabla.(\rho_{\mathrm{m}}\mathbf{u}\mathbf{u}^2))\right]\\
&= \frac{d}{dt}\int_{\Delta} dV\frac{\rho_{m}\mathbf{u}^2}{2} &(4)
\end{aligned}
$$

In a similar fashion, we can transform the RHS to write,

$$
\int_{\Delta} dV\mathbf{u}.\left[-\nabla p + \rho_{\mathrm{m}}\nabla V\right] = \int_{\Delta} dV\left[p\nabla.\mathbf{u} - K\nabla.(\rho_{\mathrm{m}}\mathbf{u})\right]
$$

From Eq.(1,3), we obtain the identities,

$$
\begin{aligned}
\frac{\partial p}{\partial t} + \mathbf{u}.\nabla p &= -\gamma p\nabla.\mathbf{u}\\
\frac{\partial p}{\partial t} + \nabla.(\mathbf{u}p) &= -(\gamma - 1)p\nabla.\mathbf{u}
\end{aligned}
$$

It follows that (we assume of course that the external potential, K depends explicitly only on position and not on t):

$$\begin{aligned} \int_\Delta dV\,[p\nabla.\mathbf{u}] &= -\frac{d}{dt}\int_\Delta (\frac{p}{\gamma-1})dV \\ \int_\Delta dV\,[-K\nabla.(\rho_\mathrm{m}\mathbf{u})] &= \frac{d}{dt}\int_\Delta dV\rho_\mathrm{m}K \end{aligned}$$

Putting all these transformations together, we see that the **total energy**, E_Δ defined by,

$$E_\Delta = \int_\Delta dV\left[(\frac{\rho_m\mathbf{u}^2}{2}) + (\frac{p}{\gamma-1}) - (\rho_\mathrm{m}K)\right] \tag{5}$$

is a constant of the motion. The first term represents the fluid kinetic energy (per unit volume), the second, the internal energy (of this isentropic fluid) and the last is the potential energy in the external field. It is of course the case that the entropy of the fluid $\sigma \propto \ln(p\rho_\mathrm{m}^{-\gamma})$ is also constant during the motion. This last statement applies only so long as the motion is continuous. There really is no guarantee that this should be so for all times since "shocks" can form in the system!

It is useful to introduce some conventional terminology here. Relations like the equation of continuity tell us that the "local" time rate of something (here ρ_m is expressed as the divergence of some "flux" ,$-\rho_\mathrm{m}\mathbf{u}$, which, in this case is the mass flux). Such relations are called **local conservation laws**. When such relations are integrated over a domain, they lead, in general to corresponding **global conservation laws**.

We can obtain the local conservation law of energy as follows. It is useful, first to state a very important and useful vector identity and its immediate corollary:

$$\nabla(\mathbf{A}.\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}.\nabla\mathbf{B} + \mathbf{B}.\nabla\mathbf{A} \tag{6}$$

$$(\frac{1}{2})\nabla(\mathbf{A}.\mathbf{A}) = \mathbf{A}\times(\nabla\times\mathbf{A}) + \mathbf{A}.\nabla\mathbf{A} \tag{7}$$

Writing the equation of motion in the form,

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} = (-\frac{1}{\rho_\mathrm{m}})\nabla p + \nabla K$$

$$\frac{\partial\mathbf{u}}{\partial t} + (\nabla\times\mathbf{u})\times\mathbf{u} = -\nabla\left[\frac{\mathbf{u}^2}{2} + \frac{\gamma}{\gamma-1}(\frac{p}{\rho_\mathrm{m}}) - K\right] \tag{8}$$

we are in a position to make two important deductions. First, take the curl of the last equation. Noticing that the curl of a gradient is zero and introducing the new vector, $\mathbf{W} \equiv \nabla \times \mathbf{u}$, we obtain the remarkable equation,

$$\frac{\partial \mathbf{W}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{W}) \tag{9}$$

This is *exactly* the same equation satisfied by **B** in ideal MHD (see lecture 1)! However, here the vector, **W** is the curl of the fluid velocity, **u** whilst **B** is not simply related to **u** or its derivatives. We will return to this result later. Taking the dot product of Eq.(8) with **u**, we obtain the equation:

$$\frac{\partial}{\partial t}(\frac{\mathbf{u}^2}{2}) = -\mathbf{u}.\nabla\left[\frac{\mathbf{u}^2}{2} + \frac{\gamma}{\gamma-1}(\frac{p}{\rho_{\rm m}}) - K\right] \tag{10}$$

Multiplying this equation by $\rho_{\rm m}$ and making use of the equations of continuity and isentropy, we obtain the "local" energy conservation law,

$$\frac{\partial}{\partial t}\left[(\frac{\rho_{\rm m}\mathbf{u}^2}{2}) + (\frac{p}{\gamma-1}) - (\rho_{\rm m}K)\right] = -\nabla.\left[\mathbf{u}(\frac{\rho_{\rm m}\mathbf{u}^2}{2} + \frac{\gamma}{\gamma-1}p - K\rho_{\rm m})\right] \tag{11}$$

Let us return to Eq.(9) and expand the RHS using the vector identity,

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla.\mathbf{B} - \mathbf{B}\nabla.\mathbf{A} + \mathbf{B}.\nabla\mathbf{A} - \mathbf{B}.\nabla\mathbf{A} \tag{12}$$

Since, $\nabla.\mathbf{W} = 0$, by definition, we obtain,

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{u}.\nabla\mathbf{W} = -\mathbf{W}\nabla.\mathbf{u} + \mathbf{W}.\nabla\mathbf{u}$$

The equation of continuity and some simple algebra shows that the **potential vorticity vector**, $\mathbf{W}^* \equiv \frac{\mathbf{W}}{\rho_{\rm m}}$, satisfies the equation,

$$\frac{\partial \mathbf{W}^*}{\partial t} + \mathbf{u}.\nabla\mathbf{W}^* = \mathbf{W}^*.\nabla\mathbf{u} \tag{13}$$

This is sometimes called **Helmholtz**' equation for the potential vorticity. Its meaning is not yet apparent, but will shortly be illustrated. Vector fields like $\mathbf{W}^*$ which satisfy this equation will be termed **Helmholtz fields.**

III.2 Kinematics of fluid flows: Eulerian and Lagrangian pictures

In order to understand the vectors, $\mathbf{W}, \mathbf{W}^*$ better, let us look a bit more closely at the *kinematics* of the flow. The fluid velocity, $\mathbf{u}(\mathbf{r}, t)$ represents the velocity of the fluid at a particular position $\mathbf{r}$, at time t. It is called the *Eulerian* velocity. Alternatively one could imagine a very light "marker" in the fluid which is "carried along" with the flow (like a small leaf or twig in a flowing stream) and measure its velocity. This motion is called "advection". Time rates of change of quantities can be of two kinds: we could envisage a function $f(\mathbf{r}, t)$ changing *at a particlular location.* This is what we mean by the *local time derivative*, $\left[\frac{\partial f}{\partial t}\right]_{\mathbf{r}}$, showing explicitly that we hold $\mathbf{r}$ *fixed* when evaluating this rate of change. Consider what happens when we are "advected" with the flow and look at the rate of change of f. Obviously, f changes with time *and also* with the change in $\mathbf{r}$ as the fluid moves about. What is kept fixed here is the *initial position* of the fluid particle. It is easily seen that this "advective time derivative" must be given by the formula,

$$\begin{aligned} \left[\frac{d}{dt}\right]_{\text{ad}} f &= \frac{Df}{Dt} \\ &= \frac{\partial f}{\partial t} + \mathbf{u}.\nabla f \end{aligned} \tag{14}$$

Plainly, we have the obvious rule, $\frac{D\mathbf{R}}{Dt} = \mathbf{u}(\mathbf{R}, t)$, where the $\mathbf{R}$ on the left refers to the position vector, $\mathbf{R}$ of a "fluid particle" or marker which happens to be at time t at the point where the Eulerian fluid velocity is $\mathbf{u}$! What this means is the following. If we imagine we *know* $\mathbf{u}$ for all $\mathbf{r}, t$, we may solve the first order ode's,

$$\frac{d\mathbf{R}}{dt} = \mathbf{u}(\mathbf{R}, t)$$

with the *initial* conditions, $\mathbf{R} = \mathbf{r}_0$ at $t = 0$. The solutions, are of course functions of t as well as the initial data, $\mathbf{r}_0$. We may thus write, $\mathbf{R} = \mathbf{R}(\mathbf{r}_0, t)$. Thus the markers placed initially at $\mathbf{r}_0$ faithfully "track" the fluid motions as time progresses. Consider now a function, $F(\mathbf{r})$ of position. At $t = 0$, it has the value, $F(\mathbf{r}_0)$ at any initial point. At a later time t, as the fluid moves, it changes to $F(\mathbf{R}(\mathbf{r}_0, t))$. Its time rate of change following the fluid is obviously, $\frac{DF}{Dt} = \frac{d\mathbf{R}}{dt}.\nabla F = \mathbf{u}.\nabla F$. If F varies with t explicitly as well, we obtain the general formula, Eq.(14). The coordinates $\mathbf{r}_0$ are called "Lagrangian positions" and time derivatives of functions holding these "initial" positions constant (ie., moving with the fluid) are called "Lagrangian derivatives". They basically tell us that a particle at $\mathbf{R}$ at time t "came from" $\mathbf{r}_0$ at the initial instant.

A surface S composed of fluid particles is called a "material surface", and moves with the fluid. If it has the equation, $F(\mathbf{r}_0) = const$, at the initial instant, evidently, at time t, it is represented by, by the equation, $F(\mathbf{R}(\mathbf{r}_0, t)) = const$. As the particles composing the surface move with the fluid, we must have, $\left[\frac{\partial F}{\partial t}\right]_{\mathbf{r}_0} = 0$. Hence, the Eulerian condition for a material surface is,

$$\begin{aligned}\left[\frac{\partial F}{\partial t}\right]_{\mathbf{r}_0} &= \frac{DF}{Dt} \\ &= \frac{\partial F}{\partial t} + \mathbf{u}.\nabla F \\ &= 0\end{aligned}$$

We are now going to prove an important formula known as Reynolds' Transport Theorem. Let $\Delta(t)$ now represent a "material volume" in a fluid bounded by a material surface S composed of fluid markers in the above sense. As the fluid moves, Δ changes with time, but its boundary is always a material surface. Let $G(\mathbf{r}, t)$ be an arbitrary function of position. Let us consider the integral,

$$I_{\Delta(t)}(G)(t) = \int_{\Delta(t)} G(\mathbf{r}, t)dV \tag{15}$$

Let us remark that at $t = 0$, $\mathbf{R}(\mathbf{r}_0, 0) = \mathbf{r}_0$, by definition. We are therefore at liberty to change variables to the *initial coordinates*, $\mathbf{r}_0$! Hence we may write,

$$\begin{aligned}I_{\Delta(t)}(G, t) &= \int_{\Delta(t)} G(\mathbf{r}, t)dV \\ &= \int_{\Delta(0)} G(\mathbf{R}(\mathbf{r}_0, t), t)\frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)}dV_0\end{aligned}$$

Now consider the time derivative, $\frac{dI_{\Delta(t)}}{dt}$. We observe that the "initial"or Lagrangian coordinates $\mathbf{r}_0$, and material surfaces are *independent* of t. We can thus "commute" the time-derivative with the integral and obtain (setting, $J \equiv \frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}$ for the Jacobian relating the Eulerian and the Lagrangian coordinates),

$$\begin{aligned}\frac{dI_{\Delta(t)}}{dt} &= \int_{\Delta(0)} \left[\frac{\partial GJ}{\partial t}\right]_0 dV_0 \\ &= \int_{\Delta(0)} \left[\frac{DG}{Dt}JdV_0 + G\frac{DJ}{Dt}dV_0\right] \\ &= \int_{\Delta(t)} \frac{DG}{Dt}dV + \int_{\Delta(t)} (\frac{1}{J}\frac{DJ}{Dt})GdV\end{aligned} \tag{16}$$

We now do a simple but extremely illuminating calculation which relates Lagrangian and Eulerian derivatives (here the Lagrangian time derivative is denoted by a dot for brevity):

$$\begin{aligned}\frac{DJ}{Dt} &= \left[\frac{\partial}{\partial t}\right]_{\mathbf{r}_0} \frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)} \\ \dot{J} &= \frac{\partial(\dot{x},y,z)}{\partial(x_0,y_0,z_0)} + \frac{\partial(x,\dot{y},z)}{\partial(x_0,y_0,z_0)} + \frac{\partial(x,y,\dot{z})}{\partial(x_0,y_0,z_0)} \qquad (17)\end{aligned}$$

Dividing this equation by J and using the well-known properties of Jacobian determinants, we get the identity,

$$\begin{aligned}\frac{1}{J}\frac{DJ}{Dt} &= \left[(\frac{\partial \dot{x}}{\partial x}) + (\frac{\partial \dot{y}}{\partial y}) + (\frac{\partial \dot{z}}{\partial z})\right] \\ &= \nabla.\mathbf{u} \qquad (18)\end{aligned}$$

remembering the relation, $\dot{\mathbf{r}} = \mathbf{u}$.

Substituting in Eq.(16), we obtain the final form of **Reynolds' Transport Theorem**:

$$\frac{dI_{\Delta(t)}}{dt} = \int_{\Delta(t)} \frac{DG}{Dt} dV + \int_{\Delta(t)} G\nabla.\mathbf{u}dV \qquad (19)$$

Comparing Eq.(18) with the equation of continuity in the form,

$$\frac{1}{\rho_{\mathrm{m}}}\frac{D\rho_{\mathrm{m}}}{Dt} = -\nabla.\mathbf{u} \qquad (20)$$

we can "integrate" the Lagrangian continuity equation thus:

$$\rho_{\mathrm{m}}(\mathbf{r},t)J(\mathbf{r},\mathbf{r}_0) = \rho_{\mathrm{m}}(\mathbf{r}_0) \qquad (21)$$

The equations of motion in the Lagrangian coordinates take the form,

$$\rho_{\mathrm{m}}\frac{D^2\mathbf{r}}{Dt^2} = -\frac{\partial p}{\partial \mathbf{r}} + \rho_{\mathrm{m}}\frac{\partial K}{\partial \mathbf{r}} \qquad (22)$$

These are very complicated despite an appearance of deceptive simplicity and resemblance to the equations of particle dynamics. This is because ρ_{m} and p are very complicated functions of the $\mathbf{r}$ via the continuity equation and the equation of state. This dependence is part of the solution and is not known beforehand! Lagrangian coordinates do however, enable us to "integrate" the equations of potential vorticity (first done by Cauchy). The derivation given here is quick but somewhat artificial; it effectively involves verifying an ansatz.

Cauchy's Transport Theorem

Let $\mathbf{Z}_0 = (Z_0^i)$ $(i = 1, 2, 3)$ be an arbitrary vector function of $\mathbf{r}_0 = (r_0^i)$. Consider the vector defined by,

$$\begin{aligned} \mathbf{Z} &= \mathbf{Z}_0 . \frac{\partial \mathbf{r}}{\partial \mathbf{r}_0} \\ Z^i &= Z_0^k \frac{\partial r^i}{\partial r_0^k} \end{aligned} \tag{23}$$

using the standard "summation convention" . Note that $\mathbf{Z}$ depends *linearly* on $\mathbf{Z}_0(\mathbf{r}_0)$, *and* on t via $\mathbf{r}(\mathbf{r}_0, t)$. We see that $\mathbf{Z}$ in fact, satisfies the differential equations,

$$\begin{aligned} \frac{D\mathbf{Z}}{Dt} &= \frac{\partial \mathbf{Z}}{\partial t} + \mathbf{u}.\nabla \mathbf{Z} \\ &= \mathbf{Z}.\nabla \mathbf{u} \end{aligned} \tag{24}$$

N.B. The proof of Cauchy's Transport Theorem is included for completeness. It may be omitted on a first reading. Of course, the Theorem itself is important and illustrates the concept of "material derivative".

[**Proof**:

We differentiate Eq.(23) with respect to t, keeping $\mathbf{r}_0$ fixed (ie., form the Lagrangian derivative) and obtain,

$$\dot{Z}^i = Z_0^k \frac{\partial \dot{r}^i}{\partial r_0^k} \tag{25}$$

We also have the "inverse" formula,

$$Z_0^i = Z^q \frac{\partial r_0^i}{\partial r^q} \tag{26}$$

Substituting in Eq.(24), we find,

$$\begin{aligned} \dot{Z}^i &= Z^q \frac{\partial r_0^k}{\partial r^q} \frac{\partial \dot{r}^i}{\partial r_0^k} \\ &= Z^q \frac{\partial \dot{r}^i}{\partial r^q} \\ &= Z^q \frac{\partial u^i}{\partial r^q} \\ \dot{\mathbf{Z}} &= \mathbf{Z}. \frac{\partial \mathbf{u}}{\partial \mathbf{r}} \\ \frac{D\mathbf{Z}}{Dt} &= \mathbf{Z}.\nabla \mathbf{u} \end{aligned}$$

This completes the proof.]

An immediate application is to Eq.(13) for the potential vorticity vector, $\mathbf{W}^*$. We see that Cauchy's Transport Theorem enables us to "integrate" this equation and write, $\mathbf{W}^* = \mathbf{W}_0^*.\frac{\partial \mathbf{r}}{\partial \mathbf{r}_0}$. Hence, if the *initial* potential vorticity distribution, $\mathbf{W}_0^*$ is *known* and the transformation matrix $\frac{\partial \mathbf{r}}{\partial \mathbf{r}_0}$ is calculated, the current distribution of vorticity is predicted! Note the resemblance of this formula to that for $\sigma_m = \rho_m^{-1}$: viz., $(\frac{\sigma_m}{\sigma_m(0)}) = J = |\frac{\partial \mathbf{r}}{\partial \mathbf{r}_0}|$.

A profound and interesting corollary of Cauchy's integral is **Lagrange's Theorem on the permanence of vorticity**. Suppose we have an ideal fluid and we *start off the motion* with zero vorticity. What can we say about the generation of vorticity? Since $\mathbf{W}_0^* \equiv 0$, Cauchy's transport theorem says, "it must stay zero for all time!". Lagranges's theorem says that if at some instant, the entire flow is **vorticity free** or **irrotational**, it must be so for *all times*. However, if at $t = 0$ there is a tiny amount of vorticity, it can certainly be amplified enormously by ideal fluid motions. Vorticity plays a fundamental role in fluid flows, especially in turbulent flows and in the presence of rotation and magnetic fields. We shall simply touch upon some of the key principles involved.

There is another, interesting interpretation to Cauchy's Transport Theorem. This is called the "frozen-in" field concept. To understand this, consider two neighbouring "initial points", $\mathbf{r}_0, \mathbf{r}'_0$, with $\delta\mathbf{r}_0 \equiv \mathbf{r}_0 - \mathbf{r}'_0$. We know that the ideal motion at some later time t takes these points to, $\mathbf{r}(\mathbf{r}_0, t)$ and $\mathbf{r}(\mathbf{r}'_0, t)$ respectively. It is plain that if $\delta\mathbf{r}_0$ is an "infinitesimal" displacement, at least for short times, we must have,

$$\begin{aligned} \mathbf{r}(\mathbf{r}_0, t) - \mathbf{r}(\mathbf{r}'_0, t) &= \delta\mathbf{r} \\ &= \delta\mathbf{r}_0.\frac{\partial \mathbf{r}}{\partial \mathbf{r}_0} \end{aligned} \tag{27}$$

The equation is exactly that satisfied by the potential vorticity. This means that the fluid particles marking a "vortex line" advect it along during their motion. Thus a vortex line is said to be "frozen-in" to the motion of an ideal fluid. Since, in ideal MHD, the $\mathbf{B}$ field satisfies the *same* kinematic equation as vorticity, we see that *magnetic field lines must also be frozen into the fluid*! Note however that in ideal MHD, vorticity *does not* satisfy the frozen-in condition due to the presence of Lorentz force in the equation of motion.

Lord Kelvin established a very important property of vorticity. This is contained in the following **circulation theorem**.

Kelvin's Circulation Theorem:

Let L be a closed "material contour" in an ideal fluid. The integral, $K_L = \int_L \mathbf{u}.d\mathbf{l}$ is a constant of the motion.

Proof: As in Reynolds' transport theorem, we refer the integral to Lagrangian coordinates, $\mathbf{r}_0$. Then,

$$\begin{aligned} K_L &= \int_L u_i dx^i \\ &= \int_{L_0} u_i \frac{\partial x^i}{\partial x_0^k} dx_0^k \end{aligned} \tag{28}$$

We can now safely differentiate with respect to time under the integral sign, since the Lagrangian coordinates are independent of time.

$$\begin{aligned} \frac{dK_L}{dt} &= \int_{L_0} \frac{Du_i}{Dt}\frac{\partial x^i}{\partial x_0^k} dx_0^k + u_i \frac{\partial u_i}{\partial x_0^k} dx_0^k \\ &= \int_{L_0} \left(-\frac{1}{\rho_{\mathrm{m}}}\frac{\partial p}{\partial x^i} + \frac{\partial K}{\partial x^i}\right)\frac{\partial x^i}{\partial x_0^k} dx_0^k + \frac{\partial}{\partial x_0^k}(\mathbf{u}^2/2) dx_0^k \\ &= 0 \end{aligned} \tag{29}$$

The last follows from the fact that p is a function of ρ_{m}; hence all three terms on the RHS are seen to be *perfect differentials,* which when integrated round a closed contour L give zero. Thus Kelvin's theorem is established.

The integral, K_L is called the "circulation" of the flow taken around the material contour, L. It is obviously equal to, $\int_S(\nabla \times \mathbf{u}).d\mathbf{S}$. However much closed material contours may get entangled, they must keep the same circulations. This "permanence" of vorticity has many applications in fluid mechanics, condensed matter physics and elsewhere. Note, however that the theorem is not valid when the pressure is *not* a function of density or the external field is not derivable from a potential. Non isentropic processes can, and do, create/destroy vorticity.

III.3 Special cases of ideal flows

Let us consider some special solutions of the equations we have developed. An important class of flows are *steady* flows in which the external forces and the boundary conditions do no depend upon time and the flow variables depend only upon position. The equations simplify considerably and admit some interesting types of solutions.

Consider the steady equation of momentum for example:

$$(\nabla \times \mathbf{u}) \times \mathbf{u} \;=\; -\nabla \left[\frac{\mathbf{u}^2}{2} + \frac{\gamma}{\gamma - 1} (\frac{p}{\rho_\mathrm{m}}) - K \right] \tag{30}$$

Taking the dot product with $\mathbf{u}$, we derive the result that the quantity, $H = \frac{\mathbf{u}^2}{2} + \frac{\gamma}{\gamma-1}(\frac{p}{\rho_\mathrm{m}}) - K$ is constant on each "streamline", defined by the equations, $\frac{d\mathbf{x}}{ds} = \mathbf{u}/|\mathbf{u}|$. This is called **Bernoulli's principle for steady flows.** It says roughly that as one moves along a stream line, the flow is fastest where the pressure is least and vice versa. It is at the bottom of why aircraft actually fly!

A less well-known result is that H is also constant along "vortex lines" defined by, the equations, $\frac{d\mathbf{x}}{ds} = \mathbf{W}/|\mathbf{W}|$. Vortex lines and stream lines become parallel for special flows called "Beltrami flows". Otherwise they do not align. In these general cases, we see that H must actually be constant on surfaces spanned by the vortex and stream lines (ie., the surface normal, $\mathbf{n}$ at each point of the surface satisfies, $\mathbf{n}.\mathbf{W} \times \mathbf{u} = 0$).

We can look upon the Bernoulli relation in the following way: it is, in fact *necessary and sufficient* for Eq.(30) to be solved for $\mathbf{W}$ consistently, given the RHS. Thus it is a *solubility condition*! The solution gives $\mathbf{W}_\perp$. The parallel vorticity must be obtained by imposing the condition, $\nabla.\mathbf{W} = 0$. Furthermore, there is a powerful mathematical analogy between the Bernoulli relation and the equilibrium MHD momentum equation! This can be used in all sorts of problems by employing the famous **Feynman Principle**: " same equations have same solutions!"

Three dimensional exact solutions to the Euler equations, even under steady flow conditions are very hard to construct. However, a very simple exact solution is the state of rest! Thus we have, $\mathbf{u} \equiv 0$. The pressure, density and the external potential K must satisfy the laws of hydrostatics:

$$\begin{aligned} \frac{\nabla p}{\rho_\mathrm{m}} &= \nabla K \\ \frac{\gamma}{\gamma - 1}(\frac{p}{\rho_\mathrm{m}}) &= K + const. \end{aligned}$$

In the absence of an external force field, a simple, thermodynamic equilibrium with $\rho_\mathrm{m} = \rho_0, p = p_0$, constant, uniform values is a valid solution. Obviously, the state of uniform motion must also be a solution! While this seems to be a pretty uninteresting solution of "fluid flow", we can find quite exciting solutions "close" to these boring solutions. Such "neighbouring solutions" can be obtained by a very general and powerful procedure known as "linearization" which has many applications elsewhere, but is best illustrated here in deriving **sound waves** in an ideal fluid.

Let us represent $\rho_{\rm m} = \rho_0 + \tilde{\rho}, p = p_0 + \tilde{p}, \mathbf{u} = \tilde{\mathbf{u}}$, where the tilde quantities are "small". This clearly means, $\frac{\tilde{\rho}}{\rho_0} \ll 1, \frac{\tilde{p}}{p_0} \ll 1$. Exactly how small $\tilde{\mathbf{u}}$ is supposed to be will become plain in a moment. Next we substitute into the continuity equation:

$$\begin{aligned} \frac{\partial \tilde{\rho}}{\partial t} + \nabla.(\rho_0 \tilde{\mathbf{u}}) &= -\nabla.(\tilde{\rho}\tilde{\mathbf{u}}) \\ \frac{\partial}{\partial t}(\frac{\tilde{\rho}}{\rho_0}) + \nabla.\tilde{\mathbf{u}} &= -\nabla.\left[(\frac{\tilde{\rho}}{\rho_0})\tilde{\mathbf{u}}\right] \end{aligned} \tag{31}$$

Note that the term on the RHS is of "second order" of smallness, as can be seen by inspection. Linearization means "neglect all terms of second and higher order of smallness". This equation therefore relates the rate of change of the density fluctuation, $\frac{\tilde{\rho}}{\rho_0}$ to the divergence of the velocity perturbation. Next we consider the equation of momentum balance:

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = -(\frac{1}{\rho_0})\nabla \tilde{p} - \tilde{\mathbf{u}}.\nabla \tilde{\mathbf{u}} \tag{32}$$

The isentropic equation of state is, $(\frac{p}{p_0}) = (\frac{\rho_{\rm m}}{\rho_0})^\gamma$. This is immediately linearized to give,

$$\frac{\tilde{p}}{p_0} = \gamma \frac{\tilde{\rho}}{\rho_0} \tag{33}$$

Substitution in the momentum equation gives, after dropping the second order (ie., nonlinear) term,

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = -(\frac{\gamma p_0}{\rho_0})\nabla(\frac{\tilde{\rho}}{\rho_0}) \tag{34}$$

Denoting the relative density, $\frac{\tilde{\rho}}{\rho_0} = \sigma$, and eliminating $\tilde{\mathbf{u}}$ in the linearized continuity equation, we obtain **D'Alembert's** famous linear Wave Equation governing the evolution of σ:

$$\frac{\partial^2 \sigma}{\partial t^2} = (\frac{\gamma p_0}{\rho_0})\nabla^2 \sigma \tag{35}$$

The positive quantity, $(\frac{\gamma p_0}{\rho_0})$ is a characteristic property of the equilibrium state and has the dimensions of velocity squared. From the theory of the wave equation, it is easily demonstrated that the speed, C_s defined by, $C_s^2 = \frac{\gamma p_0}{\rho_0}$, is the **propagation**

velocity of the sound waves. It is clear from kinetic theory of the perfect gas that C_s is of the same order as the **thermal speed** of the molecules of the gas. Note that the momentum equation says that the velocity fluctuations, $\tilde{\mathbf{u}}$ associated with a sound wave are *irrotational*: ie., have zero vorticity. We can also see that the linearization, to be valid, requires the velocity fluctuations associated with the sound wave to be small compared with C_s. To demonstrate this fact, we construct the general solution to the wave equation in unbounded domains.

Let us consider a **harmonic** wave, of the form, $\sigma \simeq \exp(i\mathbf{k}.\mathbf{x} - i\omega t)$, where the vector $\mathbf{k}$ (the "wave vector") and the frequency ω are constant parameters, to be chosen to satisfy the equation. It is reasonable to consider imaginary exponentials, since we may always consider real parts afterwards. The equation is linear, so this causes no trouble!

When we substitute this harmonic form in the equation, we find that $\mathbf{k}, \omega$ cannot be arbitrary, but must satisfy the **dispersion relation**,

$$\omega^2 = C_s^2 \mathbf{k}^2 \tag{36}$$

but the amplitude can be arbitrary. Thus, for each $\mathbf{k}$ two possible values of ω can be found to satisfy the dispersion relation. These correspond to outgoing and incoming sound waves. Since the equation is *linear*, the **Principle of Superposition** applies, and we may write the general solution as an integral,

$$\sigma(\mathbf{x}, t) = \int \left[\hat{\Sigma_+}(\mathbf{k}) \exp(i\mathbf{k}.\mathbf{x} - i\omega(\mathbf{k})t) + \hat{\Sigma_-}(\mathbf{k}) \exp(i\mathbf{k}.\mathbf{x} + i\omega(\mathbf{k})t)\right] \frac{d\mathbf{k}}{(2\pi)^3} \tag{37}$$

The theory of Fourier integrals tells us that the functions, $\hat{\Sigma_+}, \hat{\Sigma_-}(\mathbf{k})$ are uniquely determined from the initial conditions set on σ, $\frac{\partial \sigma}{\partial t}$ at the initial instant, $t = 0$. The solution also states that $\tilde{\mathbf{u}} \simeq C_s \sigma$ in order of magnitude. Thus the velocity fluctuations associated with a sound wave are "small" compared to the speed of sound.

We have seen that flows without vorticity occur naturally in sound waves. Let us take a moment to consider *general* flows with zero vorticity everywhere. Such flows are called **irrotational**. It is quite reasonable to consider such flows, since Lagrange's theorem says that if initially a flow had no vorticity, none will develop later in ideal flow.

Since, $\nabla \times \mathbf{u} = 0$, we may introduce the **velocity potential function** ϕ by requiring, $\mathbf{u} = \nabla \phi$. Let us reinstate an external force-field derivable from a potential, K. The continuity equation now reads,

$$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla.(\rho_{\mathrm{m}} \nabla \phi) = 0 \tag{38}$$

We can use the equation of momentum balance derived earlier in the form Eq.(8) and simply substitute.

$$\nabla(\frac{\partial \phi}{\partial t}) = -\nabla\left[\frac{(\nabla\phi)^2}{2} + \frac{\gamma}{\gamma-1}(\frac{p}{\rho_{\mathrm{m}}}) - K\right]$$

$$\frac{\partial \phi}{\partial t} + \frac{(\nabla\phi)^2}{2} + \frac{\gamma}{\gamma-1}(\frac{p}{\rho_{\mathrm{m}}}) - K = H(t) \qquad (39)$$

where $H(t)$ is an arbitrary function of t. Since ϕ itself does not really determine the flows (only its derivatives), there is a "gauge invariance" here. By redefining ϕ, we can even choose H to be zero!. The resultant equation is called **Bernoulli's Equation for compressible ideal irrotational fluids**. It is at once more and less general than the *steady* Bernoulli's equation derived from Eq.(31). The present equation is valid for *time-dependent, irrotational/potential flows*, whereas the steady Bernoulli equation is valid for **rotational** flows with arbitrary vorticity, *on each stream line*. The coupled Eqs.(38,39) can be used to solve all sorts of interesting problems, including nonlinear, exact treatment of soundwaves, aerodynamics, internal waves in planetary and stellar atmospheres etc.

We shall move on to a technically important special case which occurs whenever the velocities in the problem are "small" compared with C_s. Denote a typical speed in the flow by U. The nondimensional ratio, $U/C_s = Ma$ is called the **Mach number** of the flow. We are going to consider the simplifications that occur when the flow is "subsonic", ie., when $Ma \ll 1$. The following argument applies to both rotational and irrotational flows. It is convenient to present the argument in the absence of an external potential, K. Once it is grasped, it is extended, with some modifications to the general case.

N.B. The derivation of the equations of incompressible hydrodynamics from the compressible Euler equations given below is an instructive one, but is somewhat beyond the scope of the present course. It is given here for completeness and may be omitted on a first reading.

[Let us write, $\mathbf{u} = C_s Ma\mathbf{u}^*$, where $\mathbf{u}^*$ is a nondimensional velocity. This merely states that for very low Mach numbers, the flow speed is much smaller than the sound speed, C_s. Let us now compare the terms, $\nabla.(\rho_{\mathrm{m}}\mathbf{uu})$ and $-\nabla p$ in the general equation of momentum balance, Eq.(2). Writing $p \simeq p_0, \rho_{\mathrm{m}} \simeq \rho_0, \mathbf{u}^* \simeq 1$, we see that, $Ma^2 \simeq (p-p_0)/p_0$. This says in effect that the pressure *variations* about the "ambient value", p_0 must be small like Ma^2. Otherwise, the pressure gradient term cannot be balanced by inertia! This simple estimate suggests, along with the adiabatic equation of state which relates pressure variations with density variations, that when $Ma \ll 1$ we should expand the variables as follows:

$$\mathbf{u} = C_s \left[Ma\mathbf{u}_1 + (Ma)^2\mathbf{u}_2 + ... \right] \tag{40}$$

$$p = p_0 \left[1 + (Ma)^2 p_2 + ... \right] \tag{41}$$

$$\rho_{\rm m} = \rho_0 \left[1 + (Ma)^2 \rho_2 + ... \right] \tag{42}$$

Note carefully that all coefficients of the power series in Ma within the square brackets are *nondimensional*, O(1) quantities.

Substitution in the momentum balance equation gives in leading order the "hydrostatic balance":

$$\nabla p_0 = 0 \tag{43}$$

which implies that the pressure in the state of rest is uniform. It follows that ρ_0 is also uniform. From the $O(1)$ expansion of the equation of continuity, we see that $\frac{\partial \rho_0}{\partial t} = 0$ as well.

We may therefore take, ρ_0, p_0 to be the "equilibrium values", $\rho^*_{\rm m}, p^*$ used in the isentropic equation of state. The momentum balance in the order, $(Ma)^2$ gives:

$$\frac{\partial \mathbf{u}_1}{\partial t} + \nabla.(\mathbf{u}_1\mathbf{u}_1) = -(\frac{1}{\gamma})\nabla p_2 \tag{44}$$

Note that we have also adopted a "slow time" relative to the typical sound time scale, L/C_s. Thus, $t \to tL/(MaC_s)$, where L is a typical length-scale of variation of $\mathbf{u}_1$. This means that $\frac{\partial}{\partial t} \simeq |\mathbf{u}|/L$, ie., time rates of change occur on an "advection time-scale".

The same expansion in the adiabatic equation gives:

$$p_2 = \gamma \rho_2 \tag{45}$$

Putting this in the above equation, the leading order subsonic momentum balance equation reads,

$$\frac{\partial \mathbf{u}_1}{\partial t} + \nabla.(\mathbf{u}_1\mathbf{u}_1) = -\nabla \rho_2 \tag{46}$$

The equation of continuity gives in the $O(Ma)$ order, the "incompressibility condition" of subsonic flow:

$$\nabla.\mathbf{u}_1 = 0 \tag{47}$$

Having derived these equations for the four nondimensional dependent variables, $\mathbf{u}_1, \rho_2$, let us put them in their traditional "dimensional" forms.]

We simply set, $\mathbf{u} = C_s Ma\mathbf{u}_1, p = p_0 + \tilde{p}, \rho_{\rm m} = \rho_0$ and obtain the standard Euler equations of **incompressible hydrodynamics**,

$$\nabla.\mathbf{u} = 0 \tag{48}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} = -\nabla(\frac{\tilde{p}}{\rho_0}) \tag{49}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{W} \times \mathbf{u} = -\nabla\left[(\frac{\tilde{p}}{\rho_0}) + \mathbf{u}^2/2\right] \tag{50}$$

Incompressible flow is peculiar in that the pressure, $\tilde{p}$ is determined, not by the equation of state, but in fact, by the incompressibility constraint on $\mathbf{u}$. This is analogous to the electric field in quasi neutral plasmas being determined by equations of motion rather than Gauss' Law. Note that in steady state, we have the Bernoulli relation,

$$\frac{\rho_0 \mathbf{u}^2}{2} + \tilde{p} = P_s \tag{51}$$

where P_s is a constant on each stream line. It is called the "stagnation pressure". It is evident that pressure variations, $\tilde{p}$ are of order $\rho_0\mathbf{u}^2$ in incompressible flows in general. These variations are sometimes called "dynamic" pressures induced by the flows satisfying the incompressibility condition. Of course, they are much smaller (by Ma^2 in fact) than the "ambient thermodynamic pressure", p_0. The latter is constant and plays no role in the motion of the fluid. Once $\tilde{p}$ is determined, the density variations can be calculated, from the equation of state, if desired. These induce higher order flows which are generally not of interest.

Note also that we can consider, **incompressible, irrotational** flows. Then, we set, $\mathbf{u} = \nabla\phi$. Substituting in the momentum equation, we obtain the irrotational Bernoulli relation (the very original one derived by **Bernoulli himself!**),

$$\rho_0 \frac{\partial \phi}{\partial t} + \frac{\rho_0(\nabla\phi)^2}{2} + \tilde{p} = 0 \tag{52}$$

Even more remarkably, we get a *closed* equation for ϕ (ie., not involving $\tilde{p}$) when we substitute in the continuity equation!

$$\nabla^2\phi = 0 \tag{53}$$

This, of course is Laplace's equation and is *linear* in ϕ. This means that ideal, irrotational, incompressible flow can be solved using the theory of Laplace's equation! In general, however, the boundary conditions (especially free-boundary conditions) can be nonlinear. However, an enormous literature exists on these "potential flows".

It is useful to notice some facts about two-dimensional incompressible (not necessarily irrotational) flows. In two dimensions, the continuity equation becomes, in Cartesian coordinates,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{54}$$

If we introduce a new function Ψ such that, $u_x = \frac{\partial \Psi}{\partial y}, u_y = -\frac{\partial \Psi}{\partial x}$, the equation of continuity is **identically** satisfied. The function, Ψ is called the *stream function.* If the flow is *steady*, the contour lines of constant Ψ are "stream lines", or lines of flow. We also see that the vorticity vector is always perpendicular to the plane of flow: $\mathbf{W} = \nabla \times \mathbf{u} = -\mathbf{z}\nabla^2\Psi = \tilde{\omega}\mathbf{z}$, where we use the two-dimensional Laplacian. Substituting in Eq.(50) and taking the curl to eliminate the pressure term, we obtain the remarkably simple "vorticity equation" of two-dimensional incompressible flow:

$$\begin{aligned}
\frac{\partial \tilde{\omega}}{\partial t} + \mathbf{u}.\nabla\tilde{\omega} &= \frac{D\tilde{\omega}}{Dt} \\
&= 0 \\
\frac{\partial \tilde{\omega}}{\partial t} + \frac{\partial(\tilde{\omega}, \Psi)}{\partial(x, y)} &= 0 \qquad (55) \\
\tilde{\omega} &= -\nabla^2\Psi \qquad (56)
\end{aligned}$$

N.B. The following remarks may be omitted on a first reading.

[Observe that if the flow is irrotational, in addition, $\tilde{\omega} \equiv 0$. This entails, $\nabla^2\Psi = 0$. It may be shown from definitions that ϕ and Ψ are *conjugate functions* in the sense of Cauchy and Riemann and the theory of functions of the complex variable, $s = x + iy$ becomes immediately available for solving problems in two-dimensional, ideal, incompressible, potential flow theory! This powerful tool has been used with great success in aerodynamics and elsewhere.]

In the problem set associated with this lecture we explore several interesting applications of the above general principles. We now conclude our brief excusrion into the world of neutral fluid flows.

Plasma Physics

Lecture 11: Collisions, two-fluid theory and qualitative ideas of plasma turbulence

Summary

Basic concepts of two-fluid theory ("extended MHD"); qualitative aspects of Coulomb collisions. Elements of electrostatic drift wave instability. Qualitative account of the genesis and consequences of low frequency plasma turbulence.

XI.1 Coulomb collisions in plasmas

This lecture will be more descriptive than earlier ones and some later discussions. The idea is to introduce you to the concept of "non-ideal" or "dissipative" processes in quasi-neutral plasmas. You know that in ideal MHD there is no dissipation. Typically the electrons and ions "know" about each other only through the electromagnetic fields produced, and these too are "smoothed out" ones. However, it is clear, that if two charged particles are very close, they *will*, in general, "feel" the Coulomb force (at the very least!) between them. This is called a short-range "collision" and one can calculate the famous Rutherford scattering formula using classical mechanics. This formula says that two colliding charged particles will scatter each other due to their mutual electrostatic interaction in well-defined ways. In general, this scattering means that the distribution function, $f(\mathbf{r}, \mathbf{v}, t)$ will change due to such a collision, apart from changes attributable to previously discussed "smoothed out" electric and magnetic fields.

Boltzmann and Maxwell were the first to systematically study collisions and their effects on the kinetic equations in neutral gases. Their work has been extended by many workers for the case of the Coulomb interaction between charged particles, and versions of Boltzmann's equation with "collisional terms" are available for plasmas. This topic is a very extensive one (if you wish to study this thoroughly, there is no better place than Lifshitz and Pitaevski's *Physical Kinetics*. An accessible treatment of collisions and the subtleties which are specific to plasmas may be found in the book *Plasma Physics* by Goldston and Rutherford), and well outside the scope of this course. I shall merely introduce you to some the main ideas which are relatively easy to grasp in a qualitative fashion.

The principal effects of collisions can be summarized thus: 1. collisions between particles, charged or not, destroy the "memory" the particles have of their initial conditions. Thus, after a collision, the colliding particles will, in general, be "scattered" in random directions with random velocities. 2. Having said this, simple inter-particle forces (like the Coulomb force, for example) *do conserve* certain properties like the total energy of the two-colliding particles, their momentum etc. This means in practice that the collisions, although introducing a non-trivial "decorrelation" or "memory loss", do conserve the total number/mass, kinetic energy and momentum of the system, as a whole. This fact assigns a special status to the first 5 velocity moments of the distribution function. 3. Collisions "drive" the system towards thermodynamic equilibrium. Indeed, **Boltzmann** showed that his famous collisional equation, which he derived by an ingeneous mixture of dynamical and statistical arguments, implies that in a "closed" system, a certain integral involving the distribution function (in fact, $-\int f \ln f d\mathbf{r}d\mathbf{v}$) continually increases, and can be regarded as the "kinetic definition" of entropy. This quantity reaches a maximum value in thermodynamic (and mechanical) equilibrium, when f is given by the Maxwell-Boltzmann distribution. 4. Collisions induce the particles in the system to execute "random walks". As **Einstein, Langevin** and others showed, such random walks result in "diffusion" which is an extremely common and well-known *macroscopic, irreversible* phenomenon. This diffusive behaviour is "irreversible" in time and occurs both in velocity and position spaces.

Let us try to understand diffusion of particles. Suppose, for instance we consider a small blob of helium gas released in the atmosphere. The helium atoms collide with the air molecules (which are at some fixed temperature, and have a zero mean velocity say) and exchange energy and momentum with them. Let us suppose that that the air molecules exert a collisional "drag" on the helium and tend to lower the average speed of the helium to zero. We remember that the helium has a partial pressure, $p = n_{He}T$, the constant (and uniform) temperature T is in joules and n_{He} is the number density of the helium. Provided the drag is large enough compared with inertia (this is always true of diffusive transport!) the equation of motion for the blob of helium atoms is,

$$\nabla n_{He}T = -m_{He}n_{He}\nu\mathbf{v}_{He} \tag{1}$$

where, m_{He} is the mass of a helium atom, and $\mathbf{v}_{He}$ the fluid velocity. The quantity ν has the dimensions of a frequency, and can be called the "momentum relaxation rate". Note that if we write, $c_{He}^2 = T/m_{He}$, where c_{He} is the "thermal velocity" of the helium atoms, the particle flux (ie., number density times velocity), is given by **Fick's Law of Diffusion**,

$$n_{He}\mathbf{v}_{He} = -\frac{c_{He}^2}{\nu}\nabla n_{He} \tag{2}$$

where, $D_{He} = \frac{c_{He}^2}{\nu}$, is called the "diffusivity" or diffusion coefficient for helium in air. This says physically that the flux of helium will be from regions of higher to lower concentrations-a physically reasonable result!

Using the "particle continuity" equation for helium, we derive the "diffusion equation", which describes the spread of the helium atoms in air.

$$\frac{\partial n_{He}}{\partial t} = D_{He} \nabla^2 n_{He} \tag{3}$$

The *same* equation was derived by **Einstein** for "**Brownian** dust particles" (far larger than the helium atoms!) and the quantity ν was related by him to the viscous drag on the particles.

In an apparently completely different field, **Fourier** suggested that **heat** (ie., thermal energy) is transported diffusively in many simple materials. In a solid like copper, for instance, if we let C_V be the specific heat at constant volume (supposed constant), the transport of heat by *thermal conduction* is described by the *energy* conservation equation,

$$C_V \frac{\partial T}{\partial t} = -\nabla . \mathbf{q} \tag{4}$$

Here, $\mathbf{q}$ is the *heat flux vector.* Fourier postulated that temperature gradients "drive" the heat flux (ie., heat always flows down the temperature gradient and thereby tends to equalize temperatures and increase entropy) and, at least for reasonable gradients, $\mathbf{q} = -K_T \nabla T$, where the constant K_T is called the *thermal conductivity* of copper. This relation is called **Fourier's Law of Heat Conduction**. It is just Fick's law, but now applied to temperature. It follows immediately, that T satisfies (in the absence of heat sources and sinks), **Fourier's equation of heat conduction**:

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T \tag{5}$$

where, $\chi = K_T / C_V$ is the *thermal diffusivity* of copper. It is plain that diffusion always results whenever the "flux" of some quantity which is described by a conservation equation is proportional to the negative gradient of that quantity. It is also plain that if a quantity is "freely diffusing" in a bounded region from which it cannot escape, as time goes on, it is "irreversibly" spread out over all of the domain **uniformly**.

You should not think that only *scalars* like concentration or temperature can diffuse. Vector quantities like momentum, vorticity and magnetic fields can also diffuse! Essentially the process is similar in that collisions cause "momentum transfer" and tend to equalize the speeds of adjacent layers of flow. For example, in a conductor (at rest, like a copper bar), **Ohm's Law** says that the electric field $\mathbf{E}$ at a point, $\mathbf{r}, t$, and the current density, $\mathbf{j}$ are related by,

$$\mathbf{E} = \eta \mathbf{j} \tag{6}$$

where η, like, K_H is characteristic of copper (strictly calculable only by quantum theory!) and is called its *resistivity*. Assuming that all fields change slowly (ie., the displacement current is negligible), we have from Maxwell's equations,

$$\begin{aligned} \epsilon_0 c^2 \nabla \times \mathbf{B} &= \mathbf{j} \\ \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ &= D_B \nabla^2 \mathbf{B} \end{aligned} \tag{7}$$

where the "magnetic diffusivity", $D_B = \eta \epsilon_0 c^2$. We have used **Ohm's Law** and the vector identity, $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla^2 \mathbf{A}$. Thus, in a resistive medium at rest, the magnetic field (and the current density $\mathbf{j}$!) *diffuse* at a rate proportional to the resistivity of the medium.

Remember that **Ohm's Law** in a *moving medium* with a finite, *isotropic* resistivity takes the form (as in MHD),

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} \tag{8}$$

This results in the *resistive MHD, advection-diffusion equation*,

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ &= \nabla \times (\mathbf{u} \times \mathbf{B}) + D_B \nabla^2 \mathbf{B} \end{aligned} \tag{9}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}.\nabla \mathbf{B} = \mathbf{B}.\nabla \mathbf{u} + D_B \nabla^2 \mathbf{B} \tag{10}$$

The last equation applies to *incompressible* resistive MHD, where we may take, $\nabla.\mathbf{u} = 0$. This equation is virtually *exactly* mirrored by the equation satisfied by incompressible, *viscous* flow of a neutral fluid. It was shown by Navier and Stokes that viscosity of a fluid diffuses the vorticity, $\mathbf{W} = \nabla \times \mathbf{u}$ according to,

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{u}.\nabla \mathbf{W} = \mathbf{W}.\nabla \mathbf{u} + \nu^* \nabla^2 \mathbf{W} \tag{11}$$

where the constant, ν^* has the dimensions of a diffusivity (ie., L^2/T) and is called the *kinematic viscosity* of an incompressible fluid. **Maxwell** derived a famous formula for

$\rho_m \nu^* = \mu$, the *viscosity* of the fluid from kinetic considerations involving the binary collision frequency, $1/\tau_{coll}$ of the molecules of the gas.

One can show easily the following result: let a "particle" move on the average, a distance l between scattering events which, occur with a frequency, $\nu = 1/\tau_{coll}$. In one dimension, we could simply assume that that the particle moves to the right or left with a velocity $c = \pm l/\tau_{coll}$ between the "collisions". On the average, the particle has an equal chance of going to the right or left. Hence if we consider a very large ensemble of (noninteracting!) particles, their average location is the origin, asuming that at $t = 0$ they all started there. However, if we consider their RMS distance from the origin, $(< (\Delta x)^2 >)^{1/2}$, we will find that $< (\Delta x)^2 > \simeq l^2 t/2\tau_{coll}$. Thus, the "spread" of the particles is a "diffusive" process with diffusivity, $D = l^2/2\tau_{coll}$! This is the fundamental connection between "molecular" collisions and quantities like l and τ_{coll} and macroscopic coefficients like D which can be determined by experiments involving "phenomenological" laws like those of **Fick and Fourier**.

It turns out that a complete analysis of charged particle collisions leads to electron-electron, electron-ion and ion-ion collision frequencies. These rates are functions of the ion charge, mass ratios and energies of the colliding particles. The generalization of the Vlasov kinetic equations for charged particles is called the "Fokker-Planck equation" with the "Landau collision term". If the collision frequencies are sufficiently high, it is possible to solve this approximately and derive general "dissipative" relations for particle and thermal diffusivities and viscosities. This restricted theory is very similar to the Maxwell-Boltzmann kinetic theory for neutral gases and goes by the name, "Braginskii Theory". A generalization of it which is far-reaching in that it considers low collisionality (ie., the collisions are a "weak" effect on a basically collisionless system) and particle trapping is called the "neo-classical" theory. We cannot go into even an outline of these theories here. The basic differences can be stated, however. If an electron-ion plasma is immersed in a **magnetic field** and the diffusivity is sought in the direction **parallel** to the field, **Braginskii 's** theory leads to the result that $D_{\|} \simeq c_{th}^2 \tau_{coll}$, where, the *mean-free-path*, $\lambda \simeq c_{th}\tau_{coll} \ll L$. Here, c_{th} is the typical thermal velocity of the particle in question and τ_{coll} is **Braginskii's** appropriate "collision time" for the particle (essentially a function of its temperature), and L is a typical length-scale over which the fields, temperatures and the densities vary. In the direction **perpendicular** to the field however, the collisions *enhance* the transport and $D_{\perp} \simeq r_L^2/\tau_{coll}$. This strongly suggests that in the parallel direction, the mean-free-path, λ is the appropriate "step-length" whilst in the perpendicular direction, the **Larmor radius**, $r_L = c_{th}/\Omega_c$ plays the role of step length. We are discussing **strongly magnetized plasmas** wherein, $\Omega_c \tau_{coll} \gg 1$. It is then evident that such plasmas are *extremely anisotropic* as regards diffusion and viscosity. Thus $D_\perp \simeq D_\parallel/(\Omega_c\tau_{\rm coll})^2$. This is precisely why magnetic configurations with closed field lines are chosen for confinement devices like tokamaks.

Experimentally, these "classical" or Braginskii diffusion of particles, energy and momentum are *never* found! The observed rates are often many *orders of magnitude* higher, although, of course, they are still far slower than equilibration rates parallel to the field. A partial explanation, largely valid for ions, is given by the *neoclassical* theory which asserts that while $\tau_{\rm coll}$ is indeed relevant for collisions between particles which are not trapped by the spatially varying magnetic fields, the "trapped" particles must execute much larger drift motions and experience de-trapping collisions which considerably enhance their diffusion rates over the "classical" Braginskii estimate, $r_L^2/\tau_{\rm coll}$. The step-lengths relate to the radial widths of the trapped particles' complicated orbits and the higher collision rates correspond to greater impact of certain types of "small angle" collisions. It turns out however, that for electrons, the observed diffusion rates are far higher than even the neoclassical ones. The explanation of this so-called "anomalous transport" is one of the outstanding (and tantalizing!) problems of modern plasma physics. It is believed, and there is a large body of evidence to directly support this belief, that plasma "turbulence" is responsible for enhancing the transport rates far above classical or even neoclassical values by changing both the effective "step-lengths" and the "collision times" responsible for the diffusion.

XI.3 Two-fluid equilibria

Let us consider a very simple but illustrative example. Suppose we have a *uniform* magnetic field, $\mathbf{B} = B_0\mathbf{e}_z$ along the z-axis. We consider the *steady, quasi-neutral equilibrium* of an electron-proton (for simplicity, extensions to other fully ionized atoms is obvious) plasma, where we *assume* that the ions are at rest in the "laboratory frame" and are "cold" (ie., their temperature is small compared to the electrons). We further assume that all equilibrium quantities vary only with r, the radial coordinate measured from the origin of a cylindrical coordinate system (r, θ, z). In short, nothing varies with t, θ or z. Although this can be generalized, we will assume that the *only* non-zero component of the magnetic field is the z-component, which reduces to B_0 in the absence of the plasma. We adopt a fluid description, a much simplified version of Braginskii's continuum equations for a fully ionized, electron-proton plasma. While the ions are cold, the electrons will be taken to be at a *uniform* temperature, T_e measured in joules, and charge on an electron is $-e$.

Let us write down the equations of momentum balance for ions, noting that quasi-neutrality implies, $n_i \simeq n_e$:

$$m_p n_e \frac{D\mathbf{u}_i}{Dt} = en_e(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + \mathbf{R}_{ie} \tag{12}$$

where, by assumption, we may take $\mathbf{u}_i = 0$ and $\mathbf{R}_{ie} = m_e n_e \nu_f(\mathbf{u}_e - \mathbf{u}_i)$ represents the frictional drag due to collisions with electrons which have a "fluid velocity", $\mathbf{u}_e$. Note that we have used the fact that $T_i \simeq 0$ ("cold ions") and have dropped the pressure term and several other "viscosity terms" as well (that is why this is a simplified model!).

Note that quasi-neutrality says that $\nabla.\mathbf{j} = 0$. From this and the fact that things vary only with r, we see that, $j_r = en_e(u_{ir} - u_{er}) = 0$. Therefore, in the r direction, the above equation reduces to,

$$0 = en_e E_r \tag{13}$$

This says we can take $E_r = 0$, and of course, we may take, $\mathbf{E} = 0$. Let us turn to the corresponding *electron momentum balance equation*. The electrons do have a pressure, $p_e = n_e T_e$, which can vary with r, since n_e can vary with r. The electrons have low mass and their inertia can be neglected as a first approximation (we can justify this after the fact, if necessary). The "radial" electron momentum equation becomes,

$$0 = -T_e \frac{dn_e}{dr} - en_e u_{e\theta} B_z \tag{14}$$

where $B_z(r)$ is the z-component of the magnetic field which reduces to B_0 in the absence of the plasma. We see that an electron pressure/density gradient is allowed, so long as a current, $j_\theta = -en_e u_{e\theta}$ is allowed to flow. The above equation says, the radial equilibrium of the electron fluid depends upon, $j_\theta = \frac{T_e}{B_z}\frac{dn_e}{dr}$. Let us now remember that **Ampère's Law** says (defining μ_0 by the relation, $\epsilon_0\mu_0 = 1/c^2$, as is conventional),

$$\begin{aligned} -(\frac{1}{\mu_0})\frac{dB_z}{dr} &= j_\theta(r) \\ &= \frac{T_e}{B_z}\frac{dn_e}{dr} \\ T_e n_e + (\frac{1}{2\mu_0})B_z^2 &= (\frac{1}{2\mu_0})B_0^2 \end{aligned} \tag{15}$$

This equation relates the radial variation of the number density, $n_e(r)$ and that of B_z. We assume that at some radius, $r = a$, the plasma has become so tenuous that $B_z = B_0$. Note a very important fact: the magnetic field *within* the plasma, ie., $B_z(r)$, for $r < a$ is *always smaller* than B_0, the "vacuum" field. Thus, a classical plasma with a pressure gradient (ie which is *not* in strict thermodynamic equilibrium) is *diamagnetic* and tends to "expel" magnetic field, in this simple geometry. For this reason, the current, $j_z(r)$ is called the **diamagnetic** electron current. Since it is only a funtion of r, we continue to preserve quasi-neutrality. The electron equation of motion in the z direction is identically satisfied, as can be readily checked. What of the θ component? Clearly, we must have,

$$\begin{aligned} 0 &= en_e u_{er} B_z + m_e n_e \nu_f(-u_{e\theta}) \\ &= en_e u_{er} B_z + \frac{m_e}{e}\nu_f j_\theta \end{aligned} \tag{16}$$

We can solve for the "radial electron flux", $\Gamma_e(r) = n_e u_{er}$ in terms of the other quantities, substituting, $j_\theta = \frac{T_e}{B_z}\frac{dn_e}{dr}$ and obtain the remarkable relation,

$$n_e u_{er} = -\frac{m_e}{B_z e^2}\nu_f \frac{T_e}{B_z}\frac{dn_e}{dr} \tag{17}$$

This can be made more transparent by setting, $T_e = \frac{1}{2}m_e c_{\text{th}}^2$, and observing that, $\frac{m_e}{B_z e^2}\nu_f \frac{T_e}{B_z} = \frac{1}{2}\rho_e^2\nu_f = D_{\perp e}$. Thus, we have derived, upon substitution into the continuity equation, the electron particle diffusion equation,

$$\frac{\partial n_e}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(rD_{\perp e}\frac{\partial n_e}{\partial r}) \tag{18}$$

we see that this Braginskii two-fluid model results in the "classical" particle diffusivity $D_{\perp e} = \frac{1}{2}\rho_e^2\nu_f; \rho_e = c_{\text{th}}/\Omega_{ce}$ across the magnetic field (making allowance for the radial variation of $D_{\perp e}$!). This is closely related to the intuitive "random-walk" argument with ρ_e as the step length and $1/\nu_f$ as the "time-step". Note that, **Braginskii** calculates that, $\nu_f \simeq \frac{2^{1/2}n_e\Lambda e^4}{12\pi^{3/2}\epsilon_0^2 m_e^{1/2}T_e^{3/2}}$, where Λ is the so-called "Coulomb logarithm", and has a value 10-20. There is a corresponding formula for ions which will not be needed here.

You can easily check that using the *ion* momentum equation leads to the same value for the ion particle flux. This is because the "friction force", $\mathbf{R}_{ie}$ does not change the total momentum of the two species, but depends only on the current (ie., difference of the two flows). However, you might find it disturbing that we started with the assumption of steady state and yet end up with a transient diffusion equation. The answer is a very general one: in a *dissipative system*, it is necessary to have *external sources* in order to maintain the system away from thermodynamic equilibrium. Thus, to have a *true steady state* with a general n_e profile, we must have a source of particles! Thus the particle contiuity equation must really be,

$$\frac{1}{r}\frac{d}{dr}(rD_{\perp e}\frac{dn_e}{dr}) + S_p(r) = 0 \tag{19}$$

where $S_p(r)$ is an appropriate particle source. This must balance the steady losses due to classical radial diffusion to maintain the $n_e(r)$ profile. This equation and the equation for B_z, Eq.(15) must be simultaneously solved to determine the n_e, B_z, j_θ profiles.

It is of interest to note that having $u_{iz}(r) = u_{ez}(r) = U(r)$, with arbitrary U is also an allowed solution in this model! Such flows do not disturb the equilibrium determined. However, a deeper examination of the Braginskii equations reveals that unless there are *momentum* sources within the plasma, U can only be a constant, amounting to a simple Galilean transformation along the direction of the magnetic field. This simple configuration corresponds to an infinite "straight" system and early plasma devices called "Q-machines" had a similar structure. They exhibit the basic features of "confinement", and diffusion across field. The fact that electrons and ions are transported at exactly the *same* rate and quasi-neutrality is maintained is called **ambipolarity** of particle diffusion.

An important generalization is obtained by having a constant, uniform *electric field*, E_z applied by external means. Now the equation of motion in the z direction is no longer trivial (Nb. We disregard a minor complication of Braginskii theory which says that the "collision rate" ν_f must be *nonisotropic*, ie., is different parallel and perpendicular to the field). The electron and ion equations are:

$$\begin{aligned} 0 &= -en_eE_z + m_en_e\nu_f(u_{iz} - u_{ez}) \\ 0 &= en_eE_z + m_en_e\nu_f(u_{ez} - u_{iz}) \end{aligned}$$

Thus, they reduce to the *single equation*,

$$E_z = \eta_{\|} j_z \tag{20}$$

$$\eta_{\|} = \frac{m_e\nu_f}{n_e e^2} \tag{21}$$

Furthermore, the fact that j_z is now non-zero means that we have also to introduce, $B_\theta(r)$ by Ampère's Law. Indeed, unless E_z is particularly small, we have to solve the equlibrium problem of determining n_e, B_z, B_θ self-consistently, given $T_e, \nu_f, E_z, S_p, B_0$. This is not difficult, and leads to an equilibrium configuration called the "classical screw pinch". Indeed, given the *full* Braginskii equations, the sources and boundary conditions, one can construct (if necessary, by numerical integration) the self-consistent classical equilibrium, even including the energy transport equation! I mention this simply to show that the Braginskii theory is "complete" and leads to definite equilibria if sources and boundary conditions are specified. Unfortunately, the theory has only extremely limited validity, even if its basic premises that $\Omega_{ce}\tau_{\rm coll} \gg 1, \rho_e \frac{d\ln n_e}{dr} = \rho^* \ll 1$ *are* satisfied. When the field varies along its length (as it must do in any toroidal confinement configuration in general), the Braginskii theory must be extended to include neoclassical "trapped particle effects" which are very significant. More importantly, experiments conclusively show that the values of $D_{\perp e}$ etc obtained in reality are far larger than the above classical estimates.

We shall leave the subject of two-fluid equilibria with this illustrative but very simple example and turn our attention to low frequency waves and stability of such equilibria.

XI.4 Drift waves

It is rarely sufficient in plasma physics to construct an equilibrium. The *stability* of equilibria is of paramount importance. We are going to consider the simple two-fluid equilibrium constructed in the last section and study low frequency wave motions. The equilibrium is characterized by a *uniform* magnetic field, $\mathbf{B} = B\mathbf{e}_z$. Of course, we have already seen that the equilibrium **diamagnetic electric current** will modify the field slightly within the plasma. This "non-uniformity" of B_z is small, provided the plasma pressure, $n_{e0}T_e$ is small compared to the magnetic field energy, $B^2/2\mu_0$, as can be inferred from Eq.(15). Indeed, the smallness of the nondimensional pressure, $\beta_e = \frac{2\mu_0 n_{e0}T_e}{B^2} \ll 1$, is *required* for the fluctuations to be purely electrostatic. This condition is not always met in astrophysics, though in typical modern tokamak plasmas, $\beta \leq 0.1$, although in many experiments (including MAST at Culham), this value has been exceeded!

The ions are at rest and are taken to be "cold". The electrons are at a uniform temperature within a cylinder, $r \leq a$. The equilibrium electron density profile is assumed to be, $n_{e0}(r)$. Let us now consider "small oscillations" about this equilibrium. Let $\tilde{n}(r, \theta, z, t)$ be the "perturbed" density (in both electrons and ions; we are going to consider quasi-neutral dynamics!). Denote the perturbed ion velocity by, $\tilde{\mathbf{u}}_i$. The ion continuity equation, retaining only first order perturbed quantities is,

$$\begin{aligned} \frac{\partial \tilde{n}}{\partial t} + \nabla.(n_{e0}\tilde{\mathbf{u}}_i) &= 0 \\ \frac{\partial \tilde{n}}{\partial t} + \tilde{u}_{ir}\frac{dn_{e0}}{dr} + n_{e0}\nabla.\tilde{\mathbf{u}}_i &= 0 \end{aligned} \qquad (22)$$

The electron momentum balance equation is considered next. Written in full, it takes the form,

$$m_e n \frac{D\mathbf{u}_e}{Dt} = -\nabla(nT_e) - en(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + m_e n \nu_f(\mathbf{u}_i - \mathbf{u}_e) \qquad (23)$$

It can be shown that in many circumstances, the electric field is dominated by electrostatic fluctuations. It is usually a good approximation to neglect electron inertia terms in Eq.(23) (if we wish, we can always come back to it later!). Retaining only first order quantities, and dropping the collisional term (we will reinstate this later!) we find the equation becomes,

$$-T_e \nabla \tilde{n} + e n_{e0} \nabla \tilde{\Phi} - e n_{e0} \tilde{\mathbf{u}}_e \times \mathbf{B} \;=\; 0 \tag{24}$$

where we have set, $\tilde{\mathbf{E}} = -\nabla\tilde{\Phi}$, in accordance with electrostatics. Note that in this equation, T_e is a constant and $n_{e0}(r)$ is a given profile (in the range, $0 \leq r \leq a$), and $\mathbf{B} = B\mathbf{e}_z$. Taking the dot product with $\mathbf{B}$, we derive the relation,

$$\begin{aligned} \frac{\partial}{\partial z}\left[-T_e \tilde{n} + e n_{e0} \tilde{\Phi}\right] &= 0 \\ \frac{\tilde{n}}{n_{e0}} &= \frac{e\tilde{\Phi}}{T_e} \end{aligned} \tag{25}$$

This relation between electron density fluctuations and electrostatic potential fluctuations is often called the **Boltzmann** relation or the **adiabaticity** relation. Whilst it is seen here to be a consequence of *parallel* electron momentum balance under certain assumptions, it is also the expression one gets from the Maxwell-Boltzmann distribution relating density and potential. This is all that we need from the electron physics!

Let us turn our attention to the ion dynamics. The ion momentum balance equation, when linearized similarly, bearing in mind that the ion temperature is assumed to be zero yields,

$$m_i n_{e0} \frac{\partial \tilde{\mathbf{u}}_i}{\partial t} \;=\; -e n_{e0} \nabla \tilde{\Phi} + e n_{e0} \tilde{\mathbf{u}}_i \times \mathbf{e}_z B \tag{26}$$

Suppose we are interested in frequencies *small* compared with $\Omega_{ci} = eB/m_i$. It can be seen that the first approximation to the above equation is obtained by setting,

$$\tilde{\mathbf{u}}_i \;=\; \frac{1}{B}\mathbf{e}_z \times \nabla_\perp \tilde{\Phi} \tag{27}$$

where $\nabla_\perp = \nabla - \mathbf{e}_z \frac{\partial}{\partial z}$. This is none other than the statement that neglecting inertia, the ion fluid moves under the electrostatic perturbation according to the $\mathbf{E} \times \mathbf{B}$ drift! Having obtained $\tilde{\mathbf{u}}_i$, let us substitute this into the ion continuity equation, Eq.(22):

$$\frac{\partial \tilde{n}}{\partial t} + \nabla.(n_{e0}\frac{1}{B}\mathbf{e}_z \times \nabla_\perp \tilde{\Phi}) \;=\; 0 \tag{28}$$

It is now actually convenient to use scalar components. Note that in the (r, θ, z) cylindrical coordinates, $\nabla_\perp \tilde{\Phi} = (\frac{\partial \tilde{\Phi}}{\partial r}, \frac{1}{r}\frac{\partial \tilde{\Phi}}{\partial \theta}, 0)$. This leads to,

$$\tilde{u}_{ir} = -\frac{1}{Br}\frac{\partial \tilde{\Phi}}{\partial \theta} \tag{29}$$

$$\tilde{u}_{i\theta} = \frac{1}{B}\frac{\partial \tilde{\Phi}}{\partial r} \tag{30}$$

It is easy to verify from the uniformity of B that the fluctuating ion velocity, $\tilde{\mathbf{u}}_i$ as given by the above equation, is divergence-free, viz., $\nabla.\tilde{\mathbf{u}}_i = \frac{1}{r}\frac{\partial}{\partial r}(r\tilde{u}_{ir}) + \frac{1}{r}\frac{\partial u_{i\theta}}{\partial \theta} = 0$.

Using this result, it is immediately seen that the equation of continuity, Eq.(28) simplifies to,

$$\frac{\partial \tilde{n}}{\partial t} + \tilde{u}_{ir}\frac{dn_{e0}}{dr} = 0$$

$$\frac{\partial \tilde{n}}{\partial t} - \frac{1}{Br}\frac{\partial \tilde{\Phi}}{\partial \theta}\frac{dn_{e0}}{dr} = 0$$

We may *eliminate* the density fluctuation, $\tilde{n}$ using the **adiabatic** relation, Eq.(25), $\frac{\tilde{n}}{n_{e0}} = \frac{e\tilde{\Phi}}{T_e}$, to get a single "wave equation" (ie., a first-order *hyperbolic* p.d.e) for the nondimensional electrostatic potential fluctuation, $\frac{e\tilde{\Phi}}{T_e}$:

$$\frac{\partial}{\partial t}(\frac{e\tilde{\Phi}}{T_e}) = \frac{T_e}{eBr}\frac{1}{n_{e0}}\frac{dn_{e0}}{dr}\frac{\partial}{\partial \theta}(\frac{e\tilde{\Phi}}{T_e}) \tag{31}$$

This equation is easily solved, subject to the obvious condition that the potential must be periodic in θ. In fact, let us look for "harmonically varying solutions" (this is possible since the coefficients of this equation are at most functions of r and not of θ, t). In fact, let us set, $\frac{e\tilde{\Phi}}{T_e} = f(z,r)\exp[m\theta - \omega t]$, where m is an integer (non zero!) and ω is a frequency "eigenvalue" to be determined. Substitution in Eq.(31) gives,

$$\omega = \omega_*$$

$$= -\frac{T_e}{eB}(\frac{1}{n_{e0}}\frac{dn_{e0}}{dr})(\frac{m}{r}) \tag{32}$$

There are several things to be noted about this result. First of all, we see that the frequency, ω_*, called the **electron drift frequency**, is proportional to the density gradient of the equilibrium profile, $n_{e0}(r)$. Furthermore, it is also proportional to the **azimuthal wave number**, $k_\theta = m/r$. It is directly proportional to the electron temperature and inversely proportional to the magnetic field. To understand this dependence a bit better, let us introduce the "thermal speed" of the *ions, defined by the rule,* $C_s^2 = T_e/m_i$. We also introduce the "effective ion Larmor radius" (the actual one is zero, since the ions are cold!), $\rho_s = C_s/\Omega_{ci} = C_s/(eB/m_i)$. This is correct for hydrogen, deuterium and tritium, but will require obvious modifications for other nuclei. Let us also note that the "scale-length" for the variation of the equilibrium density is given by, $L_n(r) = (|\frac{1}{n_{e0}}\frac{dn_{e0}}{dr}|)^{-1}$. Typical density profiles will have negative density gradients. It follows that ω_* is positive for positive k_θ.

Using the definitions of C_s, ρ_s, we see that Eq.(32) may be written as,

$$\omega_* = (\frac{C_s}{L_n})(k_\theta \rho_s) \tag{33}$$

The frequency C_s/L_n is roughly like the time a sound wave in the plasma will take to travel the length, L_n. The nondimensional wave number, $(k_\theta \rho_s)$ determines the angular frequency, ω_* of the wave at each radius r. From Eq.(14) we know that this is the direction of the *equilibrium electron drift*, $u_{e\theta} = \frac{C_s}{L_n}\rho_s$. Indeed, we see that, $\omega_* = u_{e\theta}k_\theta(r)$, at each radius. Thus the *phase velocity* of the disturbance at each radius, is $r\dot{\theta} = u_{e\theta}$. The disturbance moves with the electron diamagnetic drift velocity! For this reason, these waves are called **electron drift waves**.

You will have noted (I hope!) some odd features: while we have determined that these waves are obviously **stable** (ie., periodic in time), they only propagate in the θ-direction, (transverse to the magnetic field). Since we always require that $\rho_* = \rho_s/L_n \ll 1$, the phase velocity is always much smaller than the thermal velocity, C_s of the ions. It is also clear that $\omega_* \ll \Omega_{ci}$, since we shall only be interested in wave lengths long compared with ρ_s, so that $k_\theta \rho_s \leq 1$, or at least is not very great compared to unity. We have also no information about propagation in the z-direction, parallel to the ambient magnetic field.

Let us try to remedy some of these defects. We start with the ion momentum equation, Eq.(26). Taking the z-component, we find that,

$$\begin{aligned} m_i n_{e0}\frac{\partial \tilde{u}_{iz}}{\partial t} &= -en_{e0}\frac{\partial \tilde{\Phi}}{\partial z} \\ \frac{\partial \tilde{u}_{iz}}{\partial t} &= -C_s^2\frac{\partial}{\partial z}(\frac{e\tilde{\Phi}}{T_e}) \end{aligned} \tag{34}$$

upon making use of the definitions above. As was stated earlier, Eq.(27) is a *first approximation* to the solution of Eq.(26) in the *perpendicular* direction to the field.

To get the next approximation, we simply substitute this first approximation on the LHS of Eq.(26) and calculate the next approximation (this is called the method of **successive approximations or iterations**).

$$\tilde{\mathbf{u}}_{\perp i} = (C_s\rho_s)\mathbf{e}_z \times \nabla_\perp(\frac{e\tilde{\Phi}}{T_e}) - \rho_s^2\nabla_\perp\frac{\partial}{\partial t}(\frac{e\tilde{\Phi}}{T_e}) \tag{35}$$

where the simple relation, $\rho_s = C_s/\Omega_{ci}$ has been used. You can check the dimensions of each term, and also the fact that the last term corresponds to the "inertial drift" of the ions in the time-dependent fluctuating electrostatic field. Note that we have still not included dissipation. We now substitute into the ion continuity equation, Eq.(22) using the adiabaticity relation, Eq.(25), as before and eliminate the density fluctuation in favour of the electric potential fluctuations to obtain,

$$\frac{\partial}{\partial t}(\frac{e\tilde{\Phi}}{T_e}) - \frac{T_e}{eBr}\frac{1}{n_{e0}}\frac{dn_{e0}}{dr}\frac{\partial}{\partial\theta}(\frac{e\tilde{\Phi}}{T_e}) = -\frac{\partial}{\partial z}(\tilde{u}_{iz}) + \rho_s^2\nabla_\perp^2\frac{\partial}{\partial t}(\frac{e\tilde{\Phi}}{T_e}) \tag{36}$$

We have to solve this equation along with Eq.(34). To get an idea of what this system might give, we make the so-called **local** approximation. Thus we assume that the wave-lengths of the disturbance are small compared to the scale-length L_n, which may be treated as a constant. We assume the "harmonic" approximation, and set,

$$\frac{e\tilde{\Phi}}{T_e} = \hat{A}(r,m,k_z,\omega)\exp\left[im\theta + ik_z z - i\omega t\right] \tag{37}$$

$$\frac{\tilde{u}_{iz}}{C_s} = \hat{B}(r,m,k_z,\omega)\exp\left[im\theta + ik_z z - i\omega t\right] \tag{38}$$

where $\hat{A}, \hat{B}$ are non-dimensional amplitudes, m, k_z are wave numbers and ω is the frequency to be determined.

Evidently, Eq.(34) gives,

$$\hat{B} = \frac{C_s k_z}{\omega}\hat{A} \tag{39}$$

whilst, Eq.(36) leads to,

$$(\omega - \omega_*)\hat{A} = \omega\rho_s^2\left[\frac{1}{r}\frac{d}{dr}(r\frac{d\hat{A}}{dr}) - (\frac{m}{r})^2\hat{A}\right] + \frac{C_s k_z^2}{\omega}\hat{A} \tag{40}$$

This is now a second-order ODE which can be solved subject to the boundary condition that $\hat{A}$ vanishes for $r = 0$ and $r = a$, for example. Given m, k_z, n_{e0}, we can determine the "eigenvalues" ω. You can immediately see that we recover Eq.(32) when the terms on the right due to the "inertial drift" and the "parallel" motion is neglected. Suppose only the inertial term is neglected, and we consider $m = 0$. We then obtain, $\omega = \pm C_s k_z$, which we have seen is typical of sound waves. Rather than attempting to solve this equation generally (not trivial for general n_{e0} profiles!) we assume that we are interested in relatively short radial wavelengths compared with L_n and write, $r = r_0(1 + x)$ and specialize to the case when all equilibrium quantities are evaluated at r_0, a reference radius, and set, $\hat{A} = \hat{C}\exp(i\lambda x)$, where λ is a nondimensional "radial" wave number. Then, we have, $\left[\frac{1}{r}\frac{d}{dr}(r\frac{d\hat{A}}{dr}) - (\frac{m}{r})^2\hat{A}\right] \simeq -\frac{1}{r_0^2}(\lambda^2 + m^2)\hat{C}$. It is reasonable to define the "perpendicular wave number", $k_\perp^2 = (\lambda^2 + m^2)/r_0^2$.We now find that ω satisfies the "local", quadratic, dispersion relation,

$$\omega^2 - \left[\frac{\omega_*}{1 + \rho_s^2 k_\perp^2}\right]\omega - \frac{C_s^2 k_z^2}{1 + \rho_s^2 k_\perp^2} = 0 \tag{41}$$

where all equilibrium quantities (including ω_*) are evaluated at r_0. The solution of the quadratic gives,

$$2\omega = \frac{1}{1 + \rho_s^2 k_\perp^2}\left[\omega_* \pm (\omega_*^2 + 4C_s^2 k_z^2)^{1/2}\right] \tag{42}$$

showing immediately that these "electrostatic" drift waves are really modified sound waves, capable of propagating both parallel to and perpendicular to the magnetic field. Note the complicated dependence on the wave numbers. When $k_z = 0$, we get a simple result for the drift wave frequency, generalizing Eq.(33):

$$\omega = \frac{C_s \rho_* k_\theta}{1 + \rho_s^2 k_\perp^2} \tag{43}$$

where $\rho_* = \rho_s/L_n$. On the other hand, if we consider purely parallel propagation (ie., $k_\perp = 0$), we simply get sound waves propagating parallel to the field with phase velocity, C_s.

The formula shows that with the inclusion of the inertial drift, the drift waves *can* propagate along the density gradient, as well as in the azimuthal direction. Note however that within the assumptions of this "dissipationless" model, drift waves are completely **stable**, as the frequencies are real for all possible choices of wave numbers!

Other effects such as collisions and ion or electron temperature (strictly *entropy* gradients) do, in fact lead to instabilities. These instabilities are extremely important, as they are very widely observed in many plasma devices and are thought to be the basis of **drift wave turbulence** in quasi-neutral plasmas. We look at this vast subject briefly.

XI.5 Qualitative notions of drift wave turbulence in plasmas

The simplest way to understand one possible mechanism of drift wave *instability* is to consider the "generalized Ohm's Law", Eq.(24). If we include a resistive drag term, this equation becomes, upon taking the z component,

$$\begin{aligned} \frac{\partial}{\partial z}(T_e\tilde{n} - en_{e0}\tilde{\Phi}) &= m_e n_{e0}\nu_{ei}(\tilde{u}_{zi} - \tilde{u}_{ze}) \\ \frac{\partial}{\partial z}(\frac{\tilde{n}}{n_{e0}} - \frac{e\tilde{\Phi}}{T_e}) &= \frac{m_e\nu_{ei}}{T_e}(\tilde{u}_{zi} - \tilde{u}_{ze}) \end{aligned} \tag{44}$$

We observe that neglecting electron inertia (and dissipation!), the perpendicular electron velocity from the *electron momentum* equation is,

$$\tilde{\mathbf{u}}_{\perp e} = C_s\rho_s\mathbf{e}_z \times \nabla_\perp\frac{e\tilde{\Phi}}{T_e} - \frac{C_s\rho_s}{n_{e0}}\mathbf{e}_z \times \nabla_\perp\tilde{n} \tag{45}$$

Note that quasi-neutrality requires, $\nabla.\left[n_{e0}(\mathbf{u}_i - \mathbf{u}_e)\right] = 0$. From Eqs.(35,44,45), we obtain the relation,

$$-\rho_s^2\frac{\partial}{\partial t}(\nabla.\left[n_{e0}\nabla_\perp(\frac{e\tilde{\Phi}}{T_e})\right]) + \frac{n_{e0}T_e}{m_e\nu_{ei}}\frac{\partial^2}{\partial z^2}(\frac{\tilde{n}}{n_{e0}} - \frac{e\tilde{\Phi}}{T_e}) = 0 \tag{46}$$

Denoting the amplitude for the density fluctuations, $\frac{\tilde{n}}{n_{e0}}$ by $\hat{C}$, and making the same "local" approximations as before (taking $\rho_s \ll L_n k_\perp \rho_s \simeq 1$), we find, from Eq.(46),

$$-i\rho_s^2\omega k_\perp^2\hat{A} - \frac{T_e k_z^2}{m_e\nu_{ei}}(\hat{C} - \hat{A}) = 0$$

It follows that the **adiabaticity relation** between density and potential fluctuations, Eq.(25) is modified by collisions to,

$$\hat{C} = \hat{A}(1 - i\Delta) \tag{47}$$

$$\begin{aligned} \Delta &= \frac{\rho_s^2\omega m_e\nu_{ei}k_\perp^2}{T_e k_z^2} \\ &= (\frac{m_e}{m_i})(\frac{\omega\nu_{ei}}{\Omega_{ci}^2})(\frac{k_\perp}{k_z})^2 \end{aligned} \tag{48}$$

This says that there is a *phase shift* between the density and potential fluctuations introduced by electron-ion drag. This a very widely observed "generic" feature of collisional effects and is also similar in character to that due to collisionless phase-mixing effects such as **Landau damping**. The instability is sensitive to this phase-shift, as will shortly appear.

Substituting from Eqs.(34,35,47,48) into the ion continuity equation yields,

$$
\begin{aligned}
\frac{\partial}{\partial t}\left[(\frac{\tilde{n}}{n_{e0}}) - \rho_s^2\nabla_\perp^2(\frac{e\tilde{\Phi}}{T_e})\right] - \left[\frac{T_e}{eBr}\frac{\partial}{\partial\theta}(\frac{e\tilde{\Phi}}{T_e})\right](\frac{1}{n_{e0}}\frac{dn_e}{dr}) + \frac{\partial\tilde{u}_{iz}}{\partial z} &= 0 \\
-i\omega(\hat{C} + \rho_s^2 k_\perp^2\hat{A}) + i\omega_*\hat{A} + i\frac{C_s^2k_z^2}{\omega}\hat{A} &= 0 \\
\omega^2(1 - i\frac{\Delta}{1+\rho_s^2k_\perp^2}) - \omega\frac{\omega_*}{1+\rho_s^2k_\perp^2} - \frac{C_s^2k_z^2}{1+\rho_s^2k_\perp^2} &= 0 \qquad (49)
\end{aligned}
$$

This dispersion relation generalizes Eq.(41) to the case when there is a finite resistivity. Typically, one has $C_sk_z \ll \omega_*$; $k_z^2 \ll k_\perp^2$. You can see that the latter is really necessary to get reasonable values of Δ. Under these circumstances, a first approximation to the solution is, $\omega_{\rm re} \simeq \frac{\omega_*}{1+\rho_s^2k_\perp^2}$, which is the "dissipationless" result. It is immediately seen that the next approximation including the Δ term gives an imaginary part to the frequency:

$$
\begin{aligned}
\frac{\omega_{\rm im}}{\omega_{\rm re}} &= \frac{\Delta(\omega_{\rm re})}{1+\rho_s^2k_\perp^2} \\
&= (\frac{m_e}{m_i})(\frac{\omega_{\rm re}\nu_{ei}}{\Omega_{ci}^2})(\frac{k_\perp}{k_z})^2\frac{1}{1+\rho_s^2k_\perp^2} \qquad (50)
\end{aligned}
$$

It is clear that a positive $\omega_{\rm re}$, implies that $\omega_{\rm im} > 0$, and this corresponds to growth, since $-i\omega t \simeq \omega_{\rm im}t$ for large t. The drift waves are **linearly unstable** in the presence of electron-ion collisions causing momentum transfer between the species. Note that the analysis *does not* apply, as it stands when $k_z = 0$!

What happens next? The growing "modes" don't simply go on growing. Rather, many nonlinear terms which are neglected in the analysis tends to stop or "saturate" the growth of the "sea" of modes produced by this drift instability from essentially "thermal noise". This sea of unstable drift waves is an example of "low frequency turbulence" observed in tokamaks and many other plasmas.There are of course many *other* instabilities than the very simple ones studied here. The net effect of this turbulence can only be really assessed by fully *electromagnetic, nonlinear* calculations which take not only the "**microscale**" structures around the size of ρ_s, but also of "**mesoscale**" fluctuations and the system-sized **macroscale** variations of density, temperature, fields and current.

The problems involved are very complicated, not only because kinetic effects such as **Landau damping, and trapped particles** play important roles, but also of the enormous disparity in temporal and spatial scales involved. You must remember that the "fluid" approach is strictly incorrect when the wave lengths of the disturbances are comparable to the Larmor radius. The ions would certainly tend to "average" out fluctuations finer in scale than their Larmor radii. Numerical simulations take fantastic computing resources to do realistic calculations. However, as in fluid turbulence, many models have been studied, both analytically and computationally. In recent years, the enormous computing power available have indeed brought us close to solving these tremendous challenges.

In essence, plasma turbulence due to drift and shear-Alfvén waves (which we have not considered here at all!) is created by the "free-energy" available in the density, temperature and current gradients due to the external sources (or fusion/gravitational heating!). This turbulence produces stochastic magnetic fields and strong $\mathbf{E} \times \mathbf{B}$ drift fluctuations which tend, as a rule, to transport energy, particles and momentum far faster than the rates calculable from Coulomb collisions, even when trapping effects are taken into account. The turbulence modifies the "profiles" and the latter influence the former! There is a "modal ecology" where modes "eat" other modes, grow and die according to the specific rules prescribed by plasma dynamics and Maxwell's equations reduced to a quasi-neutral electrodynamic system. The plasma "organizes itself" in ways which are not fully understood. Although "chaos" plays an important role, plasma turbulence, like atmospheric turbulence is a fascinating mixture of coherent and incoherent interactions among a very large number of degrees of freedom in driven dissipative systems.

The challenge is not only to "understand" the main features of this nonlinear "dance" of the plasma, but to *calculate* effectively *all the measurable properties associated with the transport and confinement* and show that the results are in agreement with measurements in experiments or astrophysical/technological observations. In this Lecture, I have outlined the basic concepts involved and the challenges ahead.