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EURATOM/UKAEA Fusion

Lectures on Plasma Physics: Problems and Solutions

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Cosmic Plasmas, Physics 418

Problem Set for Lecture 1: Elements

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Note: The problems (in this set and those for Lectures 2,3) are designed to bring out key points made in the lectures and clarify them through explicit examples. Hints for their solutions are provided in some cases. Problems which are "hard" are starred; they will be dealt with in the "problems class", at least in outline. Solutions to the problems will be handed out separately. Some additional problems are also provided for entertainment for those who wish to go deeper into the subject. They are optional extras and will not be required as a part of this course.

1. Calculate in joules the energy acquired by a deuterium nucleus (composed of a proton and a neutron) when accelerated in a straight line by an electric field of 2 kV/m over a distance of 1 m. Assuming it started from rest, calculate its speed at this point.

2. Find the spherically symmetric solution to Eq.(2) of the text which matches with the Coulomb potential of a point charge e at the origin, $\frac{e}{4\pi\epsilon_0 R}$, as $R \to 0$ and which goes to zero at infinity. (Hint: You may find it useful to use the fact that for functions $f(R)$ with, $R^2 = x^2 + y^2 + z^2$, $\nabla^2 f = \frac{1}{R^2}\frac{d}{dR}(R^2\frac{df}{dR})$. Write, $\phi = f(R) = g/R$. Solve the equation which results for g, and apply the boundary conditions stated).

3. A uniform, infinite cylindrical plasma (radius a, cylinder oriented along the z-axis) carries a steady current of I_p MA. Calculate the magnetic field at any radius r in magnitude and direction due to this current. You may assume that the current density, $\mathbf{j} = (\frac{I_p}{\pi a^2})\mathbf{z}$ MA/m^2, for $r \leq a$ and vanishes for $r > a$. (Hint: Solve "Ampère's Law", Eq.(3), for $\mathbf{B}$, using the given values for ϵ_0, c). Calculate, using the results, the magnetic field at $r = a = 1$m, when $I_p = 2$MA. Generalise this result to obtain a formula for the magnetic field in vacuum for $r > a$ due to an *arbitrary, cylindrically symmetric* distribution of the current density within the plasma (ie., when j_z is an arbitrary function of $r^2 = x^2 + y^2$, but the total current flowing is still I_p).

4*. Find the *general solution* of the 1-dimensional, collisionless, "free-particle" kinetic equation for the distribution function $f(x, v, t)$,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$$

Using this solution, or otherwise prove the following results. Assume that the domain in question is all of "velocity space" (ie., $-\infty < v < \infty$) and *periodic* position space, (ie., $f(x, v, t)$ is a periodic function of x/L with period, 2π, and L is a fixed, "periodicity length"):

1. The integrals, $I_n(t) = \int\int f^n dxdv$ are constants of the motion for any n for which the integrals exist at $t = 0$; ie., $\frac{dI_n}{dt} = 0$. Deduce that the evolution of f according to this equation conserves the total number of particles, $N = I_1$.

2. Show that if *initially* f is nonnegative (ie., $f(x, v, 0) \geq 0$), the kinetic equation preserves this positivity property for all times (both for $t > 0$ and for $t < 0$!).

3. Show that the integral, $H(t) = \int\int(f \ln f)dxdv$ (called **Boltzmann's** μ-space entropy) is also a constant of the motion.

4. Assume that at $t = 0$ $f(x, v, 0) = C.n_0(x) \exp\left[-M(v - u_0(x))^2/2T\right]$, where, C is a "normalization constant", M represents the mass of the particles, and T(joules) represents the uniform "temperature" of the particles and $n_0(x), u_0(x)$ are certain periodic functions of x, where $n_0(x) \geq 0$. First, evaluate C in terms of M, T, N given the condition that $\int_{-\infty}^{+\infty} dv \int_{-\pi}^{\pi} f(x, v, 0)\frac{dx}{L} = N$, assuming that the function, $n_0(x)$ satisfies the condition, $\int_{-\pi}^{\pi} n_0(x)\frac{dx}{L} = N$. Next, *interpret* physically the meaning of $n_0(x)$ and $u_0(x)$. Taking, $n_0(x) = \sum_{k=-\infty}^{\infty} \nu_k \cos(kx/L)$, and $u_0(x) = \sum_{k=-\infty}^{\infty} U_k \cos(kx/L)$, where the Fourier coefficients are certain constants characteristic of the initial conditions, find, using the general solution for f, the expressions for the *velocity* moments, $< f >= \int_{-\infty}^{\infty} f(x, v, t)dv; < vf >= \int_{-\infty}^{\infty} vf(x, v, t)dv; < v^2 f >= \int_{-\infty}^{\infty} v^2 f(x, v, t)dv$ as functions of x, t.

5. Show that, as $|t| \to \infty$, these velocity moments tend to *uniform and constant* values, in spite of the fact that f itself (considered as a function of x, t for fixed v) *does not*! (Hint: For this exercise, you will need to be able to evaluate "Gaussian integrals" like $\int_{-\infty}^{\infty} x^n \exp(-\beta x^2)dx$, where $\beta > 0, n$ are constants. For $n = 0$, the value of the integral is, $I_0(\beta) = (\frac{\pi}{\beta})^{1/2}$. For odd n the integral vanishes by

symmetry. For even n the values can be found by differentiation of I_0 repeatedly with respect to β).

The final result that the *velocity moments* of the solutions of the free particle kinetic equations approach constant values as $|t| \to \infty$ is called "phase mixing" and is a crucial (but not the only) element in an important phenomenon called "Landau" or "collisionless damping" which you will encounter many times.

5. Show that if at $t = 0$ the Maxwell equation, $\nabla.\mathbf{B} = 0$ is satisfied, it will hold for all t according to Faraday's Law.

6. Use the equation of continuity, Eq.(22) to derive Eq.(25) from Eq.(24) explicitly.

7. Show that in ideal MHD plasmas, $\mathbf{E}.\mathbf{B} = 0$.

8. Derive the ideal MHD equations in a form where $\mathbf{E}, \mathbf{j}$ do not appear explicitly.

9. Show that for any bounded volume V with boundary surface, S, $\int_S p\mathbf{n}dS = \int_V \nabla p dV$. (Hint: let $\mathbf{a}$ be an arbitrary constant vector. Apply Gauss' divergence theorem to $\int_V \mathbf{a}.\nabla p dV$ and infer the result from the fact that $\mathbf{a}$ is arbitrary).

Additional (optional) problems for entertainment and self-study

10. Solve the collisionless kinetic equation for particles of mass M moving in a conservative, external force-field, $F_x = \frac{dK}{dx}$ with potential, $K(x)$, a function of x in *infinite* x space, assuming that the distribution function does not vary with time:

$$v\frac{\partial f}{\partial x} + \frac{1}{M}\frac{dK}{dx}\frac{\partial f}{\partial v} = 0$$

(Hint: Consider the particle energy, $E(x, v) = \frac{1}{2}Mv^2 - K(x)$ and observe that the equation can be written as, $\frac{\partial(E,f)}{\partial(x,v)} = 0$. Infer that if you can find a particular solution to the above equation, an arbitrary function of it is also a solution! The same result can also be obtained by the *method of characteristics* which can be used more generally). If the system is *known* to be in thermodynamic equilibrium, determine the solution completely in terms of the total number of particles and the temperature. Calculate the number density as a function of x, assuming that $K(x) = -K_0x^2$ (ie., we have *harmonically bound particles*). If $K(x)$ does not grow unboundedly at infinity but goes to a constant, is thermodynamic equilibrium strictly possible at any non-zero temperature?

10*. Using the results of Problem 9, derive an equation for the time rate of change of $\mathbf{B}^2$. Assuming the potential, K to depend only on position, obtain an equation for the time rate of change of the kinetic energy per unit volume, $(1/2)\rho_{\mathrm{m}}\mathbf{u}^2$. From these two equations, obtain the local form of the law of conservation of energy for an ideal isentropic plasma.

11*. An ideal plasma is confined within an azimuthally symmetric toroidal vessel with an arbitrary cross section and a perfectly conducting, rigid wall. The electric field on the wall satisfies, $\mathbf{E}_{\mathrm{tangential}} = \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = \mathbf{0}$, where $\mathbf{n}$ is the unit outward surface normal at the wall. The plasma is assumed to be moving inside the vessel with arbitrary velocities $\mathbf{u}$ which satisfy the boundary condition , $\mathbf{u}.\mathbf{n} = 0$. Given that the plasma dynamics within this vessel is described by Eqs.(29)-(34) (Nb. The plasma motion is *not* assumed to be azimuthally symmetric or time independent!), demonstrate the following results.

1. The plasma total mass, $M_p = \int_V \rho_{\mathrm{m}} dV$ is a constant of the motion.
2. The *toroidal magnetic flux*, $\chi_{\mathrm{t}} = \int_\Psi \mathbf{B}.\mathbf{e}_\phi dRdZ$ is a constant of the motion, where the z-axis is the axis of symmetry of the toroidal vessel and cylindrical coordinates (R, Z, ϕ) are used to locate an arbitrary point within the plasma. The "wall" of the plasma is represented by a closed contour (does not vary with time), $\Psi(R, Z) = const$, and the integral is taken over the area enclosed by this contour in the "poloidal plane" with $\phi = 0$. The unit vector $\mathbf{e}_\phi = \mathbf{e}_Z \times \mathbf{e}_R$.
3. The global *helicity* of the magnetic field is defined by the integral (taken over the plasma volume bounded by the surface, $\Psi = const$), $I_B(t) = \int_V \mathbf{A}.\mathbf{B}dV$, where $\mathbf{B} = \nabla \times \mathbf{A}, \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi$. On the perfectly conducting wall, the electrostatic potential Φ may be taken to be zero. Then the vector potential $\mathbf{A}$ may be assumed to satisfy the Coulomb gauge, $\nabla.\mathbf{A} = 0$ and $\mathbf{A}_{\mathrm{tangential}} = 0$ on the wall. Show that I_B is a constant of the motion, and that $\mathbf{B}.\mathbf{n} = 0$.
4. From the *local energy conservation equation*, derive the *global* energy conservation law for the system.

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Solutions to the problems for Lecture 1

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Note: The suggested solutions here are essentially outlines or "sketches". You should check your own solutions against the suggestions made here and/or work through them if you had difficulties the first time.

1. Calculate in joules the energy acquired by a deuterium nucleus (composed of a proton and a neutron) when accelerated in a straight line by an electric field of 2 kV/m over a distance of 1 m. Assuming it started from rest, calculate its speed at this point.

Solution: We neglect the energy radiated by the accelerating charge. The work done by the field is clearly (since the charge on a deuterium nucleus is equal and opposite to that of an electron) 2 keV= 3.2×10^{-16}J. The mass, $m_D \simeq 2m_p = 3.35 \times 10^{-27}$kg. Hence, $v^2 = 9.6 \times 10^{10}(\mathrm{m/s})^2$. Hence, $v \simeq 3.1 \times 10^5$m/s. The non relativistic expression is adequate since this speed is only a thousandth of c.

2. Find the spherically symmetric solution to Eq.(2) of the text which matches with the Coulomb potential of a point charge e at the origin, $\frac{e}{4\pi\epsilon_0 R}$, as $R \to 0$ and which goes to zero at infinity. (Hint: You may find it useful to use the fact that for functions $f(R)$ with, $R^2 = x^2 + y^2 + z^2$, $\nabla^2 f = \frac{1}{R^2}\frac{d}{dR}(R^2\frac{df}{dR})$. Write, $\phi = f(R) = g/R$. Solve the equation which results for g, and apply the boundary conditions stated).

Solution: The spherically symmetric Debye equations is,

$$\frac{1}{R^2}\frac{d}{dR}(R^2\frac{d\phi}{dR}) = (\frac{2\bar{n}e^2}{\epsilon_0 T})\phi$$

Putting, $\phi = g/R$, we see that g satisfies the equation,

$$\frac{d^2 g}{dR^2} = (\frac{2\bar{n}e^2}{\epsilon_0 T})g$$

The solution of this equation which goes to zero at infinity is clearly, $g = C.\exp(-R/\lambda_{\text{Debye}})$, where, $\lambda_{\text{Debye}} = (\frac{\epsilon_0 T}{2\bar{n}e^2})^{1/2}$, and C is a constant. Since we require the solution, $\phi = g/R$ to tend to the Coulomb potential at the origin, we find that, $C = \frac{e}{4\pi\epsilon_0}$.

3. A uniform, infinite cylindrical plasma (radius a, cylinder oriented along the z-axis) carries a steady current of I_p MA. Calculate the magnetic field at any radius r in magnitude and direction due to this current. You may assume that the current density, $\mathbf{j} = (\frac{I_p}{\pi a^2})\mathbf{z}$ MA/m^2, for $r \leq a$ and vanishes for $r > a$. (Hint: Solve "Ampère's Law", Eq.(3), for $\mathbf{B}$, using the given values for ϵ_0, c). Calculate, using the results, the magnetic field at $r = a = 1$m, when $I_p = 2$MA. Generalise this result to obtain a formula for the magnetic field in vacuum for $r > a$ due to an *arbitrary, cylindrically symmetric* distribution of the current density within the plasma (ie., when j_z is an arbitrary function of $r^2 = x^2 + y^2$, but the total current flowing is still I_p).

Solution: The only nontrivial component of Ampère's equation in cylindrical polar coordinates (r, ϕ, z) is:

$$\begin{aligned} c^2\epsilon_0 \frac{1}{r}\frac{d}{dr}(rB_\phi) &= (\frac{I_p}{\pi a^2}), (r < a) \\ &= 0, (r \geq a) \end{aligned}$$

We must find a solution for which the field vanishes at infinity and is non singular at $r = 0$. Clearly, we have, for $r < a$,

$$B_\phi(r) = (\frac{I_p}{2\pi a^2 c^2 \epsilon_0})r$$

as can be verified immediately by direct substitution. Evidently, in the region, $r > a$, the equation is satisfied by, $B_\phi = C/r$, where C is a constant to be determined. This must match smoothly at $r = a$ (if not there will be a discontinuity there which will imply a "current sheet", which by the conditions of the problem does not exist!). It follows therefore that $C = (\frac{I_p}{2\pi c^2 \epsilon_0})$. In the general case of the current distribution within the cylinder being an arbitrary function of radius, we see that only the integral *within* the cylinder is modified and the interior field distribution is changed. The constant C depends *only* on the total current, I_p, and hence the *vacuum* field, outside the plasma is $B_\phi = (\frac{I_p}{2\pi r c^2 \epsilon_0})$. This formula gives the field in teslas in the numerical example.

4*. Find the *general solution* of the 1-dimensional, collisionless, "free-particle" kinetic equation for the distribution function $f(x, v, t)$,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$$

Using this solution, or otherwise prove the following results. Assume that the domain in question is all of "velocity space" (ie., $-\infty < v < \infty$) and *periodic* position space, (ie., $f(x, v, t)$ is a periodic function of x/L with period, 2π, and L is a fixed, "periodicity length"):

1. The integrals, $I_n(t) = \int\int f^n dxdv$ are constants of the motion for any n for which the integrals exist at $t = 0$; ie., $\frac{dI_n}{dt} = 0$. Deduce that the evolution of f according to this equation conserves the total number of particles, $N = I_1$.
2. Show that if *initially* f is nonnegative (ie., $f(x, v, 0) \geq 0$), the kinetic equation preserves this positivity property for all times (both for $t > 0$ and for $t < 0$!).
3. Show that the integral, $H(t) = \int\int (f \ln f)dxdv$ (called **Boltzmann's** μ-space entropy) is also a constant of the motion.
4. Assume that at $t = 0$ $f(x, v, 0) = C.n_0(x) \exp\left[-M(v - u_0(x))^2/2T\right]$, where, C is a "normalization constant", M represents the mass of the particles, and T(joules) represents the uniform "temperature" of the particles and $n_0(x), u_0(x)$ are certain periodic functions of x, where $n_0(x) \geq 0$. First, evaluate C in terms of M, T, N given the condition that $\int_{-\infty}^{+\infty} dv \int_{-\pi}^{\pi} f(x, v, 0)\frac{dx}{L} = N$, assuming that the function, $n_0(x)$ satisfies the condition, $\int_{-\pi}^{\pi} n_0(x)\frac{dx}{L} = N$. Next, *interpret* physically the meaning of $n_0(x)$ and $u_0(x)$. Taking, $n_0(x) = \sum_{k=-\infty}^{\infty} \nu_k \cos(kx/L)$, and $u_0(x) = \sum_{k=-\infty}^{\infty} U_k \cos(kx/L)$, where the Fourier coefficients are certain constants characteristic of the initial conditions, find, using the general solution for f, the expressions for the *velocity* moments, $< f >= \int_{-\infty}^{\infty} f(x, v, t)dv; < vf >= \int_{-\infty}^{\infty} vf(x, v, t)dv; < v^2 f >= \int_{-\infty}^{\infty} v^2 f(x, v, t)dv$ as functions of x, t.
5. Show that, as $|t| \to \infty$, these velocity moments tend to *uniform and constant* values, in spite of the fact that f itself (considered as a function of x, t for fixed v) *does not*! (Hint: For this exercise, you will need to be able to evaluate "Gaussian integrals" like $\int_{-\infty}^{\infty} x^n \exp(-\beta x^2)dx$, where $\beta > 0, n$ are constants. For $n = 0$, the value of the integral is, $I_0(\beta) = (\frac{\pi}{\beta})^{1/2}$. For odd n the integral vanishes by symmetry. For even n the values can be found by differentiation of I_0 repeatedly with respect to β).

The final result that the *velocity moments* of the solutions of the free particle kinetic equations approach constant values as $|t| \to \infty$ is called "phase mixing" and is a crucial (but not the only) element in an important phenomenon called "Landau" or "collisionless damping" which you will encounter many times.

Solution: Either by the *method of characteristics* or just by inspection, we see that the *general* solution to the given first order, linear pde. is, $f(x, v, t) = F(x - vt, v)$, where, $F(\alpha, \beta)$ is an "arbitrary" function of its arguments, $\alpha = x - vt, \beta = v$. From the given conditions, we see that F must be a *periodic* function of α/L with period 2π, and at $t = 0$, must be nonnegative for *values* of its arguments (since, by definition, it represents a distribution function or a probability).

1. We can prove this in two ways: without even solving the equation, if we multiply it by f^{n-1} and integrate over the x, v ranges stated, we see that,

$$\begin{aligned} \frac{dI_n}{dt} &= \int_{x,v} \frac{\partial f^n}{\partial t} \\ &= -\int_{x,v} \frac{\partial v f^n}{\partial x} dx dv \\ &= 0 \end{aligned}$$

   The last following from the periodicity of f with respect to the space variable, x. Alternatively, $I_n = \int_{x,v} F^n(x - vt, v) dx dv = \int_{\alpha,v} F^n(\alpha, v) d\alpha dv$. Clearly a periodic function integrated over its period can be "shifted" by an arbitrary amount without altering the value of the integral! Hence the time variable simply disappears from I_n and the result follows! Since, for $n = 1$, the integral equals the total number N of particles, the evolution equation clearly conserves this.

2. Nothing in the general solution says we *have* to take $t > 0$. Since the initial conditions dictate that $F(\alpha, \beta) \geq 0$, we must have that, $f(x, v, t) \equiv F(x - vt, v) \geq 0$ for *all* time.

3. Observe that if $f(x, v, t)$ is a solution of the kinetic equation, and $G(z)$ is an arbitrary (sufficiently regular, of course) function of its argument, then, $G(F) = g(x, v, t)$ is *also* a solution! This is because,

$$\begin{aligned} \frac{\partial G}{\partial t} + v\frac{\partial G}{\partial x} &= G' \left[\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x}\right] \\ &= 0 \end{aligned}$$

where G' is the derivative with respect to its argument and we have used the "chain rule" of calculus. It follows that $G = f \ln f$ is also a solution, and from the preceding result, the constancy in time of **Boltzmann's** H follows.

4. From the given relation for the "Gaussian" integrals, we have,

$$\int_{-\infty}^{+\infty} \exp\left[-M(v - u_0(x))^2/2T\right] dv \quad = \quad (\frac{2\pi T}{M})^{1/2}$$

irrespective of the function, $u_0(x)$. Since we are given the conditions that $n_0(x)$ is periodic in x/L with period 2π and, $\int_{-\pi}^{\pi} n_0(x)\frac{dx}{L} = N$, it is clear that $C = (\frac{2\pi T}{M})^{-1/2}$, in order to satisfy the normalization requirement. It is useful to think of the position coordinate, x/L as an "angle", θ, which varies from $-\pi$ to π. It is then easily seen that $n_0(\theta)$ is the number density (ie., number of particles per radian) of the system at $t = 0$. From the symmetry properties of Gaussian integrals, we know that,

$$\int_{-\infty}^{+\infty} \exp\left[-M(v - u_0(x))^2/2T\right] (v - u_0(x))dv \quad = \quad 0$$

It is clear therefore that,

$$\int_{-\infty}^{+\infty} f(x, v, 0)dv \quad = \quad n_0(x)$$
$$\int_{-\infty}^{+\infty} f(x, v, 0)vdv \quad = \quad n_0(x)u_0(x)$$

It is then clear that $u_0(x)$ is the average "fluid velocity" of the particles at the location x at $t = 0$.

5. We know from the general solution that $f(x, v, t) = C.n_0(x-vt) \exp[-M(v - u_0(x - vt))^2/2T]$, where,it is clear that $C = (\frac{2\pi T}{M})^{-1/2}$. This solution is still periodic in $\theta = x/L$ as before, but its v dependence is extremely complicated, as well as the time dependence! We have the following relations for the lowest velocity moment:

$$\int_{-\infty}^{+\infty} f(x, v, t)dv \quad = \quad C\int_{-\infty}^{+\infty} \sum_{k=-\infty}^{\infty} \nu_k \cos(kx/L - kvt/L) \exp\left[-M(v - u_0(x - vt))^2/2T\right] dv$$

The general evaluation of such series is indeed hopeless, but we are only interested in the behaviour of these averages in the limit, $|t| \to \infty$. Furthermore, to illustrate the points with the minimum of algebraic complexity, we consider the special case when $|u_0(x)| \ll (\frac{2T}{M})^{1/2}$. This simply says that the flow velocity is initially small compared to the "random" or "thermal" velocity. With this approximation, we may expand the local Maxwellian about the zero mean flow one and use the Fourier representation. We must discuss the long time behaviour of series like,

$$< f > = \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{\infty} \hat{\nu}_k(x) \exp(-ikvt/L) \exp\left[-Mv^2/2T\right] dv$$

where $\hat{\nu}_k(x)$ can be calculated from the given expressions for $n_0(x), u_0(x)$. The important thing is the *form* of this series. Now consider the integrals,

$$J_k(t) = \int_{-\infty}^{+\infty} \exp(-ikvt/L) \exp\left[-Mv^2/2T\right] dv$$

For $k \neq 0$, it is easily seen, either by explicit evaluation, or by the general **Riemann-Lebesgue** lemma of Fourier integrals, that as $|t| \to \infty$, $J_k(t) \to 0$. For, setting $V_{\text{th}} = (\frac{2T}{M})^{1/2}$, we see that,

$$\begin{aligned} J_k(t) &= V_{\text{th}} \int_{-\infty}^{+\infty} \exp\left[-\lambda^2 - i(kV_{\text{th}}t/L)\lambda)\right] d\lambda \\ &= (\pi)^{1/2} V_{\text{th}} \exp\left[-(kV_{\text{th}}t/2L)^2\right] \end{aligned}$$

from the known properties of Gaussian (complex!) integrals (consult almost any book on applied mathematics, or even the Feynman Lectures!). Hence, in the limit as $|t| \to \infty$, all the terms of the series for $< f >$, *except* the $k = 0$ term tend very rapidly to zero! Hence $< f > \to \nu_0 = \frac{N}{2\pi}$. The same proof can be clearly extended, almost with no changes to all the velocity moments. Thus, all of such moments "relax" to their spatial average values which would be the limit of thermodynamic equlibrium!

The interesting thing is that this "approach" to equilibrium by the moments of the distribution function takes place *reversibly* in time, and in the presence of an infinity of constants of the motion! Furthermore, it is very easy to verify that $f(x, v, t)$ *itself* does not show this "Landau damping" behaviour, nor is any particular "wave particle interaction" is necessary. The simplest of all possible kineticequations thus exhibits "phase mixing" and demonstrates a very generic way in which time reversible systems can show apparently irrecersible approach to equilibrium of statistical averages over an infinite velocity space under certain conditions (which are met in our example).

5. Show that if at $t = 0$ the Maxwell equation, $\nabla.\mathbf{B} = 0$ is satisfied, it will hold for all t according to Faraday's Law.

Solution: Take the divergence of Faraday's Law and obtain, $\frac{\partial}{\partial t}(\nabla.\mathbf{B}) = 0$. The result follows.

6. Use the equation of continuity, Eq.(22) to derive Eq.(25) from Eq.(24) explicitly.

Solution: Expand the terms using the product rule of differentiation.

7. Show that in ideal MHD plasmas, $\mathbf{E}.\mathbf{B} = 0$.

Solution: In ideal MHD, $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$. Taking dot product with $\mathbf{B}$ gives the result.

8. Derive the ideal MHD equations in a form where $\mathbf{E}, \mathbf{j}$ do not appear explicitly.

Solution: Trivial!

9. Show that for any bounded volume V with boundary surface, S, $\int_S p\mathbf{n}dS = \int_V \nabla p dV$. (Hint: let $\mathbf{a}$ be an arbitrary constant vector. Apply Gauss' divergence theorem to $\int_V \mathbf{a}.\nabla p dV$ and infer the result from the fact that $\mathbf{a}$ is arbitrary).

Solution: Use the hint!

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Problem Set for Lecture 2: Particle orbit theory

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Note: The problems are designed to bring out key points made in the lectures and clarify them through explicit examples. Hints for their solutions are provided in some cases. Problems which are "hard" are starred; they will be dealt with in the "problems class", at least in outline. Solutions to the problems will be handed out separately. Some additional problems are also provided for entertainment for those who wish to go deeper into the subject. They are optional extras and will not be required as a part of this course.

1. Using the nonrelativistic Lagrangian given in Eq.(6), show that Lagrange's equations are the same as Newton's equations in Cartesian coordinates.

2. Cylindrical coordinates, (R, ϕ, Z) are related to Cartesian coordinates through the transformation laws, $x = R\cos\phi, y = R\sin\phi, z = Z$. Show that the velocity components, V_R, V_ϕ, V_Z of a particle with coordinates, $(R(t), \phi(t), Z(t))$ can be expressed in terms of $\frac{dR}{dt}, \frac{d\phi}{dt}, \frac{dZ}{dt}$ according to the formula,

$$
\begin{aligned}
\mathbf{V} &= \frac{d\mathbf{R}}{dt} \\
&= \frac{dR}{dt}\mathbf{e}_R + R\frac{d\phi}{dt}\mathbf{e}_\phi + \frac{dZ}{dt}\mathbf{e}_Z
\end{aligned}
$$

where $\mathbf{e}_{R,\phi,Z}$ are, respectively the unit vectors in the corresponding directions. In cylindrical coordinates, the vector potential, $\mathbf{A} = A_R\mathbf{e}_R + A_\phi\mathbf{e}_\phi + A_z\mathbf{e}_Z$ and $\mathbf{B} = \nabla\times\mathbf{A} = (\frac{1}{R}\frac{\partial A_Z}{\partial\phi} - \frac{\partial A_\phi}{\partial Z})\mathbf{e}_R + (\frac{\partial A_R}{\partial Z} - \frac{\partial A_Z}{\partial R})\mathbf{e}_\phi + (\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \frac{1}{R}\frac{\partial A_R}{\partial\phi})\mathbf{e}_Z$. If Φ represents the scalar potential, obtain the nonrelativistic Lagrangians for a charged particle with mass M and charge e moving in the fields due to these potentials. (Hint: $\mathbf{R} = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + Z(t)\mathbf{e}_Z$,; express the unit vectors, $\mathbf{e}_R(t), \mathbf{e}_\phi(t)$ in terms of $\mathbf{e}_x, \mathbf{e}_y$. Write down the expression for the kinetic energy, $T = \frac{1}{2}M\mathbf{V}.\mathbf{V}$, using the results of Pb. 1 and the nonrelativistic Lagrangian in the Notes (see Eq.(6))).

3*. Using the nonrelativistic Lagrangian of the previous problem and Lagrange's equations, obtain the forms for the Newton's equations of motion of a charged particle in cylindrical coordinates. If the potentials of Problem 3 are independent of ϕ, the azimuthal angle, obtain the corresponding constant of motion. This constant is called the "canonical angular momentum" of the particle. Is the *ordinary angular momentum* of the particle about the z-axis a constant of the motion in the presence of a magnetic field, even if the Lagrangian has azimuthal symmetry? If, in addition, the potentials are also independent of t, what can you say about the energy, H?

4. Estimate the gyroradius r_L of an alpha particle (Helium 4 nucleus) with Mass number $A = 4$, atomic number, $Z = 2$ and energy 4 MeV in the Earth's magnetic field (approx. 10^{-4}T). Given that the typical scale length of the field is of the order of 1000 km, calculate ρ^* for this particle. Look up the data for the Sun's average field near its surface and for Jupiter and calculate the corresponding numbers.

5. Imagine a charged particle moving with a speed $v_{\|}$ along a circular B-field line, the radius of curvature R_c is assumed much larger than the Larmor radius, r_L of the particle in the field of magnitude B. Treat the centrifugal force on the particle, $Mv_{\|}^2/R_c$ as an "external" force, $\mathbf{f}_{\perp}$ (cf. Eq.(21)) and work out the resultant drift. Compare your result in magnitude and direction with the last term of Eq.(50).

Additional (optional) problems for entertainment and self-study

6. Taking $\mathbf{b} = \mathbf{e}_z$ and $\mathbf{B} = B\mathbf{e}_z, \mathbf{E} = E\mathbf{e}_z$, where B, E are constants, integrate the *relativistic* equations,

$$\frac{d\mathbf{p}_{\perp}}{dt} = (eB/M)(\mathbf{p}_{\perp} \times \mathbf{e}_z)$$
$$\frac{d\mathbf{p}_{\|}}{dt} = eE\mathbf{e}_z$$

subject to the conditions, $p_{\|}(0) = 0, \mathbf{p}_{\perp}(0) = p_0\mathbf{e}_x$. Determine the time variation of M. Obtain the expressions for the velocities, $\mathbf{v} = \mathbf{p}/M$ and solve for the trajectory of the system, given that the particle starts from $(x_0, 0, 0)$ at $t = 0$. Determine the time variation of the Larmor frequency and the Larmor radius in the *laboratory frame*.(Hint:

it is advantageous to introduce the new independent "gyrophase" variable, through, $\frac{d\zeta}{dt} = -eB/M(t)$ and write the equations in component form in Cartesian coordinates).

7.* Solve the equations,

$$\frac{d\mathbf{p}}{dt} = eE\mathbf{e}_x + (eB/M)\mathbf{p} \times \mathbf{e}_z \tag{1}$$

together with the trajectory equations, $\frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{M}$ in the relativistic case *in complete* generality, assuming only that E, B are constants and $M = (m^2 + \mathbf{p}^2/c^2)^{1/2}$, where m is the rest mass and e is the charge of the particle. In particular, consider arbitrary initial data and all possible values of E/B. What happens to the "$\mathbf{E} \times \mathbf{B}$" drift of a charged particle when $E/B > c$?

8.* A uniform $\mathbf{B}$ field along the z-axis is represented by the vector potential, $\mathbf{A} = (\frac{BR^2}{2})\mathbf{e}_\phi$. Consider the plane non relativistic motion (in the plane, $Z = 0$) of a charged particle of mass M and charge e under the action of a central gravitational potential, $G\mu/R$ where μ is the mass of the central attracting body (neglect the reaction on this body which can be assumed to be at rest at the origin). Set up the two-dimensional equations of motion and solve them after obtaining the constants of the motion. In particular, obtain the period and the trajectory of the particle's motion in the plane.

9*. Reconsider the previous problem, assuming that $\frac{dZ}{dt} = 0$ for all time, but the motion is relativistic (ignoring any radiation due to accelerations) due to the very large value of the gravitational attraction, what happens to the motion? What happens if the central force is a repulsive, electrostatic one?

10. Derive Eq.(43) of the text for the nonrelativistic averaged "drift" Lagrangian in detail.

11. Using the previous result and the suggestion made in the text, derive Eq.(26) of the text. Similarly show in detail that Eq.(48) follows by varying the drift Lagrangian and taking the parallel component.

12. Justify the statement made in the text that the terms neglected in deriving Eq.(50) are indeed smaller than the retained terms by at least ρ^*.

13*. A "wiggler" magnetic field can be specified approximately in cylindrical coordinates thus:

$$B_R = -\frac{1}{R}\frac{\partial \Psi}{\partial Z}$$

$$B_Z = \frac{1}{R}\frac{\partial \Psi}{\partial R}$$
$$B_\phi = 0$$

where the function $\Psi(R, Z) = \left[B_0 + B_1 \cos(\frac{Z}{L})\right](\frac{R^2}{2})$, where $B_0 \gg B_1$ are constants and L is a scale length, much larger than the Larmor radius r_L of a particle with mass, M and charge e moving nonrelativistically. Show that this field is indeed divergence free and determine the components of the vector potential in terms of Ψ.

Assume that there is no electric field in the system. Integrate the equation Eq.(48) and find the conditions on the initial values of $v_{||}, v_\perp$ and location for the particles to be "trapped" in the minima of the field and for them to be "passing" (ie., be "untrapped"). Assume that the probability of finding a particle with a given set of initial velocities is a Maxwellian at a constant temperature, calculate, at a radius R_0 and location, Z_0, the probability of being trapped or passing. Apart from μ and kinetic energy, what other constants of motion exist in this case, and why?

14*. A purely azimuthal field in a vacuum is of the form, $\mathbf{B} = B_\phi \mathbf{e}_\phi = B_0(\frac{R_0}{R})\mathbf{e}_\phi = -\frac{\partial A_Z}{\partial R}\mathbf{e}_\phi$. It is regular for $R > 0$ and R_0 is a "reference radius" where the value of the field is B_0. Assuming there are no electric fields and the magnetic field is stationary, solve the exact relativistic (and nonrelativistic) equations of motion by finding the three constants of the motion. Assume that at $t = 0$ the particle is at $R = R_{\text{in}}, \phi = 0, z = 0$ and $\mathbf{v} = v_R(0)\mathbf{e}_R + v_\phi(0)\mathbf{e}_\phi + v_Z(0)\mathbf{e}_Z$. Obtain the geometrical characteristics of the orbit. Generalize the solution to the case when there is an electrostatic potential, $\Phi(R)$.

15*. Reconsider the above problem non relativistically, using the "drift" equations derived for the guiding centre motion in leading order of ρ^*. Calculate the drifts of charged particles (positive and negative) in such a field, amd work out the guiding centre orbit. How does this compare with the exact solution in the nonrelativistic case obtained in the problem above?

16. Carry out the gyro averaging procedure *directly* on the nonrelativistic equations of motion and derive the drift equations for the guiding centres. (Hint: Consult the literature, particularly, Francis Chen's book, Ch. 2).

17. Derive the expressions for $\mathbf{V} = \frac{d\mathbf{R}}{dt}$ for a particle in terms of *spherical coordinates*, r, θ, ϕ (where, $x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r \cos\theta$), and using it, write down the nonrelativistic Lagrangian for a charged particle of mass M and charge e in these coordinates. Obtain the equations of motion from it in the presence of general $\mathbf{E}, \mathbf{B}$ fields (expressed as usual in terms of an electric potential, Φ and vector potential $\mathbf{A}$ in spherical coordinates) and an external gravitational force field $\mathbf{f} = M\nabla K$.

18*. Discuss, using the nonrelativistic, Newtonian guiding centre equations, the motion of a charged particle in the dipole magnetic field of a star of mass M_*. Find all the relevant constants of the motion of the exact system and reduce the equations as much as possible towards obtaining a numerical solution (the latter is *not* expected!). You may assume that the dipole is oriented along the spin axis of the star and the initial position of the particle is in the equatorial plane, well away from the "edge" of the star.

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Solutions to the problems for Lecture 2

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Note: The suggested solutions here are essentially outlines or "sketches". You should check your own solutions against the suggestions made here and/or work through them if you had difficulties the first time.

1. Using the nonrelativistic Lagrangian given in Eq.(6), show that Lagrange's equations are the same as Newton's equations in Cartesian coordinates.

Solution: The Lagrangian is,

$$L_{\rm nr} = (1/2)M\mathbf{v}^2 - e\Phi(\mathbf{r},t) + e\mathbf{A}(\mathbf{r},t).\mathbf{v}$$

Let $\mathbf{r} = (x_1, x_2, x_3), \mathbf{v} = (V_1, V_2, V_3) = (\dot{x}_1, \dot{x}_2, \dot{x}_3), \mathbf{A} = (A_1, A_2, A_3)$. Clearly, $\frac{\partial L_{\rm nr}}{\partial V_1} = MV_1 + eA_1$, $\frac{\partial L_{\rm nr}}{\partial x_1} = -e\frac{\partial \Phi}{\partial x_1} + e\frac{\partial A_1}{\partial x_1}V_1 + e\frac{\partial A_2}{\partial x_1}V_2 + e\frac{\partial A_3}{\partial x_1}V_3$.

Substituting in Lagrange's equation, $\frac{d}{dt}(\frac{\partial L_{\rm nr}}{\partial V_1}) = \frac{\partial L_{\rm nr}}{\partial x_1}$, and using the facts that $\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + V_1\frac{\partial \mathbf{A}}{\partial x_1} + V_2\frac{\partial \mathbf{A}}{\partial x_2} + V_3\frac{\partial \mathbf{A}}{\partial x_3}, \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi, \mathbf{B} = \nabla \times \mathbf{A}$, we obtain Newton's equations.

2. Cylindrical coordinates, (R, ϕ, Z) are related to Cartesian coordinates through the transformation laws, $x = R\cos\phi, y = R\sin\phi, z = Z$. Show that the velocity components, V_R, V_ϕ, V_Z of a particle with coordinates, $(R(t), \phi(t), Z(t))$ can be expressed in terms of $\frac{dR}{dt}, \frac{d\phi}{dt}, \frac{dZ}{dt}$ according to the formula,

$$\begin{aligned} \mathbf{V} &= \frac{d\mathbf{R}}{dt} \\ &= \frac{dR}{dt}\mathbf{e}_R + R\frac{d\phi}{dt}\mathbf{e}_\phi + \frac{dZ}{dt}\mathbf{e}_Z \end{aligned}$$

where $\mathbf{e}_{R,\phi,Z}$ are, respectively the unit vectors in the corresponding directions. In cylindrical coordinates, the vector potential, $\mathbf{A} = A_R\mathbf{e}_R + A_\phi\mathbf{e}_\phi + A_z\mathbf{e}_Z$ and $\mathbf{B} =$

$\nabla \times \mathbf{A} = (\frac{1}{R}\frac{\partial A_Z}{\partial \phi} - \frac{\partial A_\phi}{\partial Z})\mathbf{e}_R + (\frac{\partial A_R}{\partial Z} - \frac{\partial A_Z}{\partial R})\mathbf{e}_\phi + (\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \frac{1}{R}\frac{\partial A_R}{\partial \phi})\mathbf{e}_Z$. If Φ represents the scalar potential, obtain the nonrelativistic Lagrangians for a charged particle with mass M and charge e moving in the fields due to these potentials. (Hint: $\mathbf{R} = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + Z(t)\mathbf{e}_Z$,; express the unit vectors, $\mathbf{e}_R(t), \mathbf{e}_\phi(t)$ in terms of $\mathbf{e}_x, \mathbf{e}_y$. Write down the expression for the kinetic energy, $T = \frac{1}{2}M\mathbf{V}.\mathbf{V}$, using the results of Pb. 1 and the nonrelativistic Lagrangian in the Notes (see Eq.(6))).

Solution: $\mathbf{R} = R\cos\phi\mathbf{e}_x + R\sin\phi\mathbf{e}_y + Z\mathbf{e}_z$. Differentiate w.r.t t, bearing in mind that the basis vectors $\mathbf{e}_{x,y,z}$ are constants. We obtain, $\mathbf{V} = (\dot{R}\cos\phi - R\sin\phi\dot{\phi})\mathbf{e}_x + (\dot{R}\sin\phi + R\cos\phi\dot{\phi})\mathbf{e}_y + \dot{Z}\mathbf{e}_Z = \frac{dR}{dt}\mathbf{e}_R + R\frac{d\phi}{dt}\mathbf{e}_\phi + \frac{dZ}{dt}\mathbf{e}_Z$, from the definitions of the unit vectors in the cylindrical system.

Using these results, we see that, $L_{\rm nr}(\dot{R}, \dot{\phi}, \dot{Z}, R, \phi, Z, t)$ can be written as,

$$L_{\rm nr} = \frac{1}{2}M\left[\dot{R}^2 + R^2\dot{\phi}^2 + \dot{Z}^2\right] - e\Phi + e\dot{R}A_R + eR\dot{\phi}A_\phi + e\dot{Z}A_Z$$

3*. Using the nonrelativistic Lagrangian of the previous problem and Lagrange's equations, obtain the forms for the Newton's equations of motion of a charged particle in cylindrical coordinates. If the potentials of Problem 3 are independent of ϕ, the azimuthal angle, obtain the corresponding constant of motion. This constant is called the "canonical angular momentum" of the particle. Is the *ordinary angular momentum* of the particle about the z-axis a constant of the motion in the presence of a magnetic field, even if the Lagrangian has azimuthal symmetry? If, in addition, the potentials are also independent of t, what can you say about the energy, H?

Solution: Lagrange's equations give the equations of motion in cylindrical coordinates. The method is exactly similar to that used in Pb. 1 and will not be repeated. Note however that they automatically include the "centrifugal" and Coriolis terms in the radial and azimuthal directions.

If Φ and $\mathbf{A}$ are independent of the azimuthal angle, ϕ, the **cyclicity theorem** says that the "conjugate canonical momentum", $P_\phi = \frac{\partial L_{\rm nr}}{\partial \dot{\phi}}$ is a constant of the motion. We see easily that, $P_\phi = MR^2\dot{\phi} + eRA_\phi$. If A_ϕ is identically zero, we would indeed have, $P_\phi = MR^2\dot{\phi} = constant$, and this implies the constancy of the *mechanical* angular momentum. If A_ϕ is a general function of R, Z, t, however, the mechanical angular momentum *is not constant*! If Φ and $\mathbf{A}$ are independent of t, the Notes give the proof that the energy, $H = T + e\Phi$ is also a constant of the motion.

4. Estimate the gyroradius r_L of an alpha particle (Helium 4 nucleus) with Mass number $A = 4$, atomic number, $Z = 2$ and energy 4 MeV in the Earth's magnetic field (approx. 10^{-4}T). Given that the typical scale length of the field is of the order of 1000 km, calculate ρ^* for this particle. Look up the data for the Sun's average field near its surface and for Jupiter and calculate the corresponding numbers.

Solution: The rest mass of the alpha is, $Am_p = 4 \times 1.67 \times 10^{-27}$kg. The charge is $Z.e = 2 \times 1.6 \times 10^{-19}$C. Now, 4 MeV corresponds to 6.4×10^{-13}J. Since this is low compared to the rest energy, we may use nonrelativistic formulae. We may assume that the "perpendicular" energy, $E_\perp$, of the particle is equal to its "parallel" energy, and take the former to be 2MeV. Thus, $c_\perp = (2E_\perp/M)^{1/2} = (4\times1.6\times10^{-13}/4\times1.67\times10^{-27})^{1/2} \simeq 10^7$m/s. The gyro frequency, $\Omega_c = ZeB/Am_p = 10^{-4}(2 \times 1.6 \times 10^{-19})/(4 \times 1.67 \times 10^{-27}) = 5000$rads/s. Then, $r_L = c_\perp/\Omega_c = 2$km. Hence, $\rho^* \simeq 2 \times 10^{-3}$, and the particle is indeed "stuck" to the field line, in comparison with the length scale over which the field varies. I leave the reference work on the Sun and Jupiter to you.

5. Imagine a charged particle moving with a speed $v_\parallel$ along a circular B-field line, the radius of curvature R_c is assumed much larger than the Larmor radius, r_L of the particle in the field of magnitude B. Treat the centrifugal force on the particle, $Mv_\parallel^2/R_c$ as an "external" force, $\mathbf{f}_\perp$ (cf. Eq.(21)) and work out the resultant drift. Compare your result in magnitude and direction with the last term of Eq.(50).

Solution: The centrifugal force $\mathbf{f} = (Mv_\parallel^2/R_c)\mathbf{e}_R$. From the drift formula, we see that $\mathbf{f} \times \mathbf{b}/eB = (Mv_\parallel^2/eBR_c)\mathbf{e}_Z = (v_\parallel^2/\Omega_c R_c)\mathbf{e}_Z = (v_\parallel^2/\Omega_c)\mathbf{b} \times \mathbf{k}$, since, $\mathbf{k} = -(\frac{1}{R_c})\mathbf{e}_R$, for a circle with centre at the origin and radius R_c, at any point where the radius vector is along $\mathbf{e}_R(\phi)$. Furthermore, $\mathbf{b}$ is the unit tangent vector, $= \mathbf{e}_\phi(\phi)$, and we obviously have, $\mathbf{e}_Z = \mathbf{b} \times \mathbf{k}$. This proves the result required. You should draw a diagram to illustrate the relevant vectors in both magnitude and direction. Note incidentally that this drift depends only upon the parallel kinetic energy of the particle and its charge.

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Problem Set for Lecture 3: Applications of the fluid description

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Note: The problems are designed to bring out key points made in the lectures and clarify them through explicit examples. Hints for their solutions are provided in some cases. Problems which are "hard" are starred; they will be dealt with in the "problems class", at least in outline. Solutions to the problems will be handed out separately. Some additional problems are also provided for entertainment for those who wish to go deeper into the subject. They are optional extras and will not be required as a part of this course.

1. Show that Eqs.(1-3) imply that the internal energy per unit volume, $e = \frac{p}{\gamma-1}$, of a perfect gas satisfies the equation,

$$\frac{\partial e}{\partial t} + \nabla.(\mathbf{u}e) + p\nabla.\mathbf{u} \;=\; 0$$

If m is the mass of a molecule, the *number density* of the gas is related to the mass density $\rho_{\rm m}$ through, $n = \frac{\rho_{\rm m}}{m}$. Given that in a perfect gas, with temperature measured in joules, $p = nT$, find the equation satisfied by the temperature T.

2. Find the equation satisfied by the *entropy per particle*, $s = -\ln p + \frac{\gamma}{\gamma-1}\ln T$ of a perfect gas. Deduce the conservation equation satisfied by the *entropy per unit volume*, $\Sigma = ns = \rho_{\rm m}s/m$.

3. Show that Eq.(11) follows from Eq.(10).

4. Show that Eq.(13) follows from Eq.(9) and the equations of continuity, Eq.(1).

5. In deriving the wave equation for small amplitude sound waves, Eq.(35), we used the *isentropic relation* between density and pressure. **Newton** assumed a perfect gas and related pressure and density by treating the *temperature* (rather than the entropy, as we and **Laplace** did) as a constant. Derive the analogue of Eq.(35) with Newton's isothermal relation. How could you experimentally distinguish between the theory of sound waves due to **Laplace** (isentropic) and **Newton** (isothermal)?

6. If a source emits sound with a constant wavelength in vessels filled with hydrogen and helium respectively at the same temperature, where would one hear a higher frequency and why?

7. Show from Eq.(30) that $\mathbf{W}.\nabla H = 0$, in steady ideal flow. In Eq.(39) show why it is permissible to set $H(t) = 0$.

8. Consider a steady, incompressible ideal flow through a convergent-divergent "nozzle" formed by rotating a parabola ($y = (y_{\max} - y_{\min})(\frac{x}{L})^2 + y_{\min}$) about the x-axis. Apply the mass conservation and the Bernoulli equations to obtain the variation of the pressure along the symmetry axis. Assume that the "inlet" is located at $x = -L$ and the centreline velocity there is u_0. Calculate the difference in pressure between the inlet and the point where the pressure is a minimum (where is this point?).

9. We have derived the sound wave equation in a gas at rest. If the gas were moving with a *uniform and constant velocity*, $\mathbf{u}_0$ with respect to an observer, what is the equation of satisfied by the density fluctuations in that frame? Derive the famous "Doppler shift" formula from the corresponding dispersion equation (analogue of Eq.(36)).(Hint: Observe that $\rho_m = \rho_0, p = p_0, \mathbf{u} = \mathbf{u}_0$ is an exact solution of the Euler equations. Discuss "small oscillations" about this state as has been done in the Notes about the state of "rest" where, $\mathbf{u}_0 = 0$.)

Additional (optional) problems for entertainment and self-study

10. In deriving Eq.(29) ("Kelvin's circulation theorem"), justify the statement that all the terms on the right vanish. If p is *not* a function of ρ_m, is the theorem true?

11. Show that, in incompressible flow, the pressure can always be "eliminated" by solving the Poisson equation for it in terms of the velocity and vorticity.,

$$\nabla^2 \tilde{p} = -\rho_0 \nabla.(\mathbf{W} \times \mathbf{u}) - \rho_0 \nabla^2(\mathbf{u}^2/2)$$

If we further assume that the flow is *irrotational*, what does the resultant equation simplify to and how is it related to the Bernoulli relation, Eq.(52)?

12. Show that in *steady, two dimensional, incompressible, but rotational* flow, the vorticity $\tilde{\omega}$ is a function of the stream function, Ψ.(Hint: If f, g are two functions of (x, y) such that one is a function of the other, it is necessary and sufficient that, the Jacobian, $\frac{\partial(f,g)}{\partial(x,y)} = 0$, identically).

13.* ("Water waves"): Consider water filling an infinite (in the x, y plane) lake of uniform depth $-H$, with uniform gravity acting downwards (ie., the gravitational acceleration is, $-g\mathbf{z}$). At rest, the water surface is taken to be the $z = 0$ plane, above which we assume there is a gas at constant pressure, p_∞.

1. Solve for the hydrostatic pressure equilibrium within the lake, assuming the water density to be uniform and constant at ρ_0.

2. Next consider "small" oscillations of the water surface, given by the equation, $z = \eta(x, y, t)$. Assuming the flow due to the water wave to be small amplitude and irrotational, discuss the wave motion of the surface and the velocity fields. Assume that the "free surface", $F(x, y, z, t) \equiv z - h(x, y, t)$ is a "material surface" (this is called the kinematic boundary condition) and the pressure $\tilde{p} = p_\infty$ on this surface as the boundary conditions at the free surface. The bottom of the lake is assumed to be "impenetrable" (ie., $\mathbf{u}.\mathbf{z} = 0$). Periodic boundary conditions may be assumed in the x and y directions. Thus you may assume that all quantities vary like, $F(z) \exp[ik_x x + ik_y y - i\omega t]$, where k_x, k_y are "wave numbers" in the respective directions and ω is the frequency. Determine, in particular, the function F for the perturbed velocity potential ϕ and the pressure, and the dispersion relation which connectsω with k_x, k_y.

3. What is the primary difference between this "water wave" and the sound wave as regards the propagation speed of the disturbances of a given wave number?

14*. This problem is typical of the so-called "exterior flow problem" in ideal incompressible, irrotational flow theory and illustrates the utility of the velocity potential: Let $\mathbf{x} = (x, y, z)$.

1. Show that the velocity potential defined by, $\phi_0 = \mathbf{U}.\mathbf{x}$ is a possible, 3-d exact solution of the incompressible, irrotational hydrodynamic equations of motion, where $\mathbf{U}(t)$ is a spatially uniform but time-dependent vector.(Hint. Verify that, $\nabla^2(\phi_0) = 0$).

2. Calculate the pressure at every point of the flow and interpret the solution physically.

3. Show also that $\phi_1 = \frac{\mathbf{U}.\mathbf{x}}{r^3}; r^2 = x^2 + y^2 + z^2$ is a solution, as is the sum, $\phi_{a,\mathbf{U}}(\mathbf{x}) = \phi_0 + (\frac{a^3}{2})\phi_1$, where a is a positive constant.(Hint: $1/r$ is a solution of Laplace's equation, as are its spatial derivatives. The solution is simplified if at any instant t, you take $\mathbf{U} = U(t)\mathbf{e}_z$ and use spherical polar coordinates, r, θ, ϕ, noting that only r, θ enter the problem. The time is merely a parameter in this problem!)

4. Show that $\mathbf{n}.\nabla\phi_{a,\mathbf{U}} = 0$ on the spherical surface, $r = a$, where $\mathbf{n}$ is the unit normal to the surface. Interpret the solution as the flow past a solid sphere of radius a (in the region outside the sphere) which becomes uniform and equal to $\mathbf{U}$ at infinity. (Hint: calculate the fluid velocity on the surface of the sphere and at infinity from the velocity potential).

5. When $\mathbf{U}$ is independent of time, work out the pressure everywhere in the fluid and by integrating it over the surface of the sphere, and show that the total force exerted by the fluid on the sphere vanishes. In particular, show that the sphere does not experience a "drag" force parallel to the flow direction at infinity. This is a special case of an important theorem of steady, ideal, incompressible, irrotational flow known as **D'Alembert's Theorem** (sometimes mistakenly called "D'Alembert's paradox").

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Solutions to the problems for Lecture 3

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Note: The suggested solutions here are essentially outlines or "sketches". You should check your own solutions against the suggestions made here and/or work through them if you had difficulties the first time.

1. Show that Eqs.(1-3) imply that the internal energy per unit volume, $e = \frac{p}{\gamma-1}$, of a perfect gas satisfies the equation,

$$\frac{\partial e}{\partial t} + \nabla.(\mathbf{u}e) + p\nabla.\mathbf{u} = 0$$

If m is the mass of a molecule, the *number density* of the gas is related to the mass density ρ_{m} through, $n = \frac{\rho_{\mathrm{m}}}{m}$. Given that in a perfect gas, with temperature measured in joules, $p = nT$, find the equation satisfied by the temperature T.

Solution: The continuity equation, upon expansion and using the fact that $\rho_{\mathrm{m}} = m.n$ gives,

$$\frac{\partial n}{\partial t} + \mathbf{u}.\nabla n + n\nabla.\mathbf{u} = 0$$

Multiply this equation by $n^{\gamma-1}$ to get,

$$\begin{aligned}\frac{\partial n^\gamma}{\partial t} + \mathbf{u}.\nabla n^\gamma + \gamma n^\gamma \nabla.\mathbf{u} &= \frac{\partial n^\gamma}{\partial t} + \nabla.(\mathbf{u}n^\gamma) + (\gamma-1)n^\gamma\nabla.\mathbf{u} \\ &= 0 \qquad (1)\end{aligned}$$

Now, $p = p^*(n/n^*)^\gamma$. The required equation for $e = \frac{p}{\gamma-1}$ follows upon multiplying by the constant, $p^*(n^*)^{-\gamma}$ and using the definitions. Substituting, $e = nT/(\gamma-1)$ in the equation for e (just derived), we find that T satisfies,

$$\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T + (\gamma-1)T\nabla.\mathbf{u} = 0$$

2. Find the equation satisfied by the *entropy per particle*, $s = -\ln p + \frac{\gamma}{\gamma-1}\ln T$ of a perfect gas. Deduce the conservation equation satisfied by the *entropy per unit volume*, $\Sigma = ns = \rho_{\mathrm{m}} s/m$.

Solution: From the definitions, and the preceding results, observe that,

$$\begin{aligned}
\frac{Ds}{Dt} &= \frac{\partial s}{\partial t} + \mathbf{u}.\nabla s \\
&= -\frac{1}{p}\frac{Dp}{Dt} + \frac{\gamma}{\gamma-1}\frac{1}{T}\frac{DT}{Dt} \\
&= -(\gamma-1)\nabla.\mathbf{u} + \gamma\nabla.\mathbf{u} \\
&= \nabla.\mathbf{u}
\end{aligned}$$

Since, we have from the continuity equation, the relation, $\frac{Dn}{Dt} + n\nabla.\mathbf{u} = 0$, it follows that the entropy per unit volume, $\Sigma = ns$ satisfies,

$$\frac{D\Sigma}{Dt} = 0$$

We see that in an ideal (ie., isentropic) fluid, the material derivative of the entropy per unit volume vanishes. This is in fact what is meant by *isentropic* flow. Note that Σ need not be uniform in space or constant in time.

3. Show that Eq.(11) follows from Eq.(10).

Solution: It is easy to verify that Eq.(11) *reduces* to Eq.(10). Firstly, since K is only a function of position, using the equation of continuity and preceding results, we get the following:

$$\begin{aligned}
\frac{\partial \rho_{\mathrm{m}} K}{\partial t} + \nabla.(\rho_{\mathrm{m}} K \mathbf{u}) &= \rho_{\mathrm{m}} \mathbf{u}.\nabla K \\
\frac{\partial}{\partial t}\left(\frac{p}{\gamma-1}\right) + \nabla.\left(\mathbf{u}\frac{p}{\gamma-1}\right) &= -p\nabla.\mathbf{u}
\end{aligned}$$

We expand the LHS of Eq.(11):

$$
\begin{aligned}
\frac{\partial}{\partial t}\left[(\frac{\rho_{\rm m}\mathbf{u}^2}{2})+(\frac{p}{\gamma-1})-(\rho_{\rm m}K)\right] &= \frac{\partial\rho_{\rm m}}{\partial t}\frac{\mathbf{u}^2}{2}+\rho_{\rm m}\frac{\partial}{\partial t}(\frac{\mathbf{u}^2}{2})-\nabla.\left[(\frac{p}{\gamma-1})\mathbf{u}\right]-p\nabla.\mathbf{u} \\
&\quad +\nabla.(\rho_{\rm m}K\mathbf{u})-\rho_{\rm m}\mathbf{u}.\nabla K \\
&= -\nabla.\left[\mathbf{u}(\frac{1}{\gamma-1}p-K\rho_{\rm m})\right]-\nabla.(\rho_{\rm m}\frac{\mathbf{u}^2}{2}\mathbf{u})+\rho_{\rm m}\mathbf{u}.\nabla(\frac{\mathbf{u}^2}{2}) \\
&\quad +\rho_{\rm m}\frac{\partial}{\partial t}(\frac{\mathbf{u}^2}{2})-\nabla.(p\mathbf{u})+\mathbf{u}.\nabla p-\rho_{\rm m}\mathbf{u}.\nabla K \\
&= -\nabla.\left[\mathbf{u}(\frac{\rho_{\rm m}\mathbf{u}^2}{2}+\frac{\gamma}{\gamma-1}p-K\rho_{\rm m})\right] \\
&\quad +\rho_{\rm m}\frac{\partial}{\partial t}(\frac{\mathbf{u}^2}{2})+\rho_{\rm m}\mathbf{u}.\nabla(\frac{\mathbf{u}^2}{2})+\mathbf{u}.\nabla p-\rho_{\rm m}\mathbf{u}.\nabla K
\end{aligned}
$$

By virtue of Eq.(8) (it is equivalent to Eq.(10)!), the terms in the last row cancel, and we have arrived at Eq.(11).

4. Show that Eq.(13) follows from Eq.(9) and the equations of continuity, Eq.(1).

Solution: We have seen in the Notes that Eq.(9) can be expanded to,

$$\frac{\partial\mathbf{W}}{\partial t}+\mathbf{u}.\nabla\mathbf{W} = -\mathbf{W}\nabla.\mathbf{u}+\mathbf{W}.\nabla\mathbf{u}$$

From the equation of continuity, we find that $\frac{1}{\rho_{\rm m}}$ satisfies the equation,

$$\frac{\partial}{\partial t}(\frac{1}{\rho_{\rm m}})+\mathbf{u}.\nabla(\frac{1}{\rho_{\rm m}}) = (\frac{1}{\rho_{\rm m}})\nabla.\mathbf{u}$$

It follows readily that $\mathbf{W}^*=\mathbf{W}/\rho_{\rm m}$ satisfies Eq.(13).

5. In deriving the wave equation for small amplitude sound waves, Eq.(35), we used the *isentropic relation* between density and pressure. **Newton** assumed a perfect gas and related pressure and density by treating the *temperature* (rather than the entropy, as we and **Laplace** did) as a constant. Derive the analogue of Eq.(35) with Newton's isothermal relation. How could you experimentally distinguish between the theory of sound waves due to **Laplace** (isentropic) and **Newton** (isothermal)?

Solution: The linearized isentropic relation says, $\frac{\tilde{p}}{p_0}=\gamma\frac{\tilde{\rho}}{\rho_0}$. If we take the *isothermal assumption* of **Newton**, $p=\rho(T_0/m)$, where T_0 is the temperature (in joules). Then, linearization yields, $\frac{\tilde{p}}{p_0} = \frac{\tilde{\rho}}{\rho_0}$. In effect, we will get **Newton's** theory, simply by replacing $\gamma = 1$ in all the formulae! We get the same wave equation for small disturbances, but $C_s^2 = p_0/\rho_0$. It is now obvious how we distinguish between the two theories. We simply measure the speed of sound at fixed ambient conditions. Since $C_s^2(Laplace) = \gamma C_s^2(Newton)$ we have a clear experimental verdict. Indeed, Laplace modified Newton's theory *precisely* because Newton's prediction of the speed of sound was considerably lower than the actual value!

6. If a source emits sound with a constant wavelength in vessels filled with hydrogen and helium respectively at the same temperature, where would one hear a higher frequency and why?

Solution: Hydrogen being the lighter gas would have a larger sound speed than Helium at the same temperature. Since $\omega = C_s k$, at given wave length, the frequency would be higher in Hydrogen.

7. Show from Eq.(30) that $\mathbf{W}.\nabla H = 0$, in steady ideal flow. In Eq.(39) show why it is permissible to set $H(t) = 0$.

Solution: Taking the dot product of Eq.(30) with $\mathbf{W}$ gives the first result. Since $\mathbf{u} = \nabla\phi$, we may add an arbitrary function of t to ϕ without altering any physics! Hence we can simply "absorb" an arbitrary function $H(t)$ into the definition of the velocity potential ϕ. This new ϕ will give the same velocities as the old one and will have $H = 0$, identically.

8. Consider a steady, incompressible ideal flow through a convergent-divergent "nozzle" formed by rotating a parabola ($y = (y_{\text{max}} - y_{\text{min}})(\frac{x}{L})^2 + y_{\text{min}}$) about the x-axis. Apply the mass conservation and the Bernoulli equations to obtain the variation of the pressure along the symmetry axis. Assume that the "inlet" is located at $x = -L$ and the centreline velocity there is u_0. Calculate the difference in pressure between the inlet and the point where the pressure is a minimum (where is this point?).

Solution: Since the flow is stated to be steady and incompressible, we may assume the density to be ρ_{m}. Let all the inlet quantities be denoted by the subscript 0 and the "throat" ($y = y_{\text{min}}$) quantities, by the subscript, "t". Thus the inlet area is $A_0 = \pi y_{\text{max}}^2$, $A_t = \pi y_{\text{min}}^2$. The steady mass continuity relation reads,

$$\rho_{\text{m}} \bar{u}_0 A_0 = \rho_{\text{m}} \bar{u}_t A_t$$

The steady, incompressible Bernoulli relation is,

$$\rho_{\rm m}\frac{\bar{u}_0^2}{2}+\bar{p}_0 \;=\; \rho_{\rm m}\frac{\bar{u}_t^2}{2}+\bar{p}_t$$

It follows that,

$$\begin{aligned}\frac{\bar{p}_0-\bar{p}_t}{(\frac{1}{2}\rho_{\rm m}\bar{u}_0{}^2)} &= (\frac{\bar{u}_t}{\bar{u}_0})^2-1\\ &= (\frac{A_0}{A_t})^2-1\\ &= (\frac{y_{\rm max}}{y_{\rm min}})^4-1\end{aligned}$$

It is evident that $\bar{p}_0 > \bar{p}_t$. It is evident that the pressure at the throat is lower than at any other point, and correspondingly the velocity is the greatest there. Note that this solution is only valid if the velocity at the throat, $\bar{u}_t$is very much smaller than the speed of sound there (this is the condition for incompressible flow). It is a problem for the "advanced" student to consider the same general situation when the flow is*not assumed to be incompressible*, but still under steady, continuous and isentropic conditions (the general case of 1-d compressible "nozzle flow").

9. We have derived the sound wave equation in a gas at rest. If the gas were moving with a *uniform and constant velocity*, $\mathbf{u}_0$ with respect to an observer, what is the equation of satisfied by the density fluctuations in that frame? Derive the famous "Doppler shift" formula from the corresponding dispersion equation (analogue of Eq.(36)).(Hint: Observe that $\rho_{\rm m} = \rho_0, p = p_0, \mathbf{u} = \mathbf{u}_0$ is an exact solution of the Euler equations. Discuss "small oscillations" about this state as has been done in the Notes about the state of "rest" where, $\mathbf{u}_0 = 0$.)

Solution: Firstly, note that $\mathbf{u}_0, \rho_0, p_0$ do, indeed, satisfy the compressible, steady Euler equations. Next, we "linearize" the system, exactly as before, substituting, $\rho_{\rm m} = \rho_0 + \tilde{\rho}, p = p_0 + \tilde{p}, \mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$ in the full equations of motion and get (upon dropping terms quadratic in the disturbances)

$$\begin{aligned}\frac{\partial\tilde{\rho}}{\partial t}+\mathbf{u}_0.\nabla\tilde{\rho}+\rho_0\nabla.\tilde{\mathbf{u}} &= 0\\ \frac{\partial\tilde{\mathbf{u}}}{\partial t}+\mathbf{u}_0.\nabla\tilde{\mathbf{u}} &= -(\frac{1}{\rho_0})\nabla\tilde{p}\\ \frac{\tilde{p}}{p_0} &= \gamma(\frac{\tilde{\rho}}{\rho_0})\end{aligned}$$

It is easily seen that the relative density fluctuation, $\sigma = \frac{\tilde{\rho}}{\rho_0}$ satisfies the modified wave equation,

$$(\frac{\partial}{\partial t} + \mathbf{u}_0.\nabla)^2\sigma = C_s^2\nabla^2\sigma$$

in place of Eq.(35). As before, we try the "harmonic plane wave", $\sigma \simeq \exp(i\mathbf{k}.\mathbf{x} - i\omega t)$. We find that the dispersion relation relating $\mathbf{k}, \omega$ to be,

$$(\omega - \mathbf{u}_0.\mathbf{k})^2 \;\; = \;\; C_s^2\mathbf{k}^2$$

The frequency is no longer given by, $\omega = \pm C_s|\mathbf{k}|$, as in the case of sound waves propagating in "still" air, but is in fact, $\omega = \mathbf{u}_0.\mathbf{k} \pm C_s|\mathbf{k}|$. Suppose the source of sound moves "with the fluid", ie., with the velocity, $\mathbf{u}_0$. To a stationary observer, aligned along the direction of this velocity, sound will be appearing to come towards him with speed, $C_s + V$ and as the source recedes away from him, the wave will seem to be travelling at speed, $C_s - V$. Since the wave length remains unchanged, he will hear the higher pitched Doppler "up shift" on approach of the source towards him and the "down shift" as the source moves past and away from him. Note that the **D'Alembert** wave equation for sound is *not* form invariant under Galilean transformations. This says that sound speed depends upon the relative motion of the observer, the "medium" (ie., the gas), and the source in general and is *not absolute* like the speed of light which is invariant for all observers and requires time and space to be relative for relatively moving observers.

This simple analysis can be generalized to situations when $\mathbf{u}_0$ is nonuniform (ie varies in space) but still satisfies the equations of motion. In this "sheared flow" case, the wave equation is more complex (involving equations with spatially variable coefficients as opposed to Eq.(35) with its constant coefficients), as it also is when ρ_0, p_0 are spatially variable, steady exact solutions of the Euler equations. Such modified wave propagation problems in stratified/moving gases are of great importance in meteorology, astro/geophysics and in engineering.

Cosmic Plasmas, Physics 418

Problem Set for Lecture 4: Collisions, two-fluid theory and qualitative ideas of plasma turbulence

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Note: The problems are designed to bring out key points made in the lectures and clarify them through explicit examples. Hints for their solutions are provided in some cases. Problems which are "hard" are starred; they will be dealt with in the "problems class", at least in outline. Solutions to the problems will be handed out separately. Some additional problems are also provided for entertainment for those who wish to go deeper into the subject. They are optional extras and will not be required as a part of this course.

1. Consider the 1-dimensional diffusion equation,

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}$$

where, $-\infty < x < +\infty$. At $t = 0$, $n(x,0) = n_0(x)$, where n_0 is an even, positive function of x which tends to zero like an exponential as $x \to \infty$. Show, **without solving the equation** that,

1. $\int_{-\infty}^{+\infty} n(x,t)dx = I_0(t)$ is a constant equal to $I_0(0) = \int_{-\infty}^{+\infty} n_0(x)dx$
2. $<(\Delta x)^2> \equiv \frac{\int_{-\infty}^{+\infty} n(x,t)x^2 dx}{I_0}$ satisfies the equation, $\frac{d<(\Delta x)^2>}{dt} = D$, and integrate this equation.
3. Interpret $p(x,t) = n(x,t)/I_0$ as a "probablility distribution" and work out the physical meaning of the above two results.

2. Consider heat conduction in a copper bar (cf. Eqs.(4,5) of the lectures) to proceed according to Fourier's hypothesis. Since the volume doe not change, the change in the entropy, $TdS = C_V dT$, according to thermodynamics. Since entropy-flux can be defined as, $\frac{\mathbf{q}}{T}$, show that heat conduction equation, Eq.(5) implies that S continually increases within a bounded volume Δ of copper whose boundaries are "adiabatic" (ie., $\mathbf{q}.\mathbf{n} = 0$ on Σ, the bounding surface of Δ, where, $\mathbf{n}$ is the unit normal to Σ), provided $K_T > 0$ everywhere. If you start with an arbitrary initial distribution of temperature within Δ, determine the temperature distribution within the region Δ as $t \to \infty$. How does this asymptotic distribution depend on the thermal diffusivity χ, on the *initial distribution*, $T(\mathbf{r}, 0)$? Explain how Fourier's Law of heat conduction embodies the Second Law of Thermodynamics on the basis of this example.

3. Consider the two-fluid equilibrium obtained in the Notes (Eqs.(13-19)). If $S_p = (\frac{r}{a})S^*$, for $0 \leq r \leq a$ and S^* is a constant (units: electrons. $\mathrm{m}^{-3}.\mathrm{s}^{-1}$), obtain $n_e(r)$ and $B_z(r)$, assuming that ν_f is given by Braginskii's formula mentioned in the text. T_e, B_0, Λ are to be treated as known (constant) quantities.

4. Find the condition on L_n, k_θ, ρ_s for the ordering, $\omega_* \ll \Omega_{ci}$ to be valid.

5. Show that the solution to Eq.(31) of the Notes is given by, $\frac{e\tilde{\Phi}}{T_e} = F(\theta - \omega_* t, r, z)$, where $\frac{e\tilde{\Phi}}{T_e}(\theta, r, z, 0) = F(\theta, r, z)$, at $t = 0$ and F is periodic in θ, but may be an "arbitrary" function of its other arguments.

6*. Show that if $n_{e0} = n^* \exp\left[-(r/L_n)^2\right]$, where n^*, L_n are constants, ω_* is a constant. Solve Eq.(40) of the text for $\hat{A}$ and determine the eigenvalues ω and eigenfunctions. (Hint: a knowledge of Bessel functions is needed for this problem!)

7. Derive Eq.(46) of the Notes in detail.

8. Taking ν_{ei} to be given by Braginskii's formula for ν_f and estimate how large $\frac{k_\perp}{k_z}$ must be to get $\omega_{\mathrm{im}} = 0.01\omega_{\mathrm{re}}$. Assume that $T_e \simeq 1\mathrm{KeV}$, $B \simeq 1\mathrm{T}$, $L_n \simeq 1\mathrm{m}$, $\rho_s k_\perp = 0.1$. These are typical "tokamak" values. Take $\Lambda = 20$.

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Solutions to the problems for Lecture 11

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Note: The suggested solutions here are essentially outlines or "sketches". You should check your own solutions against the suggestions made here and/or work through them if you had difficulties the first time.

Some important corrections to the Lecture Notes for Lecture 11 and the problem set distributed. In the lecture Notes, a formula was given for the **Braginskii** collision frequency, ν_f Unfortunately, the formula quoted was in Gaussian units! The correct SI formula is the following:

$$\nu_f = \frac{2^{1/2} n_e \Lambda e^4}{12\pi^{3/2}\epsilon_0^2 m_e^{1/2} T_e^{3/2}}$$

The notation and the symbols are exactly as in the lecture Notes.

Corrections to the problems: Problem 1 needs a correction in Part 2, as indicated below. In the last problem, the number density, n_e was not specified. It should be taken as $5 \times 10^{19}\mathrm{m}^{-3}$.

1. Consider the 1-dimensional diffusion equation,

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}$$

where, $-\infty < x < +\infty$. At $t = 0$, $n(x, 0) = n_0(x)$, where n_0 is an even, positive function of x which tends to zero like an exponential as $x \to \infty$. Show, **without solving the equation** that,

1. $\int_{-\infty}^{+\infty} n(x,t)dx = I_0(t)$ is a constant equal to $I_0(0) = \int_{-\infty}^{+\infty} n_0(x)dx$

2. $< (\Delta x)^2 > \equiv \frac{\int_{-\infty}^{+\infty} n(x,t)x^2 dx}{I_0}$ satisfies the equation, $\frac{d<(\Delta x)^2>}{dt} = 2D$ (this was wrongly given in the distributed sheet!), and integrate this equation.

3. Interpret $p(x,t) = n(x,t)/I_0$ as a "probablility distribution" and work out the physical meaning of the above two results.

Solution: 1. Integrate the equation from $-\infty$ to ∞ with respect to x. From the boundary conditions at infinity, the rhs vanishes. Hence, we find, $\frac{dI_0}{dt} = 0$. Evaluating it at $t = 0$ gives the required result.

2. Multiply the diffusion equation by x^2 and integrate w.r.t x over the whole domain. We obtain,

$$\begin{aligned} \frac{d}{dt}\left(\int_{-\infty}^{+\infty} n(x,t)x^2 dx\right) &= D\int_{-\infty}^{+\infty} x^2 \frac{\partial^2 n}{\partial x^2} dx \\ &= -2D\int_{-\infty}^{+\infty} x\frac{\partial n}{\partial x} dx \\ &= 2D\int_{-\infty}^{+\infty} n dx \\ &= 2DI_0(0) \end{aligned}$$

where we use integration by parts (twice!) and the b.c's. Note that we also use the result obtained above for the constancy of I_0, incidentally correcting a typo in the problem. The rhs should be $2 \times DI_0$, *not* DI_0 as the problem originally stated! The integration is straight forward: $< (\Delta x)^2 > (t) = 2Dt + < (\Delta x)^2 > (0)$. This shows that the mean-square displacement is *linearly proportional to the time*, a typical result of **random walk theory**.

3. It is evident that $p(x,t) = n/I_0$ is a positive probability density, as it is clearly true that $\int_{-\infty}^{+\infty} p(x,t)dx = 1$. Thus $p(x,t)dx$ is the probability of finding a particle in the interval, $x, x+dx$ at time t. Obviously, $< (\Delta x)^2 > (t)$ is the mean square displacement of the random walking particle ("drunkard"!) at time t. We have shown that this increases linearly with time, although $< \Delta x >= 0$ for all time. Thus, even though the drunkard may have had a zero probability of being outside a finite interval *initially*, as $t \to \infty$, there will be a small likelihood of finding him arbitrarily far from the starting point.

2. Consider heat conduction in a copper bar (cf. Eqs.(4,5) of the lectures) to proceed according to Fourier's hypothesis. Since the volume doe not change, the change in the entropy, $TdS = C_V dT$, according to thermodynamics. Since entropy-flux can be defined as, $\frac{\mathbf{q}}{T}$, show that heat conduction equation, Eq.(5) implies that S continually increases within a bounded volume Δ of copper whose boundaries are "adiabatic" (ie., $\mathbf{q}.\mathbf{n} = 0$ on Σ, the bounding surface of Δ, where, $\mathbf{n}$ is the unit normal to Σ), provided $K_T > 0$ everywhere. If you start with an arbitrary initial distribution of temperature within Δ, determine the temperature distribution within the region Δ as $t \to \infty$. How does this asymptotic distribution depend on the thermal diffusivity χ, on the *initial distribution*, $T(\mathbf{r}, 0)$? Explain how Fourier's Law of heat conduction embodies the Second Law of Thermodynamics on the basis of this example.

Solution: We convert the equation for T into one for S!

$$\begin{aligned} C_V \frac{\partial T}{\partial t} &= -\nabla.\mathbf{q} \\ T\frac{\partial S}{\partial t} &= -\nabla.\left[T\frac{\mathbf{q}}{T}\right] \\ &= -T\nabla.(\frac{\mathbf{q}}{T}) - \nabla T.\frac{\mathbf{q}}{T} \end{aligned}$$

We divide this equation by T and use Fourier's Law, $\mathbf{q} = -K_T \nabla T$, to get,

$$\frac{\partial S}{\partial t} + \nabla.(\frac{\mathbf{q}}{T}) = K_T(\frac{\nabla T}{T})^2$$

Next we integrate this equation over the domain Δ and make use of the divergence theorem and the b.c., to obtain,

$$\begin{aligned} \frac{d}{dt}(\int_\Delta SdV) + (\int_\Delta \nabla.\frac{\mathbf{q}}{T}dV) &= \int_\Delta K_T(\frac{\nabla T}{T})^2 dV \\ \frac{d}{dt}(\int_\Delta SdV) + \int_\Sigma \frac{\mathbf{q}}{T}.\mathbf{n}dA &= \int_\Delta K_T(\frac{\nabla T}{T})^2 dV \\ \frac{d}{dt}(\int_\Delta SdV) &= \int_\Delta K_T(\frac{\nabla T}{T})^2 dV \end{aligned}$$

where the surface integral vanishes from the "adiabatic wall condition". For $K_T > 0$ everywhere, the rhs is evidently positive (unless $\nabla T \equiv 0$ *everywhere* within Δ!). Hence the total entropy of the system, $\int_\Delta SdV$ increases monotonically during the process of thermal conduction. Let us observe that as in problem 1, Fourier's heat conduction equation in Δ with the adiabatic b.c conserves the total internal energy. Thus integrating the basic equauation over Δ and making use of the divergence theorem and the b.c, we find that $E(t) = C_V \int_\Delta TdV$ is a constant. Since it is plain that in the "final state" as t tends to infinity, the entropy is a maximum if and only if T is *uniform* throughout Δ, we see that this uniform temperature must be equal to, $T_{\text{final}} = \frac{E(0)}{V(\Delta)C_V}$, where $V(\Delta)$ is the volume of the domain Δ. This *does not* depend on K_T! In fact it only depends upon the *total* initial internal energy, $E(0)$, the specific heat, C_V and the volume of the domain.

Fourier's Law clearly implies that a nonequilibrium temperature distribution in an adiabatic enclosure (ie., a thermally "insulated" system) will evolve in such a manner as to continually and irreversibly increase the entropy, which reaches its maximum value in the thermodynamic equilibrium state when the temperature becomes uniform everywhere. This evolution must of course be consistent with the *First Law of Thermodynamics*, ie., the Law of Conservation of Energy, and it indeed is, as shown by our simple calculations. The thermal conductivity must be everywhere positive, but its size is irrelevant as far as qualitative consistency with Thermodynamic Laws are concerned.

3. Consider the two-fluid equilibrium obtained in the Notes (Eqs.(13-19)). If $S_p = (\frac{r}{a})S^*$, for $0 \leq r \leq a$ and S^* is a constant (units: electrons. $\mathrm{m}^{-3}.\mathrm{s}^{-1}$), obtain $n_e(r)$ and $B_z(r)$, assuming that ν_f is given by Braginskii's formula mentioned in the text. T_e, B_0, Λ are to be treated as known (constant) quantities.

Solution: Consider Eq.(19):

$$\frac{1}{r}\frac{d}{dr}(rD_{\perp e}\frac{dn_e}{dr}) + (\frac{r}{a})S^* \quad = \quad 0$$

multiplying by r and integrating once from the origin, we find, quite generally, $D_{\perp e}\frac{dn_e}{dr} = -\frac{r^2}{3a}S^*$, where we use the fact the particle flux must be zero at the origin (ie., the axis). Now, we are given that $\nu_f \simeq \frac{2^{1/2}n_e\Lambda e^4}{12\pi^{3/2}\epsilon_0^2 m_e^{1/2}T_e^{3/2}}$. Note that all quantities except n_e are constant. We may therefore write, $D_{\perp e} = D^*\frac{n_e(r)}{n_e(0)}$, $D^* = \frac{1}{2}\rho_e^2\nu_0$, where, $\nu_0 = \frac{2^{1/2}n_e(0)\Lambda e^4}{12\pi^{3/2}\epsilon_0^2 m_e^{1/2}T_e^{3/2}}$. We then find that $n_e(r)$ satisfies the equation,

$$n_e\frac{dn_e}{dr} \quad = \quad -\frac{r^2}{3a}\frac{n_e(0)S^*}{D^*}$$

This equation is readily integrated to give, $n_e^2 = C - \frac{2r^3}{9a}\frac{n_e(0)S^*}{D^*}$. We can obtain the constant C from the condition, $n_e(a) = 0$. In fact, we get, $C = \frac{2a^2 n_e(0)S^*}{9D^*}$. Since, at $r = 0, n_e = n_0(0)$, by definition, we have, $9D^* = 2a^2S^*$. This then gives, $n_e(0) = (\frac{4a^2S^*}{9\rho_e^2})(\frac{12\pi^{3/2}\epsilon_0^2 m_e^{1/2}T_e^{3/2}}{2^{1/2}\Lambda e^4})$. The density profile can be written more transparently in the form, $n_e(r/a) = n_e(0)\left[1 - (\frac{r}{a})^3\right]$. It follows from Eq.(15) that, $B_z(r) =$

$B_0\left[1-\frac{2\mu_0 T_e n_e}{B_0^2}\right]^{1/2}$. It is an interesting problem to figure out how big S^* can be to have a consistent solution, and what happens if this is exceeded.

4. Find the condition on L_n, k_θ, ρ_s for the ordering, $\omega_* \ll \Omega_{ci}$ to be valid.

Solution: From Eq.(33) of the Notes, we have, $\frac{\omega_*}{\Omega_{ci}} = (\frac{C_s}{\Omega_{ci}L_n})(k_\theta\rho_s) = \frac{\rho_s}{L_n}(k_\theta\rho_s) = \rho^* k_\theta\rho_s$. Hence the condition is $\rho^* \ll 1$, for $k_\theta\rho_s \simeq 1$.

5. Show that the solution to Eq.(31) of the Notes is given by, $\frac{e\tilde{\Phi}}{T_e} = F(\theta-\omega_* t, r, z)$, where $\frac{e\tilde{\Phi}}{T_e}(\theta, r, z, 0) = F(\theta, r, z)$, at $t=0$ and F is periodic in θ, but may be an "arbitrary" function of its other arguments.

Solution: Simply substituting the given functional form into Eq.(31), we get the result! It is evident that at $t=0$, we match the initial condition. The periodicity of F wrt. θ is a simple consequence of the single-valuedness of the electrostatic potential.

6*. Show that if $n_{e0} = n^* \exp[-(r/L_n)^2]$, where n^*, L_n are constants, ω_* is a constant. Solve Eq.(40) of the text for $\hat{A}$ and determine the eigenvalues ω and eigenfunctions. (Hint: a knowledge of Bessel functions is needed for this problem!)

Solution: Substituting for n_{e0} and its derivative in Eq.(32), we see immediately that ω_* for this profile is independent of r and satisfies, $\omega_* = \frac{2T_e}{eB}\frac{m}{L_n}$. We may plainly write Eq.(40) in the form,

$$\frac{1}{r}\frac{d}{dr}(r\frac{d\hat{A}}{dr}) - (\frac{m}{r})^2\hat{A} = -K\hat{A}$$

$$K = \frac{\left[\frac{\omega_*}{\omega} - 1 + \frac{C_s k_z^2}{\omega^2}\right]}{\rho_s^2}$$

Suppose we set, $z = rK^{1/2}$. The equation then becomes (with $\hat{A} = f(z)$),

$$\frac{d^2 f}{dz^2} + \frac{1}{z}\frac{df}{dz} + (1 - \frac{m^2}{z^2}) = 0$$

We recognize this as Bessel's equation of integer order m in the variable z (see, for example, *Handbook of Mathematical Functions, eds. Abramowitz and Stegun, p. 358*). It has two linearly independent solutions, $J_m(z), Y_m(z)$, of which only $J_m(z)$ is regular at the origin. We require also that $\hat{A}(r=a) = 0$. This implies that, $J_m(aK^{1/2}) = 0$. Now, the zeros of Bessel functions are well-documented (see above reference or books on Bessel functions). They are all real and increasing numbers. We denote them by $\alpha_1^{(m)}, \alpha_2^{(m)}, ...$ We then obtain the result that $aK^{1/2} = \alpha_1^{(m)}$, for example. The "eigenvalue equation" determining the frequency, ω then becomes,

$$\left[\frac{\omega_*}{\omega} - 1 + \frac{C_s k_z^2}{\omega^2}\right] = \frac{(\alpha_1^{(m)}\rho_s)^2}{a^2}$$

$$(\frac{\omega}{\omega_*})^2 - \frac{1}{1 + \frac{(\alpha_1^{(m)}\rho_s)^2}{a^2}}\frac{\omega_*}{\omega} - \frac{1}{1 + \frac{(\alpha_1^{(m)}\rho_s)^2}{a^2}}\frac{C_s k_z^2}{\omega_*^2} = 0$$

It is entirely reasonable to regard, $\frac{\alpha_1^{(m)}}{a}$ as a typical "perpendicular wavenumber" associated with the zero,$\alpha_1^{(m)}$ of J_m. Setting this to be, $\kappa_1^{(m)} = \frac{\alpha_1^{(m)}}{a}$, we see that the above eigenvalue relation for the "dimensionless frequency", $\frac{\omega}{\omega_*}$ becomes,

$$(\frac{\omega}{\omega_*})^2 - \frac{1}{1 + (\kappa_1^{(m)}\rho_s)^2}(\frac{\omega_*}{\omega}) - \frac{1}{1 + (\kappa_1^{(m)}\rho_s)^2}(\frac{C_s k_z^2}{\omega_*^2}) = 0$$

Note that this is virtually identical with the "local dispersion relation", Eq.(41) derived in the text, upon dividing the latter by ω_*^2! Incidentally, we see that the "normal modes" are labelled by the "azimuthal quantum number", m and the "radial quantum number", $\alpha_k^{(m)}, k = 1, 2, ...$ This completes the exact solution of the drift wave poblem for this "Gaussian" density profile.

7. Derive Eq.(46) of the Notes in detail.

Solution: Using Eq.(35) for $\tilde{\mathbf{u}}_{\perp i}$, and Eq.(45) for $\tilde{\mathbf{u}}_{\perp e}$, we find that the perturbed perpendicular current density,

$$\begin{aligned}
\tilde{\mathbf{j}}_\perp &= en_{e0}(\tilde{\mathbf{u}}_{\perp i} - \tilde{\mathbf{u}}_{\perp e}) \\
&= en_{e0}\left[(C_s\rho_s)\mathbf{e}_z \times \nabla_\perp(\frac{e\tilde{\Phi}}{T_e}) - \rho_s^2\nabla_\perp\frac{\partial}{\partial t}(\frac{e\tilde{\Phi}}{T_e}) - C_s\rho_s\mathbf{e}_z \times \nabla_\perp\frac{e\tilde{\Phi}}{T_e} - \frac{C_s\rho_s}{n_{e0}}\mathbf{e}_z \times \nabla_\perp\tilde{n}\right] \\
&= en_{e0}\left[-\rho_s^2\nabla_\perp\frac{\partial}{\partial t}(\frac{e\tilde{\Phi}}{T_e})\right] - eC_s\rho_s\mathbf{e}_z \times \nabla_\perp\tilde{n}
\end{aligned}$$

Note that the $\mathbf{E} \times \mathbf{B}$ drifts in both species cancel. This drift does not ever drive perpendicular *currents* in quasi neutral conditions, only *flows*. Now, Eq.(44) gives an expression for the fluctuating *parallel* current density:

$$\begin{aligned}
\tilde{\mathbf{j}}_z &= \mathbf{e}_z en_{e0}(\tilde{u}_{zi} - \tilde{u}_{ze}) \\
&= \mathbf{e}_z\frac{n_{e0}T_e}{m_e\nu_{ei}}\frac{\partial}{\partial z}(\frac{\tilde{n}}{n_{e0}} - \frac{e\tilde{\Phi}}{T_e})
\end{aligned}$$

We are almost done! Let us recall that "quasi-neutrality" requires that $\nabla.\tilde{\mathbf{j}} = 0$. This means, $\nabla.(\tilde{\mathbf{j}}_\perp + \tilde{\mathbf{j}}_z) = 0$. It is elementary to note that the second term in the expression for the perpendicular current is divergence-free and substitution gives Eq.(46) of the text. This completes the derivation.

It is an excellent exercise (left to the interested student!) to *repeat the above analysis of drift waves carefully* in the case when the diamagnetic variation of B_z with r is fully taken into account. Some terms we have approximated away (can you identify which?) will enter this analysis.

8. Taking ν_{ei} to be given by Braginskii's formula for $\nu_{ei} \simeq \frac{2^{1/2} n_e \Lambda e^4}{12\pi^{3/2}\epsilon_0^2 m_e^{1/2} T_e^{3/2}}$ and estimate how large $\frac{k_\perp}{k_z}$ must be to get $\omega_{\rm im} = 0.01\omega_{\rm re}$. Assume that $T_e \simeq 1$KeV, $n_e \simeq 5\times10^{19}\text{m}^{-3}$, $B \simeq 1$T, $L_n \simeq 1$m, $\rho_s k_\perp = 0.1$. These are typical "tokamak" values. Take $\Lambda = 20$.

Solution: Taking the ions to be protons, we know that $\frac{m_e}{m_i} = 5.4 \times 10^{-4}$ and $\Omega_{ci} = eB/m_i = 4.8 \times 10^{-19}/1.67 \times 10^{-27} \simeq 3 \times 10^8$rads/s, for $B = 1$T. Similarly, $C_s = (T_e/m_i)^{1/2} = (1.6 \times 10^{-16}/1.67 \times 10^{-27})^{1/2} \simeq 3.2 \times 10^5$m/s. It follows that, $\rho_s = C_s/\Omega_{ci} \simeq 10^{-3}$m. Taking $\epsilon_0 = 8.85 \times 10^{-12}$ (SI) , $\Lambda = 20$, we calculate $\nu_{ei} \simeq \frac{2^{1/2}\times5\times10^{19}\times20\times6.55\times10^{-76}}{12\times5.56\times78.3\times10^{-24}\times0.95\times10^{-15}\times2\times10^{-24}} = \frac{9.3\times10^{-55}}{9.9\times10^{-60}} \simeq 10^5\text{s}^{-1}$.

We approximate $\omega_{\rm re} \simeq (C_s/L_n)(\rho_s k_\perp) \simeq 3.2 \times 10^4$. Then,

$$\begin{aligned}\frac{\omega_{\rm im}}{\omega_{\rm re}} &\simeq (\frac{m_e}{m_i})(\frac{\omega_{\rm re}\nu_{ei}}{\omega_{ci}^2})(\frac{k_\perp}{k_z})^2 \\ &\simeq (5.4\times10^{-4})(\frac{3.2\times10^4\times10^5}{3^2\times10^{16}})(\frac{k_\perp}{k_z})^2 \\ &\simeq 1.9\times10^{-11}(\frac{k_\perp}{k_z})^2 \qquad (1)\end{aligned}$$

Thus we find that $(\frac{k_\perp}{k_z})^2 \simeq (\frac{1}{1.9}) \times 10^9$. This implies that $|\frac{k_\perp}{k_z}| \simeq 2.3 \times 10^4$! Since $\rho_s k_\perp \simeq 0.1$, $k_z \simeq 0.1/(2.3 \times 10^4 \rho_s) \simeq 4 \times 10^{-3}\text{m}^{-1}$. This is, even compared to $1/L_n$ a very long wave length! This example illustrates the sort of relatively simple estimates one can make in drift wave physics.